Output Regulation Theory for Distributed Parameter Systems with Unbounded Control and Observation

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***Abstract*— In this paper we consider the theory of robust output regulation for distributed parameter systems with infinite-dimensional exosystems. The main purpose of the paper is to extend selected results of the existing state space theory to allow plants with unbounded control and observation operators. In particular, we show that under suitable assumptions on the closed-loop system, the solvability of the output regulation problem can be characterized using the solvability of the regulator equations. The theoretic results are illustrated with an example where we consider output tracking for a heat equation with point observation.**

I. INTRODUCTION

Asymptotic output tracking and disturbance rejection — known together as the output regulation problem — have been studied for linear infinite-dimensional systems since the early 1980's [10], [8], [1]. Recently, there have been development in the theory in a situation where the reference and disturbance signals are generated using an infinite-dimensional exosystem [4], [3], [5]. This extension allows studying nonsmooth periodic and almost periodic signals in the output regulation problem. However, in the previous references the considered systems have been assumed to have bounded input and output operators. This is a limitation for the applicability of the results, since many actual control systems incorporate, for example, boundary control or measurements at a single point. The purpose of this paper is to extend some of the main results in [3], [5] to cover systems with unbounded control and observation operators. In the frequency domain the output regulation problem for such systems has been studied in [9], [12].

We consider linear distributed parameter systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \quad x(0) = x_0 \in X \tag{1a}$$

$$y(t) = Cx(t) + Du(t). \tag{1b}$$

In this paper the input and output operators may be unbounded in such a way that $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, where X_1 and X_{-1} are scale spaces related to the operator A.

In extending the output regulation theory we work under the assumption that the closed-loop system consisting of the plant and the controller (which may also have unbounded input and output operators) has a well-defined state. More precisely, we assume that the closed-loop system operator with maximal domain generates a strongly continuous semigroup. The unboundedness of the operator C requires that we slightly modify the definition of the output regulation problem, since the output of the plant (1) may no longer be well-defined for all initial states $x_0 \in X$.

As our main result we show that under the standing assumptions on the systems, the solvability of the output regulation problem can be characterized using the solvability of the *regulator equations* [1]

$$\Sigma S = A_e \Sigma + B_e \tag{2a}$$

$$0 = C_e \Sigma + D_e. \tag{2b}$$

This extends the theory presented in [3], [5].

In Section V we derive concrete conditions for the closed-loop system operator to generate a strongly continuous semigroup. For this we use the perturbation theory for semigroups under unbounded perturbations [2, Sec. III.3].

The theoretic results are illustrated in Section VI where we study a one-dimensional heat equation with point observation. We design a one-dimensional feedback controller that achieves robust output tracking of constant reference signals.

II. MATHEMATICAL PRELIMINARIES

In this section we introduce the notation and state the assumptions on the plant, the exosystem and the controller. Our main assumption is that while the input and output operators of the plant and the controller are allowed to be unbounded operators, we assume that the closed-loop system is well-defined in the sense that the closed-loop system operator (with maximal domain) generates a strongly continuous semigroup.

For Banach spaces X and Y and $A : X \to Y$, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators is $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ are the spectrum, the point spectrum and the resolvent set of A. For $\lambda \in \rho(A)$ the resolvent operator is $R(\lambda, A) = (\lambda I - A)^{-1}$. Inner products are denoted by $\langle \cdot, \cdot \rangle$.

In this paper we consider a linear system (1), where $x(t) \in X$ is the state of the system, $y(t) \in Y$ is the output, and $u(t) \in U$ the input. The space X is a Banach space, and $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$. Here $w(t) \in X$ denotes the disturbance signal to the state of the plant. We assume that $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$ on X. For a fixed $\lambda_0 > \omega_0(T(t))$ we define the scale spaces $X_1 = (\mathcal{D}(A), \|(\lambda_0 - A)\cdot\|)$ and $X_{-1} = \overline{(X, \|R(\lambda_0, A)\cdot\|)}$ (the completion of X with respect to the norm $\|R(\lambda_0, A)\cdot\|$) [2, Sec. II.5]. We assume the input and

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output operators of the plant are such that $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$, and the feedthrough operator satisfies $D \in \mathcal{L}(U, Y)$. We denote by $A_{-1} : X \subset X_{-1} \to X_{-1}$ the extension of the operator A to the the space X_{-1}.

The reference signal $y_{ref}(t)$ to be tracked and the disturbance signal $w(t)$ are generated by an exosystem

$$\dot{v}(t) = Sv(t) \qquad v(0) = v_0 \in W \tag{3a}$$
$$y_{ref}(t) = -Fv(t) \tag{3b}$$
$$w(t) = Ev(t) \tag{3c}$$

on a separable Hilbert space W (the minus sign is for notational convenience).

