Reduced Order Internal Models in Robust Output Regulation

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In this paper we consider robust output regulation and the internal model principle for infinite-dimensional linear systems. We concentrate on a problem where the control law is required to be robust with respect to a restricted class of perturbations. We show that depending on the class of admissible perturbations, it is often possible to construct a robust controller with a smaller internal model than the one given by the internal model principle. In addition, we also look for minimal classes of perturbations that make the full internal model necessary. We introduce a straightforward way of testing for robustness of the control law for a given set of perturbations. The test in particular shows that the robustness is only dependent on the way the perturbations affect the transfer function of the plant at the frequencies of the exosystem. The theoretic results are applied to designing controllers for a one-dimensional wave equation and for a system consisting of three independent shock absorber models.

***Index Terms*—Robust control, linear systems, distributed parameter systems.**

I. Introduction

The concept of an internal model, referring to a part of a controller copying the dynamics of the exosystem, was introduced to the vocabulary of robust control in the 1970's by Francis and Wonham, and Davison. Their well-known *internal model principle* of output regulation theory states that the necessary and sufficient condition for a controller to be robust with respect to uncertainties and changes in the parameters of the plant is that the controller itself must contain p copies of the exosystem's dynamics, where p is the dimension of the output space [1], [2]. Since the pioneering work of Francis and Wonham, and Davison in finite-dimensional control, the problems of robust output tracking and disturbance rejection have been studied actively also for distributed parameter systems, see [3]–[10] and references therein. Recently, also the internal model principle was generalized for infinite-dimensional linear systems and for infinite-dimensional exosystems [10], [11].

In this paper we consider the control of linear infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) + Du(t). \tag{1b}$$

Our main intention is to take a closer look at the reasons that make the copies of the exosystems dynamics necessary and sufficient for the control law to be robust with respect to uncertainties in the parameters of the system (1). The question we aim to answer is that if we restrict the class of admissible perturbations of the operators (A, B, C, D), is it then possible to construct a robust controller with less than p copies of the exosystem's dynamics? To the authors' knowledge, this problem has not been studied earlier even in the case of finite-dimensional systems.

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Motivation for our study arises from the fact that in many applications the control law is not required to be robust with respect to *all* perturbations to the parameters of the system (1). Instead, the admissible perturbations often have a very special structure. As a simple example, we can consider a second order differential equation $\ddot{w}(t) + a_1\dot{w}(t) + a_0 w(t) = 0$ written as a first order system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where $x_1(t) = w(t)$ and $x_2(t) = \dot{w}(t)$. The structure of the system matrix in particular shows that the changes in the coefficients a_0 and a_1 of the original equation only affect the elements on the second line of the matrix.

Studying robustness of controllers with respect to restricted classes of perturbations leads to several new results presented in this paper. The most important one of these is a simple method for testing the robustness of a feedback controller with respect to perturbations in the operators A, B, C, and D of the system (1). For finite-dimensional systems this test for robustness reduces to verifying the solvability of certain matrix equations. Applying these results to the controller design for a finite-dimensional linear plant is illustrated in the example considered in Section VI. In addition to the practical value of this method, the new robustness conditions in particular reveal that the perturbations in the parameters of the plant affect the tracking of reference signals only through the change of the values $P(i\omega_k)$ of the transfer function $P(\lambda) = CR(\lambda, A)B + D$ at the frequencies $i\omega_k$ of the signal generator. Because of this, any uncertainties that do not change the behaviour of $P(\lambda)$ at these frequencies, do not affect the regulation property. We see an example of this phenomenon in Section IV, where we consider output tracking for a damped wave equation. In this example the uncertainties in the damping coefficient of the wave equation do not change the value of the transfer function at the frequency $\lambda = 0$, and thus do not affect the tracking of constant reference signals.

In this paper we also look for the smallest class of admissible perturbations such that the controller must contain a full p-copy internal model in order for the control law to be robust with respect to the perturbations in this class. In particular, earlier state space proofs of the internal model principle rely heavily on the controller being robust with respect to uncertainties also in the output operators of the exosystem. Robustness with respect to perturbations in these operators is often unnecessary, since the reference signals are

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usually known accurately. Therefore it is natural to ask if the internal model in the controller can be reduced if we do not require robustness with respect to such uncertainties. Our main result on this topic, stated in Theorem 1 below, shows that the requirement of robustness with respect to all arbitrarily small rank one perturbations in any one of the operators of the plant (1) is enough to necessitate the existence of a full internal model in the controller. In particular this concludes that a full internal model is necessary even if the control law is not required to be robust with respect to uncertainties in the output operators of the exosystem.

Theorem 1. *If the feedback control law is robust with respect to all arbitrarily small rank one perturbations in any one of the operators A, B, C, or D, then it necessarily incorporates a p-copy internal model of the exosystem.*

The proof of Theorem 1 also yields a lower bound for the number of copies of the individual frequencies of the exosystem that have to be included in the controller in order to ensure robustness with respect to given perturbations. For plants with an equal number of inputs and outputs this lower bound can in particular be computed directly from the chosen class of perturbations.

We illustrate using our results as a set of tools for testing the robustness of a controller with two examples. In the first example we consider output tracking of constant reference signals for a one-dimensional damped wave equation. We solve the output regulation problem with a one-dimensional feedback controller, and use our theoretic results to derive conditions for the preservation of the output tracking property. In the second example we consider output tracking for a system consisting of three identical and independent shock absorber models. We begin by building a one-dimensional controller to achieve tracking of reference signals with a single frequency component $i\beta$. We continue by using our results to examine the robustness properties of the control law. Finally, we augment the initial controller in such a way that it becomes robust with respect to a predetermined class of uncertainties.

In Section II we state the standing assumptions on the plant, exosystem, and the controller considered in this paper. In Section III we formulate the robust output regulation problem. The method for testing robustness with respect to given perturbations is introduced in Section IV. Theorem 1 is proved in Section V. In Section VI we consider an example of designing controllers for a system consisting of three independent shock absorber models. Section VII contains concluding remarks.

II. Mathematical Preliminaries

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1}$. The dual pairing on a Banach space and the inner product on a Hilbert space are both denoted by $\langle \cdot, \cdot \rangle$.

For $n \in \mathbb{N}$ we denote $X^n = X \times X \times \cdots \times X$ and $\mathcal{D}(A)^n = \mathcal{D}(A) \times \cdots \times \mathcal{D}(A)$ where a Banach space X and the domain $\mathcal{D}(A)$, respectively, are repeated n times. If $T \in \mathcal{L}(X, Y)$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T \in X^n$ for some $n \in \mathbb{N}$, then by $T\boldsymbol{x}$ we mean that the operator T is applied to all of the components of $\boldsymbol{x}$, i.e. $T\boldsymbol{x} = (Tx_1, \ldots, Tx_n)^T \in Y^n$.

