Robust Output Regulation and the Preservation of Polynomial Closed-Loop Stability

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Abstract

In this paper we study the robust output regulation problem for distributed parameter systems with infinite-dimensional exosystems. The main purpose of this paper is to demonstrate the several advantages of using a controller that achieves polynomial closed-loop stability, instead of a one stabilizing the closed-loop system strongly. In particular, the most serious unresolved issue related to strongly stabilizing controllers is that they do not possess any known robustness properties. In this paper we apply recent results on the robustness of polynomial stability of semigroups to show that, on the other hand, many controllers achieving polynomial closed-loop stability are robust with respect to large and easily identifiable classes of perturbations to the parameters of the plant. We construct an observer based feedback controller that stabilizes the closed-loop system polynomially and solves the robust output regulation problem. Subsequently, we derive concrete conditions for finite rank perturbations of the plant's parameters to preserve the closed-loop stability and the output regulation property. The theoretic results are illustrated with an example where we consider the problem of robust output tracking for a one-dimensional heat equation.

KEY WORDS: Robust output regulation, distributed parameter system, polynomial stability, internal model principle

1 Introduction

Robust tracking of reference signals and rejection of disturbance signals — known together as robust output regulation — for a linear system form a fundamental problem in engineering, as well as an actively studied theoretical problem for mathematical control systems. For infinite-dimensional linear systems the theory of robust output regulation has been developed since the early 1980's, see [18, 20, 3, 19, 7, 11] and references therein. More recently, there has also been interest in considering robust output tracking and disturbance rejection of signals generated by an infinite-dimensional exosystem [9, 8, 14, 15]. Such a setting allows considering tracking and rejection signals that are nonsmooth periodic or almost periodic functions. In particular, the well-known internal model principle by Francis and Wonham [6], and Davison [5] was extended for linear systems with infinite-dimensional exosystems in [14]. Although the internal model principle provides a conclusive answer concerning certain properties of a robust controller, there is freedom in stabilizing the closed-loop system consisting of the plant and the controller. Because of this, designing actual control laws for infinite-dimensional exosystems is still in many ways an open research problem. This paper is devoted to addressing some of the most important open questions, namely, choosing and achieving an appropriate type of closed-loop stability, and studying the robustness properties of the resulting control law.

The infinite-dimensional systems considered in this paper are of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + w_s(t), \qquad x(0) = x_0 \in X \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where the state $x(t)$ belongs to a Hilbert space X. Our aim is to design a feedback controller that is capable of steering the output $y(t)$ of the plant to given nonsmooth periodic and almost periodic reference

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trajectories, despite disturbance signals $w_s(t)$ of the same type. In order to study this class of signals, we assume the reference signals $y_{ref}(t)$ and disturbance signals are generated by an infinite-dimensional exosystem

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{2a}$$
$$w_s(t) = Ev(t) \tag{2b}$$
$$y_{ref}(t) = -Fv(t), \tag{2c}$$

where the state $v(t)$ belongs to a separable Hilbert space W (the minus sign is for notational convenience). The operator S is allowed to be a diagonal or, more generally, a block diagonal operator with eigenvalues on the imaginary axis. For example, if S is chosen to be a diagonal operator with eigenvalues $i\frac{2\pi k}{\tau}$ for $k \in \mathbb{Z}$, then the exosystem (2) will be capable of generating the class of continuous τ-periodic signals [15].

The robust output regulation requires choosing the parameters of a feedback controller in such a way that

- The closed-loop system consisting of the plant and the controller is stable and the output asymptotically tracks the reference signal.
- If the operators (A, B, C, D) of the plant (1) are perturbed in such a way that the closed-loop stability is preserved, the tracking of the reference signal is preserved.

One of the main difficulties in studying output regulation with an infinite-dimensional signal generator is that exponential stabilization of the closed-loop system is no longer possible. This follows from the fact that by the internal model principle, a controller achieving robust output regulation must be able to reduplicate the dynamics of the exosystem. In particular, the system operator of the controller must therefore have an infinite number of eigenvalues on the imaginary axis. These unstable eigenvalues will inevitably make the exponential stabilization of the closed-loop an impossible task. Due to this limitation, the previous papers considering infinite-dimensional exosystems redefine the robust output regulation problem in such a way that the closed-loop semigroup is required to be strongly stable [8, 15]. After this small modification, the robust output regulation problem becomes solvable with a suitable choice of an observer based dynamic error feedback controller. However, there are two serious downsides to using strong closed-loop stability. First of all, the existing theory must be modified by including an additional assumption on the solvability of a Sylvester equation

$$\Sigma S = A_e \Sigma + B_e \tag{3}$$

where A_e and B_e are operators of the closed-loop system. For an exponentially stable closed-loop system this equation will always have a solution, but this is no longer true for strongly stable closed-loop systems. The solvability of (3) can be characterized using the properties of the closed-loop system, but this leads to conditions that are often difficult to verify [8, Lem. 6]. Moreover, same condition on the solvability of (3) must also be verified for any perturbations we wish to consider.

The second drawback is that the robustness properties of a strongly stable closed-loop are very difficult to study. This is due to the fact that general strongly stable semigroups do not have any known robustness properties. The situation is in contrast with the case of exponential stability, which is preserved in particular for all bounded perturbations of small enough norms. This difference between the two stability types has a concrete effect on robust output regulation, since the second part of the problem requires determining classes of perturbations that preserve the closed-loop stability.

In this paper we show that we can overcome the above difficulties by modifying the robust output regulation problem in such a way that we aim at *polynomial stability* of the closed-loop system. First of all, we will show that the polynomial stability will immediately imply the solvability of the Sylvester equation (3) in an appropriate sense, and we can therefore remove the extraneous assumptions from the theory. However, the real main advantage of polynomial stability over strong stability is that recent results have shown polynomially stable semigroups to possess certain robustness properties [16, 12, 13]. In this paper we use the robustness results in [13] to characterize classes of finite rank perturbations of the operators (A, B, C, D) under which the polynomial stability of the closed-loop system and the output tracking are preserved. The perturbation results require conditions on the degree of polynomial stability of the closed-loop system. Nevertheless, these conditions can be met for large classes of systems, and are usually easy to verify. For example, the results in Section 4.1 show that for a stable single-input single-output system together with a diagonal exosystem the highest achievable degree of polynomial closed-loop stability can be determined based on the behavior of the transfer function of the plant on the imaginary axis. As we pointed out earlier, analyzing the robustness properties of a controller with an infinite-dimensional internal model has until now been impossible due to lack of results on preservation of nonexponential stability types.

As another main result of this paper we show that an observer based controller of the same form as the one used in [8, 15] can be used in polynomial stabilizing the closed-loop system. The construction of the controller generalizes techniques from [8], where the exosystem was diagonal, and from [15], where only the single-input single-output situation was studied. We begin by showing that the proposed controller stabilizes the closed-loop polynomially provided that the copy of the exosystem's dynamics — the internal model — in the controller is stabilized polynomially. In particular, this first result does not depend on the way the dynamics of the exosystem are reduplicated in the controller. This is convenient, since we will also see that the choice of the operators of the internal model can have a crucial effect on its stabilization. For systems with multiple outputs the internal model principle implies that the internal model will have an infinite number of repeated eigenvalues on the imaginary axis. Polynomial stabilization of such semigroups is in general an open research problem. We complete construction of the controller by presenting a method for the polynomial stabilization of the internal model in the case where the exosystem is diagonal and the plant has an equal number of inputs and outputs.

Finally, as another desirable consequence of the polynomial stability we show that the stability of the closed-loop system also immediately yields a polynomial decay rate for the regulation error. This result extends the results in [2], where such a decay rate for the regulation error was presented in connection to output regulation with a feedforward control law.

The construction of a controller solving the robust output regulation problem, as well as the conditions for the preservation of the closed-loop stability, are illustrated in an example where we consider output tracking for a stable one-dimensional heat equation. We design controllers for the robust output regulation problem for two different infinite-dimensional exosystems, and derive concrete conditions for the preservation of the closed-loop stability and output tracking under rank one perturbations of the plant's parameters.

In Section 2 we introduce the standing assumptions on the plant, the controller, the resulting closed-loop system, and the classes of perturbations we consider. In Section 3 we formulate the robust output regulation problem mathematically, and show that the polynomial closed-loop stability implies a polynomial decay rate for the regulation error. In Section 4 we introduce the form of the observer based controller and choose its parameters in such a way that the closed-loop system is stabilized polynomially. Section 5 contains the analysis for the preservation of the stability of the closed-loop system. In Section 6 we study the robust output regulation problem for a one-dimensional heat equation. Section 7 contains concluding remarks. In Appendix A we collect some helpful lemmata used in proving the main results of the paper.

2 Mathematical Preliminaries

In this section we state the basic assumptions on the system, the exosystem and the controller. We begin by introducing the notation used in this paper.

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. If $f(\cdot) : \mathbb{R} \to \mathbb{C}$ and $\alpha > 0$, we denote $f(\omega) = \mathcal{O}(|\omega|^\alpha)$ if there exist constants $M > 0$ and and $\omega_0 > 0$ such that

$$|f(\omega)| \leq M|\omega|^\alpha$$

for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$. The following definition introduces the terminology we use when discussing the polynomial stability of a semigroup. It should be noted that the definition we use requires a polynomially stable semigroup to be uniformly bounded. This immediately implies that every polynomially stable semigroup is also strongly stable.

Definition 1. *Let $\alpha > 0$. A semigroup $T(t)$ on a Hilbert space X generated by $A : \mathcal{D}(A) \subset X \to X$ is* polynomially stable with α, *if $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$, and if there exists $M \geq 1$ such that*

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

We use the following characterizations for the polynomial stability of a semigroup on a Hilbert space [1, Lem. 2.3, Thm. 2.4], [10, Lem. 3.2].

Lemma 2. *Assume A generates a uniformly bounded semigroup on a Hilbert space X, and $i\mathbb{R} \subset \rho(A)$. For a fixed $\alpha > 0$ the following are equivalent.*

$$\text{(a)} \quad \|T_A(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0$$

$$\text{(b)} \quad \|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha).$$

2.1 The Plant and the Infinite-Dimensional Exosystem

In this paper we consider the control of a linear distributed parameter system of form (1) on a Hilbert space X. Here $x(t) \in X$ is the state of the system, $u(t) \in U$ the input and $y(t) \in Y$ the output. The input space U and the output space Y are finite-dimensional Hilbert spaces. We assume that A generates a strongly continuous semigroup on X and that the rest of the operators are bounded in such a way that $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$. For $\lambda \in \rho(A)$ the transfer function of the plant is given by $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$.

The considered reference signals as well as the disturbance signals $w_s(t)$ to the state are assumed to be generated by an infinite-dimensional exosystem (the minus sign is for notational convenience)

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{4a}$$

$$w_s(t) = Ev(t), \tag{4b}$$

$$y_{ref}(t) = -Fv(t). \tag{4c}$$

The Hilbert space W and the operators $S : \mathcal{D}(S) \subset W \to W$, $E \in \mathcal{L}(W, X)$, and $F \in \mathcal{L}(W, Y)$ satisfy the assumptions stated below. In particular, in the following we choose the system operator S to be an infinite-dimensional block diagonal operator consisting of finite-dimensional Jordan blocks.

The state space W of the exosystem is chosen to be a separable Hilbert space with an orthonormal basis $\{\, \phi_k^l \in W \mid k \in \mathbb{Z},\ l = 1, \ldots, n_k \,\}$. By this we mean that

$$W = \overline{\text{span}} \left\{ \phi_k^l \right\}_{kl} \quad \text{and} \quad \langle \phi_k^l, \phi_n^m \rangle = \begin{cases} 1 & k = n,\, l = m \\ 0 & \text{otherwise.} \end{cases}$$

The lengths $n_k \in \mathbb{N}$ of the subsequences are assumed to be uniformly bounded. For a given ordered sequence of frequencies $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ the operators $S_k \in \mathcal{L}(W)$ representing the finite-dimensional Jordan blocks are defined as

$$S_k = i\omega_k \langle \cdot, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle \cdot, \phi_k^l \rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1} \right).$$

We assume the frequencies $\{i\omega_k\}_{k\in\mathbb{Z}}$ have no finite accumulation points and that $\omega_k \neq \omega_l$ for $k \neq l$. The system operator S of the infinite-dimensional exosystem (4) on the space W is defined by

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \qquad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty \right\}$$

and the output operators E and F are assumed to be Hilbert–Schmidt operators, i.e. they satisfy

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|E\phi_k^l\|^2 < \infty, \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

Defining the regulation error as $e(t) = y(t) - y_{ref}(t)$, the plant can be written in a standard form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t), \quad x(0) = x_0 \in X \\ e(t) &= Cx(t) + Du(t) + Fv(t) \end{aligned}$$

where $v(t) \in W$ is the state of the exosystem (4). We further assume that $\sigma(A) \cap \sigma(S) = \varnothing$ and that the transfer function of the plant at the frequencies $i\omega_k$ of the exosystem, $P(i\omega_k)$, is surjective for all $k \in \mathbb{Z}$.