The block diagonal exosystem with eigenvalues $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ is constructed by first choosing the space W to be a separable Hilbert space with an orthonormal basis $\{\phi_k^l\}_{kl} := \{\, \phi_k^l \in W \mid k \in \mathbb{Z},\ l = 1, \dots, n_k \,\}$. By this we mean

$$W = \overline{\operatorname{span}}\,\{\phi_k^l\}_{kl}, \qquad \langle \phi_k^l, \phi_n^m \rangle = \begin{cases} 1 & k = n,\, l = m \\ 0 & \text{otherwise.} \end{cases}$$

The lengths $n_k \in \mathbb{N}$ of the subsequences are uniformly bounded. For given $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ the operators $S_k \in \mathcal{L}(W)$ representing n_k-dimensional Jordan blocks are defined as

$$S_k = i\omega_k \langle \cdot, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle \cdot, \phi_k^l \rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1} \right).$$

The operators S_k have the property that $(i\omega_k I - S_k)\phi_k^1 = 0$, and $(S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1}$ for all $l \in \{2, \dots, n_k\}$. The system operator S is defined as

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \quad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty \right\}.$$

The spectrum of the operator S satisfies

$$\sigma(S) = \overline{\sigma_p(S)} = \overline{\{i\omega_k\}_{k\in\mathbb{Z}}}.$$

The operator S generates a strongly continuous group $T_S(t)$ on W, and

$$T_S(t)v = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v, \phi_k^l \rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!} \phi_k^j,$$

for all $v \in W$, and $t \in \mathbb{R}$. For any $n_S \in \mathbb{N}$ such that $n_S \geq n_k$ for all $k \in \mathbb{Z}$ there exists $M_S \geq 1$ such that

$$\|T_S(t)\| \leq M_S(|t|^{n_S} + 1), \qquad \forall t \in \mathbb{R}.$$

The output operators E, and F of the exosystem are assumed to be Hilbert–Schmidt operators, i.e.,

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|E\phi_k^l\|^2 < \infty, \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

For $k \in \mathbb{Z}$ we define the orthogonal projection

$$P_k = \sum_{l=1}^{n_k} \langle \cdot, \phi_k^l \rangle \phi_k^l$$

onto the finite-dimensional subspace $\operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ of W. With this notation the domain of the operator S satisfies

$$\mathcal{D}(S) = \left\{\, v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} (1 + \omega_k^2) \|P_k v\|^2 < \infty \,\right\}.$$

We define scale spaces $W_\alpha \subset W$ related to the exosystem.

Definition 2.1: For $\alpha \geq 0$ we denote by $(W_\alpha, \|\cdot\|_\alpha)$ the space

$$W_\alpha = \left\{\, v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} (1 + \omega_k^2)^\alpha \|P_k v\|^2 < \infty \,\right\}$$

with norm $\|\cdot\|_\alpha$ defined by

$$\|v\|_\alpha^2 = \sum_{k\in\mathbb{Z}} (1 + \omega_k^2)^\alpha \|P_k v\|^2, \qquad v \in W_\alpha.$$

For all $\alpha \geq 0$ the spaces $(W_\alpha, \|\cdot\|_\alpha)$ are Hilbert spaces, and for $0 \leq \beta \leq \alpha$ we have $W_\alpha \subset W_\beta$. For nonnegative integer values $m \in \mathbb{N}_0$ the spaces W_m coincide with the domains $\mathcal{D}((S - I)^m)$ and the norms $\|\cdot\|_m$ are equivalent to the norms defined by the mappings $v \mapsto \|(S - I)^m v\|$ on W_m. The spaces W_α are invariant under the group $T_S(t)$, the restrictions $T_S(t)|_{W_\alpha}$ are strongly continuous groups on W_α and the generators of these groups are $S|_{W_\alpha} : \mathcal{D}(S|_{W_\alpha}) \subset W_\alpha \to W_\alpha$ with domains $\mathcal{D}(S|_{W_\alpha}) = W_{\alpha+1}$.

We consider an error feedback controller of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z,$$
$$u(t) = Kz(t)$$

on a Banach space Z. The operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup $T_{\mathcal{G}_1}(t)$ on Z, and the scale spaces Z_1 and Z_{-1} are defined similarly as for the plant. We assume $\mathcal{G}_2 \in \mathcal{L}(Y, Z_{-1})$ and $K \in \mathcal{L}(Z_1, U)$. The extension of $\mathcal{G}_1$ to the space Z_{-1} is denoted by $\mathcal{G}_{1,-1} : Z \subset Z_{-1} \to Z_{-1}$.