In this paper we consider the control of a linear distributed parameter system of the form (1) on a Banach space X. The function $x(t) \in X$ is the state of the system, $u(t) \in U = \mathbb{C}^m$ the input and $y(t) \in Y = \mathbb{C}^p$ the output. The dimensions of the input space and the output space satisfy $p \leq m$. We assume that A generates a strongly continuous semigroup on X and that the rest of the operators are bounded in such a way that $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$. We further assume $\mathcal{R}(C) = Y$ and $\mathcal{N}(B) = \{0\}$. For $\lambda \in \rho(A)$ the transfer function of the plant is given by $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$.

The considered reference signals are assumed to be generated by a finite-dimensional exosystem

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{2a}$$
$$y_{ref}(t) = -Fv(t) \tag{2b}$$

on $W = \mathbb{C}^{n_W}$ with $S \in \mathcal{L}(W)$ and $-F \in \mathcal{L}(W, Y)$. The spectrum of $S \in \mathcal{L}(W)$ lies on the imaginary axis, and we can without loss of generality assume that S is in its Jordan canonical form. We denote $S = \operatorname{diag}(S_1, \ldots, S_q)$, where S_k is a Jordan block associated to an eigenvalue $i\omega_k$, and q is the number of Jordan blocks in the canonical form. In $W = \mathbb{C}^{n_W}$ we denote the standard Euclidean basis $(e_1, \ldots, e_{n_W})$ as $\{ \phi_k^l \in W \mid k \in 1, \ldots, q, \; l = 1, \ldots, n_k \}$, in such a way that

$$(\phi_1^1, \ldots, \phi_1^{n_1}, \phi_2^1, \ldots, \phi_2^{n_2}, \ldots, \phi_q^1, \ldots, \phi_q^{n_q}) = (e_1, e_2, \ldots, e_{n_W}),$$

where $n_k \in \mathbb{N}$ is the dimension of the Jordan block S_k. The form of S implies that for all $k \in \{1, \ldots, q\}$ the sequence $(\phi_k^l)_{l=1}^{n_k}$ is a Jordan chain of S, i.e. $(i\omega_k I - S)\phi_k^1 = 0$, and

$$(S - i\omega_k I)\phi_k^l = \phi_k^{l-1} \quad \forall\, l \in \{2, \ldots, n_k\}.$$

We assume the output operator $-F$ of the exosystem satisfies $F\phi_k^1 \neq 0$ for all $k \in \{1, \ldots, q\}$.

The plant can be written in standard form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{3a}$$
$$e(t) = Cx(t) + Du(t) + Fv(t) \tag{3b}$$

where $e(t) = y(t) - y_{ref}(t) \in Y$ is the regulation error and $v(t) \in W$ is the state of the exosystem (2). We assume that $\sigma(A) \cap \sigma(S) = \varnothing$ and that $P(i\omega_k)$ is surjective for all $k \in \{1, \ldots, q\}$.

In this paper we consider dynamic error feedback controllers of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z$$
$$u(t) = Kz(t)$$

on a Banach space Z. Here $z(t) \in Z$ is the state of the controller, the operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$. The closed-loop system consisting of the plant and the controller on the

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space $X_e = X \times Z$ with state $x_e(t) = (x(t), z(t))^T$ is given by

$$\begin{aligned}\dot{x}_e(t) &= A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0} = (x_0, z_0)^T \\ e(t) &= C_e x_e(t) + D_e v(t),\end{aligned}$$

where $C_e = (C, \quad DK)$, $D_e = F$,

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} \quad \text{and} \quad B_e = \begin{pmatrix} 0 \\ \mathcal{G}_2 F \end{pmatrix}.$$

The operator $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ generates a semigroup $T_{A_e}(t)$ on X_e.

A. The Classes of Perturbations to the Plant

In this paper we consider a situation where parameters of the plant (3) are perturbed in such a way that the operators A, B, C, and D are changed into $\tilde{A} : \mathcal{D}(\tilde{A}) \subset X \to X$, $\tilde{B} \in \mathcal{L}(U, X)$, $\tilde{C} \in \mathcal{L}(X, Y)$, and $\tilde{D} \in \mathcal{L}(U, Y)$, respectively. For $\lambda \in \rho(\tilde{A})$ we denote by $\tilde{P}(\lambda) = \tilde{C}R(\lambda, \tilde{A})\tilde{B} + \tilde{D}$ the transfer function of the perturbed plant. We likewise denote the operators of the closed-loop system consisting of the perturbed plant and the controller by $\tilde{C}_e = \begin{pmatrix} \tilde{C} & \tilde{D}K \end{pmatrix}$, and

$$\tilde{A}_e = \begin{pmatrix} \tilde{A} & \tilde{B}K \\ \mathcal{G}_2 \tilde{C} & \mathcal{G}_1 + \mathcal{G}_2 \tilde{D} K \end{pmatrix}.$$

The perturbation of the operators A, B, C, and D do not affect the operators B_e and D_e of the closed-loop system.

The perturbations $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in $\mathcal{O}$ are assumed to satisfy the following conditions:

(a) The perturbed system operator $\tilde{A}$ generates a strongly continuous semigroup on X and satisfies $i\omega_k \in \rho(\tilde{A})$ for all $k \in \{1, \ldots, q\}$.
(b) The perturbed closed-loop system is exponentially stable, i.e., $\tilde{A}_e$ generates an exponentially stable semigroup on X_e.

If the unperturbed closed-loop system is exponentially stable, then the above conditions are satisfied, in particular, for any bounded perturbations of small enough norms.

B. Special Operators

To state the main results of the paper, we need some additional notation. For $k \in \{1, \ldots, q\}$ and $n \in \mathbb{N}$ we define the operator $J_{\mathcal{G}_1}(i\omega_k) : \mathcal{D}(\mathcal{G}_1)^n \subset Z^n \to Z^n$ to be a block upper triangular operator with diagonal elements $i\omega_k - \mathcal{G}_1$ and identity operators I on the first superdiagonal, i.e.,

$$J_{\mathcal{G}_1}(i\omega_k) = \begin{pmatrix} i\omega_k - \mathcal{G}_1 & I & & \\ & i\omega_k - \mathcal{G}_1 & & \\ & & \ddots & I \\ & & & i\omega_k - \mathcal{G}_1 \end{pmatrix}.$$

The form of the operator $J_{\mathcal{G}_1}(i\omega_k)$ immediately implies that for all $\boldsymbol{z} = (z_{n_k}, \ldots, z_1) \in \mathcal{D}(\mathcal{G}_1)^{n_k}$ such that $\boldsymbol{z} \neq 0$ the condition $J_{\mathcal{G}_1}(i\omega_k)\boldsymbol{z} = 0$ is equivalent to $(z_l)_{l=1}^{n_k}$ forming a Jordan chain of $\mathcal{G}_1$ associated to the eigenvalue $i\omega_k$, i.e. $(i\omega_k - \mathcal{G}_1)z_1 = 0$ and $(\mathcal{G}_1 - i\omega_k)z_l = z_{l-1}$ for $l \in \{2, \ldots, n_k\}$.

