Feedforward Output Regulation for Distributed Parameter Systems with Infinite-Dimensional Exosystems

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***Abstract*— In this paper we study the asymptotic output tracking for distributed parameter systems with general infinite-dimensional exosystems. We present conditions for the solvability of the problem and construct the appropriate open loop control law using the states of the system and the exosystem. In particular the results do not assume the exosystem to have a block diagonal structure. As an example we consider asymptotic output tracking for a heat equation.**

I. Introduction

In this paper we consider the problem of asymptotic output tracking for linear infinite-dimensional systems [9], [1], [3], [5]. In particular we are interested in solving the problem in a situation where the plant is exponentially stabilizable, the reference signal is generated with a very general infinite-dimensional exosystem, and the states of both the plant and the exosystem are available for feedback. Motivation for considering infinite-dimensional exosystems arises from applications such as control of robot arms, disk drive systems, and magnetic power supplies for proton synchrotrons [10, and references therein], where the goal is to track nonsmooth functions with high accuracy. Generating such reference signals is not possible with finite-dimensional exosystems.

The general problem formulation states that for a linear distributed parameter system $\mathcal{P}$ with output $y(t)$ and for a given reference signal $y_{ref}(t)$ we are to design a control law $\mathcal{C}$ producing a control signal $u(t)$ in such a way that

$$\lim_{t\to\infty} \|y(t) - y_{ref}(t)\| = 0.$$

The reference signal $y_{ref}(t)$ is obtained as an output of another linear system called the *exosystem* $\mathcal{S}$

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \tag{1a}$$
$$y_{ref}(t) = Fv(t). \tag{1b}$$

The open loop control scheme is depicted in Figure 1.

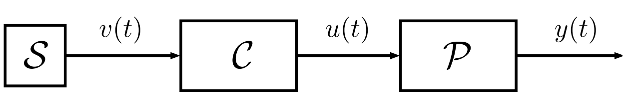

Fig. 1. The open loop control scheme

Most of the results concerning asymptotic output tracking for distributed parameter systems require that the exosystem is a finite-dimensional linear system and S and F are matrices with appropriate dimensions. Recently in [4], [3], [5] the theory has been extended to a situation where also the exosystem (1) is allowed to be an infinite-dimensional system on a Hilbert space W. Using such exosystems allows us to consider asymptotic tracking of a larger class of signals, e.g., nonsmooth periodic signals and *almost periodic* functions. In the previous references the system operator S of the considered infinite-dimensional exosystem has been of some particular form, e.g., a diagonal [4], [3] or a block diagonal [5], [7] operator. In this paper we consider the asymptotic output tracking without assuming any such structure, but instead only assuming that the generated reference signals do not decay to zero as $t \to \infty$. The main benefits of not making assumptions on the structure of the exosystem are that the results on the solvability of the output tracking problem are simplified and the presentation of the theory for different types of exosystems can be unified. More general conditions on the exosystems also gives us hope to further enlarge the classes of reference signals for asymptotic output tracking, and also to be able to distinguish the appropriate classes of signals and exosystems from the less suitable ones.

As our main results we present conditions under which the asymptotic output tracking problem can be solved using an open loop control law employing the states of the system to be controlled and the exosystem. We also derive expressions for the parameters in the control law. The presented results generalize the conditions familiar from the output tracking of linear systems with finite-dimensional and infinite-dimensional block diagonal exosystems.

It has been shown in [3], [5], [7] that when considering *error feedback control* of distributed parameter systems with a block diagonal infinite-dimensional exosystem, it is possible to choose the controller in such a way that the control structure is robust with respect to pertubrations preserving the stability of the closed-loop system. However, the internal model principle of control theory [2], [5] implies that any such controller must contain a copy of the dynamics of the exosystem. In particular, it was also observed in [3], [7] that one of the main difficulties in the construction of robust error feedback controllers is that this copy of the exosystem's dynamics must be appropriately stabilized in order to achieve output tracking. Whereas in the previous references it was shown that in the case of diagonal and some block diagonal exosystems there are ways of achieving strong closed-loop stability, this is not in general possible if the exosystem does not have such special structure.

Our main motivation for considering open loop control

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instead of error feedback control is that the control scheme is extremely simple, and the design does not require knowledge on how to stabilize the dynamics of the exosystem. The drawback of open loop control, on the other hand, is that the resulting control structure will not be robust with respect to perturbations in the parameters of the system.