The system and the controller can be written together as a closed-loop system on the Banach space $X_e = X \times Z$. This composite system with state $x_e(t) = (x(t), z(t))^T$ can be written formally on $X_{-1} \times Z_{-1}$ as

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \quad x_e(0) = x_{e0}, \tag{4a}$$
$$e(t) = C_e x_e(t) + D_e v(t), \tag{4b}$$

where $e(t) = y(t) - y_{ref}(t)$ is the *regulation error*, $x_{e0} = (x_0, z_0)^T$, $C_e = (C,\ DK)$, $D_e = F$,

$$A_e = \begin{pmatrix} A_{-1} & BK \\ \mathcal{G}_2 C & \mathcal{G}_{1,-1} + \mathcal{G}_2 DK \end{pmatrix}, \quad B_e = \begin{pmatrix} E \\ \mathcal{G}_2 F \end{pmatrix}.$$

Due to the unboundedness of the operators B, C, $\mathcal{G}_2$, and K the domain of the operators A_e will not be $\mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1)$ as in references [3], [6]. Instead, we consider the domain

$$\mathcal{D}(A_e) = \left\{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{D}(C) \times \mathcal{D}(K) \;\middle|\; \begin{pmatrix} A_{-1}x + BKz \\ \mathcal{G}_2 Cx + (\mathcal{G}_{1,-1} + \mathcal{G}_2 DK)z \end{pmatrix} \in X \times Z \right\}.$$

This is the maximal domain such that A_e is an operator on X_e, i.e., maximal domain such that $\mathcal{R}(A_e) \subset X_e$. Also the operator C_e is unbounded with domain $\mathcal{D}(C_e) = \mathcal{D}(C) \times \mathcal{D}(K) \supset \mathcal{D}(A_e)$ and $B_e \in \mathcal{L}(W, X \times Z_{-1})$

Assumption 2.2: Throughout the paper (A, B, C, D) and $(\mathcal{G}_1, \mathcal{G}_2, K)$ are such that A_e with the given domain generates a strongly continuous semigroup $T_e(t)$ on X_e, and that C_e is relatively bounded with respect to A_e.

If $\lambda_0 \in \rho(A_e)$, then the A_e-boundedness of C_e is equivalent to the condition $C_e(A_e - \lambda_0 I)^{-1} \in \mathcal{L}(X_e, Y)$.

The results in this paper are presented using only the parameters of the closed-loop system. Because of this, they are also applicable for any type of controller for which the closed-loop system can be written in the form (4).

III. The Output Regulation Problem

The output regulation problem on W_α consists of choosing the controller parameters in such a way that the controlled system can track the reference signals and reject the disturbance signals originating from the initial states $v_0 \in W_\alpha$ of the infinite-dimensional exosystem. As shown in [6], in the case of the periodic reference and disturbance signals the choices of the initial states of the exosystem are directly related to the level of smoothness of the signals to be tracked and rejected.

The Output Regulation Problem on W_α: Let $\alpha \geq 0$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:

1) The closed-loop system operator A_e generates a strongly stable semigroup on X_e.
2) For all initial states $v_0 \in W_{\alpha+1}$ and $x_{e0} \in \mathcal{D}(A_e)$ the regulation error goes to zero asymptotically, i.e.,
$$\lim_{t\to\infty} e(t) = 0.$$

The following theorem shows that solvability of the output regulation problem can be characterized using the solvability of the regulator equations. For this result we need to assume that the operator $\mathcal{G}_2$ of the controller is bounded. The main difficulty resulting from unboundedness of this operator is that then also the operator B_e will be unbounded, and the Sylvester equation in Theorem 3.1 must be interpreted on a space larger than X_e.

Theorem 3.1: Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is such that $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, that A_e generates a strongly stable semigroup on X_e, and that the Sylvester equation $\Sigma S = A_e \Sigma + B_e$ on $W_{\alpha+1}$ has a solution $\Sigma \in \mathcal{L}(W_\alpha, X_e)$. Then the following are equivalent:

(a) The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the output regulation problem on W_α.
(b) The regulator equations
$$\Sigma S = A_e \Sigma + B_e \tag{5a}$$
$$0 = C_e \Sigma + D_e \tag{5b}$$
on $W_{\alpha+1}$ have a solution $\Sigma \in \mathcal{L}(W_\alpha, X_e)$.

For the proof of the theorem we need some auxiliary results. In particular, Theorem 3.3 shows that the state of the closed-loop system and the regulation error can be expressed using the solution Σ of the Sylvester equation (5a).

Lemma 3.2: If $1 \in \rho(A_e)$ and if $\Sigma \in \mathcal{L}(W_\alpha, X_e)$ is the solution of (5a), then $C_e\Sigma \in \mathcal{L}(W_{\alpha+1}, Y)$.