For $k \in \{1, \ldots, q\}$ and for an operator $\tilde{A}$ we define a block triangular operator $\mathbb{R}(i\omega_k, \tilde{A}) \in \mathcal{L}(X^{n_k})$ by

$$\mathbb{R}(i\omega_k, \tilde{A}) = \begin{pmatrix} R(i\omega_k, \tilde{A}) & -R(i\omega_k, \tilde{A})^2 & \cdots & (-1)^{n_k-1} R(i\omega_k, \tilde{A})^{n_k} \\ & R(i\omega_k, \tilde{A}) & \cdots & (-1)^{n_k-2} R(i\omega_k, \tilde{A})^{n_k-1} \\ & & \ddots & \vdots \\ & & & R(i\omega_k, \tilde{A}) \end{pmatrix}.$$

For $k \in \{1, \ldots, q\}$ and for operators $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ satisfying $i\omega_k \in \rho(\tilde{A})$, we denote by $\tilde{\mathbb{P}}(i\omega_k) \in \mathcal{L}(U^{n_k}, Y^{n_k})$ the operator

$$\begin{aligned}\tilde{\mathbb{P}}(i\omega_k) &= \begin{pmatrix} \tilde{P}(i\omega_k) & -\tilde{C}R(i\omega_k, \tilde{A})^2 \tilde{B} & \cdots & (-1)^{n_k-1} \tilde{C} R(i\omega_k, \tilde{A})^{n_k} \tilde{B} \\ & \tilde{P}(i\omega_k) & \cdots & (-1)^{n_k-2} \tilde{C} R(i\omega_k, \tilde{A})^{n_k-1} \tilde{B} \\ & & \ddots & \vdots \\ & & & \tilde{P}(i\omega_k) \end{pmatrix} \\ &= \tilde{C}\mathbb{R}(i\omega_k, \tilde{A})\tilde{B} + \tilde{D}.\end{aligned}$$

For the operators A, B, C, and D of the nominal plant, we use the notation $\mathbb{P}(i\omega_k)$. Finally, for $k \in \{1, \ldots, q\}$ we use notation

$$\Phi_k = (\phi_k^{n_k}, \phi_k^{n_k-1}, \ldots, \phi_k^1)^T \in W^{n_k}. \tag{5}$$

In particular, it should be noted that if for some $k \in \{1, \ldots, q\}$ we have $n_k = 1$, then the above operators reduce to $J_{\mathcal{G}_1}(i\omega_k) = i\omega_k - \mathcal{G}_1$, $\mathbb{R}(i\omega_k, \tilde{A}) = R(i\omega_k, \tilde{A})$, and $\tilde{\mathbb{P}}(i\omega_k) = \tilde{P}(i\omega_k)$.

III. The Robust Output Regulation Problem

We begin by formulating our main control problem consisting of robust asymptotic output tracking. The main difference in our definition compared to corresponding problems encountered elsewhere in the literature is that we do not require the control law to be robust with respect to perturbations to the operator F.

The Robust Output Regulation Problem. Choose $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

1) The closed-loop system operator A_e generates an exponentially stable semigroup on X_e.
2) For all initial states $v_0 \in W$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.
3) If the operators (A, B, C, D) of the plant are changed to $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$, then $\lim_{t\to\infty} e(t) = 0$ for all initial states $v_0 \in W$ and $x_{e0} \in X_e$.

Parts 1. and 2. of the problem, i.e., the control problem without the robustness aspect, are called the *output regulation problem*. The following definition clarifies the terminology used throughout the paper.

Definition 2. If the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies parts 1. and 2. of the robust output regulation problem, it is said to *solve the output regulation problem*. If the controller solves the robust output regulation problem (with respect to given perturbations), it is called *robust* (with respect to given perturbations).

Robustness of a controller with respect to given perturbations can be characterized using the Sylvester type regulator equations [1], [6].

Theorem 3. *A controller solving the output regulation problem is robust with respect to given perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}$ if and only if the perturbed regulator equations*

$$\Sigma S = \tilde{A}_e \Sigma + B_e \tag{6a}$$
$$0 = \tilde{C}_e \Sigma + D_e \tag{6b}$$

have a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e)$.

Proof. This is a direct consequence of Theorem 3.1 in [11]. □

IV. Characterizing the Robustness Properties of a Controller

In this section we lay the groundwork for the theoretic consideration in the rest of the paper. As a first step, we present a new way of characterizing robustness of a controller. In particular, this characterization has a special property that the effect of the perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}$ is visible only through the perturbation of the transfer function $P(i\omega_k)$ of the plant at the frequencies $i\omega_k$ of the exosystem. The characterization also has an advantage that the the frequencies of the controller can be considered separately, and the conditions can be used as a test for the robustness of a controller with respect to given perturbations. Theorem 4 makes use of the operators defined in Section II-B.

Theorem 4. *A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solving the output regulation problem is robust with respect to given perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}$ if and only if the equations*

$$\tilde{\mathbb{P}}(i\omega_k) K \boldsymbol{z}^k = -F\Phi_k \tag{7a}$$
$$J_{\mathcal{G}_1}(i\omega_k)\boldsymbol{z}^k = 0 \tag{7b}$$

have a solution $\boldsymbol{z}^k = (z_{n_k}^k, \ldots, z_1^k) \in \mathcal{D}(\mathcal{G}_1)^{n_k}$ for all $k \in \{1, \ldots, q\}$. Moreover, for every $k \in \{1, \ldots, q\}$ the solution of (7) is unique.

It should be noted that Theorem 4 requires the perturbed closed-loop system to be exponentially stable. This is not stated explicitly, since the perturbations in $\mathcal{O}$ preserve the closed-loop stability by the standing assumptions made in Section II-A.