In addition to considering the asymptotic output tracking problem for general infinite-dimensional exosystem, we introduce more easily verifiable conditions for the solvability of the problem for an infinite-dimensional block diagonal exosystem. The conditions relate in a very concrete way the high frequency behavior of the transfer function of the stabilized plant and the smoothness of the considered reference signals.

As an example we consider asymptotic output tracking for a stable heat equation together with diagonal exosystem whose eigenmodes consist of the rational points on the interval $[-i, i] \subset i\mathbb{R}$. Such an exosystem is, in particular, capable of producing any sinusoidal reference signal whose frequency is a rational number on the interval $[-1, 1]$, whereas for a finite-dimensional exosystem the generated signals may only contain components of prespecified fixed frequencies. In applications, such an exosystem would correspond to a situation where, e.g., the frequencies of disturbance signals to the heat equation are not known accurately.

II. Mathematical Preliminaries

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle\cdot,\cdot\rangle$.

In this paper we consider a linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where $x(t) \in X$ is the state of the system, $y(t) \in Y$ is the output, and $u(t) \in U$ the input. The spaces X, U and Y are general Banach spaces. We assume that $A : \mathcal{D}(A) \subset X \to X$ generates a C_0-semigroup on X and the other operators are bounded, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$. The transfer function of the plant is given by $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$ for all $\lambda \in \rho(A)$. For the solvability of the output tracking problem we assume that $i\mathbb{R} \subset \rho(A)$ and that the transfer function $P(\lambda)$ is boundedly invertible for all $\lambda \in i\mathbb{R}$.

The reference signal $y_{ref}(t)$ to be tracked is generated using a possibly infinite-dimensional exosystem

$$\dot{v}(t) = Sv(t) \qquad v(0) = v_0 \in W \tag{2a}$$
$$y_{ref}(t) = Fv(t) \tag{2b}$$

on a Banach space W. We assume S generates a strongly continuous group $T_S(t)$ on W and that $F \in \mathcal{L}(W, Y)$. We assume the group $T_S(t)$ is polynomially bounded, i.e., there exist $N \in \mathbb{N}$ and $M \geq 1$ such that $\|T_S(t)\| \leq M(1 + |t|^N)$ for all $t \in \mathbb{R}$. This implies that the growth bound of the group is zero, i.e., $\omega_0(T_S(t)) = 0$, and that $\sigma(S) \subset i\mathbb{R}$.

To be able to use the results presented in [6] make the following assumption on the exosystem. The content of this nondecay condition is that regardless of the choice of the operator F, the exosystem (2) may not generate reference signals that decay to zero asymptotically.

Assumption 2.1: The exosystem is such that for all $Q \in \mathcal{L}(W, Y)$ and all $v_0 \in W$ we have

$$QT_S(t)v_0 \overset{t\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad QT_S(t_0)v_0 = 0 \ \ \forall t_0 \geq 0. \tag{3}$$

As a control law we consider a state feedback (K, L) of the form

$$u(t) = Kx(t) + Lv(t),$$

where $K \in \mathcal{L}(X, U)$ and $L \in \mathcal{L}(W, U)$.

The closed-loop system consisting of the plant and the controller is a linear system on X described by equations

$$\dot{x}(t) = A_e x(t) + B_e v(t), \qquad x(0) = x_0 \in X \tag{4a}$$
$$e(t) = C_e x(t) + D_e v(t), \tag{4b}$$

where $A_e = A + BK$, $B_e = BL$, $C_e = C + DK$, and $D_e = DL - F$.

III. The Output Regulation Problem

The main control problem studied in this paper is stated as follows.

The Output Tracking Problem Choose the parameters (L, K) of the state feedback law in such a way that the following are satisfied:

(a) The closed-loop system operator A_e generates a strongly stable semigroup on X;
(b) For all initial states $v_0 \in W$ and $x_0 \in X$ the output of the plant asymptotically tracks the reference signal $y_{ref}(t)$, i.e.,

$$\lim_{t\to\infty} \|y(t) - y_{ref}(t)\| = 0.$$

It is well-known that in the case of finite-dimensional and certain infinite-dimensional exosystems the solvability of the output tracking problem can be characterized using the solvability of the so-called *regulator equations* [1], [3], [5]. It was shown in [6] that if the exosystem satisfies Assumption 2.1, we can use the following theorem.

Theorem 3.1: Assume the exosystem satisfies Assumption 2.1, the state feedback control law is such that $A_e = A + BK$ generates a strongly stable C_0-semigroup on X and that the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e \tag{5}$$

has a solution $\Sigma \in \mathcal{L}(W, X)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$. Then the following are equivalent:

(a) The control law (K, L) solves the output tracking problem.