Proof: Let $v \in W_{\alpha+1}$. The Sylvester equation (5a) implies $\Sigma(S-I)v = (A_e-I)\Sigma v + B_e v$. Now $\Sigma v \in \mathcal{D}(A_e) \subset \mathcal{D}(C_e)$ and using (5a) we have
$$\begin{aligned}
&\|C_e\Sigma v\| = \|C_e(A_e - I)^{-1}(A_e - I)\Sigma v\| \\
&\quad= \|C_e(A_e - I)^{-1}(\Sigma(S - I)v - B_e v)\| \\
&\quad\leq \|C_e(A_e - I)^{-1}\|_{\mathcal{L}(X_e,Y)}\big(\|\Sigma\|_{\mathcal{L}(W_\alpha,X_e)}\|(S - I)v\|_\alpha \\
&\qquad + \|B_e\|_{\mathcal{L}(W_\alpha,X_e)}\|v\|_\alpha\big),
\end{aligned}$$
which implies that $C_e\Sigma \in \mathcal{L}(W_{\alpha+1}, Y)$. ■

Theorem 3.3: Let $\Sigma \in \mathcal{L}(W_\alpha, X_e)$ be a solution of the Sylvester equation (5a). For all $x_{e0} \in X_e$ and $v_0 \in W$ and for all $t \geq 0$ the state of the closed-loop system satisfies
$$x_e(t) = T_e(t)(x_{e0} - \Sigma v_0) + \Sigma v(t), \tag{6a}$$
and for all $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W_{\alpha+1}$ the regulation error is given by
$$e(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(t). \tag{6b}$$
If $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W_{\alpha+1}$, then the regulation error $e(t)$ is continuous and satisfies
$$\|e(t) - (C_e\Sigma + D_e)T_S(t)v_0\| \to 0$$
as $t \to \infty$.

Proof: Let $v \in W_{\alpha+1}$. Then $\Sigma v \subset \mathcal{D}(A_e)$ and for all $t > s$ we have
$$\begin{aligned}
&T_e(t - s)B_e T_S(s)v = T_e(t - s)(\Sigma S - A_e\Sigma)T_S(s)v \\
&= \ -T_e(t - s)A_e\Sigma T_S(s)v + T_e(t - s)\Sigma S T_S(s)v \\
&= \ \frac{d}{ds}\left(T_e(t - s)\Sigma T_S(s)v\right).
\end{aligned}$$
Integrating both sides of this equation from 0 to $t > 0$ gives
$$\int_0^t T_e(t - s)B_e T_S(s)v ds = \Sigma T_S(t)v - T_e(t)\Sigma v. \tag{7}$$
Since the operators on both sides are in $\mathcal{L}(W_\alpha, X_e)$ and since $W_{\alpha+1}$ is dense in W_α, equation (7) holds for all $v \in W_\alpha$ and $t > 0$.

For all $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ the mild state of the closed-loop system is given by
$$x_e(t) = T_e(t)x_{e0} + \int_0^t T_e(t - s)B_e T_S(s)v_0 ds.$$
We can now use (7) to conclude that
$$\begin{aligned}
x_e(t) &= T_e(t)x_{e0} + \Sigma T_S(t)v_0 - T_e(t)\Sigma v_0 \\
&= T_e(t)(x_{e0} - \Sigma v_0) + \Sigma T_S(t)v_0.
\end{aligned}$$
If $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W_{\alpha+1}$, then $\Sigma T_S(t)v_0 \in \mathcal{D}(A_e) \subset \mathcal{D}(C_e)$ for all $t \geq 0$ and the regulation error is given by
$$\begin{aligned}
e(t) &= C_e x_e(t) + D_e v(t) \\
&= C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)T_S(t)v_0.
\end{aligned}$$

The function $t \mapsto T_S(t)v_0 \in W_{\alpha+1}$ is continuous and by Lemma 3.2 we have $C_e\Sigma + D_e \in \mathcal{L}(W_{\alpha+1}, Y)$. Therefore $t \mapsto (C_e\Sigma + D_e)T_S(t)v_0$ is continuous. Since we have

$$\begin{aligned} &C_eT_e(t)(x_{e0} - \Sigma v_0) \\ &\quad = C_e(A_e - I)^{-1}(A_e - I)T_e(t)(x_{e0} - \Sigma v_0) \\ &\quad = C_e(A_e - I)^{-1}T_e(t)(A_e - I)(x_{e0} - \Sigma v_0), \end{aligned}$$

where $C_e(A_e - I)^{-1} \in \mathcal{L}(X_e, Y)$, we can conclude that $e(t)$ is continuous. Moreover, we have

$$\begin{aligned} &\|e(t) - (C_e\Sigma + D_e)T_S(t)v_0\| = \|C_eT_e(t)(x_{e0} - \Sigma v_0)\| \\ &\quad \le \|C_e(A_e - I)^{-1}\|\|T_e(t)(A_e - I)(x_{e0} - \Sigma v_0)\| \to 0 \end{aligned}$$

as $t \to \infty$ due to the strong stability of $T_e(t)$. ■

Lemma 3.4: Let $\tilde{X}$ be a Banach space, and let $\alpha \ge 0$. If $Q : W_\alpha \to \tilde{X}$ is such that

$$QT_S(t)v_0 \to 0$$

for all $v_0 \in W_\alpha$, then $Q = 0$.