Although the operators involved in equations (7) may seem complicated, they consist of well-known elements. In particular, the diagonal elements of the operator $\tilde{\mathbb{P}}(i\omega_k)$ are equal to the transfer function of the perturbed plant at the frequency $i\omega_k$, i.e., $\tilde{P}(i\omega_k) = \tilde{C}R(i\omega_k, \tilde{A})\tilde{B} + \tilde{D}$, and its other elements are derivatives of $\tilde{P}(\lambda)$ (up to multiplication by scalars) at this same frequency. In practical applications the values of these matrix-valued functions can be approximated numerically. The verification of the conditions (7) reduces to computations with matrices for finite-dimensional systems, as well as for infinite-dimensional systems with finite-dimensional controllers. Using Theorem 4 to analyze the robustness properties of a controller in these situations is illustrated in Example 6 and in Section VI.

The proof of the theorem is based on the following properties of the regulator equations. The proof of Lemma 5 is presented in the Appendix.

Lemma 5. *Let $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}$. An operator $\Sigma = (\Pi, \Gamma)^T \in \mathcal{L}(W, X_e)$ with $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e) = \mathcal{D}(\tilde{A}) \times \mathcal{D}(\mathcal{G}_1)$ satisfies the Sylvester equation $\Sigma S = \tilde{A}_e \Sigma + B_e$ if and only if*

$$J_{\mathcal{G}_1}(i\omega_k)\Gamma\Phi_k = \mathcal{G}_2\left(\tilde{\mathbb{P}}(i\omega_k)K\Gamma\Phi_k + F\Phi_k\right) \tag{8a}$$
$$\Pi\Phi_k = \mathbb{R}(i\omega_k, \tilde{A})\tilde{B}K\Gamma\Phi_k \tag{8b}$$

for all $k \in \{1, \ldots, q\}$. If $\Sigma = (\Pi, \Gamma)^T$ is a solution of the equation $\Sigma S = \tilde{A}_e \Sigma + B_e$, then

$$\tilde{C}_e \Sigma \Phi_k + D_e \Phi_k = \tilde{\mathbb{P}}(i\omega_k)K\Gamma\Phi_k + F\Phi_k \tag{8c}$$

for all $k \in \{1, \ldots, q\}$.

Proof of Theorem 4. Let $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}$. We will first show that robustness of a controller with respect to the given perturbations implies that the equations (7) have solutions for all $k \in \{1, \ldots, q\}$. Since the perturbed closed-loop system is exponentially stable, the robustness of the controller together with Theorem 3 implies that the perturbed regulator equations (6) have a solution $\Sigma = (\Pi, \Gamma)^T \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Pi) \subset \mathcal{D}(\tilde{A})$ and $\mathcal{R}(\Gamma) \subset \mathcal{D}(\mathcal{G}_1)$. Let $k \in \{1, \ldots, q\}$. We now have from (8a) and (8c) in Lemma 5 that the perturbed regulator equations (6) in particular imply

$$J_{\mathcal{G}_1}(i\omega_k)\Gamma\Phi_k = \mathcal{G}_2\left(\tilde{\mathbb{P}}(i\omega_k)K\Gamma\Phi_k + F\Phi_k\right)$$
$$0 = \tilde{\mathbb{P}}(i\omega_k)K\Gamma\Phi_k + F\Phi_k.$$

If we choose $\boldsymbol{z}^k = \Gamma\Phi_k \in \mathcal{D}(\mathcal{G}_1)^{n_k}$, then (7a) follows immediately from the second equation. Furthermore, substituting the second equation into the right-hand side of the first further concludes $J_{\mathcal{G}_1}(i\omega_k)\boldsymbol{z}^k = 0$, and thus $\boldsymbol{z}^k$ is the solution of the equations (7). Since $k \in \{1, \ldots, q\}$ was arbitrary, this concludes the first part of the proof.

Now assume that for all $k \in \{1, \ldots, q\}$ the equations (7) have solutions $\boldsymbol{z}^k = (z_{n_k}^k, \ldots, z_1^k) \in \mathcal{D}(\mathcal{G}_1)^{n_k}$. Define operators $\Pi \in \mathcal{L}(W, X)$, $\Gamma \in \mathcal{L}(W, Z)$, and $\Sigma \in \mathcal{L}(W, X_e)$ by

$$\Gamma = \sum_{k=1}^{q}\sum_{l=1}^{n_k} \langle \cdot, \phi_k^l \rangle z_l^k, \tag{9a}$$
$$\Pi = \sum_{k=1}^{q}\sum_{l=1}^{n_k} \langle \cdot, \phi_k^l \rangle \sum_{j=0}^{l-1} (-1)^j R(i\omega_k, \tilde{A})^{j+1}\tilde{B}K z_{l-j}^k, \tag{9b}$$

and $\Sigma = (\Pi, \Gamma)^T$. We will show that Σ is a solution of the perturbed regulator equations (6). First of all, we have $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}) \times \mathcal{D}(\mathcal{G}_1) = \mathcal{D}(\tilde{A}_e)$. Let $k \in \{1, \ldots, q\}$. For $l \in \{1, \ldots, n_k\}$ we have $\Gamma\phi_k^l = z_l^k$, which together with the definition of Π and the definition of Φ_k in (5) implies that (8b) is satisfied. Furthermore, since $\Gamma\Phi_k = \boldsymbol{z}^k$, we have from (7) that

$$J_{\mathcal{G}_1}(i\omega_k)\Gamma\Phi_k = J_{\mathcal{G}_1}(i\omega_k)\boldsymbol{z}^k = 0 = \mathcal{G}_2\left(\tilde{\mathbb{P}}(i\omega_k)K\boldsymbol{z}^k + F\Phi_k\right)$$
$$= \mathcal{G}_2\left(\tilde{\mathbb{P}}(i\omega_k)K\Gamma\Phi_k + F\Phi_k\right),$$

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which is precisely (8a). Lemma 5 now implies that Σ is a solution of the Sylvester equation $\Sigma S = \tilde{A}_e\Sigma + B_e$. Furthermore, Lemma 5 and equation (7a) imply that we have

$$\begin{aligned}\tilde{C}_e\Sigma\Phi_k + D_e\Phi_k &= \tilde{\mathbb{P}}(i\omega_k)K\Gamma\Phi_k + F\Phi_k\\ &= \tilde{\mathbb{P}}(i\omega_k)K\boldsymbol{z}^k + F\Phi_k = 0.\end{aligned}$$

This concludes that Σ is a solution of the perturbed regulator equations, and thus by Theorem 3 the controller is robust with respect to the given perturbations.