(b) The solution Σ of the Sylvester equation (5) satisfies

$$C_e\Sigma + D_e = 0. \tag{6}$$

Together the operators equations (5) and (6) are called the regulator equations.

Before constructing the control law to solve the asymptotic output tracking problem we will further illustrate the nondecay condition in Assumption 2.1. We will do this by showing that it is not in general sufficient that the group generated by S is unstable or even isometric. In the following example the group $T_S(t)$ is completely unstable, i.e., $T_S(t)v_0 \to 0$ as $t \to \infty$ if and only if $v_0 = 0$, but for all reference signals generated by the exosystem we have $y_{ref}(t) = FT_S(t)v_0 \to 0$ as $t \to \infty$.

Counterexample 3.2: Let $W = L^2(a,b)$ for some $a < b$ and let $S \in \mathcal{L}(W)$ be a multiplication operator defined by $(Sv)(\xi) = i\xi v(\xi)$ for all $v \in W$. It is easy to see that $\sigma(S) = \sigma_c(S) = [ia, ib] \subset i\mathbb{R}$ and that S is skew-adjoint. The operator S generates a multiplication group $T_S(t)$ defined by $(T_S(t)v)(\xi) = e^{i\xi t}v(\xi)$ on W and this group is isometric, since

$$\|T_S(t)v\|_{L^2}^2 = \int_a^b |e^{i\xi t}v(\xi)|^2 d\xi = \int_a^b |v(\xi)|^2 d\xi = \|v\|_{L^2}^2$$

for all $v \in W$.

Let $Y = \mathbb{C}$ and $Q \in \mathcal{L}(W, \mathbb{C})$. By the Riesz Representation Theorem there exists $w \in W$ such that

$$Qv = \int_a^b v(\xi)w(\xi)d\xi, \qquad \forall v \in W.$$

For any initial state $v_0 \in W$ we then have

$$QT_S(t)v_0 = \int_a^b e^{i\xi t}v_0(\xi)w(\xi)d\xi \to 0$$

as $t \to \infty$ due to the Riemann-Lebesgue Lemma, since $v_0(\cdot)w(\cdot) \in L^1(a,b)$. This concludes that this exosystem does not satisfy the nondecay condition in Assumption 2.1.

IV. Solution of the Output Regulation Problem

Theorem 4.1: Assume the pair (A, B) is exponentially stabilizable and that the constrained Sylvester equation

$$\Pi S = A\Pi + B\Gamma \tag{7a}$$
$$0 = C\Pi + D\Gamma - F \tag{7b}$$

has a solution $\Pi \in \mathcal{L}(X, W)$ with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, and $\Gamma \in \mathcal{L}(W, U)$. The output tracking problem is solved by a state feedback law with parameters $K \in \mathcal{L}(X, U)$ and $L \in \mathcal{L}(W, U)$, where K is chosen in such a way that the operator $A+BK$ generates an exponentially stable C_0-semigroup and $L = \Gamma - K\Pi$. The state feedback law is thus given by

$$u(t) = Kx(t) + (\Gamma - K\Pi)v(t).$$

Proof: Since $A_e = A + BK$, we have from the choice of the operator $K \in \mathcal{L}(X, U)$ that the closed-loop system is exponentially stable. We have from [8, Cor. 8] that since S generates a group with growth bound $\omega_0(T_S(t)) = 0$, the Sylvester equation (5) has a unique solution $\Sigma \in \mathcal{L}(W, X)$. Therefore we have from Theorem 3.1 that the state feedback law solves the output tracking problem if this solution satisfies the regulation constraint (6).

We will first show that Π is the unique solution of (5). If we write $L = \Gamma - K\Pi$, we have that for all $v \in \mathcal{D}(S)$ we have

$$\begin{aligned}(A_e\Pi + B_e)v &= (A + BK)\Pi v + BLv \\ &= A\Pi v + BK\Pi v + B(\Gamma - K\Pi)v \\ &= A\Pi v + B\Gamma v = \Pi S v\end{aligned}$$

since $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and Π satisfies (7a). Since $v \in \mathcal{D}(S)$ was arbitrary, we have $\Pi S = A_e\Pi + B_e$. Since the solution of this equation is unique, we must have $\Sigma = \Pi$.