Proof: Let $k \in \mathbb{Z}$ and $v_0 \in P_kW$ be arbitrary. Then for all $t \in \mathbb{R}$ we have $T_S(t)v_0 \in P_kW \subset W_\alpha$, and

$$QT_S(t)v_0 = e^{i\omega_k t}\sum_{l=1}^{n_k}\langle v_0, \phi_k^l\rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!}Q\phi_k^j \tag{8a}$$

$$= e^{i\omega_k t}\sum_{j=0}^{n_k-1} t^j \cdot \frac{1}{j!}\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle Q\phi_k^{l-j} \tag{8b}$$

depends continuously on t. If $QT_S(t)v_0 \to 0$, it is easy to see that we must have

$$\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle Q\phi_k^{l-j} = 0 \qquad \forall j \in \{0, \ldots, n_k - 1\}.$$

Moreover, for $j = 0$ we in particular have $0 = \sum_{l=1}^{n_k}\langle v_0, \phi_k^l\rangle Q\phi_k^l = Qv_0$. Since $k \in \mathbb{Z}$ and $v_0 \in P_kW$ were arbitrary, we can conclude that $Q\phi_k^l = 0$ for all $k \in \mathbb{Z}$ and $l \in \{1, \ldots, n_k\}$. Since $\{\, \phi_k^l \mid k \in \mathbb{Z}, l = 1, \ldots, n_k \,\}$ is a basis of W_α, this concludes $Q = 0$. ■

We can now collect the above results to prove Theorem 3.1.

Proof of Theorem 3.1 We will first show that (b) implies (a). Assume the regulator equations (5) have a solution $\Sigma \in \mathcal{L}(W_\alpha, X_e)$. Since $T_e(t)$ is strongly stable, we have from Theorem 3.3 that for all initial states $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W_{\alpha+1}$

$$\lim_{t\to\infty}\|e(t)\| = \lim_{t\to\infty}\|e(t) - (C_e\Sigma + D_e)v(t)\| = 0,$$

since $C_e\Sigma + D_e = 0$ on $W_{\alpha+1}$. Thus the controller solves the output regulation problem on W_α.

It remains to prove that (a) implies (b). Assume the controller solves the output regulation problem on W_α and $\Sigma \in \mathcal{L}(W_\alpha, X_e)$ is a solution of the Sylvester equation (5a) on $W_{\alpha+1}$. Since the regulation error decays to zero asymptotically for all initial states of the closed-loop system and the exosystem, Theorem 3.3 implies that for all $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W_{\alpha+1}$ we must have

$$\begin{aligned} &\|(C_e\Sigma + D_e)T_S(t)v_0\| \\ &\qquad \le \|(C_e\Sigma + D_e)T_S(t)v_0 - e(t)\| + \|e(t)\| \xrightarrow{t\to\infty} 0, \end{aligned}$$

and thus $\lim_{t\to\infty}(C_e\Sigma + D_e)T_S(t)v_0 = 0$ for every $v_0 \in W_{\alpha+1}$. This together with Lemma 3.4 concludes that Σ also satisfies equation (5b). ■

IV. The Robust Output Regulation Problem

In this section we define the robust output regulation problem. The problem consists of choosing a controller that solves the output regulation problem in such a way that the decay of the regulation error is robust with respect to a suitable class of perturbations of the operators of the plant.

The Robust Output Regulation Problem on W_α: Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

(a) The closed-loop system operator A_e generates a strongly stable semigroup on X;
(b) For all initial states $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W_{\alpha+1}$ the regulation error decays to zero asymptotically, i.e.

$$\lim_{t\to\infty} e(t) = 0.$$

(c) If the operators (A, B, C, D, E, F) are perturbed to (A', B', C', D', E', F') in such a way that the perturbed closed-loop system operator A'_e with maximal domain generates a strongly stable semigroup on X_e, C'_e is A'_e-bounded, and the Sylvester equation $\Sigma' S = A'_e\Sigma' + B'_e$ has a solution, then for all initial states $x_{e0} \in \mathcal{D}(A'_e)$ and $v_0 \in W_{\alpha+1}$ the regulation error satisfies $e(t) \to 0$ as $t \to \infty$.

The regulator equations can also be used to characterize the controllers solving the robust output regulation problem.

Theorem 4.1: Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ with $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ solves the output regulation problem on W_α. The controller also solves the robust output regulation problem on W_α if and only if for all perturbations for which the perturbed closed-loop system is strongly stable and the Sylvester equation $\Sigma' S = A'_e\Sigma' + B'_e$ has a solution $\Sigma' \in \mathcal{L}(W_\alpha, X_e)$, we have

$$C'_e\Sigma' + D'_e = 0 \tag{9}$$

on $W_{\alpha+1}$.