The operators S_k in the definition of the infinite-dimensional exosystem satisfy

$$(i\omega_k I - S_k)\phi_k^1 = 0, \qquad (S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1} \quad \forall l \in \{2, \dots, n_k\}$$

and thus they can indeed be viewed as single Jordan blocks of dimensions n_k associated to eigenvalues $i\omega_k$. Since the operator S is an infinite block diagonal operator consisting of operators the S_k, it can be considered to be a generalization of a matrix in a Jordan canonical form. Since $\{i\omega_k\}_k$ has no finite accumulation points, it is straightforward to verify that the spectrum of the operator S satisfies

$$\sigma(S) = \sigma_p(S) = \{i\omega_k\}_{k\in\mathbb{Z}},$$

Moreover, the operator S generates a C_0-group $T_S(t)$ satisfying

$$T_S(t)v = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v, \phi_k^l\rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!}\phi_k^j, \qquad v \in W,\ t \in \mathbb{R}.$$

This group is polynomially bounded forward and backwards in time. More precisely, for any $n \in \mathbb{N}$ such that $n_k \leq n$ for all $k \in \mathbb{Z}$ there exists $M_S \geq 1$ such that

$$\|T_S(t)\| \leq M_S(|t|^n + 1), \qquad \forall t \in \mathbb{R}.$$

This implies that the growth bound of the C_0-group is $\omega_0(T_S(t)) = 0$. We define $n_\infty \in \mathbb{N}$ as

$$n_\infty = \lim_{N\to\infty} \big(\max\{\, n_k \mid |k| \geq N \,\}\big)$$

(or equivalently $n_\infty = \limsup_{k\to\infty} n_k$). The value n_∞ can be thought of as the asymptotic maximum of the dimension of the Jordan blocks of S. For $k \in \mathbb{Z}$ we denote by P_k the orthogonal projection

$$P_k = \sum_{l=1}^{n_k} \langle \cdot, \phi_k^l\rangle \phi_k^l$$

onto the finite-dimensional subspace $\operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ of W. With this notation the domain of the operator S satisfies

$$\mathcal{D}(S) = \Big\{\, v \in W \;\Big|\; \sum_{k\in\mathbb{Z}} \omega_k^2 \|P_k v\|^2 < \infty \,\Big\} = \Big\{\, v \in W \;\Big|\; \sum_{k\in\mathbb{Z}} (1+\omega_k^2)\|P_k v\|^2 < \infty \,\Big\}.$$

We also define a set of scale spaces $W_\alpha \subset W$ related to the system operator S of the exosystem.

Definition 3. *For $\alpha \geq 0$ we denote by $(W_\alpha, \|\cdot\|_\alpha)$ the space*

$$W_\alpha = \Big\{\, v \in W \;\Big|\; \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v\|^2 < \infty \,\Big\}$$

with norm $\|\cdot\|_\alpha$ defined by

$$\|v\|_\alpha^2 = \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v\|^2, \qquad v \in W_\alpha.$$

For all $\alpha \geq 0$ the spaces $(W_\alpha, \|\cdot\|_\alpha)$ are Hilbert spaces, and for $0 \leq \beta \leq \alpha$ we have $W_\alpha \subset W_\beta$ and

$$\|v\|_\beta \leq \|v\|_\alpha \qquad \forall v \in W_\alpha.$$

For nonnegative integer values $m \in \mathbb{N}_0$ the spaces W_m coincide with the domains $\mathcal{D}((S+I)^m)$ and the norms $\|\cdot\|_m$ are equivalent to the norms defined by the mappings $v \mapsto \|(S+I)^m v\|$ on W_m. It can also be verified that the spaces W_α are invariant under the group $T_S(t)$, the restrictions $T_S(t)|_{W_\alpha}$ are strongly continuous groups on W_α and the generators of these groups are $S|_{W_\alpha} : \mathcal{D}(S|_{W_\alpha}) \subset W_\alpha \to W_\alpha$ with domains $\mathcal{D}(S|_{W_\alpha}) = W_{\alpha+1}$.

2.2 The Controller and the Closed-Loop System

We consider the dynamic error feedback controller

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z$$
$$u(t) = Kz(t)$$

on a Hilbert space Z. Here $z(t) \in Z$ is the state of the controller, $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$. The closed-loop system consisting of the plant and the controller on the Hilbert space $X_e = X \times Z$ (with norm $\|(x, z)^T\|^2 = \|x\|^2 + \|z\|^2$) with state $x_e(t) = (x(t), z(t))^T$ is given by

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \quad x_e(0) = x_{e0} = (x_0, z_0)^T$$
$$e(t) = C_e x_e(t) + D_e v(t),$$

where $C_e = (C \;\; DK)$, $D_e = F$,

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} \quad \text{and} \quad B_e = \begin{pmatrix} E \\ \mathcal{G}_2 F \end{pmatrix}.$$

The operator $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ generates a C_0-semigroup $T_{A_e}(t)$ on X_e. Furthermore, since E and F are Hilbert-Schmidt operators and $\mathcal{G}_2$ is bounded, it is easy to see that also B_e is Hilbert-Schmidt, i.e. $(B_e \phi_k^l)_{kl} \in \ell^2(X_e)$.

2.3 Classes of Perturbations

In this paper we consider a situation where parameters of the plant are perturbed in such a way that the operators A, B, C, D, E, and F are changed into $A' : \mathcal{D}(A') \subset X \to X$, $B' \in \mathcal{L}(U, X)$, $C' \in \mathcal{L}(X, Y)$, and $D' \in \mathcal{L}(U, Y)$, $E' \in \mathcal{L}(W, X)$, and $F' \in \mathcal{L}(W, Y)$, respectively. We assume the operator A' generates a semigroup on X and the difference $A' - A$ is relatively bounded with respect to A, i.e. $\mathcal{D}(A') \supset \mathcal{D}(A)$ and there exist constants $c_1, c_2 \geq 0$ such that

$$\|(A' - A)x\| \leq c_1 \|Ax\| + c_2 \|x\|, \qquad \forall x \in \mathcal{D}(A).$$

Finally, we assume that E' and F' are Hilbert-Schmidt operators. We denote the operators of the closed-loop system consisting of the perturbed plant and the controller by

$$A_e' = \begin{pmatrix} A' & B'K \\ \mathcal{G}_2 C' & \mathcal{G}_1 + \mathcal{G}_2 D'K \end{pmatrix}, \qquad B_e' = \begin{pmatrix} E' \\ \mathcal{G}_2 F' \end{pmatrix},$$

$C_e' = \begin{pmatrix} C' & D'K \end{pmatrix}$, and $D_e' = F'$. The operator A_e' generates a strongly continuous semigroup and it is straightforward to verify that $(B_e' \phi_k^l)_{kl} \in \ell^2(X_e)$.

3 The Robust Output Regulation Problem

We begin this section by presenting a more precise mathematical definition for the robust output regulation problem. It should be noted that the problem studied in this paper differs from the one considered in [8, 14]. In particular, we are required to choose the parameters of the controller in such a way that the closed-loop system is polynomially stable.

The statement of the problem depends on a parameter $\alpha > 0$, the value of which affects two things. First of all, we require that the closed-loop system is polynomially stable with α. Moreover, in the second and third parts we consider regulation for initial states of the exosystem that are in a scale space of S. It was shown in [15, Sec. 2] that for periodic exogeneous signals the choice of the initial state v_0 corresponds to the degree of smoothness of the generated signals. In the problem statement also the asymptotic Jordan block structure of S affects the initial states of the exosystem for which the controller is required to achieve output tracking and disturbance rejection. If all but a finite number of the Jordan blocks of S are trivial, we then have $n_\infty = 1$ and the problem is solvable for all initial states $v_0 \in W_\alpha$.

The Robust Output Regulation Problem. *Let $\alpha > 0$ and denote $\alpha_0 = n_\infty \alpha$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:*

(1) *The semigroup $T_e(t)$ generated by the closed-loop system operator A_e is polynomially stable with* α.

(2) *For all initial states $v_0 \in W_{\alpha_0}$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e.,* $\lim_{t\to\infty} e(t) = 0$.

(3) *If the parameters (A, B, C, D, E, F) are perturbed to (A', B', C', D', E', F') in such a way that the new closed-loop system (A'_e, B'_e, C'_e, D'_e) is polynomially stable with α, then we have* $\lim_{t\to\infty} e(t) = 0$ *for all initial states $v_0 \in W_{\alpha_0}$ and $x_{e0} \in X_e$.*

We saw in Section 2 that a higher value of α in polynomial stability means that the decay is slower. Therefore, if the value of $\alpha > 0$ in the robust output regulation problem is increased, it means that (1) the closed-loop stability becomes weaker and that (2) the class of reference and disturbance signals we can consider becomes smaller.

In this paper we use the following definition for an internal model of an infinite-dimensional exosystem in the controller [14].

Definition 4 (The $\mathcal{G}$-conditions)**.** *A controller $(\mathcal{G}_1, \mathcal{G}_2)$ is said to satisfy the* $\mathcal{G}$-conditions *related to the infinite-dimensional exosystem if*

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \forall k \in \mathbb{Z}, \tag{6a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{6b}$$

and

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1) \qquad \forall k \in \mathbb{Z}. \tag{6c}$$

The following theorem shows that a controller that stabilizes the closed-loop system polynomially and satisfies the $\mathcal{G}$-conditions solves the robust output regulation problem. It should be noted that unlike in [8, 14], we do not need additional assumptions to ensure the solvability of the Sylvester equation in the regulator equations, but instead this is already guaranteed by the polynomial stability of the closed-loop system.

Theorem 5. *Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions and it stabilizes the closed-loop system polynomially with some $\alpha > 0$. Then the controller solves the robust output regulation problem.*

In particular, for any perturbations of the parameters of the plant there exists $M' > 0$ such that

$$\|e(t)\| \le \frac{M'}{t^{1/\alpha}} \left(\|A_e x_{e0}\| + \|x_{e0}\| + \|S v_0\| + \|v_0\| \right) \qquad \forall t > 0. \tag{7}$$

for all initial states satisfying $x_{e0} \in \mathcal{D}(A'_e)$ and $v_0 \in W_{\alpha_0 + 1}$, where $\alpha_0 = n_\infty \alpha$.

Proof. It is sufficient to show that the regulation error decays to zero asymptotically for the nominal system as well as for all systems where the operators have been perturbed in such a way that the closed-loop system is polynomially stable with α, and that for all perturbations we get the polynomial decay rate (7).

Let A', B', C', D', E' and F' be the perturbed operators of the system. We denote the closed-loop system operators related to the perturbed system as A'_e, B'_e, C'_e, and D'_e. We begin by showing that the *regulator equations*

$$\Sigma'_e S = A'_e \Sigma'_e + B'_e \tag{8a}$$

$$0 = C'_e \Sigma'_e + D'_e \tag{8b}$$

have a solution $\Sigma'_e \in \mathcal{L}(W_{\alpha_0}, X_e)$ satisfying $\Sigma'_e(W_{\alpha_0+1}) \subset \mathcal{D}(A'_e)$. For this, we will use results in Sections 3 and 4 in [14]. In [14] the scale spaces W_{α_0} were only defined for $\alpha_0 = m \in \mathbb{N}_0$. However, the choice of the space only affects the conditions for the operator Σ'_e solving the Sylvester equation (8a) to be in $\mathcal{L}(W_{\alpha_0}, X_e)$, and the results in [14] can be used as they are once we replace $m \in \mathbb{N}_0$ with $\alpha_0 \ge 0$.

We begin by verifying the conditions in Assumption 1 in [14]. Since $\sigma(A'_e) \cap \sigma_p(S) = \varnothing$, we have $\mathcal{R}(i\omega_k - A'_e)^l = X_e$ for all $l \in \{1, \ldots, n_k\}$. Let $N \in \mathbb{N}$ be such that $n_k \le n_\infty$ and $|\omega_k| \ge 1$ whenever $|k| \ge N$. Such a choice is always possible due to the definition of n_∞ and because $\{\omega_k\}$ has no finite accumulation points. Then for any $k \in \mathbb{Z}$ with $|k| \geq N$ and for all $x_e' \in X_e$ with $\|x_e'\| \leq 1$ we have

$$\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R(i\omega_k, A_e')^{l+1-j}B_e'\phi_k^j, x_e'\rangle\right| \leq \sum_{j=1}^{l}\|R(i\omega_k, A_e')\|^{l+1-j}\|B_e'\phi_k^j\| \cdot \|x_e'\|$$
$$\leq \max\{\|R(i\omega_k, A_e')\|, \|R(i\omega_k, A_e')\|^{n_\infty}\} \cdot \sum_{j=1}^{n_k}\|B_e'\phi_k^j\|$$
$$\leq \max\{1, \|R(i\omega_k, A_e')\|^{n_\infty}\} \cdot \sqrt{n_\infty} \cdot \left(\sum_{j=1}^{n_k}\|B_e'\phi_k^j\|^2\right)^{\frac{1}{2}}.$$

Since the closed-loop system is polynomially stable with α, we have from Lemma 2 that there exists $\tilde{M} > 0$ such that $\|R(i\omega_k, A_e')\| \leq \tilde{M}(1 + |\omega_k|^\alpha)$. Using Lemma 18 we thus have for $\alpha_0 = n_\infty\alpha$

$$\sup_{|k|\geq N}\frac{\|R(i\omega_k, A_e')\|^{2n_\infty}}{(1+\omega_k^2)^{\alpha_0}} \leq \sup_{|k|\geq N}\frac{\tilde{M}^{2n_\infty}(1+|\omega_k|^\alpha)^{2n_\infty}}{(1+\omega_k^2)^{n_\infty\alpha}}$$
$$\leq \sup_{|k|\geq N}\frac{\tilde{M}^{2n_\infty}(2|\omega_k|^\alpha)^{2n_\infty}}{(\omega_k^2)^{n_\infty\alpha}} = (2\tilde{M})^{2n_\infty} < \infty.$$

This implies that there exists $M \geq 0$ such that

$$\frac{\max\{1, \|R(i\omega_k, A_e')\|^{2n_\infty}\}}{(1+\omega_k^2)^{\alpha_0}} \leq M$$

for all $k \in \mathbb{Z}$ with $|k| \geq N$. Since by assumption the operator B_e' satisfies $(B_e'\phi_k^l)_{kl} \in \ell^2(X_e)$, we have

$$\sup_{\|x_e'\|\leq 1}\sum_{|k|\geq N}\frac{1}{(1+\omega_k^2)^{\alpha_0}}\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R(i\omega_k, A_e')^{l+1-j}B_e'\phi_k^j, x_e'\rangle\right|^2$$
$$\leq \sum_{|k|\geq N}\frac{\max\{1, \|R(i\omega_k, A_e')\|^{2n_\infty}\}}{(1+\omega_k^2)^{\alpha_0}}\sum_{l=1}^{n_k}n_\infty\sum_{j=1}^{n_k}\|B_e'\phi_k^j\|^2 \leq Mn_\infty^2\sum_{k\in\mathbb{Z}}\sum_{j=1}^{n_k}\|B_e'\phi_k^j\|^2$$

and

$$\sup_{\|x_e'\|\leq 1}\sum_{k\in\mathbb{Z}}\frac{1}{(1+\omega_k^2)^{\alpha_0}}\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R(i\omega_k, A_e')^{l+1-j}B_e'\phi_k^j, x_e'\rangle\right|^2$$
$$\leq \sup_{\|x_e'\|\leq 1}\sum_{|k|<N}\frac{1}{(1+\omega_k^2)^{\alpha_0}}\sum_{l=1}^{n_k}\left(\sum_{j=1}^{l}\|R(i\omega_k, A_e')^{l+1-j}B_e'\phi_k^j\|\|x_e'\|\right)^2$$
$$+ Mn_\infty^2\sum_{k\in\mathbb{Z}}\sum_{j=1}^{n_k}\|B_e'\phi_k^j\|^2 < \infty$$

because the sum over $k \in \mathbb{Z}$ satisfying $|k| < N$ consists of finitely many terms. The above condition together with Lemma 3.2 in [14] implies that the Sylvester equation (8a) has a unique solution $\Sigma_e' \in \mathcal{L}(W_{\alpha_0}, X_e)$ satisfying $\Sigma_e'(W_{\alpha_0+1}) \subset \mathcal{D}(A_e')$.