Using (7b) we can now see that

$$\begin{aligned}&C_e\Sigma + D_e = (C + DK)\Pi + DL - F \\ = \; &C\Pi + D(K\Pi + L) - F = C\Pi + D\Gamma - F = 0.\end{aligned}$$

As stated above, Theorem 3.1 now implies that the state feedback law $u(t) = Kx(t) + (\Gamma - K\Pi)v(t)$ solves the output tracking problem. ■

V. Output Regulation for Block Diagonal Exosystems

In this section present conditions for asymptotic output tracking in the case where the system operator of the exosystem consists of finite-dimensional Jordan blocks. Such exosystems have been studied in [5], [7]. It should be noted that we make very few assumptions, e.g., on the eigenvalues of exosystem. In the next section we will solve an example for an exosystem whose eigenvalues are the rational points on the interval $[-i, i] \subset i\mathbb{R}$.

The block diagonal exosystem with eigenvalues $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ is constructed by first choosing the state space W of the exosystem in such a way that it is a separable Hilbert space with an orthonormal basis

$$\{\phi_k^l\}_{kl} := \{\, \phi_k^l \in W \mid k \in \mathbb{Z},\ l = 1, \ldots, n_k \,\}.$$

In other words, we have

$$W = \overline{\text{span}}\,\{\phi_k^l\}_{kl}, \qquad \langle \phi_k^l, \phi_n^m \rangle = \begin{cases} 1 & k = n,\, l = m \\ 0 & \text{otherwise.}\end{cases}$$

Here the lengths $n_k \in \mathbb{N}$ of the subsequences are uniformly bounded. For given $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ the operators $S_k \in \mathcal{L}(W)$ representing the finite-dimensional Jordan blocks are defined as

$$S_k = i\omega_k \langle \cdot, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle \cdot, \phi_k^l \rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1} \right).$$

The operators S_k have the property that $(i\omega_k I - S_k)\phi_k^1 = 0$, and $(S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1}$ for all $l \in \{2, \ldots, n_k\}$, and thus they represent single Jordan blocks of dimensions n_k associated to eigenvalues $i\omega_k$. Finally, the system operator S of the exosystem is chosen as

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \quad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty \right\}.$$

It is straightforward to show that the spectrum of the operator S satisfies

$$\sigma(S) = \overline{\sigma_p(S)} = \overline{\{i\omega_k\}_{k\in\mathbb{Z}}}.$$

The operator S generates a C_0-group $T_S(t)$ on W, and

$$T_S(t)v = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v, \phi_k^l\rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!}\phi_k^j,$$

for all $v \in W$, and $t \in \mathbb{R}$. For any $n_S \in \mathbb{N}$ such that $n_S \geq n_k$ for all $k \in \mathbb{Z}$ there exists $M_S \geq 1$ such that

$$\|T_S(t)\| \leq M_S(|t|^{n_S} + 1), \qquad \forall t \in \mathbb{R}.$$

This also implies that the growth bound of the C_0-group is $\omega_0(T_S(t)) = 0$.

We will first show that this exosystem satisfies Assumption 2.1.

Lemma 5.1: The block diagonal exosystem satisfies the nondecay condition in Assumption 2.1.

Proof: Since the spaces $W_k := \operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ are $T_S(t)$-invariant for all $k \in \mathbb{Z}$, we can consider $v_0 \in W_k$ separately for $k \in \mathbb{Z}$. For any $Q \in \mathcal{L}(W, Y)$ and for all $k \in \mathbb{Z}$ and $v_0 \in W_k$ we have

$$QT_S(t)v_0 = e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v_0, \phi_k^l\rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!} Q\phi_k^j \tag{8a}$$

$$= e^{i\omega_k t} \sum_{j=0}^{n_k-1} t^j \cdot \frac{1}{j!} \sum_{l=j+1}^{n_k} \langle v_0, \phi_k^l\rangle Q\phi_k^{l-j} \tag{8b}$$

If $QT_S(t)v_0 \to 0$, it is easy to see that we must have

$$\sum_{l=j+1}^{n_k} \langle v_0, \phi_k^l\rangle Q\phi_k^{l-j} = 0 \qquad \forall j \in \{0, \ldots, n_k - 1\}.$$

However, by (8) this also implies $QT_S(t_0)v_0 = 0$ for all $t_0 \in \mathbb{R}$. ■