Proof: Since the controller solves the output regulation problem on W_α, it remains to verify the third part of the robust output regulation problem. This part requires that the controller solves the output regulation problem for the perturbed operators (A', B', C', D', E', F'). However, since A'_e generates a strongly stable semigroup and since $\Sigma' S = A'_e\Sigma' + B'_e$ has a solution $\Sigma' \in \mathcal{L}(W_\alpha, X_e)$, we have from Theorem 3.1 that this is true if and only if (9) is true. ■

V. Conditions for A_e Generating a Semigroup

If $x_e = (x,z)^T \in \mathcal{D}(A_e)$, we can write (on $X_{-1}\times Z_{-1}$)

$$\begin{aligned} A_e x_e &= \begin{pmatrix} A_{-1}x + BKz \\ \mathcal{G}_2 Cx + \mathcal{G}_{1,-1}z + \mathcal{G}_2 DKz \end{pmatrix} \\ &= \begin{pmatrix} A_{-1} & 0 \\ 0 & \mathcal{G}_{1,-1} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \\ &\quad + \begin{pmatrix} B & 0 \\ 0 & \mathcal{G}_2 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \\ &= (\mathcal{A}_{-1} + \mathcal{B}\mathcal{D}\mathcal{C})\, x_e. \end{aligned}$$

Here we have denoted $\mathcal{A} = \left(\begin{smallmatrix} A & 0 \\ 0 & \mathcal{G}_1 \end{smallmatrix}\right) : \mathcal{D}(A)\times\mathcal{D}(\mathcal{G}_1) \subset X\times Z \to X\times Z$. Then $X_{-1}\times Z_{-1}$ is the extended space corresponding to $\mathcal{A}$, and $\mathcal{A}_{-1}$ is the extension of $\mathcal{A}$. Our assumptions imply that we have $\mathcal{B}\in\mathcal{L}(U\times Y, X_{-1}\times Z_{-1})$, $\mathcal{C}\in\mathcal{L}(X_1\times Z_1, Y\times U)$, and $\mathcal{D}\in\mathcal{L}(Y\times U, U\times Y)$. In fact, due to the above equations we can see that for $x_e \in \mathcal{D}(C)\times\mathcal{D}(K)$ the requirement $A_e x_e \in X_e$ is equivalent to requiring $(\mathcal{A}_{-1}+\mathcal{B}\mathcal{D}\mathcal{C})x_e \in X\times Z$. Thus we have that A_e with maximal domain satisfies

$$A_e = (\mathcal{A}_{-1} + \mathcal{B}\mathcal{D}\mathcal{C})\,|_{X\times Z}$$

(the restriction of $\mathcal{A}_{-1}+\mathcal{B}\mathcal{D}\mathcal{C}$ to $X\times Z$). These observations can be used to derive conditions for the operator A_e to generate a semigroup on X_e. The following theorem states conditions in the situation where the operators C and K are bounded and where B and $\mathcal{G}_2$ are admissible control operators (in the sense of [11, Sec. 4.2]).

Theorem 5.1: Assume C and K are bounded operators. If B is $T(t)$-admissible and $\mathcal{G}_2$ is $T_{\mathcal{G}_1}(t)$-admissible, then A_e with maximal domain generates a strongly continuous semigroup on X_e.

Proof: Let $t>0$ and $(f,g)^T \in L^2([0,t], X\times Z)$. Since C, D, and K are bounded, we have $Kg(\cdot)\in L^2([0,t],U)$ and $Cf(\cdot)+DKg(\cdot)\in L^2([0,t],Y)$. By Proposition 4.2.2 in [11], the admissibility of B and $\mathcal{G}_2$ imply

$$\begin{aligned} &\int_0^t T_{\mathcal{A},-1}(t-s)\mathcal{B}\mathcal{D}\mathcal{C}\begin{pmatrix} f(s) \\ g(s) \end{pmatrix} ds \\ &\quad = \int_0^t \begin{pmatrix} T_{A,-1}(t)BKg(s) \\ T_{\mathcal{G}_1,-1}\mathcal{G}_2(Cf(s)+DKg(s)) \end{pmatrix} ds \in X\times Z. \end{aligned}$$

Corollary III.3.4 in [2] concludes that A_e generates a strongly continuous semigroup on X_e. ∎

VI. Robust Output Tracking for a 1D Heat Equation

In this section we consider robust output tracking of constant reference signals for a stable one-dimensional heat equation with point observation.

We choose $X = L^2(0,1)$,

$$Ax = \sum_{k=1}^{\infty} -k^2\pi^2 \langle x, \varphi_k\rangle_{L^2}\varphi_k(\cdot)$$

$$x\in\mathcal{D}(A) = \Big\{\, x\in X \;\Big|\; \sum_{k=1}^{\infty} k^4 |\langle x,\varphi_k\rangle_{L^2}|^2 < \infty \,\Big\},$$

where $\varphi_k(z) = \sqrt{2}\sin(2\pi k)$. With this choice of a system operator, the plant (1) becomes a one-dimensional heat equation on the interval $(0,1)$ with homogeneous Dirichlet boundary conditions. The operator A is boundedly invertible and generates an exponentially stable analytic semigroup on X. As a control operator we choose $Bu = bu$ with $b = \frac{1}{\sqrt{2}\pi}\sum_{k=1}^{\infty}\frac{1}{k}\varphi_k(\cdot)$ (which corresponds to distributed control $(Bu)(z) = (1-z)u$ on $(0,1)$).