It remains to prove that if the equations (7) have solutions for all $k \in \{1,\ldots,q\}$, then these solutions are unique. We first note that since the perturbed closed-loop system is exponentially stable and since S is a finite-dimensional operator with $\sigma(S) \subset i\mathbb{R}$, the solution of the Sylvester equation (6a) is unique [12].

Assume that for some $k_0 \in \{1,\ldots,q\}$ the equations (7) have solutions $\boldsymbol{z}^{k_0}, \tilde{\boldsymbol{z}}^{k_0} \in \mathcal{D}(\mathcal{G}_1)^{n_k}$. Let $\boldsymbol{z}^k$ be solutions of (7) for $k \neq k_0$. As in the second part of this proof, we can define $\Sigma = (\Pi,\Gamma)^T$ and $\tilde{\Sigma} = (\tilde{\Pi},\tilde{\Gamma})^T$ with the formulas in (9) using the sets $(\boldsymbol{z}^1,\ldots,\boldsymbol{z}^{k_0},\ldots,\boldsymbol{z}^q)$ and $(\boldsymbol{z}^1,\ldots,\tilde{\boldsymbol{z}}^{k_0},\ldots,\boldsymbol{z}^q)$ of vectors, respectively. As above, we have from Lemma 5 that Σ and $\tilde{\Sigma}$ are solutions of the Sylvester equation (6a). Since the solution of this equation is unique, we must have $\Sigma = \tilde{\Sigma}$. Due to the definitions of Γ and $\tilde{\Gamma}$ this is only possible if $\boldsymbol{z}^{k_0} = \tilde{\boldsymbol{z}}^{k_0}$. □

Before moving on in the theory, we consider the control of a one-dimensional wave equation. The example illustrates the use of Theorem 4 in testing the robustness properties of the controller, and in particular in identifying classes of perturbations that do not destroy the controllers ability to track the reference signals.

Example 6. We consider output tracking for a damped one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2}(z,t) + \alpha\frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + (b_1(z),\ b_2(z))u(t)$$

on the interval $[0,1]$, with boundary conditions $w(0,t) = w(1,t) = 0$ and initial conditions $w(z,0) = w_0(z)$, and $\frac{\partial w}{\partial t}(z,0) = w_1(z)$. The damping coefficient α is assumed to be positive. The control input satisfies $u(t) \in U = \mathbb{C}^2$ and $b_1(\cdot), b_2(\cdot) \in L^2(0,1)$. The output of the system is of the form

$$y(t) = \int_0^1 \begin{pmatrix} c_1(z)\\ c_2(z)\end{pmatrix} w(z,t)dz,$$

where $c_1(\cdot), c_2(\cdot) \in L^2(0,1)$ and $y(t) \in Y = \mathbb{C}^2$.

We denote $A_0x(z) = x''(z)$ with domain

$$\mathcal{D}(A_0) = \big\{x \in L^2(0,1) \ \big|\ x,\ x' \text{ abs. cont.},\ x'' \in X,\ x(0) = x(1) = 0\big\}$$

and $x(t) = \big(w(\cdot,t), \frac{\partial w}{\partial t}(\cdot,t)\big)^T$,

$$A = \begin{pmatrix} 0 & I\\ A_0 & -\alpha\end{pmatrix}, \qquad B = \begin{pmatrix} 0\\ B_0\end{pmatrix}, \qquad C = (C_0,\ 0),$$

where B_0 and C_0 are such that

$$B_0u = \big(b_1(z),\ b_2(z)\big)\begin{pmatrix} u_1\\ u_2\end{pmatrix}, \quad C_0x = \begin{pmatrix} \int_0^1 c_1(z)x(z)dz\\ \int_0^1 c_2(z)x(z)dz\end{pmatrix}.$$

Now $-A_0$ is a positive operator, the system to be controlled is of the form (1) on the Hilbert space $X = \mathcal{D}((-A_0)^{1/2}) \times L^2(0,1)$, and the operator A generates an exponentially stable semigroup on X.

We want to consider output tracking of constant reference signals of the form $y_{ref}(t) \equiv c\cdot(1,\ 2)^T$. To this end, we choose an exosystem on $W = \mathbb{C}$ with parameters $S = 0 \in \mathbb{C}$ and $F = (-1,\ -2)^T$. The transfer function of the system at $i\omega_0 = 0$ is given by

$$\begin{aligned}P(0) &= CR(0,A)B = -CA^{-1}B\\ &= -(C_0,\ 0)\begin{pmatrix} \alpha A_0^{-1} & A_0^{-1}\\ I & 0\end{pmatrix}\begin{pmatrix} 0\\ B_0\end{pmatrix} = -C_0A_0^{-1}B_0.\end{aligned}$$

If the functions $b_1(\cdot), b_2(\cdot), c_1(\cdot), c_2(\cdot)$ are such that $\operatorname{rank} C_0A_0^{-1}B_0 = 2$, then $P(0)$ is invertible.

As the controller, we choose a one-dimensional dynamic error feedback controller

$$\begin{aligned}\dot{z}(t) &= 0\cdot z(t) + \varepsilon\left(1,\ 0\right)e(t), \qquad z(0) = z_0 \in Z\\ u(t) &= P(0)^{-1}Fz(t).\end{aligned}$$

on $Z = \mathbb{C}$, i.e., where $\mathcal{G}_1 = 0$, $\mathcal{G}_2 = \varepsilon(1,\ 0)$, and $K = P(0)^{-1}F$ with $\varepsilon > 0$. For $z = -1 \in \mathcal{N}(0 - \mathcal{G}_1) = \mathbb{C}$ we have

$$P(0)Kz = P(0)(P(0)^{-1}F)\cdot(-1) = -F,$$

and similarly as in Theorem 4 we can conclude that the controller solves the output regulation problem provided that the closed-loop system is exponentially stable.