Since the Sylvester equation (8a) has a solution Σ_e', the controller satisfies the $\mathcal{G}$-conditions, and $\sigma(S) \cap \sigma(A_e) = \varnothing$, we have from Theorem 4.3, Theorem 5.2, and Lemma 5.7 in [14] that $C_e'\Sigma_e' + D_e' = 0$, and thus Σ_e' is a solution of the regulator equations (8). By Lemma 3.3 in [14] for initial states $x_{e0} \in X_e$ and $v_0 \in W_{\alpha_0}$ the regulation error is given by

$$e(t) = C_e'T_e(t)(x_{e0} - \Sigma_e'v_0) + (C_e'\Sigma_e' + D_e')T_S(t)v_0 = C_e'T_e(t)(x_{e0} - \Sigma_e'v_0). \tag{9}$$

Since $T_e(t)$ is strongly stable, we immediately get

$$\|e(t)\| \leq \|C_e'\|\|T_e(t)(x_{e0} - \Sigma_e'v_0)\| \to 0$$

as $t \to \infty$.

It remains to prove the polynomial decay rate for the regulation error. This can be done by extending techniques used in [2], where a similar decay rate was shown for feedforward output regulation. We begin by proving a couple of helpful estimates. Since the difference $A' - A$ is relatively bounded, there exist $\tilde{c}_1, \tilde{c}_2 \geq 0$ such that $\|(A'-A)x\| \leq \tilde{c}_1\|Ax\| + \tilde{c}_2\|x\|$ for all $x \in \mathcal{D}(A)$.

We will first show that the relative boundedness of $A' - A$ also implies that $A'_e - A_e$ is relatively bounded with respect to A_e. Since $\mathcal{D}(A'_e) = \mathcal{D}(A') \times \mathcal{D}(\mathcal{G}_1)$, we have $\mathcal{D}(A'_e - A_e) = \mathcal{D}(A_e)$. For any $x_e = (x, z)^T \in \mathcal{D}(A_e)$ we have

$$\|(A'_e - A_e)x_e\| = \left\| \begin{pmatrix} (A'-A)x + (B'-B)Kz \\ \mathcal{G}_2(C'-C)x + \mathcal{G}_2(D'-D)Kz \end{pmatrix} \right\|.$$

Since $(A' - A)$ is relatively bounded with respect to A and the other operators are bounded, it is straightforward to verify that there exist $c_1, c_2 \geq 0$ such that

$$\|(A'_e - A_e)x_e\| \leq c_1\|A_e x_e\| + c_2\|x_e\|.$$

We therefore also have

$$\begin{aligned} \|A'_e x_{e0}\| &= \|(A'_e - A_e)x_{e0} + A_e x_{e0}\| \leq \|(A'_e - A_e)x_{e0}\| + \|A_e x_{e0}\| \\ &\leq c_1\|A_e x_{e0}\| + c_2\|x_{e0}\| + \|A_e x_{e0}\| = (c_1+1)\|A_e x_{e0}\| + c_2\|x_{e0}\|. \end{aligned}$$

Finally, for all initial states $x_{e0} \in \mathcal{D}(A'_e)$ and $v_0 \in W_{\alpha_0+1}$ we have $x_{e0} - \Sigma'_e v_0 \in \mathcal{D}(A'_e)$, and using $\Sigma'_e S = A'_e \Sigma'_e + B'_e$ we get an estimate

$$\begin{aligned} \|e(t)\| &= \|C'_e T_e(t)(A'_e)^{-1}(A'_e x_{e0} - A'_e \Sigma'_e v_0)\| \leq \|C'_e\|\|T_e(t)(A'_e)^{-1}\| \left(\|A'_e x_{e0}\| + \|A'_e \Sigma'_e v_0\|\right) \\ &= \|C'_e\|\|T_e(t)(A'_e)^{-1}\| \left(\|A'_e x_{e0}\| + \|(\Sigma'_e S - B'_e)v_0\|\right) \\ &\leq \|C'_e\| \frac{M''}{t^{1/\alpha}} \left(\|A'_e x_{e0}\| + \|\Sigma'_e\|\|Sv_0\| + \|B'_e\|\|v_0\|\right) \\ &\leq \|C'_e\| \frac{M''}{t^{1/\alpha}} \left((c_1+1)\|A_e x_{e0}\| + c_2\|x_{e0}\| + \|\Sigma'_e\|\|Sv_0\| + \|B'_e\|\|v_0\|\right) \\ &\leq \frac{M''\|C'_e\| \max\{c_1+1, c_2, \|\Sigma'_e\|, \|B'_e\|\}}{t^{1/\alpha}} \left(\|A_e x_{e0}\| + \|x_{e0}\| + \|Sv_0\| + \|v_0\|\right). \end{aligned}$$

Choosing $M' = M''\|C'_e\| \max\{c_1+1, c_2, \|\Sigma'_e\|, \|B'_e\|\} > 0$ concludes the proof. □

4 Construction of a Robust Controller

In this section we construct an observer based feedback controller that solves the robust output regulation problem. We being by showing how the closed-loop system can be stabilized polynomially provided that the internal model in the controller is polynomially stable. At this point we do not make any specific assumptions on the structure of the internal model, but instead assume the internal model satisfies the $\mathcal{G}$-conditions in Definition 4. The internal model can be chosen to be of an appropriate form depending on requirements of the case at hand. We continue in Section 4.1 where we present a method for constructing and stabilizing the internal model for a system with an equal number of inputs and outputs together with a diagonal exosystem.

Assumption 6. *Assume that the pair (A, B) is exponentially stabilizable and the pair (C, A) exponentially detectable.*

We begin by introducing the general structure of the feedback controller used in solving the robust output regulation problem.

Definition 7. *The parameters of the error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ on the Hilbert space $Z = X \times Z_2$ are chosen such that*

$$\mathcal{G}_1 = \begin{pmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & G_1 \end{pmatrix}, \qquad \mathcal{G}_2 = \begin{pmatrix} -L \\ G_2 \end{pmatrix}, \qquad K = \begin{pmatrix} K_1 & K_2 \end{pmatrix},$$

where the operators $G_1 : \mathcal{D}(G_1) \subset Z_2 \to Z_2$, $G_2 \in \mathcal{L}(Y, Z_2)$, and where $K_1 \in \mathcal{L}(X, U)$, $K_2 \in \mathcal{L}(Z_2, U)$, and $L \in \mathcal{L}(Y, X)$ have the following properties.

- *G_1 generates a polynomially bounded group on Z_2.*
- *(G_1, G_2) satisfy the $\mathcal{G}$-conditions.*

Our task is to show that the free parameters in the controller can be fixed in such a way that the controller solves the robust output regulation problem. Since the *internal model* (G_1, G_2) in the controller satisfies the $\mathcal{G}$-conditions, Lemma 20 shows that also the full controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions provided that the closed-loop system is polynomially stable. The main task is therefore to choose appropriate K_1, K_2 and L to achieve closed-loop stability.

The following theorem states that if the internal model in the controller can be stabilized polynomially, then also the full closed-loop system will be polynomially stable with the same exponent.

Theorem 8. *Choose $K_{11} \in \mathcal{L}(X, U)$ and $L \in \mathcal{L}(Y, X)$ such that $A+BK_{11}$ and $A+LC$ are exponentially stable. Then the Sylvester equation*

$$G_1 H_{e1} = H_{e1}(A + BK_{11}) + G_2(C + DK_{11})$$

on $\mathcal{D}(A)$ has a unique solution $H_{e1} \in \mathcal{L}(X, Z_2)$ satisfying $H_{e1}(\mathcal{D}(A)) \subset \mathcal{D}(G_1)$.

Denote $B_1 = H_{e1}B + G_2 D$ and assume $K_2 \in \mathcal{L}(Z_2, U)$ can be chosen in such a way that the semigroup generated by the operator $G_1 + B_1 K_2$ is polynomially stable with $\alpha > 0$. Then for the choice $K_1 = K_{11} + K_2 H_{e1}$ the closed-loop system is polynomially stable with α.

Finally, the controller solves the robust output regulation problem.

Proof. Since $A + BK_{11}$ generates an exponentially stable semigroup and since the semigroup generated by $-G_1$ is polynomially bounded, we have from [17] that the Sylvester equation has a unique solution $H_{e1} \in \mathcal{L}(X, Z_2)$ and $H_{e1}(\mathcal{D}(A)) \subset \mathcal{D}(G_1)$.

It was shown in the proof of Theorem 13 in [8] that the operator A_e is similar to an operator

$$\begin{pmatrix} A + BK_1 & BK_2 & BK_1 \\ G_2(C + DK_1) & G_1 + G_2 DK_2 & G_2 DK_1 \\ 0 & 0 & A + LC \end{pmatrix}.$$

Since $A + LC$ generates an exponentially stable semigroup, the similarity and Lemma 19 imply that the closed-loop system is polynomially stable with α if the same is true for the semigroup generated by the operator

$$A_{e1} = \begin{pmatrix} A + BK_1 & BK_2 \\ G_2(C + DK_1) & G_1 + G_2 DK_2 \end{pmatrix}.$$

It was further shown in the proof of Theorem 13 in [8] that with the given choices of operators K_{11}, H_{e1}, and K_1, the operator A_{e1} is similar to a block triangular operator

$$\begin{pmatrix} A + BK_{11} & BK_2 \\ 0 & G_1 + B_1 K_2 \end{pmatrix}.$$

Since $A + BK_{11}$ is exponentially stable and $G_1 + B_1 K_2$ is polynomially stable with α by assumption, we have from Lemma 19 that the semigroup generated by the above operator is polynomially stable with α. This concludes that the closed-loop system is polynomially stable with α.

Since the operators (G_1, G_2) satisfy the $\mathcal{G}$-conditions and $i\mathbb{R} \subset \rho(A_e)$, we have from Lemma 20 that the controller $(\mathcal{G}_1, \mathcal{G}_2)$ satisfies the $\mathcal{G}$-conditions. Theorem 5 therefore concludes that the controller solves the robust output regulation problem. □

4.1 Constructing an Internal Model for Diagonal Exosystems

In this section we demonstrate how to choose the internal model (G_1, G_2) and how to stabilize it polynomially in the case where the exosystem is diagonal, the frequencies $\{\omega_k\}_{k\in\mathbb{Z}}$ have a uniform gap, and the plant has an equal number of inputs and outputs.

Throughout this section we assume $\dim U = \dim Y = p$, and that $n_k = 1$ for all $k \in \mathbb{Z}$. Since we assumed $P(i\omega_k)$ are surjective, the first assumption implies that $P(i\omega_k)$ are invertible for all $k \in \mathbb{Z}$. We denote the transfer function of the stabilized plant by $P_K(\lambda) = (C + DK_{11})R(\lambda, A + BK_{11})B + D$ for all $\lambda \in \rho(A + BK_{11})$. Since the invertibility of the transfer function is invariant under bounded feedback, we have that $P_K(i\omega_k)$ are invertible for all $k \in \mathbb{Z}$.

Choose $Z_2 = W \times W \times \cdots \times W = W^p$ and choose the appropriate inner product such that it is a Hilbert space. The space Z_2 has an orthonormal basis $\{ \varphi_k^l \mid k \in \mathbb{Z},\ l = 1, \dots, p \} \subset Z_2$ defined in such a way that

$$\varphi_k^1 = (\phi_k^1, 0, \dots, 0), \qquad \varphi_k^2 = (0, \phi_k^1, \dots, 0), \qquad \cdots \qquad \varphi_k^p = (0, \dots, 0, \phi_k^1)$$

for all $k \in \mathbb{Z}$. Define $G_1 = \operatorname{diag}(S, S, \dots, S) : \mathcal{D}(S)^p \subset W^p \to W^p$ and let $g_2 \in W$ be such that $\langle g_2, \phi_k^1 \rangle \neq 0$ for all $k \in \mathbb{Z}$. Finally, define $G_2 \in \mathcal{L}(Y, Z_2)$ by

$$G_2 y = \sum_{l=1}^{p} \sum_{k \in \mathbb{Z}} \frac{\langle g_2, \phi_k^1 \rangle (P_K(i\omega_k)^{-1} y)_l}{\|P_K(i\omega_k)^{-1}\|} \varphi_k^l,$$

where $(P_K(i\omega_k)^{-1} y)_l$ denotes the lth element of the vector $P_K(i\omega_k)^{-1} y \in \mathbb{C}^p$.