We consider an unbounded observation operator $Cx = x(1/\sqrt{2})$ with domain $\mathcal{D}(C) = \{\, x\in X \mid x(\cdot) \text{ is cont.}\,\}$. Then $\mathcal{D}(A)\subset\mathcal{D}(C)$ and for all $x\in\mathcal{D}(A)$ we have

$$\begin{aligned} \|Cx\| &= \left|\sum_{k=1}^{\infty}\langle x,\varphi_k\rangle C\varphi_k\right| \le \sum_{k=1}^{\infty}|\langle x,\varphi_k\rangle||\varphi_k(1/\sqrt{2})| \\ &\le \sqrt{2}\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}k^2\pi^2|\langle x,\varphi_k\rangle| \\ &\le \sqrt{2}\left(\sum_{k=1}^{\infty}\frac{1}{k^4\pi^4}\right)^{\frac12}\left(\sum_{k=1}^{\infty}k^4\pi^4|\langle x,\varphi_k\rangle|^2\right)^{\frac12} \le \frac{\|Ax\|}{3\sqrt{5}}. \end{aligned}$$

This concludes that $C\in\mathcal{L}(X_1,\mathbb{C})$. Since $\mathcal{D}(A)\subset\mathcal{D}(C)$, the transfer function $P(\lambda) = CR(\lambda,A)B$ is well-defined for all $\lambda\in\rho(A)$. It can also be verified that $P(0) = -CA^{-1}B \neq 0$.

We consider tracking of constant reference signals. To this end, we choose the operators of exosystem as $W=\mathbb{C}$, $S = 0\in\mathbb{C}$, $E = 0\in X$, $F = -1\in\mathbb{C}$. Then for the initial state $v_0\in\mathbb{C}$ the reference signal generated by the exosystem is

$$y_{ref}(t) = -Fe^{St}v_0 = -(-1)\cdot 1\cdot v_0 = v_0.$$

We begin by constructing a one-dimensional feedback controller such that the closed-loop system is well-posed and the controller solves the output regulation problem. Subsequently, we use Theorem 4.1 to show that the controller also solves the robust output regulation problem.

We choose the parameters of the controller on $Z=\mathbb{C}$ in such a way that $\mathcal{G}_1 = 0\in\mathbb{C}$, $\mathcal{G}_2 = \varepsilon > 0$, and $K = -P(0)^{-1} = \frac{1}{CA^{-1}B}\in\mathbb{C}$.

Since the operators B, $\mathcal{G}_2$, and K are bounded, the maximal domain of the operator A_e is $\mathcal{D}(A_e) = \mathcal{D}(A)\times\mathbb{C}$. If we denote $H = \varepsilon CA^{-1}\in\mathcal{L}(X,\mathbb{C})$, and $T = T^{-1} = \left(\begin{smallmatrix} I & 0 \\ H & -1 \end{smallmatrix}\right)\in\mathcal{L}(X\times\mathbb{C})$, then $HBK = \varepsilon CA^{-1}B\frac{1}{CA^{-1}B} = \varepsilon$ and

$$\begin{aligned} TA_eT^{-1} &= \begin{pmatrix} I & 0 \\ H & -1 \end{pmatrix}\begin{pmatrix} A & BK \\ \varepsilon C & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ H & -1 \end{pmatrix} \\ &= \begin{pmatrix} A+BKH & -BK \\ HA-\varepsilon C + HBKH & -HBK \end{pmatrix} \\ &= \begin{pmatrix} A+\varepsilon BKCA^{-1} & -BK \\ 0 & -\varepsilon \end{pmatrix} + \varepsilon^2\begin{pmatrix} 0 & 0 \\ CA^{-1} & 0 \end{pmatrix}. \end{aligned}$$

Since the operator CA^{-1} is bounded, it is clear that TA_eT^{-1} generates a strongly continuous semigroups on X_e, and the same is therefore also true for A_e. Moreover, since $T_A(t)$ is exponentially stable, standard perturbation theory for exponentially stable semigroups can be used to deduce that there exists $\varepsilon^* > 0$ such that for all $0<\varepsilon<\varepsilon^*$ the semigroups generated by TA_eT^{-1} and A_e are exponentially stable.