It remains to show that $\varepsilon > 0$ can be chosen in such a way that the closed-loop system is exponentially stable. For this we will use techniques similar to the ones in [13, App. B]. Denote $\mathcal{G}_2^0 = (1,\ 0)$ and choose $H = \varepsilon\mathcal{G}_2^0CA^{-1}$ and $T = T^{-1} = \left(\begin{smallmatrix} I & 0\\ H & -I\end{smallmatrix}\right)$. The operator A_e is similar to

$$\begin{aligned}&\begin{pmatrix} I & 0\\ H & -I\end{pmatrix}\left[\begin{pmatrix} A & 0\\ \varepsilon\mathcal{G}_2^0C & 0\end{pmatrix} + \begin{pmatrix} B\\ \varepsilon\mathcal{G}_2^0D\end{pmatrix}\begin{pmatrix}0 & K\end{pmatrix}\right]\begin{pmatrix} I & 0\\ H & -I\end{pmatrix}\\
&= \begin{pmatrix} A & 0\\ \varepsilon\mathcal{G}_2^0C - \varepsilon\mathcal{G}_2^0C & 0\end{pmatrix}\\
&\quad + \begin{pmatrix} B\\ -\varepsilon\mathcal{G}_2^0C(-A)^{-1}B - \varepsilon\mathcal{G}_2^0D\end{pmatrix}\begin{pmatrix}\varepsilon K\mathcal{G}_2^0CA^{-1} & -K\end{pmatrix}\\
&= \begin{pmatrix} A + \varepsilon BK\mathcal{G}_2^0CA^{-1} & -BK\\ 0 & \varepsilon\mathcal{G}_2^0P(0)K\end{pmatrix}\\
&\quad + \varepsilon^2\begin{pmatrix} 0 & 0\\ -\mathcal{G}_2^0P(0)K\mathcal{G}_2^0CA^{-1} & 0\end{pmatrix}.\end{aligned}$$

Since A generates an exponentially stable semigroup and since we have $\varepsilon\mathcal{G}_2^0P(0)K = \varepsilon(1,\ 0)P(0)(P(0)^{-1}F) = -\varepsilon$, we can use standard perturbation theory to conclude that for small enough $\varepsilon > 0$ the closed-loop system is exponentially stable.

We can now investigate the robustness of the control law. Since $\dim Y = 2 > 1 = \dim\mathcal{N}(0 - \mathcal{G}_1)$, the controller is not guaranteed to be robust with respect to all perturbations in the parameters of the plant. However, Theorem 4 tells us that the control law is robust with respect to all perturbations

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that preserve the stability of the closed-loop system and for which the equations (7) have a solution. In particular, if the damping coefficient α of the original system is changed by a small amount, then for the perturbed transfer function we have

$$\begin{aligned}\tilde{P}(0) &= CR(0,\tilde{A})B = -\left(C_0,\ 0\right)\begin{pmatrix}\tilde{\alpha}A_0^{-1} & A_0^{-1}\\ I & 0\end{pmatrix}\begin{pmatrix}0\\ B_0\end{pmatrix}\\ &= -C_0A_0^{-1}B_0 = P(0).\end{aligned}$$

Because the perturbation is not visible in the value of the transfer function at 0, we can conclude using Theorem 4 that small changes in α do not affect the tracking of constant reference signals.

In a more general case, the effect of the perturbations to the transfer function of the plant can be written as $\tilde{P}(\lambda) = P(\lambda) + \Delta(\lambda)$ where $\Delta(\lambda) = \left(\begin{smallmatrix}\delta_{11}(\lambda) & \delta_{12}(\lambda)\\ \delta_{21}(\lambda) & \delta_{22}(\lambda)\end{smallmatrix}\right)$. If we write $\mu = 1/z \in \mathbb{C}$ for $z \in \mathcal{N}(0 - \mathcal{G}_1) = \mathbb{C}$, then the solvability of the equations in Theorem 4 is equivalent to

$$\begin{aligned} & \tilde{P}(0)Kz = -F \\ \Leftrightarrow\quad & \left(P(0) + \Delta(0)\right)\left(P(0)^{-1}F\right) = -\mu F \\ \Leftrightarrow\quad & \Delta(0)P(0)^{-1}F = -(1+\mu)F \end{aligned}$$

for some $\mu \in \mathbb{C}$. This concludes that the controller is robust with respect to any small enough perturbations to the parameters of the plant if and only if the perturbation $\Delta(0) = \tilde{P}(0) - P(0)$ is such that multiplication by $\Delta(0)P(0)^{-1}$ does not change the direction of F, i.e., $\Delta(0)P(0)^{-1}F \subset \operatorname{span} F$. This condition, in turn, can be written as a set of conditions on the relationships between the components $\delta_{ij}(0)$ of the perturbing function $\Delta(\cdot)$ evaluated at $\lambda = 0$.

V. Necessity of the p-Copy Internal Model

In this section we study situations in which the controller is required to incorporate a p-copy internal model in order for it to achieve robustness with respect to the given class of perturbations. In particular, we are interested in finding minimal classes of perturbations necessitating a full internal model, i.e., the dynamics of the exosystem must be copied p times in the controller. The following theorem states the main result of this section.

Theorem 7. *Let $\varepsilon > 0$. If the controller is robust with respect to all rank one perturbations of norm smaller than ε in any one of the operators A, B, C, or D, then it incorporates a p-copy internal model of the exosystem*

The proof of the theorem is based on choosing the perturbations to the operators in such a way that the equations (7) have a sufficient number of linearly independent solutions. We begin by proving a general lower bound for the numbers of copies of the exosystem's dynamics that must be included in a robust controller.

Theorem 8. *Let $k \in \{1,\dots,q\}$, let $P_K^\dagger$ be the Moore-Penrose pseudoinverse of the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k-\mathcal{G}_1)}$, and denote $\Delta = \tilde{P}(i\omega_k) - P(i\omega_k)$. Define*

$$p_k = \dim\operatorname{span}\left\{\, (I + \Delta K P_K^\dagger)^{-1}F\phi_k^1 \;\middle|\; (\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}_0 \,\right\},$$

where $\mathcal{O}_0 \subset \mathcal{O}$ is a given class of perturbations for which the operators $I + \Delta K P_K^\dagger$ are invertible. If the controller is robust with respect to perturbations in $\mathcal{O}_0$, then

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq p_k$$

and $\mathcal{G}_1$ has at least p_k independent Jordan chains of lengths greater than or equal to n_k associated to the eigenvalue $i\omega_k$.

Proof. Let $k \in \{1,\dots,q\}$. We have from [11, Lem. 6.3] that $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k-\mathcal{G}_1)}$ is injective, and therefore its pseudoinverse $P_K^\dagger$ is a left inverse, i.e., $P_K^\dagger P(i\omega_k)Kz = z$ for all $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$.

Theorem 4 implies that if the controller is robust with respect to given perturbations, then the equations (7) have a solution $\boldsymbol{z} = (z_{n_k},\dots,z_1) \in \mathcal{D}(\mathcal{G}_1)^{n_k}$. The equation $J_{\mathcal{G}_1}(i\omega_k)\boldsymbol{z} = 0$ and the form of the operator $J_{\mathcal{G}_1}(i\omega_k)$ now immediately imply that any such $\boldsymbol{z}$ satisfies

$$(i\omega_k - \mathcal{G}_1)z_1 = 0, \quad (i\omega_k - \mathcal{G}_1)z_l = z_{l-1}, \quad l \in \{2,\dots,n_k\}.$$

Therefore, if $z_1 \neq 0$, then $\{z_l\}_{l=1}^{n_k}$ is a Jordan chain of $\mathcal{G}_1$ associated to the eigenvalue $i\omega_k$. In order to prove the theorem, it is thus sufficient to show that the number of solutions $\boldsymbol{z}$ of equations (7) with linearly independent elements z_1 is at least the number of linearly independent elements

$$(I + \Delta K P_K^\dagger)^{-1}F\phi_k^1 \tag{10}$$

given by different perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}_0$.