We can now turn to conditions for existence of a state feedback law solving the asymptotic output tracking problem. In order to state the conditions, we denote by $P_K(\lambda) = (C + DK)R(\lambda, A + BK)B + D \in \mathcal{L}(U, Y)$ for $\lambda \in \rho(A + BK)$ the transfer function of the plant stabilized with a feedback $u = Kx + \tilde{u}$. We then have that $P_K(i\omega_k)$ is invertible for all $k \in \mathbb{Z}$ because of the assumption on invertibility of $P(i\omega_k)$, and the fact that this property is preserved under state feedback. Furthermore, the differentiability properties of the resolvent operator show that for all $n \in \mathbb{N}$ we have

$$P_K^{(n)}(\lambda) = \frac{d^n}{d\lambda^n} P_K(\lambda) = \frac{d^n}{d\lambda^n}\left[(C+DK)R(\lambda, A+BK)B+D\right]$$
$$= (-1)^n n!(C+DK)R(\lambda, A+BK)^{n+1}B.$$

Theorem 5.2: Let $K \in \mathcal{L}(U, X)$ be chosen in such a way that $A+BK$ generates an exponentially stable semigroup. If $P_K^{(l)}(i\omega_k) \in \mathcal{L}(U, Y)$ are boundedly invertible for all $k \in \mathbb{Z}$ and $l \in \{1, \ldots, n_k - 1\}$ and if $F \in \mathcal{L}(W, Y)$ is such that $(F\phi_k^l)_{k,l} \in \ell^2(Y)$ and

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|P_K(i\omega_k)^{-1}\|^{2(n_k-l)} \|P_K(i\omega_k)^{-1}F\phi_k^l\|^2 < \infty, \tag{9}$$

then there exists a state feedback law solving the output tracking problem for the block diagonal exosystem.

Proof: For brevity, we denote $P_k = P_K(i\omega_k)$ and $P_k^{(j)} = P_K^{(j)}(i\omega_k)$. We will first need to show that the constrained Sylvester equation (7) has a solution $\Pi \in \mathcal{L}(W, X)$ and $\Gamma \in \mathcal{L}(W, U)$.

If we denote $\Gamma = K\Pi + L$ with $L \in \mathcal{L}(W, U)$, we can rewrite (7) as

$$\Pi S = (A + BK)\Pi + BL \tag{10a}$$

$$0 = (C + DK)\Pi + DL - F. \tag{10b}$$

Since $\omega_0(T_S(t)) = 0$ and since $A + BK$ generates an exponentially stable semigroup, we have from [8, Cor. 8] that for any $L \in \mathcal{L}(W, U)$ the equation (10a) has a unique solution $\Pi \in \mathcal{L}(W, X)$.

Let $k \in \mathbb{Z}$. If we apply both sides of (10a) to $\phi_k^1, \phi_k^2, \ldots, \phi_k^{n_k}$, we obtain

$$BL\phi_k^1 = (i\omega_k I - A - BK)\Pi\phi_k^1,$$
$$BL\phi_k^2 = (i\omega_k I - A - BK)\Pi\phi_k^2 + \Pi\phi_k^1$$
$$\vdots$$
$$BL\phi_k^{n_k} = (i\omega_k I - A - BK)\Pi\phi_k^{n_k} + \Pi\phi_k^{n_k-1}$$

Since $i\omega_k \in \rho(A + BK)$, a direct computation yields

$$\Pi\phi_k^l = \sum_{j=1}^{l} (-1)^{l-j} R(i\omega_k, A + BK)^{l+1-j} BL\phi_k^j \tag{11}$$

for all $l \in \{1, \ldots, n_k\}$. Applying both sides of (10b) to ϕ_k^l and substituting the above expression we obtain

$$F\phi_k^l = DL\phi_k^l + \sum_{j=1}^{l} (-1)^{l-j}(C+DK)R(i\omega_k, A+BK)^{l+1-j}BL\phi_k^j$$

$$= \sum_{j=1}^{l} \frac{1}{(l-j)!} P_k^{(l-j)} L\phi_k^j.$$

Collecting the equations for $l \in \{1, \ldots, n_k\}$ we obtain a triangular system of equations for $(L\phi_k^l)_{l=1}^{n_k}$

$$\begin{pmatrix} \frac{1}{0!}P_k & & & \\ \frac{1}{1!}P_k^{(1)} & \frac{1}{0!}P_k & & \\ \vdots & & \ddots & \\ \frac{1}{(n_k-1)!}P_k^{(n_k-1)} & \cdots & \frac{1}{1!}P_k^{(1)} & \frac{1}{0!}P_k \end{pmatrix} \begin{pmatrix} L\phi_k^1 \\ L\phi_k^2 \\ \vdots \\ L\phi_k^{n_k} \end{pmatrix} = \begin{pmatrix} F\phi_k^1 \\ F\phi_k^2 \\ \vdots \\ F\phi_k^{n_k} \end{pmatrix}.$$