It is immediate that G_1 generates a uniformly bounded group on Z_2. The following lemma shows that the internal model (G_1, G_2) satisfies the $\mathcal{G}$-conditions.

Lemma 9. *The operators (G_1, G_2) satisfy the $\mathcal{G}$-conditions.*

Proof. Because $n_k = 1$ for all $k \in \mathbb{Z}$, the condition (6c) is trivially satisfied. Let $y \in Y$ be such that $G_2 y = 0$. From the definition of G_2 we can see that this immediately implies $\langle g_2, \phi_k^1 \rangle (P_K(i\omega_k)^{-1} y)_l = 0$ for all $k \in \mathbb{Z}$ and $l \in \{1, \dots, p\}$. Let $k \in \mathbb{Z}$ be fixed. Since $\langle g_2, \phi_k^1 \rangle \neq 0$ by assumption, we must have $P_K(i\omega_k)^{-1} y = 0$. Since $P_K(i\omega_k)^{-1}$ is invertible, we have $y = 0$. The element $y \in \mathcal{N}(G_2)$ was arbitrary, and thus (6b) is satisfied.

To prove (6a), let $k \in \mathbb{Z}$, and let $w \in W^p$ be such that $w = (i\omega_k - G_1)v = G_2 y$ for some $v \in \mathcal{D}(G_1)$ and $y \in Y$. Since G_1 is skew-adjoint, we have

$$\langle w, \varphi_k^l \rangle = \langle (i\omega_k - G_1)v, \varphi_k^l \rangle = \langle v, (-i\omega_k + G_1)\varphi_k^l \rangle = \langle v, (-i\omega_k + i\omega_k)\varphi_k^l \rangle = 0,$$

and on the other hand

$$\langle w, \varphi_k^l \rangle = \langle G_2 y, \varphi_k^l \rangle = \frac{\langle g_2, \phi_k^1 \rangle (P_K(i\omega_k)^{-1} y)_l}{\|P_K(i\omega_k)^{-1}\|}.$$

Combining these equations we have

$$\frac{\langle g_2, \phi_k^1 \rangle (P_K(i\omega_k)^{-1} y)_l}{\|P_K(i\omega_k)^{-1}\|} = 0 \qquad \forall l \in \{1, \dots, p\},$$

which implies $P_K(i\omega_k)^{-1} y = 0$ since $\langle g_2, \phi_k^1 \rangle \neq 0$ by assumption. Because $P_K(i\omega_k)^{-1}$ is invertible, we further have $y = 0$. This concludes $w = G_2 y = 0$, and thus (6a) is satisfied. □

The next theorem concludes that the internal model we have constructed can be stabilized polynomially. For brevity, we make use of the results in [15] to prove the existence of a stabilizing feedback K_2. The procedure for choosing an appropriate feedback operator is illustrated in greater detail in Section 6, where we also see that the conclusion of Theorem 10 is not optimal, if $|\omega_k|$ grow faster than at a constant rate with respect to $|k|$.

Theorem 10. *Assume the operator S is diagonal, the frequencies $\{\omega_k\}_{k \in \mathbb{Z}}$ have a uniform gap, and $U = Y = \mathbb{C}^p$. If there exist $\beta, c > 0$ such that*

$$\frac{|\langle g_2, \phi_k^1 \rangle|}{\|P_K(i\omega_k)^{-1}\|} \geq \frac{c}{|k|^\beta} \tag{10}$$

for large enough $|k|$, then for any $\alpha > \beta + \frac{1}{2}$ the operator $K_2 \in \mathcal{L}(W^p, Y)$ can be chosen in such a way that the semigroup generated by the operator $G_1 + B_1 K_2$ is polynomially stable with α.

Proof. Let $\alpha > \beta + \frac{1}{2}$. If 0 is an eigenvalue of S, we can without loss of generality assume $\omega_0 = 0$.

For the stabilization of the internal model we need the solution H_{e1} of the Sylvester equation $G_1 H_{e1} = H_{e1}(A + BK_{11}) + G_2(C + DK_{11})$. Since the operator G_1 is diagonal, we can see as in [8, Lem. 19] or in [15, Lem. 14] that the unique solution H_{e1} is given by

$$H_{e1} x = \sum_{l=1}^{p} \sum_{k \in \mathbb{Z}} \langle G_2(C + DK_{11}) R(i\omega_k, A + BK_{11}) x, \varphi_k^l \rangle \varphi_k^l$$

for all $x \in X$. For $u = (u_1, \dots, u_p) \in U = \mathbb{C}^p$ and for $k \in \mathbb{Z}$ and $l \in \{1, \dots, p\}$, we therefore have

$$\begin{aligned}\langle B_1 u, \varphi_k^l \rangle &= \langle H_{e1} Bu + G_2 Du, \varphi_k^l \rangle = \langle G_2 (C + DK_{11}) R(i\omega_k, A + BK_{11}) Bu + G_2 Du, \varphi_k^l \rangle \\ &= \langle G_2 P_K(i\omega_k) u, \varphi_k^l \rangle = \frac{\langle g_2, \phi_k^1 \rangle (P_K(i\omega_k)^{-1} P_K(i\omega_k) u)_l}{\|P_K(i\omega_k)^{-1}\|} = \frac{\langle g_2, \phi_k^1 \rangle}{\|P_K(i\omega_k)^{-1}\|} u_l.\end{aligned}$$

This shows that for an operator $K_2 \in \mathcal{L}(W^p, \mathbb{C}^p)$ we can write

$$G_1 + B_1 K_2 = \begin{pmatrix} S & & & \\ & S & & \\ & & \ddots & \\ & & & S \end{pmatrix} + \begin{pmatrix} b_1 & & & \\ & b_1 & & \\ & & \ddots & \\ & & & b_1 \end{pmatrix} \begin{pmatrix} K_2^{11} & K_2^{12} & \cdots & K_2^{1p} \\ K_2^{21} & K_2^{22} & & \\ \vdots & & \ddots & \\ K_2^{p1} & & & K_2^{pp} \end{pmatrix}$$

where $K_2^{kl} \in \mathcal{L}(W, \mathbb{C})$ and b_1 is such that $\langle b_1, \phi_k^1 \rangle = \langle g_2, \phi_k^1 \rangle / \|P_K(i\omega_k)^{-1}\|$ for $k \in \mathbb{Z}$. We have $b_1 \in W$, since

$$1 \le \|P_K(i\omega_k)^{-1}\| \|P_K(i\omega_k)\| \qquad \Leftrightarrow \qquad \frac{1}{\|P_K(i\omega_k)^{-1}\|} \le \|P_K(i\omega_k)\|$$

and $\|P_K(i\omega_k)\|$ are uniformly bounded with respect to $k \in \mathbb{Z}$ due to the exponential stability of $A + BK_{11}$. If we choose $K_2^{kl} = 0$ for $k \neq l$ and $K_2^{kk} = K_2^{11}$ for $k \in \{2, \dots, p\}$, we have

$$G_1 + B_1 K_2 = \begin{pmatrix} S + b_1 K_2^{11} & & & \\ & S + b_1 K_2^{11} & & \\ & & \ddots & \\ & & & S + b_1 K_2^{11} \end{pmatrix}.$$

It is clear that if $K_2^{11} \in \mathcal{L}(W, \mathbb{C})$ is chosen in such a way that the semigroup generated by $S + b_1 K_2^{11}$ is polynomially stable with α, then the same is true for the semigroup generated by $G_1 + B_1 K_2$. Since S is a diagonal operator with simple eigenvalues that have a uniform gap, we can use pole placement in choosing K_2^{11} [21]. In fact, we can directly apply Theorem 15 in [15] (where the single-input single-output case was considered) once we replace $|P_K(i\omega_k)|$ by $1/\|P_K(i\omega_k)^{-1}\|$. In particular, with this modification the proof of [15, Thm. 15] implies that we can choose K_2^{11} in such a way that $\sigma(S + b_1 K_2^{11}) = \{\mu_k\}_{k \in \mathbb{Z}}$, where

$$\mu_0 = -1 \qquad \text{and} \qquad \mu_k = -\frac{1}{|k|^\alpha} + i\omega_k.$$

Moreover, $S + b_1 K_2^{11}$ is a Riesz-spectral operator and all but a finite number of its eigenvalues μ_k are simple. First of all, this concludes that the semigroup generated by $S + b_1 K_2^{11}$ is uniformly bounded. Since the eigenvalues $\{i\omega_k\}_{k \in \mathbb{Z}}$ have a uniform gap, we can use some geometric analysis to conclude that there exists $c > 0$ such that for $\omega \in \mathbb{R}$ with large enough $|\omega|$ we have $|i\omega - \mu_k| \ge c|\omega|^{-\alpha}$. The properties of $S + b_1 K_2^{11}$ imply that there exists a constant $M > 0$ such that for large $|\omega|$ we have

$$\|R(i\omega, S + b_1 K_2^{11})\| \le \frac{M}{\min_k |i\omega - \mu_k|} \le \frac{M}{c|\omega|^{-\alpha}} = \frac{M}{c}|\omega|^\alpha.$$

Since W is a Hilbert space, we have from Lemma 2 that $S + b_1 K_2^{11}$ is polynomially stable with α. Due to the diagonal structure this finally concludes that the semigroup generated by $G_1 + B_1 K_2$ is polynomially stable with α. □

5 Preservation of the Polynomial Stability of the Closed-Loop System

In this section we present classes of perturbations that preserve the stability of the closed-loop system. For this we use the general perturbation results in [13]. We consider situations where the closed-loop system is polynomially stable with α, where either $\alpha = 1$ or $\alpha = 2$. In these situations it is possible to present easily verifiable conditions for the preservation of the closed-loop stability.

We begin by showing that for any bounded finite rank perturbations to the operators A, B, C, and D, the change in the closed-loop system can be written in the form

$$A_e' = A_e + \Delta_e \tilde{\Delta}_e, \tag{11}$$

where $\Delta_e \in \mathcal{L}(\mathbb{C}^m, X_e)$, and $\tilde{\Delta}_e \in \mathcal{L}(X_e, \mathbb{C}^m)$ for some $m \in \mathbb{N}$.

Lemma 11. *If any one of the operators of the plant is perturbed, then the closed-loop system can be written in the form* (11) *where $\Delta_e \in \mathcal{L}(\mathbb{C}^m, X_e)$, and $\tilde{\Delta}_e \in \mathcal{L}(X_e, \mathbb{C}^m)$ for some $m \in \mathbb{N}$. In particular,*

1. *If $A' = A + \Delta_A \tilde{\Delta}_A$ with $\Delta_A \in \mathcal{L}(\mathbb{C}^m, X)$ and $\tilde{\Delta}_A \in \mathcal{L}(X, \mathbb{C}^m)$, then we can choose $\Delta_e = \begin{pmatrix} \Delta_A \\ 0 \end{pmatrix}$, $\tilde{\Delta}_e = \begin{pmatrix} \tilde{\Delta}_A & 0 \end{pmatrix}$.*
2. *If $B' = B + \Delta_B$ with $\Delta_B \in \mathcal{L}(U, X)$, then we can choose $\Delta_e = \begin{pmatrix} \Delta_B \\ 0 \end{pmatrix}$, $\tilde{\Delta}_e = \begin{pmatrix} 0 & K \end{pmatrix}$ and $m = \dim U$.*
3. *If $C' = C + \Delta_C$ with $\Delta_C \in \mathcal{L}(X, Y)$, then we can choose $\Delta_e = \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix}$, $\tilde{\Delta}_e = \begin{pmatrix} \Delta_C & 0 \end{pmatrix}$ and $m = \dim Y$.*
4. *If $D' = D + \Delta_D$ with $\Delta_D \in \mathcal{L}(U, Y)$, then we can choose $\Delta_e = \begin{pmatrix} 0 \\ \mathcal{G}_2 \Delta_D \end{pmatrix}$, $\tilde{\Delta}_e = \begin{pmatrix} 0 & K \end{pmatrix}$ and $m = \dim U$.*

Proof. The conclusions of the lemma follow immediately from

$$\begin{aligned}
A_e' &= \begin{pmatrix} A + \Delta_A \tilde{\Delta}_A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} = A_e + \begin{pmatrix} \Delta_A \\ 0 \end{pmatrix} \begin{pmatrix} \tilde{\Delta}_A & 0 \end{pmatrix} \\
A_e' &= \begin{pmatrix} A & (B + \Delta_B)K \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} = A_e + \begin{pmatrix} \Delta_B \\ 0 \end{pmatrix} \begin{pmatrix} 0 & K \end{pmatrix} \\
A_e' &= \begin{pmatrix} A & BK \\ \mathcal{G}_2 (C + \Delta_C) & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} = A_e + \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \begin{pmatrix} \Delta_C & 0 \end{pmatrix} \\
A_e' &= \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 (D + \Delta_D)K \end{pmatrix} = A_e + \begin{pmatrix} 0 \\ \mathcal{G}_2 \Delta_D \end{pmatrix} \begin{pmatrix} 0 & K \end{pmatrix}.
\end{aligned}$$

□

For simplicity, in the following we will only consider the case where the operators of the plant are perturbed one at a time. If two operators are perturbed at the same time, then the perturbed closed-loop system operator can be written as

$$A_e' = A_e + \Delta_{e1} \tilde{\Delta}_{e1} + \Delta_{e2} \tilde{\Delta}_{e2} = A_e + \begin{pmatrix} \Delta_{e1} & \Delta_{e2} \end{pmatrix} \begin{pmatrix} \tilde{\Delta}_{e1} \\ \tilde{\Delta}_{e2} \end{pmatrix} = A_e + \Delta_e \tilde{\Delta}_e,$$

and analogously for perturbation of three or all four operators at the same time. Therefore, the same methods for determining conditions for the preservation of the closed-loop stability can also be applied in these situations.