If $\boldsymbol{z} = (z_{n_k},\dots,z_1) \in \mathcal{N}(J_{\mathcal{G}_1}(i\omega_k))$ is the unique solution of (7), then the last line of the matrix equation (7a) implies that

$$\begin{aligned} & \tilde{P}(i\omega_k)Kz_1 = -F\phi_k^1 \\ \Leftrightarrow\quad & (P(i\omega_k)K + \Delta K)z_1 = -F\phi_k^1 \\ \Leftrightarrow\quad & (I + \Delta K P_K^\dagger)P(i\omega_k)Kz_1 = -F\phi_k^1 \\ \Leftrightarrow\quad & P(i\omega_k)Kz_1 = -(I + \Delta K P_K^\dagger)^{-1}F\phi_k^1 \end{aligned}$$

if the perturbations are such that the operator $I + \Delta K P_K^\dagger$ is invertible.

From the above it is immedate that the number of linearly independent solutions of the above equation must be at least the number of linearly independent elements (10). Since Jordan chains originating from linearly independent vectors are independent, this concludes the proof. □

For the proof of Theorem 7 we also need the following two lemmas. The latter in particular contains a version of a well-known result, which states that if the feedback controller solves the output regulation problem, then it must contain a copy of the dynamics of the exosystem.

Lemma 9. *If the controller solves the output regulation problem, then for all $k \in \{1,\dots,q\}$ we have $KP_K^\dagger F\phi_k^1 \neq 0$, where $P_K^\dagger$ is the Moore-Penrose pseudoinverse of $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k-\mathcal{G}_1)}$.*

Proof. Let $k \in \{1,\dots,q\}$. Due to the standing assumptions made in Section II we have $F\phi_k^1 \neq 0$. The fact that the controller solves the output regulation problem implies that the regulator equations (6) for the nominal plant have a unique

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solution $\Sigma = (\Pi, \Gamma)^T$. Lemma 5 and the forms of the operators $J_{\mathcal{G}_1}(i\omega_k)$ and $\mathbb{P}(i\omega_k)$ imply that in particular the operator Γ satisfies

$$P(i\omega_k)K\Gamma\phi_k^1 = -F\phi_k^1 \tag{11a}$$
$$(i\omega_k - \mathcal{G}_1)\Gamma\phi_k^1 = 0. \tag{11b}$$

Since $F\phi_k^1 \neq 0$, we must clearly have $K\Gamma\phi_k^1 \neq 0$. We have from [11, Lem. 6.3] that the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k - \mathcal{G}_1)}$ is injective, and therefore its pseudoinverse $P_K^\dagger$ is a left inverse, i.e. $P_K^\dagger P(i\omega_k)Kz = z$ for all $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$. Because of this, applying $-KP_K^\dagger$ to both sides of (11a) implies

$$KP_K^\dagger F\phi_k^1 = -KP_K^\dagger P(i\omega_k)K\Gamma\phi_k^1 = -K\Gamma\phi_k^1 \neq 0,$$

since $\Gamma\phi_k^1 \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$ by (11b). This concludes the proof. □

Lemma 10. *If the controller solves the output regulation problem, then for all $k \in \{1, \ldots, q\}$ we have $i\omega_k \in \sigma_p(\mathcal{G}_1)$ and $\mathcal{G}_1$ has a Jordan chain of length greater than or equal to n_k associated to $i\omega_k$.*

Proof. Let $k \in \{1, \ldots, q\}$. Since the controller solves the output regulation problem, the regulator equations (6) for the nominal plant have a unique solution $\Sigma = (\Pi, \Gamma)^T$. Lemma 5 further implies that the operator Γ satisfies

$$\mathbb{P}(i\omega_k)K\Gamma\Phi_k = -F\Phi_k$$
$$J_{\mathcal{G}_1}(i\omega_k)\Gamma\Phi_k = 0.$$

The form of the operator $J_{\mathcal{G}_1}(i\omega_k)$ and the definition of Φ_k in (5) imply $(i\omega_k - \mathcal{G}_1)\Gamma\phi_k^1 = 0$, and

$$(i\omega_k - \mathcal{G}_1)\Gamma\phi_k^l = \Gamma\phi_k^{l-1}, \quad \forall\, l \in \{2, \ldots, n_k\}.$$

Therefore $\{\Gamma\phi_k^l\}_{l=1}^{n_k}$ is a Jordan chain with the desired qualities provided that $\Gamma\phi_k^1 \neq 0$. However, the first equation above implies that we must have $P(i\omega_k)K\Gamma\phi_k^1 = -F\phi_k^1$. Since $F\phi_k^1 \neq 0$ by the standing assumptions made in Section II, we necessarily have $\Gamma\phi_k^1 \neq 0$. This concludes the proof. □

Proof of Theorem 7. We will show that robustness of a controller with respect to arbitrarily small rank one perturbations to any one of the operators A, B, C, or D implies that the controller must incorporate a full p-copy internal model. The frequencies $i\omega_k$ of the exosystem can be considered separately, and we can therefore let $k \in \{1, \ldots, q\}$ be fixed for the rest of the proof.

Our aim is to show that any one of the operators A, B, C, or D can be perturbed in such a way that the perturbations produce p linearly independent elements

$$\tilde{y}_n = (I + \Delta K P_K^\dagger)^{-1} F\phi_k^1, \tag{12}$$

where $\Delta = \tilde{P}(i\omega_k) - P(i\omega_k)$. Once we have shown this, Theorem 8 concludes that $\mathcal{G}_1$ must have p independent Jordan chains of lengths greater than or equal to n_k associated to the eigenvalue $i\omega_k$.