Since by assumptions the operators $P_K^{(l)}(i\omega_k) \in \mathcal{L}(U, Y)$ are invertible for all $l \in \{0, \ldots, n_k\}$, this system can be solved by forward substitution. The general form of the solution itself becomes very complicated, but in order to prove our result we only need to verify that the operator L defined in this manner is bounded. In order to do this, we will estimate the norms of the terms $L\phi_k^l$. This can be accomplised fairly easily.

We will first note that since $A + BK$ generates an exponentially stable semigroup, the operator norm of the resolvent operator $R(i\omega, A + BK)$ and its powers are uniformly bounded for $\omega \in \mathbb{R}$. This also implies that for all

$l \in \{1,\dots,n_k-1\}$ the norms $\|P_K^{(l)}(i\omega)\|$ are uniformly bounded with respect to $\omega \in \mathbb{R}$. We will first show that there exist constants $M_l \geq 1$ independent of $k \in \mathbb{Z}$ such that

$$\|L\phi_k^l\| \leq M_l \sum_{j=1}^{l} \max\{1, \|P_k^{-1}\|^{l-j}\} \|P_k^{-1} F \phi_k^j\|, \qquad (12)$$

for all $l \in \{1,\dots,n_k\}$. We can show this using induction. Choose $\tilde{M} \geq 1$ and $N \in \mathbb{N}$ such that $n_k \leq N$ for all $k \in \mathbb{Z}$ and

$$\sup_{\omega\in\mathbb{R}} \left\| P_K^{(l)}(i\omega) \right\| \leq \tilde{M}, \qquad \forall l \in \{1,\dots,n_k-1\}.$$

Let $k \in \mathbb{Z}$. For $l = 1$ we have $\|L\phi_k^1\| = \|P_k^{-1}F\phi_k^1\|$ and thus (12) is satisfied with $M_1 = 1$, independent of k. On the other hand, if (12) is satisfied for all $l \in \{1,\dots,m-1\}$, then

$$\begin{aligned}
&\|L\phi_k^m\| = \left\| P_k^{-1} \left(F\phi_k^m - \sum_{j=1}^{m-1} \frac{1}{j!} P_k^{(j)} L\phi_k^{m-j} \right) \right\| \\
&\leq \|P_k^{-1}F\phi_k^m\| + \|P_k^{-1}\| \sum_{j=1}^{m-1} \frac{1}{(m-j)!} \|P_k^{(m-j)}\| \|L\phi_k^j\| \\
&\leq \|P_k^{-1}F\phi_k^m\| \\
&\quad + \tilde{M}\|P_k^{-1}\| \sum_{j=1}^{m-1} M_n \sum_{n=1}^{j} \max\{1, \|P_k^{-1}\|^{j-n}\} \|P_k^{-1}F\phi_k^n\| \\
&= \|P_k^{-1}F\phi_k^m\| \\
&\quad + \tilde{M}\|P_k^{-1}\| \sum_{n=1}^{m-1} \|P_k^{-1}F\phi_k^n\| \sum_{j=n}^{m-1} M_j \max\{1, \|P_k^{-1}\|^{j-n}\} \\
&\leq \|P_k^{-1}F\phi_k^m\| \\
&\quad + \tilde{M}N \max_{1\leq j\leq m-1}\{M_j\} \sum_{n=1}^{m-1} \|P_k^{-1}F\phi_k^n\| \max\{1, \|P_k^{-1}\|^{m-n}\} \\
&\leq \tilde{M}N \max_{1\leq j\leq m-1}\{M_j\} \sum_{n=1}^{m} \max\{1, \|P_k^{-1}\|^{m-n}\} \|P_k^{-1}F\phi_k^n\|.
\end{aligned}$$