5.1 General Perturbation Results

We will now present conditions for the preservation of stability of the closed-loop system under general finite-rank perturbations $A_e + \Delta_e \tilde{\Delta}_e$, and subsequently apply these results for consider the preservation of stability under perturbation of the operators A, B, C, and D of the plant. The closed-loop system and the perturbations of A_e are assumed to satisfy the following assumptions.

Assumption 12. *Assume that the closed-loop system and the operators $\Delta_e \in \mathcal{L}(\mathbb{C}^m, X_e)$, and $\tilde{\Delta}_e \in \mathcal{L}(X_e, \mathbb{C}^m)$ satisfy the following for some $\alpha \in \{1, 2\}$, and $\beta, \gamma \in \{0, 1\}$.*

1. *The semigroup $T_e(t)$ generated by A_e is polynomially stable with α.*
2. *We have $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e^\beta)$ and $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}((A_e^*)^\gamma)$*

The following theorem gives sufficient conditions for the preservation of the polynomial stability of the closed-loop system in terms of perturbations of A_e. For the powers of the operators we use the convention that $A_e^0 = I$ and $(A_e^*)^0 = I$. This means that in the case $\beta = 0$ the condition $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e^\beta)$ becomes redundant and $\|A_e^\beta \Delta_e\| = \|\Delta_e\|$, and similarly for $\gamma = 0$. We do not consider situations $\alpha > 2$, because for $\beta > 1$ or $\gamma > 1$ the powers A_e^β and $(A_e^*)^\gamma$ become difficult to compute, and the analysis does not yield any useful perturbation results.

Theorem 13. *Let Assumption 12 be satisfied for some $\beta, \gamma \in \mathbb{N}_0$ such that $\beta + \gamma = \alpha$. Then there exists $\delta_e > 0$ such that if*

$$\|A_e^\beta \Delta_e\| \cdot \|(A_e^*)^\gamma \tilde{\Delta}_e^*\| < \delta_e, \tag{12}$$

then the semigroup generated by $A_e' = A_e + \Delta_e \tilde{\Delta}_e$ is polynomially stable with α.

Proof. The conclusion of the theorem is a direct consequence of [13, Thm. 2]. □

Remark 1. *From the proofs of Theorems 2 and 6 in [13] we can see that as the bound for the graph norms (12) we can choose any $\delta_e > 0$ satisfying*

$$\delta_e < \frac{1}{\sup_{\lambda \in \overline{\mathbb{C}^+}} \|R(\lambda, A_e) A_e^{-\alpha}\|}.$$

The adjoint of the system operator of the closed-loop system is given by

$$A_e^* = \begin{pmatrix} A^* & C^* \mathcal{G}_2^* \\ K^* B^* & \mathcal{G}_1^* + K^* D^* \mathcal{G}_2^* \end{pmatrix} : \mathcal{D}(A_e^*) = \mathcal{D}(A^*) \times \mathcal{D}(\mathcal{G}_1^*) \subset X_e \to X_e.$$

5.2 Perturbations in the Parameters of the Plant

Throughout the rest of Section 5 we assume that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions and stabilizes the closed-loop system polynomially with α. Due to Theorem 5 the controller solves the robust output regulation problem.

In the following we present conditions on the perturbations of the individual operators A, B, C, and D. Our main goal is to derive conditions that only involve range conditions on the components of the perturbation. In the case where $\alpha = 2$ this is only possible for the perturbation of the operator A. In all other cases we in addition encounter conditions of the form

$$\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1) \qquad \text{and/or} \qquad \mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*).$$

Such conditions are inconvenient, because the operators K and $\mathcal{G}_2$ are fixed parameters of the controller. However, we will see that in the case $\alpha = 1$ we can remove such conditions when perturbing operators B and C.

The next two theorems summarize the main results of this section.

Theorem 14. *Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions and the closed-loop system is polynomially stable with $\alpha = 2$.*

(a) *If $A' = A + \Delta_A \tilde{\Delta}_A$ with $\Delta_A \in \mathcal{L}(\mathbb{C}^m, X)$, $\tilde{\Delta}_A \in \mathcal{L}(X, \mathbb{C}^m)$, then there exists $\delta_A > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that*

$$\mathcal{R}(\Delta_A) \subset \mathcal{D}(A) \qquad \text{and} \qquad \mathcal{R}(\tilde{\Delta}_A^*) \subset \mathcal{D}(A^*), \tag{13}$$

and $(\|A\Delta_A\| + \|\Delta_A\|) \cdot (\|A^* \tilde{\Delta}_A^*\| + \|\tilde{\Delta}_A\|) < \delta_A$.

(b) *If $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$ and if $B' = B + \Delta_B$ with $\Delta_B \in \mathcal{L}(U, X)$, then there exists $\delta_B > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that*

$$\mathcal{R}(\Delta_B) \subset \mathcal{D}(A) \tag{14}$$

and $\|A\Delta_B\| + \|\Delta_B\| < \delta_B$.

(c) *If $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$ and if $C' = C + \Delta_C$ with $\Delta_C \in \mathcal{L}(X, C)$, then there exists $\delta_C > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that*

$$\mathcal{R}(\Delta_C^*) \subset \mathcal{D}(A^*), \tag{15}$$

and $\|A^*\Delta_C^*\| + \|\Delta_C\| < \delta_C$.

(d) *If $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$, $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$, and if $D' = D + \Delta_D$ with $\Delta_D \in \mathcal{L}(U, Y)$, then there exists $\delta_D > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that $\|\Delta_D\| < \delta_D$.*

In each of the above cases the regulation error decays asymptotically as in Theorem 5.

Theorem 15. *Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions and the closed-loop system is polynomially stable with $\alpha = 1$.*

(a_1) *If $A' = A + \Delta_A\tilde{\Delta}_A$, there exists $\delta_A > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that $\mathcal{R}(\Delta_A) \subset \mathcal{D}(A)$ and $(\|A\Delta_A\| + \|\Delta_A\|)\|\tilde{\Delta}_A\| < \delta_A$.*

(a_2) *If $A' = A + \Delta_A\tilde{\Delta}_A$, there exists $\delta_A > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that $\mathcal{R}(\tilde{\Delta}_A^*) \subset \mathcal{D}(A^*)$, and $(\|A^*\tilde{\Delta}_A^*\| + \|\tilde{\Delta}_A\|)\|\Delta_A\| < \delta_A$.*

(b) *If $B' = B + \Delta_B$, there exists $\delta_B > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that $\mathcal{R}(\Delta_B) \subset \mathcal{D}(A)$ and $(\|A\Delta_B\| + \|\Delta_B\|) < \delta_B$.*

(c) *If $C' = C + \Delta_C$, there exists $\delta_C > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that $\mathcal{R}(\Delta_C^*) \subset \mathcal{D}(A^*)$, and $(\|A^*\Delta_C^*\| + \|\Delta_C\|) < \delta_C$.*

(d) *If $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$ or $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$, and if $D' = D + \Delta_D$, there exists $\delta_D > 0$ such that the perturbed closed-loop system is polynomially stable with α provided that $\|\Delta_D\| < \delta_D$.*

In each of the above cases the regulation error decays asymptotically as in Theorem 5.

The proofs of Theorems 14 and 15 are based on straightforward norm estimates, but they require some work due to the large number of different cases we need to consider. For the proofs we need some preliminary estimates on the graph norms of the closed-loop system. We saw in Lemma 11 that for perturbations in the parameters operators A, B, C, and D the resulting perturbations in the closed-loop system operator A_e can be written in forms

$$\Delta_e = \begin{pmatrix} \Delta_{e1} \\ 0 \end{pmatrix} \quad \text{or} \quad \Delta_e = \begin{pmatrix} 0 \\ \mathcal{G}_2\Delta_{e2} \end{pmatrix}, \qquad \text{and} \qquad \tilde{\Delta}_e = \begin{pmatrix} \tilde{\Delta}_{e1} & 0 \end{pmatrix}, \quad \text{or} \quad \tilde{\Delta}_e = \begin{pmatrix} 0 & K \end{pmatrix},$$

where $\Delta_{e1} \in \mathcal{L}(\mathbb{C}^m, X)$, $\Delta_{e2} \in \mathcal{L}(\mathbb{C}^m, Y)$, and $\tilde{\Delta}_{e1} \in \mathcal{L}(X, \mathbb{C}^m)$. Therefore, the range conditions in Assumption 12 can be given in terms of the perturbations of the operators A, B, C, and D, as shown in the following lemma.

Lemma 16. *For perturbations $A' = A + \Delta_A\tilde{\Delta}_A$ we have*

$$\begin{aligned} \mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e) \quad &\Leftrightarrow \quad \mathcal{R}(\Delta_A) \subset \mathcal{D}(A) \\ \mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*) \quad &\Leftrightarrow \quad \mathcal{R}(\tilde{\Delta}_A^*) \subset \mathcal{D}(A^*). \end{aligned}$$

For perturbations $B' = B + \Delta_B$ we have

$$\begin{aligned} \mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e) \quad &\Leftrightarrow \quad \mathcal{R}(\Delta_B) \subset \mathcal{D}(A) \\ \mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*) \quad &\Leftrightarrow \quad \mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*). \end{aligned}$$

For perturbations $C' = C + \Delta_C$ we have

$$\begin{aligned} \mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e) \quad &\Leftrightarrow \quad \mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1) \\ \mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*) \quad &\Leftrightarrow \quad \mathcal{R}(\Delta_C^*) \subset \mathcal{D}(A^*). \end{aligned}$$

For perturbations $D' = D + \Delta_D$ we have

$$\begin{aligned} \mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e) \quad &\Leftrightarrow \quad \mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1) \\ \mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*) \quad &\Leftrightarrow \quad \mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*). \end{aligned}$$

Proof. The equivalences are direct consequences of Lemma 11. □

The estimates in the following lemma form the basis for the perturbation results in the rest of this section.

Lemma 17. *The perturbations of the closed-loop satisfy the following estimates.*

- *If* $\mathcal{R}(\Delta) \subset \mathcal{D}(A)$*, then*

$$\left\| A_e \begin{pmatrix} \Delta \\ 0 \end{pmatrix} \right\| \leq \sqrt{2} \max\{1, \|\mathcal{G}_2 C\|\} \left(\|A\Delta\| + \|\Delta\| \right). \tag{16a}$$

- *If* $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$*, then*

$$\left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| \leq \sqrt{2} \left(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\| \right). \tag{16b}$$

- *If* $\mathcal{R}(\tilde{\Delta}^*) \subset \mathcal{D}(A^*)$*, then*

$$\left\| A_e^* \begin{pmatrix} \tilde{\Delta}^* \\ 0 \end{pmatrix} \right\| \leq \sqrt{2} \max\{1, \|BK\|\} \left(\|A^* \tilde{\Delta}^*\| + \|\tilde{\Delta}\| \right). \tag{16c}$$

- *If* $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$*, then*

$$\left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \leq \sqrt{2} \left(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^* D^* \mathcal{G}_2^*)K^*\| \right). \tag{16d}$$

Proof. The estimates employ the scalar inequalities in Lemma 18. If $\mathcal{R}(\Delta) \subset \mathcal{D}(A)$, then

$$\begin{aligned} \left\| A_e \begin{pmatrix} \Delta \\ 0 \end{pmatrix} \right\| &= \left\| \begin{pmatrix} A\Delta \\ \mathcal{G}_2 C \Delta \end{pmatrix} \right\| \leq \left(\|A\Delta\|^2 + \|\mathcal{G}_2 C\|^2 \|\Delta\|^2 \right)^{1/2} \\ &\leq \max\{1, \|\mathcal{G}_2 C\|\} \left(\|A\Delta\|^2 + \|\Delta\|^2 \right)^{1/2} \leq \sqrt{2} \max\{1, \|\mathcal{G}_2 C\|\} \left(\|A\Delta\| + \|\Delta\| \right). \end{aligned}$$

If $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$, then

$$\begin{aligned} \left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| &= \left\| \begin{pmatrix} BK\mathcal{G}_2 \\ (\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2 \end{pmatrix} \right\| \leq \left(\|BK\mathcal{G}_2\|^2 + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|^2 \right)^{1/2} \\ &\leq \sqrt{2} \left(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\| \right). \end{aligned}$$

If $\mathcal{R}(\tilde{\Delta}^*) \subset \mathcal{D}(A^*)$, then

$$\begin{aligned} \left\| A_e^* \begin{pmatrix} \tilde{\Delta}^* \\ 0 \end{pmatrix} \right\| &= \left\| \begin{pmatrix} A^* \tilde{\Delta}^* \\ K^* B^* \tilde{\Delta}^* \end{pmatrix} \right\| \leq \left(\|A^* \tilde{\Delta}^*\|^2 + \|K^* B^*\| \|\tilde{\Delta}^*\|^2 \right)^{1/2} \\ &\leq \sqrt{2} \max\{1, \|BK\|\} \left(\|A^* \tilde{\Delta}^*\| + \|\tilde{\Delta}\| \right). \end{aligned}$$

If $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$, then

$$\left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| = \left\| \begin{pmatrix} C^* \mathcal{G}_2^* K^* \\ (\mathcal{G}_1^* + K^* D^* \mathcal{G}_2^*) K^* \end{pmatrix} \right\| \leq \sqrt{2} \left(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^* D^* \mathcal{G}_2^*)K^*\| \right).$$

□

Proof of Theorem 14. We have $\alpha = 2$, and we choose $\beta = \gamma = 1$ in Assumption 12. Since $\beta + \gamma = \alpha$, we have from Theorem 13 that there exists $\delta_e > 0$ such that the perturbed closed-loop system is polynomially stable whenever $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$, $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$, and

$$\|A_e \Delta_e\| \cdot \|A_e^* \tilde{\Delta}_e^*\| < \delta_e. \tag{17}$$

It therefore suffices to show that in each of the cases the range conditions are satisfied, and the bounds on the perturbations can be chosen in such a way that (17) is satisfied.