We will now verify that the operator C_e is A_e-bounded. For any $x_e = (x,z)^T \in \mathcal{D}(A_e) = \mathcal{D}(A) \times \mathbb{C}$ we have

$$\|C_e x_e\|^2 = \|Cx + DKz\|^2 = \|Cx\|^2 \\ \leq \|Ax + BKz\|^2 + \frac{1}{\varepsilon^2}\|\varepsilon Cx\|^2 \leq \max\{1, \frac{1}{\varepsilon^2}\}\|A_e x_e\|^2.$$

This concludes C_e is relatively bounded with respect to A_e.

To show that the controller solves the output regulation problem, it is now sufficient to show that the regulator equations (5) have a solution. To this end, choose $\Gamma = -(P(0)K)^{-1}F = -1 \in \mathbb{C} = \mathcal{L}(W,Z)$, $\Pi = -A^{-1}BK\Gamma \in X = \mathcal{L}(W,X)$, and $\Sigma = (\Pi,\Gamma)^T \in \mathcal{L}(W,X_e)$. Then for all $v \in W$ we have $\Sigma S v = \Sigma \cdot 0 \cdot v = 0 \in X_e$, and

$$A_e\Sigma v + B_e v = \begin{pmatrix} A\Pi v + BK\Gamma v \\ \mathcal{G}_2\Pi v + (\mathcal{G}_1 + \mathcal{G}_2 DK)\Gamma v \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{G}_2 F v \end{pmatrix} \\ = \begin{pmatrix} -AA^{-1}BK\Gamma v + BK\Gamma v \\ -\mathcal{G}_2 CA^{-1}BK\Gamma v \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{G}_2 F v \end{pmatrix} \\ = \begin{pmatrix} 0 \\ \mathcal{G}_2(P(0)K\Gamma v + Fv) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{G}_2 \cdot 0 \end{pmatrix} = 0.$$

This shows that Σ is a solution of the Sylvester equation $\Sigma S = A_e\Sigma + B_e$. Moreover, we have

$$C_e\Sigma v + D_e v = C\Pi v + DK\Gamma v + Fv = P(0)K\Gamma v + Fv \\ = -Fv + Fv = 0,$$

which concludes that Σ solves the regulator equations (5).

We will now show that the controller solves the robust output regulation problem. Since the original closed-loop system is exponentially stable, it is natural to consider perturbations for which also the perturbed closed-loop system is exponentially stable.

Assume that (A,B,C,D,E,F) are perturbed to (A',B',C',D',E',F') in such a way that A_e' generates an exponentially stable semigroup. Under this assumption the Sylvester equation $\Sigma' S = A_e'\Sigma' + B_e'$ has a unique bounded solution [7]. The solution is of the form $\Sigma' = (\Pi',\Gamma')^T$, and for all $v \in W$ we have that the Sylvester equation is equivalent to

$$0 = \begin{pmatrix} A'_{-1}\Pi' v + B'K\Gamma' v \\ \mathcal{G}_2 C'\Pi' v + \mathcal{G}_1\Gamma' v + \mathcal{G}_2 D'K\Gamma' v \end{pmatrix} + \begin{pmatrix} E'v \\ \mathcal{G}_2 F' v \end{pmatrix} \\ = \begin{pmatrix} A'_{-1}\Pi' v + B'K\Gamma' v + E'v \\ \varepsilon(C'\Pi' v + D'K\Gamma' v + F'v) \end{pmatrix}.$$

In particular, the second line requires that $C\Pi' v + D'K'\Gamma' v + F'v = 0$ for all $v \in W$. However, this immediately implies

$$C_e'\Sigma' v + D_e' v = C'\Pi' v + D'K\Gamma' v + F'v = 0$$

for all $v \in W$, and thus the solution Σ' of the Sylvester equation satisfies $C_e'\Sigma' + D_e' = 0$. Since the perturbations were arbitrary, Theorem 4.1 concludes that the controller solves the robust output regulation problem.

VII. Conclusions

In this paper we have studied the output regulation problem for infinite-dimensional systems with unbounded control and observation operators. As our main result we have shown that under suitable assumptions the solvability of the output regulation problem can be characterized using the solvability of the regulator equations. For this result we needed the input operator $\mathcal{G}_2$ of the controller to be bounded. In the case where $\mathcal{G}_2 \in \mathcal{L}(Y, Z_{-1})$ also the operator B_e would become unbounded, and consequently the Sylvester equation in the regulator equations would have to be considered on a space larger than X_e. An alternative approach would be to instead consider the weak form

$$\langle \Sigma S v, \varphi \rangle = \langle \Sigma v, A_e^* \varphi \rangle + \langle B_e v, \varphi \rangle$$

of the equation, where $v \in W_{\alpha+1}$ and $\varphi \in \mathcal{D}(A_e^*)$.

The results presented in this paper provide a basis for extending other results in [3], [5] for systems with unbounded control and observation operators. Most notably, one of the main objectives for future research is the extension of the *internal model principle* [5], which characterizes the controllers solving the robust output regulation problem.

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