Since the constant $M_m := \tilde{M}N \max_{1\leq j\leq m-1}\{M_j\} \geq 1$ is independent of k, this concludes the proof. If we denote $M = \max\{M_1,\dots,M_N\}$, then for any $k \in \mathbb{Z}$ such that $\|P_k^{-1}\| \leq 1$ we have

$$\sum_{l=1}^{n_k} \|L\phi_k^l\|^2 \leq M^2 \sum_{l=1}^{n_k} \left(\sum_{j=1}^{l} \|F\phi_k^j\| \right)^2 \leq M^2N^2 \sum_{l=1}^{n_k} \|F\phi_k^l\|^2$$

and if $\|P_k^{-1}\| > 1$, then a direct computation shows that

$$\begin{aligned}
\sum_{l=1}^{n_k} \|L\phi_k^l\|^2 &\leq M^2N \sum_{l=1}^{n_k} \sum_{j=1}^{l} \left(\|P_k^{-1}\|^{l-j} \|P_k^{-1}F\phi_k^j\| \right)^2 \\
&\leq M^2N \sum_{l=1}^{n_k} \sum_{j=1}^{n_k} \left(\|P_k^{-1}\|^{n_k-j} \|P_k^{-1}F\phi_k^j\| \right)^2 \\
&\leq M^2N^2 \sum_{j=1}^{n_k} \|P_k^{-1}\|^{2(n_k-j)} \|P_k^{-1}F\phi_k^j\|^2
\end{aligned}$$

These two estimates together with (9) and $(F\phi_k)_{k,l} \in \ell^2(Y)$ conclude that we have

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|L\phi_k^l\|^2 < \infty,$$

and thus $L \in \mathcal{L}(W,U)$. The expression for $L\phi_k^l$ can now be substituted back into the expression (11) for $\Pi\phi_k^l$, and the appropriate resolvent identities can be used to further verify that we indeed have $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$. ■

For diagonal exosystems, i.e., if $n_k = 1$ for all $k \in \mathbb{Z}$, the condition (9) for the existence of the state feedback control law becomes

$$\sum_{k\in\mathbb{Z}} \|P_K(i\omega_k)^{-1}F\phi_k\|^2 < \infty,$$

which is in particular always satisfied if $\dim Y < \infty$ (then $(F\phi_k)_{k\in\mathbb{Z}} \in \ell^2(Y)$) and $\sup_{k\in\mathbb{Z}} \|P_K(i\omega_k)^{-1}\| < \infty$. In this case the formulas for the solution operators Π and L are also easier to write out, and we have

$$\begin{aligned}
Lv &= \sum_{k\in\mathbb{Z}} \langle v, \phi_k^1 \rangle P_K(i\omega_k)^{-1} F\phi_k^1, \\
\Pi v &= \sum_{k\in\mathbb{Z}} \langle v, \phi_k^1 \rangle R(i\omega_k, A+BK) B P_K(i\omega_k)^{-1} F\phi_k^1,
\end{aligned}$$

for all $v \in W$.

VI. Asymptotic Output Tracking for a Stable Heat Equation

In this section we consider output tracking of an exponentially stable heat equation

$$\frac{dw}{dt}(z,t) = \frac{d^2w}{dz^2}(z,t) + \chi_{[\frac{1}{2},1]}(z)u(t)$$

with Dirichlet boundary conditions $w(0,t) = w(1,t) = 0$ and initial state $w(z,0) = w_0(z) \in L^2(0,1)$ along with a measurement

$$y(t) = \int_0^{\frac{1}{2}} w(z,t)dz.$$

The controlled heat equation can be written as a linear system on $X = L^2(0,1)$ if we choose

$$Ax = x'', \quad x \in \mathcal{D}(A) = \left\{ x \in L^2(0,1) \;\middle|\; x, x' \text{ abs. cont.}, \; x'' \in L^2(0,1) \right\},$$

and

$$Bu = b(\cdot)u, \qquad Cx = \int_0^{\frac{1}{2}} x(z)dz.$$

As in [3], for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\pi^2$ the resolvent operator $R(s,A)$ has the form

$$\begin{aligned}
R(\lambda,A)x(z) &= \int_0^z \frac{\sinh(\xi\sqrt{\lambda})\sinh\left((1-z)\sqrt{\lambda}\right)}{\sqrt{\lambda}\sinh(\sqrt{\lambda})} x(\xi)d\xi \\
&\quad + \int_z^1 \frac{\sinh(z\sqrt{\lambda})\sinh\left((1-\xi)\sqrt{\lambda}\right)}{\sqrt{\lambda}\sinh(\sqrt{\lambda})} x(\xi)d\xi,
\end{aligned}$$

and the transfer function of the plant is given by

$$P(\lambda) = CR(\lambda, A)B = \frac{4\sinh^4(\sqrt{\lambda}/4)}{\lambda\sqrt{\lambda}\sinh(\sqrt{\lambda})}.$$

This function has a removable singularity at $\lambda = 0$, and $P(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$.