(a) If $A' = A + \Delta_A\tilde{\Delta}_A$ with $\Delta_A \in \mathcal{L}(\mathbb{C}^m, X)$, $\tilde{\Delta}_A \in \mathcal{L}(X, \mathbb{C}^m)$ and the perturbation satisfies (13), we then have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$ and $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose $\delta_A > 0$ in such a way that

$$\delta_A = \frac{\delta_e}{2\max\{1, \|\mathcal{G}_2 C\|\}\max\{1, \|BK\|\}}.$$

Using Lemmas 11 and 17 we can now see that for any perturbations satisfying $(\|A\Delta_A\| + \|\Delta_A\|) \cdot (\|A^*\tilde{\Delta}_A^*\| + \|\tilde{\Delta}_A\|) < \delta_A$ we have

$$\begin{aligned}\|A_e\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e^*\| &= \left\| A_e \begin{pmatrix} \Delta_A \\ 0 \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} \tilde{\Delta}_A^* \\ 0 \end{pmatrix} \right\| \\ &\le \sqrt{2}\max\{1, \|\mathcal{G}_2 C\|\}(\|A\Delta_A\| + \|\Delta_A\|) \cdot \sqrt{2}\max\{1, \|BK\|\}(\|A^*\tilde{\Delta}_A^*\| + \|\tilde{\Delta}_A\|) < \delta_e.\end{aligned}$$

(b) If $B' = B + \Delta_B$ with $\Delta_B \in \mathcal{L}(U, X)$ and the perturbation satisfies (14), we then have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$ and $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose $\delta_B > 0$ in such a way that

$$\delta_B = \frac{\delta_e}{2\max\{1, \|\mathcal{G}_2 C\|\}(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^*D^*\mathcal{G}_2^*)K^*\|)}.$$

Using Lemmas 11 and 17 we can now see that for any perturbations satisfying $\|A\Delta_B\| + \|\Delta_B\| < \delta_B$ we have

$$\begin{aligned}\|A_e\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e^*\| &= \left\| A_e \begin{pmatrix} \Delta_B \\ 0 \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \\ &\le \sqrt{2}\max\{1, \|\mathcal{G}_2 C\|\}(\|A\Delta_B\| + \|\Delta_B\|) \cdot \sqrt{2}(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^*D^*\mathcal{G}_2^*)K^*\|) < \delta_e.\end{aligned}$$

(c) If $C' = C + \Delta_C$ with $\Delta_C \in \mathcal{L}(X, Y)$ and the perturbation satisfies (15), we then have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$ and $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose $\delta_C > 0$ in such a way that

$$\delta_C = \frac{\delta_e}{2(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|)\max\{1, \|BK\|\}}.$$

Using Lemmas 11 and 17 we can now see that for any perturbations satisfying $\|A^*\Delta_C^*\| + \|\Delta_C\| < \delta_C$ we have

$$\begin{aligned}\|A_e\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e^*\| &= \left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} \Delta_C^* \\ 0 \end{pmatrix} \right\| \\ &\le \sqrt{2}(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|) \cdot \sqrt{2}\max\{1, \|BK\|\}(\|A^*\Delta_C^*\| + \|\Delta_C\|) < \delta_e.\end{aligned}$$

(d) If $D' = D + \Delta_D$ with $\Delta_D \in \mathcal{L}(U, Y)$, we have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$ and $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose $\delta_D > 0$ in such a way that

$$\delta_D = \frac{\delta_e}{2(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|)(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^*D^*\mathcal{G}_2^*)K^*\|)}.$$

Using Lemmas 11 and 17 we can now see that for any perturbations satisfying $\|\Delta_D\| < \delta_D$ we have

$$\begin{aligned}\|A_e\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e^*\| &= \left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2\Delta_D \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \le \left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| \cdot \|\Delta_D\| \cdot \left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \\ &\le 2(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|)\|\Delta_D\|(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^*D^*\mathcal{G}_2^*)K^*\|) < \delta_e.\end{aligned}$$

□

Proof of Theorem 15. We have $\alpha = 1$, and we choose either $\beta = 1$ and $\gamma = 0$ or $\beta = 0$ and $\gamma = 1$ in Assumption 12. Since $\beta + \gamma = \alpha$, we have from Theorem 13 that there exists $\delta_e > 0$ such that the closed-loop system is polynomially stable whenever we have either $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$ and $\|A_e\Delta_e\| \cdot \|\tilde{\Delta}_e\| < \delta_e$, or alternatively $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$ and $\|\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e^*\| < \delta_e$. It is therefore again sufficient to check that in each of the cases one of the range conditions is satisfied, and the bounds on the perturbations can be chosen in such a way that the corresponding norm bound is satisfied.

(a_1) If $A' = A + \Delta_A\tilde{\Delta}_A$ and $\mathcal{R}(\Delta_A) \subset \mathcal{D}(A)$ we have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$. Choose $\delta_A = \delta_e/(\sqrt{2}\max\{1, \|\mathcal{G}_2 C\|\}) > 0$. Using Lemmas 11 and 17 we can see that if $(\|A\Delta_A\| + \|\Delta_A\|) \cdot \|\tilde{\Delta}_A\| < \delta_A$, then

$$\|A_e\Delta_e\| \cdot \|\tilde{\Delta}_e\| = \left\| A_e \begin{pmatrix} \Delta_A \\ 0 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} \tilde{\Delta}_A^* \\ 0 \end{pmatrix} \right\| \leq \sqrt{2}\max\{1, \|\mathcal{G}_2 C\|\}(\|A\Delta_A\| + \|\Delta_A\|)\|\tilde{\Delta}_A\| < \delta_e.$$

(a_2) If $A' = A + \Delta_A\tilde{\Delta}_A$ and $\mathcal{R}(\tilde{\Delta}_A^*) \subset \mathcal{D}(A^*)$ we have from Lemma 16 that $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose $\delta_A = \delta_e/(\sqrt{2}\max\{1, \|BK\|\}) > 0$. Using Lemmas 11 and 17 we can see that if $\|\Delta_A\| \cdot (\|A^*\tilde{\Delta}_A\| + \|\tilde{\Delta}_A\|) < \delta_A$, then

$$\|\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e\| = \left\| \begin{pmatrix} \Delta_A \\ 0 \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} \tilde{\Delta}_A^* \\ 0 \end{pmatrix} \right\| \leq \|\Delta_A\|\sqrt{2}\max\{1, \|BK\|\}(\|A^*\tilde{\Delta}_A^*\| + \|\tilde{\Delta}_A\|) < \delta_e.$$

(b) If $B' = B + \Delta_B$ and $\mathcal{R}(\Delta_B) \subset \mathcal{D}(A)$ we have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$. Choose $\delta_B = \delta_e/(\sqrt{2}\max\{1, \|\mathcal{G}_2 C\|\}\|K\|) > 0$. Using Lemmas 11 and 17 we can see that if $\|A\Delta_B\| + \|\Delta_B\| < \delta_B$, then

$$\|A_e\Delta_e\| \cdot \|\tilde{\Delta}_e\| = \left\| A_e \begin{pmatrix} \Delta_B \\ 0 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \leq \sqrt{2}\max\{1, \|\mathcal{G}_2 C\|\}(\|A\Delta_A\| + \|\Delta_A\|)\|K\| < \delta_e.$$

(c) If $C' = C + \Delta_C$ and $\mathcal{R}(\Delta_C^*) \subset \mathcal{D}(A^*)$ we have from Lemma 16 that $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose $\delta_C = \delta_e/(\sqrt{2}\|\mathcal{G}_2\|\max\{1, \|BK\|\}) > 0$. Using Lemmas 11 and 17 we can see that if $\|A^*\Delta_C\| + \|\Delta_C\| < \delta_C$, then

$$\|\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e\| = \left\| \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} \tilde{\Delta}_C^* \\ 0 \end{pmatrix} \right\| \leq \|\mathcal{G}_2\|\sqrt{2}\max\{1, \|BK\|\}(\|A^*\Delta_C^*\| + \|\Delta_C\|) < \delta_e.$$

(d) Assume $D' = D + \Delta_D$. If $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$ we have from Lemma 16 that $\mathcal{R}(\Delta_e) \subset \mathcal{D}(A_e)$. Choose

$$\delta_D = \frac{\delta_e}{\sqrt{2}(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|)\|K\|} > 0.$$

Using Lemmas 11 and 17 we can see that if $\|\Delta_D\| < \delta_D$, then

$$\begin{aligned} \|A_e\Delta_e\| \cdot \|\tilde{\Delta}_e\| &= \left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2\Delta_D \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \leq \left\| A_e \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| \cdot \|\Delta_D\| \cdot \left\| \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \\ &\leq \sqrt{2}(\|BK\mathcal{G}_2\| + \|(\mathcal{G}_1 + \mathcal{G}_2 DK)\mathcal{G}_2\|) \cdot \|\Delta_D\| \cdot \|K\| < \delta_e. \end{aligned}$$

On the other hand, if $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$ we have from Lemma 16 that $\mathcal{R}(\tilde{\Delta}_e^*) \subset \mathcal{D}(A_e^*)$. Choose

$$\delta_D = \frac{\delta_e}{\sqrt{2}\|\mathcal{G}_2\|(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^* D^* \mathcal{G}_2^*)K^*\|)} > 0.$$

Using Lemmas 11 and 17 we can see that if $\|\Delta_D\| < \delta_D$, then

$$\begin{aligned} \|\Delta_e\| \cdot \|A_e^*\tilde{\Delta}_e^*\| &= \left\| \begin{pmatrix} 0 \\ \mathcal{G}_2\Delta_D \end{pmatrix} \right\| \cdot \left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} 0 \\ \mathcal{G}_2 \end{pmatrix} \right\| \cdot \|\Delta_D\| \cdot \left\| A_e^* \begin{pmatrix} 0 \\ K^* \end{pmatrix} \right\| \\ &\leq \|\mathcal{G}_2\| \cdot \|\Delta_D\| \cdot \sqrt{2}(\|K\mathcal{G}_2 C\| + \|(\mathcal{G}_1^* + K^* D^* \mathcal{G}_2^*)K^*\|) < \delta_e. \end{aligned}$$

□

6 Robust Output Tracking for a Stable Heat Equation

In this section we consider an example where we design a controller to achieve robust output tracking for a heated metal bar. The system can be written as an exponentially stable one-dimensional heat equation. We design robust controllers for tracking of signals generated by two different infinite-dimensional exosystems. Finally, we use the theory presented in Section 5 to characterize classes of perturbations preserving the polynomial closed-loop stability and the output regulation property.

6.1 The Controlled Heat Equation

We consider a heat equation

$$\frac{dw}{dt}(z,t) = \frac{d^2w}{dz^2}(z,t) + \chi_{[\frac{1}{2},1]}(z)u(t)$$

on the interval $(0,1)$ with Dirichlet boundary conditions $w(0,t) = w(1,t) = 0$ and initial state $w(z,0) = w_0(z) \in L^2(0,1)$. The output of the system is given by

$$y(t) = \int_0^{\frac{1}{2}} w(z,t)dz + u(t).$$

The controlled heat equation can be written as a linear system on $X = L^2(0,1)$, $U = \mathbb{C}$ and $Y = \mathbb{C}$ if we choose the operators A, B, C, and D in such a way that

$$Ax = x'', \quad x \in \mathcal{D}(A) = \left\{ x \in L^2(0,1) \;\middle|\; x, x' \text{ abs. cont.},\; x'' \in L^2(0,1),\; x(0) = x(1) = 0 \right\},$$

and

$$Bu = b(\cdot)u, \qquad Cx = \int_0^{\frac{1}{2}} x(z)dz, \qquad D = 1.$$

As in [8] we can see that the system is exponentially stable, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\pi^2$ the resolvent operator $R(\lambda, A)$ has the form

$$\begin{aligned}(R(\lambda,A)x)(z) = &\int_0^z \frac{\sinh(\xi\sqrt{\lambda})\sinh\left((1-z)\sqrt{\lambda}\right)}{\sqrt{\lambda}\sinh(\sqrt{\lambda})} x(\xi)d\xi \\ &+ \int_z^1 \frac{\sinh(z\sqrt{\lambda})\sinh\left((1-\xi)\sqrt{\lambda}\right)}{\sqrt{\lambda}\sinh(\sqrt{\lambda})} x(\xi)d\xi,\end{aligned}$$

and the transfer function of the plant is given by

$$P(\lambda) = CR(\lambda,A)B = \frac{4\sinh^4(\sqrt{\lambda}/4)}{\lambda\sqrt{\lambda}\sinh(\sqrt{\lambda})} + 1.$$

The first term in $P(\lambda)$ has a removable singularity at $\lambda = 0$, and $P(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. In particular, $\frac{1}{2} \leq |P(i\omega)| \leq \frac{3}{2}$ for all $\omega \in \mathbb{R}$.

6.2 The Two Exosystems

We consider output tracking for two different exosystems. In both cases the exosystem is an infinite-dimensional diagonal operator on the space $W = \ell^2(\mathbb{C})$. Choosing $\phi_k = e_k$, the natural basis of $\ell^2(\mathbb{C})$, we define

$$S = \sum_{k\in\mathbb{Z}} i\omega_k \langle \cdot, \phi_k\rangle \phi_k, \qquad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} |\omega_k|^2 |\langle v, \phi_k\rangle|^2 < \infty \right\}$$

and $F \in \mathcal{L}(W, \mathbb{C})$ is chosen in such a way that $F\phi_0 = -1$ and $F\phi_k = -1/|k|^{3/5}$ for all $k \neq 0$.

The difference between the two exosystems we consider is that they have different sets of frequencies $(i\omega_k)_k$. For the first exosystem we choose $\omega_k = k$ for all $k \in \mathbb{Z}$. With this choice the reference signals to be tracked are of the form

$$y_{ref}(t) = -FT_S(t)v_0 = -\sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0, \phi_k\rangle F\phi_k = \langle v_0, \phi_0\rangle + \sum_{k\neq 0} e^{ikt}\frac{\langle v_0, \phi_k\rangle}{|k|^{3/5}},$$

which are precisely the continuous 2π-periodic signals f with Fourier coefficients $\hat{f}_0 = \langle v_0, \phi_0\rangle$ and $\hat{f}_k = \langle v_0, \phi_k\rangle |k|^{-3/5}$ for $k \neq 0$. As was shown in [15, Sec. 3], the choice of the space W_α of the initial state $v_0 \in W_\alpha$ determines the smoothness properties of the signal $y_{ref}(t)$.

For the second exosystem we choose $\omega_k = k^2$ for all $k \in \mathbb{Z}$. Such frequency distributions are encountered when studying vibration of undamped beams [4]. The reference signals generated by this exosystem are given by

$$y_{ref}(t) = -FT_S(t)v_0 = \langle v_0, \phi_0 \rangle + \sum_{k\in\mathbb{Z}} e^{ik^2 t} \frac{\langle v_0, \phi_k \rangle}{|k|^{3/5}}$$

for $v_0 \in W$.

For both choices of exosystems the frequencies $\{\omega_k\}$ have no finite accumulation points and $\inf_{k\neq l}|\omega_k - \omega_l| = 1$.

6.3 Choosing the Parameters of the Controller

Because the operator A generates an exponentially stable semigroup, we can choose $K_{11} = 0$ and $L = 0$ in the feedback controller. Therefore we also have $P_K(\lambda) = P(\lambda)$ for all $\lambda \in \rho(A)$.

Since $\dim Y = 1$, we choose $Z_2 = W$, $G_1 = S$, and $g_2 \in W$ such that $\langle g_2, \phi_0 \rangle = 1$ and $\langle g_2, \phi_k \rangle = 1/|k|^{3/5}$ for all $k \neq 0$. We then have from Section 4.1 that $G_2 \in \mathcal{L}(\mathbb{C}, W)$ is defined by

$$G_2 y = \sum_{k\in\mathbb{Z}} \frac{\langle g_2, \phi_k \rangle P(i\omega_k)^{-1} y}{\|P(i\omega_k)^{-1}\|} \phi_k = y \sum_{k\in\mathbb{Z}} \frac{\langle g_2, \phi_k \rangle |P(i\omega_k)|}{P(i\omega_k)} \phi_k$$

Since G_1 and G_2 are of the same form as in Section 4.1, we have from Lemma 9 that the internal model (G_1, G_2) satisfies the $\mathcal{G}$-conditions.

As in the proof of Theorem 10 we have that the operator $B_1 \in \mathcal{L}(\mathbb{C}, W)$ is such that $B_1 u = b_1 u$, where $b_1 \in W$ satisfies

$$\langle b_1, \phi_k \rangle = \frac{\langle g_2, \phi_k \rangle}{\|P(i\omega_k)^{-1}\|} = \langle g_2, \phi_k \rangle |P(i\omega_k)|.$$

In the following we will stabilize the internal model by choosing $K_2 = K_2^{11}$ in such a way that $G_1 + b_1 K_2$ is generates a polynomially stable semigroup. We demonstrate how to do this using pole placement [21]. Our aim is to choose $K_2 = \langle \cdot, h \rangle \in \mathcal{L}(W, \mathbb{C})$ in such a way that $G_1 + b_1 K_2$ is a Riesz-spectral operator with eigenvalues

$$\sigma(G_1 + b_1 K_2) = \{\mu_k\}_{k\in\mathbb{Z}}$$

where $\mu_0 = -1$ and $\mu_k = -\frac{1}{k^2} + i\omega_k$ for $k \neq 0$. We use Theorem 1 in [21] to show that such K_2 exists. In order to do this, we need to verify convergence of three series. For all $\lambda \in \mathbb{C}$ such that $\mathrm{dist}(\lambda, \{i\omega_k\}_k) > \frac{1}{3} \inf_{k\neq l}|i\omega_k - i\omega_l| = 1/3$ we have

$$\sum_{k\in\mathbb{Z}} \left| \frac{\langle b_1, \phi_k \rangle}{\lambda - i\omega_k} \right|^2 \leq 9 \sum_{k\in\mathbb{Z}} |\langle b_1, \phi_k \rangle|^2 = 9\|b_1\|^2 < \infty$$

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq l}} \left| \frac{\langle b_1, \phi_k \rangle}{i\omega_k - i\omega_l} \right|^2 \leq \sum_{k\in\mathbb{Z}} |\langle b_1, \phi_k \rangle|^2 = \|b_1\|^2 < \infty.$$

Finally, we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left| \frac{\mu_k - i\omega_k}{\langle b_1, \phi_k \rangle} \right|^2 &= \frac{1}{|\langle g_2, \phi_k \rangle|^2 |P(i\omega_0)|^2} + \sum_{k\neq 0} \frac{k^{-4}}{|\langle g_2, \phi_k \rangle|^2 |P(i\omega_k)|^2} \\ &\leq \frac{1}{|P(i\omega_0)|^2} + \frac{1}{\inf_{k\neq 0}|P(i\omega_k)|^2} \cdot \sum_{k\neq 0} \frac{1}{k^4 \cdot k^{-6/5}} < \infty. \end{aligned}$$

Theorem 1 in [21] now concludes that there exists $K_2 = \langle \cdot, h \rangle \in \mathcal{L}(W, \mathbb{C})$ such that $G_1 + b_1 K_2$ is a Riesz-spectral operator with eigenvalues $\sigma(G_1 + b_1 K_2) = \{\mu_k\}_k \subset \mathbb{C}^-$. Furthermore, all but a finite number of these eigenvalues are simple. This immediately concludes that the semigroup generated by $G_1 + b_1 K_2$ is uniformly bounded and that for $\omega \in \mathbb{R}$ with large $|\omega|$ the norm $\|R(i\omega, G_1 + b_1 K_2)\|$ satisfies

$$\|R(i\omega, G_1 + b_1 K_2)\| \leq \frac{M}{\mathrm{dist}(i\omega, \sigma(G_1 + b_1 K_2))} = \frac{M}{\min_{k\in\mathbb{Z}}|i\omega - \mu_k|}$$

for some $M > 0$. We have from [21, Thm. 1] that the appropriate feedback $K_2 = \langle \cdot, h\rangle$ is achieved by choosing $h \in W$ in such a way that

$$h = \sum_{k\in\mathbb{Z}} \frac{\overline{\mu_k} - \overline{i\omega_k}}{\langle \phi_k, b_1\rangle} \overline{h_k}\phi_k = \frac{1}{\langle \phi_0, g_2\rangle |P(0)|} h_0\phi_0 + \sum_{k\neq 0} \frac{-\frac{1}{k^2}}{\langle \phi_k, g_2\rangle |P(i\omega_k)|}\overline{h_k}\phi_k$$
$$= \frac{1}{|P(0)|} h_0\phi_0 - \sum_{k\neq 0} \frac{1}{|k|^{7/5}|P(i\omega_k)|}\overline{h_k}\phi_k,$$

where

$$h_k = \prod_{l\neq k} \frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} = \frac{i\omega_k - (-1)}{i\omega_k} \prod_{l\neq k,0} \frac{i\omega_k + \frac{1}{l^2} - i\omega_l}{i\omega_k - i\omega_l} = \left(1 - \frac{i}{\omega_k}\right) \prod_{l\neq k,0} \left(1 + i\frac{1}{l^2(\omega_l - \omega_k)}\right).$$

In the case of the first exosystem, we have $i\omega_k = ik$ for $k \in \mathbb{Z}$. Using some geometric analysis, we can show that there exists $c > 0$ such that for $\omega \in \mathbb{R}$ with large enough $|\omega|$ we have $|i\omega - \mu_k| \geq c|\omega|^{-2}$. Therefore the resolvent of the stabilized internal model satisfies

$$\|R(i\omega, G_1 + b_1K_2)\| \leq \frac{M}{\min_k |i\omega - \mu_k|} \leq \frac{M}{c|\omega|^{-2}} = \frac{M}{c}|\omega|^2.$$

This concludes that for this exosystem the internal model is polynomially stable with $\alpha = 2$.

On the other hand, in the case of the second exosystem we have $i\omega_k = ik^2$ for $k \in \mathbb{Z}$. Similarly we can show that there exists $c > 0$ such that for $\omega \in \mathbb{R}$ with large enough $|\omega|$ we have $|i\omega - \mu_k| \geq c|\omega|^{-1}$. In this case the resolvent of the stabilized internal model satisfies

$$\|R(i\omega, G_1 + b_1K_2)\| \leq \frac{M}{\min_k |i\omega - \mu_k|} \leq \frac{M}{c|\omega|^{-1}} = \frac{M}{c}|\omega|,$$

and thus the internal model is polynomially stable with $\alpha = 1$.

We have from Theorem 8 that the closed-loop system with the controller we have constructed is polynomially stable with α, and Theorem 5 concludes that the controller solves the robust output regulation problem, where $\alpha = 2$ in the case of the first exosystem, and $\alpha = 1$ for the second exosystem.

6.4 Perturbation of the Parameters of the Plant

We can now use Theorems 14 and 15 to study the preservation of the closed-loop stability and the output regulation property with respect to perturbations in the parameters of the plant. In the case of the first exosystem the closed-loop system is polynomially stable with $\alpha = 2$, and we can therefore use Theorem 14. In order to consider perturbation of operators B, C, or D, we would need to verify

$$\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*) \qquad \text{and/or} \qquad \mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1).$$

For the second condition to be true, we would in particular need $\mathcal{R}(G_2) \subset \mathcal{D}(G_1)$, or equivalently $g_2 \in \mathcal{D}(S)$. However, since

$$\sum_{k\neq 0} |\omega_k|^2 |\langle g_2, \phi_k\rangle|^2 = \sum_{k\neq 0} \frac{k^2}{|k|^{6/5}} = \sum_{k\neq 0} |k|^{4/5} = \infty,$$

we have $g_2 \notin \mathcal{D}(S)$, and thus the condition $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$ in Theorem 14 is not satisfied.

On the other hand, for the first condition to be true, we would need $\mathcal{R}(K_2^*) \subset \mathcal{D}(G_1^*)$, or equivalently $h \in \mathcal{D}(S^*) = \mathcal{D}(S)$. We can estimate

$$|h_k|^2 = \left| \left(1 - \frac{i}{\omega_k}\right) \prod_{l\neq k,0} \left(1 + i\frac{1}{l^2(\omega_l - \omega_k)}\right) \right|^2 = \left|1 - \frac{i}{\omega_k}\right|^2 \prod_{l\neq k,0} \left|1 + i\frac{1}{l^2(\omega_l - \omega_k)}\right|^2$$
$$= \left(1 + \frac{1}{\omega_k^2}\right) \prod_{l\neq k,0} \left(1 + \frac{1}{l^4(\omega_l - \omega_k)^2}\right) \geq 1$$

and since $|P(i\omega)| \leq \frac{3}{2}$, we have

$$\sum_{k\neq 0} |\omega_k|^2 |\langle h, \phi_k\rangle|^2 = \sum_{k\neq 0} k^2 \frac{|h_k|^2}{|k|^{14/5}|P(i\omega_k)|^2} \geq \frac{4}{9} \sum_{k\neq 0} \frac{1}{|k|^{4/5}} = \infty.$$

This implies that $h \notin \mathcal{D}(S)$, which concludes that the condition $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$ is not satisfied.

6.4.1 The Case $\alpha = 2$

The above properties conclude that in the case of the first exosystem we can only apply the first part of Theorem 14. We will use it to study the preservation of the closed-loop stability with respect to rank one perturbations of the operator A. Such perturbations are of the form

$$(A'x)(z) = x''(z) + f_1(z) \int_0^1 x(\xi)\overline{f_2(\xi)} dz = \big[(A + \langle \cdot, f_2 \rangle_{L^2} f_1)\, x\big](z), \qquad x \in \mathcal{D}(A)$$

where $f_1, f_2 \in L^2(0,1)$. Since $\Delta_A = f_1$ and $\tilde{\Delta}_A^* = f_2$, and since the operator A is selfadjoint, the conditions (13) are equivalent to $f_1, f_2 \in \mathcal{D}(A)$, i.e., f_1, f_2, f_1', f_2' are required to be absolutely continuous and $f_1(0) = f_1(1) = 0$, and $f_2(0) = f_2(1) = 0$. The appropriate graph norms in Theorem 14 are given by

$$\|A\Delta_A\| + \|\Delta_A\| = \|Af_1\|_{L^2} + \|f_1\|_{L^2} = \|f_1''\|_{L^2} + \|f_1\|_{L^2} \tag{18a}$$

$$\|A^*\tilde{\Delta}_A^*\| + \|\tilde{\Delta}_A\| = \|Af_2\|_{L^2} + \|f_2\|_{L^2} = \|f_2''\|_{L^2} + \|f_2\|_{L^2} \tag{18b}$$

Therefore, Theorem 14 concludes that the polynomial closed-loop stability as well as the decay of the regulation error are preserved for all perturbations of the form $A + \langle \cdot, f_2 \rangle f_1$, where $f_1, f_2 \in \mathcal{D}(A)$ and for which the norms

$$\|f_1\|_{L^2}, \quad \|f_2\|_{L^2}, \quad \|f_1''\|_{L^2}, \quad \|f_2''\|_{L^2}$$

are small enough.