The operator A generates an exponentially stable semigroup, and we can therefore choose $K = 0$ in the feedback. Therefore we also have $P_K(\lambda) = P(\lambda)$ for all $\lambda \in \rho(A)$.

Let $\omega_0 = 0$, let $(\omega_k)_{k=1}^{\infty}$ be the rational points on the interval $(0, 1]$ (appropriately enumerated), and let $\omega_{-k} = -\omega_k$ for $k \in \mathbb{N}$. As the system operator of the exosystem we choose

$$S = \sum_{k\in\mathbb{Z}} i\omega_k \langle \cdot, \phi_k\rangle \phi_k \in \mathcal{L}(W)$$

on the space $W = \ell^2(\mathbb{C})$, where $\phi_k = (\delta_{kl})_{l\in\mathbb{Z}}$ are the unit vectors. Then S generates a bounded group $T_S(t)$ on $W = \overline{\operatorname{span}}\{\phi_k\}_{k\in\mathbb{Z}}$ and by Theorem 5.1 it satisfies the nondecay assumption. As the output operator we can consider any bounded linear functional $F \in \mathcal{L}(W, \mathbb{C})$. We remark that with a proper choice of F, our exosystem is in particular capable of generating any sinusoidal signal with a rational frequency $\omega \in [-1, 1]$. This is in contrast with a situation of a finite-dimensional exosystem, whose signals may only contain components with prespecified fixed frequencies. Indeed, if we choose $F \in \mathcal{L}(W, \mathbb{C})$ in such a way that

$$F\phi_k = \frac{1}{k}, \qquad k \in \mathbb{Z},$$

then for any $\omega \in [-1, 1] \cap \mathbb{Q}$ there exists $n \in \mathbb{Z}$ in such a way that $\omega_n = \omega$, and the reference signal $y_{ref}(t) = ce^{i\omega t}$ can be generated by choosing the initial state v_0 of the exosystem as $v_0 = cn\phi_n$. Indeed, we then have

$$\begin{aligned} y_{ref}(t) &= FT_S(t)v_0 = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0, \phi_k\rangle F\phi_k \\ &= e^{i\omega_n t} cn \langle \phi_n, \phi_n\rangle \frac{1}{n} = ce^{i\omega t}. \end{aligned}$$

The condition (9) for the existence of a state feedback law solving the asymptotic output tracking problem for the heat equation and the diagonal exosystem is now satisfied automatically, since

$$\sum_{k\in\mathbb{Z}} \|P_K(i\omega_k)^{-1}F\phi_k\|^2 \leq \sup_{\omega\in[-1,1]} |P(i\omega)^{-1}|^2 \sum_{k\in\mathbb{Z}} |F\phi_k|^2 < \infty.$$

The state feedback solving the asymptotic output tracking problem is given by

$$\begin{aligned} u(t) &= Kx(t) + (L - K\Pi)v(t) = Lv(t) \\ &= LT_S(t)v_0 = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0, \phi_k\rangle L\phi_k \\ &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0, \phi_k\rangle P_K(i\omega_k)^{-1}F\phi_k \\ &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0, \phi_k\rangle \frac{i\omega_k\sqrt{i\omega_k}\sinh(\sqrt{i\omega_k})}{4\sinh^4(\sqrt{i\omega_k}/4)} F\phi_k. \end{aligned}$$

VII. Conclusions

In this paper we have studied the theory of asymptotic output tracking for distributed parameter systems in the case where the exosystem itself is a very general infinite-dimensional linear system. In particular we have considered the conditions for existence of an open loop control law solving the problem of output tracking for such systems and exosystems.

As was mentioned in the introduction, the asymptotic output tracking problem can be solved using open loop control even without knowledge on how to stabilize the dynamics of the exosystem. This is no longer possible when considering error feedback control, because any controller of this type must itself contain a copy of the dynamics of the exosystem. However, the solutions of these two control problems are otherwise very similar. In fact, the state feedback used in the open loop control problem is also present when using error feedback control, but in the latter case the states $x(t)$ and $v(t)$ of the system and the exosystem are replaced with estimates produced by an asymptotic observer. In this way the solution to the asymptotic output tracking problem with open loop control also gives us new insights on how to design error feedback controllers solving the problem, provided we can find a way to stabilize the dynamics of the exosystem. As already mentioned, such controllers have the highly desirable property that the asymptotic output tracking is robust with respect to perturbations in the parameters of system.

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