6.4.2 The Case $\alpha = 1$

In the case of the second exosystem we have $\alpha = 1$, and we can apply Theorem 15 to study the preservation of the closed-loop stability. If we now perturb the operator A as $A + \langle \cdot, f_2 \rangle_{L^2} f_1$ where $f_1, f_2, \in L^2(0,1)$, then the conditions of Theorem 15 require that either

$$f_1 \in \mathcal{D}(A), \qquad \text{or} \qquad f_2 \in \mathcal{D}(A).$$

This means that f_i, f_i' must be absolutely continuous and $f_i(0) = f_i(1) = 0$ for $i = 1$ or $i = 2$. We thus have directly from (18) that closed-loop stability is preserved in the following situations:

- $f_1 \in \mathcal{D}(A)$, and the norms $\|f_1\|_{L^2}$, $\|f_1''\|_{L^2}$, and $\|f_2\|_{L^2}$ are small enough.
- $f_2 \in \mathcal{D}(A)$, and the norms $\|f_1\|_{L^2}$, $\|f_2\|_{L^2}$, and $\|f_2''\|_{L^2}$ are small enough.

On the other hand, if we perturb B in such a way that

$$B'u = \big(\chi_{[1/2,1]}(\cdot) + f_3(\cdot)\big)\, u = Bu + f_3(\cdot)u,$$

where $f_3 \in L^2(0,1)$, then $\Delta_B = f_3$. The conditions in Theorem 15 require that $f_3 \in \mathcal{D}(A)$ and the corresponding graph norms is given by

$$\|A\Delta_B\| + \|\Delta_B\| = \|f_3''\|_{L^2} + \|f_3\|_{L^2}.$$

We therefore have that the closed-loop stability is preserved if f_3, f_3' are absolutely continuous, if $f_3(0) = f_3(1) = 0$, and if the norms $\|f_3\|_{L^2}$ and $\|f_3''\|_{L^2}$ are small enough.

Finally, if we perturb C in such a way that

$$C'x = \int_0^1 \Big(\chi_{[0,1/2]}(z) + \overline{f_4(z)}\Big)\, x(z) dz = Cx + \langle x, f_4 \rangle_{L^2},$$

where $f_4 \in L^2(0,1)$, then $\tilde{\Delta}_C^* = f_4$. The conditions in Theorem 15 require that $f_4 \in \mathcal{D}(A)$ and the corresponding graph norms is given by

$$\|A^*\tilde{\Delta}_C^*\| + \|\tilde{\Delta}_C^*\| = \|f_4''\|_{L^2} + \|f_4\|_{L^2}.$$

We therefore have that the closed-loop stability is preserved if f_4, f_4' are absolutely continuous, if $f_4(0) = f_4(1) = 0$, and if the norms $\|f_4\|_{L^2}$ and $\|f_4''\|_{L^2}$ are small enough.

Similarly as in the beginning of Section 6.4 we can verify that neither $\mathcal{R}(K^*) \subset \mathcal{D}(\mathcal{G}_1^*)$ nor $\mathcal{R}(\mathcal{G}_2) \subset \mathcal{D}(\mathcal{G}_1)$. Because of this, Theorem 15 can not be applied in studying preservation of the closed-loop stability under perturbations in the operator D.

7 Conclusions

In this paper we have studied the robust output regulation problem for infinite-dimensional linear systems with reference and disturbance signals generated by an infinite-dimensional block-diagonal signal generator. In such a situation the internal model principle in particular implies that any robust controller must necessarily have an infinite number of eigenvalues on the imaginary axis, and therefore the closed-loop can not be stabilized exponentially. In the previous papers concerning this topic, the closed-loop system has been stabilized either strongly or weakly. However, if the closed-loop system is not exponentially stable, we need extraneous assumptions to ensure the solvability of the Sylvester equation in the regulator equations. Moreover, in the case of strong and weak closed-loop stabilities it is not in general possible to derive any concrete robustness properties for the control law.

In this paper we have demonstrated that the above difficulties can, for the most part, be overcome by instead aiming at polynomial stability of the closed-loop system. In particular, if the closed-loop system is polynomially stable we immediately have the appropriate solvability of the Sylvester equation in the regulator equations, and we can therefore remove some of the technical assumptions. Moreover, the recent results concerning the robustness properties of polynomially stable semigroups enabled us to derive concrete conditions under which the closed-loop stability and the output regulation property are preserved when the parameters of the plant are perturbed.

In this paper we have also constructed an observer based feedback controller that stabilizes the closed-loop system polynomially. We first showed that the closed-loop stability is achieved if the internal model in the controller can be stabilized polynomially. Moreover, in the case of a square plant and a diagonal exosystem we presented a method for stabilizing the internal model. Extending the polynomial stabilization of the internal model for plants with more inputs than outputs and for nondiagonal exosystems are topics for further research.

Further research topics also include extending the theory to allow unbounded control and observation operators in the plant.

A Required Lemmata

The following three lemmas are used in proving the main results of the paper.

Lemma 18. *If $a_k \geq 0$ for $k \in \{1, \dots, n\}$, then*

$$\sum_{k=1}^{n} a_k^2 \leq \left(\sum_{k=1}^{n} a_k\right)^2 \leq n \cdot \sum_{k=1}^{n} a_k^2$$

and if $a, b \geq 0$, and $\alpha > 0$, then

$$(a+b)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha)$$

Lemma 19. *Let X_1 and X_2 be Hilbert spaces. Let $A_1 : \mathcal{D}(A_1) \subset X_1 \to X_1$ and $A_2 : \mathcal{D}(A_2) \subset X_2 \to X_2$ generate strongly continuous semigroups $T_1(t)$ and $T_2(t)$, respectively, and $B \in \mathcal{L}(X_2, X_1)$. Consider the semigroup $T(t)$ generated by*

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} : \mathcal{D}(A_1) \times \mathcal{D}(A_2) \subset X \to X$$

on the Hilbert space $X = X_1 \times X_2$. If one of the semigroups $T_1(t)$ and $T_2(t)$ is exponentially stable, and the other polynomially stable with $\alpha > 0$, then the semigroup $T(t)$ is polynomially stable with α.

Proof. Since Definition 1 implies that a polynomially stable semigroup is strongly stable, we have from [8, Lem. 20] that the semigroup $T(t)$ is strongly stable. In particular this implies that $T(t)$ is uniformly bounded. By Lemma 2 it remains to show that $i\mathbb{R} \subset \rho(A)$ and $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$.

Let $i\omega \in i\mathbb{R}$. Since $i\omega \in \rho(A_1)$ and $i\omega \in \rho(A_2)$, a direct computation shows that

$$(i\omega - A)^{-1} = \begin{pmatrix} R(i\omega, A_1) & R(i\omega, A_1) B R(i\omega, A_2) \\ 0 & R(i\omega, A_2) \end{pmatrix} \in \mathcal{L}(X).$$

We therefore have $i\omega \in \rho(A)$. Since $i\omega \in i\mathbb{R}$ was arbitrary, this implies $i\mathbb{R} \subset \rho(A)$.

Let $x = (x_1, x_2)^T \in X$ be such that $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 = 1$. For brevity denote $R_1 = R(i\omega, A_1)$ and $R_2 = R(i\omega, A_2)$. Now

$$\begin{aligned}\|R(i\omega, A)x\|^2 &= \left\| \begin{pmatrix} R_1 & R_1BR_2 \\ 0 & R_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 = \|R_1x_1 + R_1BR_2x_2\|^2 + \|R_2x_2\|^2 \\ &\leq 2\left(\|R_1\|^2\|x_1\|^2 + \|R_1\|^2\|B\|^2\|R_2\|^2\|x_2\|^2\right) + \|R_2\|^2\|x_2\|^2 \\ &\leq 2\left(\|x_1\|^2 + \|x_2\|^2\right)\left(\|R_1\|^2 + \|R_1\|^2\|B\|^2\|R_2\|^2 + \|R_2\|^2\right) \\ &\leq 2\max\left\{\|B\|^2, 1\right\}\left(\|R_1\|^2(1 + \|R_2\|^2) + \|R_2\|^2\right) \\ &\leq 2(\|B\|^2 + 1)(\|R(i\omega, A_1)\|^2 + 1)(\|R(i\omega, A_2)\|^2 + 1)\end{aligned}$$

Due to the assumptions and Lemma 2 one of the norms $\|R(i\omega, A_1)\|$ and $\|R(i\omega, A_2)\|$ is of order $\mathcal{O}(|\omega|^\alpha)$, and the other is uniformly bounded. This together with the above estimate concludes that $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$. □

Lemma 20. *Let Z_1, Z_2, and $Z = Z_1 \times Z_2$ be Hilbert spaces and assume the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ of the controller are of the form*

$$\mathcal{G}_1 = \begin{pmatrix} R_1 & R_2 \\ 0 & G_1 \end{pmatrix}, \qquad \mathcal{G}_2 = \begin{pmatrix} R_3 \\ G_2 \end{pmatrix},$$

where $R_1 : \mathcal{D}(R_1) \subset Z_1 \to Z_1$, $R_2 \in \mathcal{L}(Z_2, Z_1)$, $R_3 \in \mathcal{L}(Y, Z_1)$, $G_1 : \mathcal{D}(G_1) \subset Z_2 \to Z_2$, and $G_2 \in \mathcal{L}(Y, Z_2)$. If $i\mathbb{R} \subset \rho(A_e)$ and the operators (G_1, G_2) satisfy the $\mathcal{G}$-conditions in Definition 4*, then $(\mathcal{G}_1, \mathcal{G}_2)$ satisfy the $\mathcal{G}$-conditions.*

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$. Then we clearly also have $G_2y = 0$. Since the operator G_2 satisfies the $\mathcal{G}$-conditions, this implies $y = 0$. This concludes $\mathcal{N}(\mathcal{G}_2) = \{0\}$.

Let $k \in \mathbb{Z}$ and $z = (z_1, z_2)^T \in \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. There exist $(z_1^1, z_2^1)^T \in \mathcal{D}(R_1) \times \mathcal{D}(G_1)$ and $y \in Y$ such that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} i\omega_k - R_1 & -R_2 \\ 0 & i\omega_k - G_1 \end{pmatrix} \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \begin{pmatrix} R_3 \\ G_2 \end{pmatrix} y.$$

The second line of the above equation implies that $z_2 = (i\omega_k - G_1)z_2^1 = G_2y$. Since (G_1, G_2) satisfy the $\mathcal{G}$-conditions, we must have $G_2y = 0$, and further that $y = 0$ due to $\mathcal{N}(G_2) = \{0\}$. This also implies $z = \mathcal{G}_2y = 0$. Since $k \in \mathbb{Z}$ was arbitrary, we have $\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all $k \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$. Since $i\omega_k \in \rho(A_e)$, we have from [14, Lem. 5.7] that $Z = \mathcal{R}(i\omega_k - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$. This together with the triangular form of the operator $i\omega_k - \mathcal{G}_1$ further imply that we similarly have $Z_2 = \mathcal{R}(i\omega_k - G_1) + \mathcal{R}(G_2)$. Since (G_1, G_2) satisfy the $\mathcal{G}$-conditions, the intersection of the ranges is trivial, and $Z_2 = \mathcal{R}(i\omega_k - G_1) \oplus \mathcal{R}(G_2)$. In particular this means that for every $z_2 \in Z_2$ there exist unique elements $z_2^1 \in \mathcal{D}(G_1)$ and $y \in Y$ such that $z_2 = (i\omega_k - G_1)z_2^1 + G_2y$.

Now let $z = (z_1, z_2)^T \in \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k}$. Since $Z = \mathcal{R}(i\omega_k - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$, there exist $(z_1^1, z_2^1)^T \in \mathcal{D}(R_1) \times \mathcal{D}(G_1)$ and $y \in Y$ such that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} i\omega_k - R_1 & -R_2 \\ 0 & i\omega_k - G_1 \end{pmatrix} \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} + \begin{pmatrix} R_3 \\ G_2 \end{pmatrix} y.$$

In order to show $z \in \mathcal{R}(i\omega_k - \mathcal{G}_1)$, it is clearly sufficient to prove that $y = 0$. The second line implies that $z_2 = (i\omega_k - G_1)z_2^1 + G_2y$. However, due to the triangular structure of $\mathcal{G}_1$ and $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k}$, we have $z_2 \in \mathcal{N}(i\omega_k - G_1)^{n_k} \subset \mathcal{R}(i\omega_k - G_1)$. Since $Z_2 = \mathcal{R}(i\omega_k - G_1) \oplus \mathcal{R}(G_2)$, we must necessarily have $y = 0$. This concludes that $z \in \mathcal{R}(i\omega_k - \mathcal{G}_1)$. Since $k \in \mathbb{Z}$ and $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k}$ were arbitrary, we have $\mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k} \subset \mathcal{R}(i\omega_k - \mathcal{G}_1)$. □

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