Later in this proof we will show that if we consider rank one perturbations to the parameters of the plant, then the change of the operator $P(i\omega_k)$ is of the form

$$\Delta = \tilde{P}(i\omega_k) - P(i\omega_k) = \Delta_n\Delta_0 = \varepsilon_n\langle\cdot, u^*\rangle y_n,$$

where $\Delta_n = \varepsilon_n y_n \in \mathcal{L}(\mathbb{C}, Y)$, $\Delta_0 = \langle\cdot, u^*\rangle \in \mathcal{L}(U, \mathbb{C})$. For such perturbations with small enough $\varepsilon_n > 0$ we can use the Sherman-Morrison formula to show

$$\tilde{y}_n = (I + \Delta K P_K^\dagger)^{-1} F\phi_k^1 \tag{13a}$$
$$= \left(I - \Delta_n(1 + \Delta_0 K P_K^\dagger \Delta_n)^{-1}\Delta_0 K P_K^\dagger\right) F\phi_k^1 \tag{13b}$$
$$= \left(I - \varepsilon_n y_n \frac{\langle K P_K^\dagger \cdot, u^*\rangle}{1 + \varepsilon_n\langle K P_K^\dagger y_n, u^*\rangle}\right) F\phi_k^1 \tag{13c}$$
$$= F\phi_k^1 - \varepsilon_n y_n \frac{\langle K P_K^\dagger F\phi_k^1, u^*\rangle}{1 + \varepsilon_n\langle K P_K^\dagger y_n, u^*\rangle}. \tag{13d}$$

If u^* is chosen in such a way that $\langle KP_K^\dagger F\phi_k^1, u^*\rangle \neq 0$, then for different y_n the fractions in the second term are nonzero scalar coefficients. Therefore, the maximum possible number of linearly independent $\tilde{y}_n$ is equal to the number of linearly independent y_n.

This concludes that in order to prove the theorem, it suffices to show that with suitable perturbations of the operators A, B, C, and D we can achive p perturbations $\Delta = \tilde{P}(i\omega_k) - P(i\omega_k) = \varepsilon_n\langle\cdot, u^*\rangle y_n$ for $n \in \{1, \ldots, p\}$, where u^* satisfies $\langle KP_K^\dagger F\phi_k^1, u^*\rangle \neq 0$ and $\{y_n\}_{n=1}^p$ are linearly independent. In choosing the suitable elements u^* and $\{y_n\}_{n=1}^p$ we will use the standing assumptions $\mathcal{R}(C) = Y$ and $\mathcal{N}(B) = \{0\}$ (which implies $\mathcal{R}(B^*) = U$) made in Section II.

If we perturb the operator D as $\tilde{D} = D + \varepsilon_n\langle\cdot, u^*\rangle y_n$, then

$$\Delta = \tilde{P}(i\omega_k) - P(i\omega_k) = \tilde{D} - D = \varepsilon_n\langle\cdot, u^*\rangle y_n.$$

Therefore we can in this case choose u^* such that $\langle KP_K^\dagger F\phi_k^1, u^*\rangle \neq 0$ (this is always possible due to Lemma 9), and $\{y_n\}_{n=1}^p$ to be linearly independent elements of Y.

If we perturb the operator B as $\tilde{B} = B + \varepsilon_n\langle\cdot, u^*\rangle x_n$, then

$$\Delta = \tilde{P}(i\omega_k) - P(i\omega_k) = CR(i\omega_k, A)(\tilde{B} - B) = \varepsilon_n\langle\cdot, u^*\rangle y_n.$$

We can choose u^* such that $\langle KP_K^\dagger F\phi_k^1, u^*\rangle \neq 0$, and $\{x_n\}_{n=1}^p$ such that $y_n = CR(i\omega_k, A)x_n$ are linearly independent. The maximum number of linearly independent y_n is equal to $\dim\mathcal{R}(CR(i\omega_k, A)) = \dim Y = p$.

If we perturb the operator C as $\tilde{C} = C + \varepsilon_n\langle\cdot, x^*\rangle y_n$, then

$$\Delta = \tilde{P}(i\omega_k) - P(i\omega_k) = (\tilde{C} - C)R(i\omega_k, A)B$$
$$= \varepsilon_n\langle R(i\omega_k, A)B\cdot, x^*\rangle y_n = \varepsilon_n\langle\cdot, B^*R(i\omega_k, A)^*x^*\rangle y_n.$$

We choose $x^* \in X^*$ such that $\langle KP_K^\dagger F\phi_k^1, B^*R(i\omega_k, A)^*x^*\rangle \neq 0$. This is possible due to Lemma 9 and the fact that $\mathcal{R}(B^*R(i\omega_k, A)^*) = U$. We can then take $u^* = B^*R(i\omega_k, A)^*x^*$, and choose $\{y_n\}_{n=1}^p$ to be linearly independent elements of Y.

If we perturb the operator A as $\tilde{A} = A + \tilde{\varepsilon}_n\langle\cdot, x^*\rangle x_n$, then

$$\Delta = \tilde{P}(i\omega_k) - P(i\omega_k) = C\left(R(i\omega_k, \tilde{A}) - R(i\omega_k, A)\right)B$$
$$= CR(i\omega, A)\left[(I - \varepsilon_n\langle R(i\omega_k, A)\cdot, x^*\rangle x_n)^{-1} - I\right]B$$
$$= CR(i\omega, A)\left[I + \tilde{\varepsilon}_n x_n \frac{\langle R(i\omega_k, A)\cdot, x^*\rangle}{1 - \tilde{\varepsilon}_n\langle R(i\omega_k, A)x_n, x^*\rangle} - I\right]B$$
$$= \frac{\tilde{\varepsilon}_n\langle R(i\omega_k, A)B\cdot, x^*\rangle CR(i\omega_k, A)x_n}{1 - \tilde{\varepsilon}_n\langle R(i\omega_k, A)x_n, x^*\rangle}$$
$$= \varepsilon_n\langle\cdot, B^*R(i\omega_k, A)^*x^*\rangle y_n = \varepsilon_n\langle\cdot, u^*\rangle y_n.$$

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It is clear that $\varepsilon_n = \frac{\tilde{\varepsilon}_n}{1-\tilde{\varepsilon}_n \langle R(i\omega_k,A)x_n, x^* \rangle} > 0$ can be made arbitrarily small by choosing $\tilde{\varepsilon}_n > 0$ to be small enough. We can take $x^* \in X^*$ satisfying $\langle KP_K^\dagger F\phi_k^1, B^*R(i\omega_k,A)^*x^* \rangle \neq 0$, and choose $u^* = B^*R(i\omega_k,A)^*x^*$, and $\{x_n\}_{n=1}^p$ such that $y_n = CR(i\omega_k,A)x_n$ are linearly independent. The maximum number of linearly independent y_n is equal to $\dim \mathcal{R}(CR(i\omega_k,A)) = \dim Y = p$. □

As we saw in this section, Theorem 8 is well suited to the theoretical task of proving Theorem 7. On the other hand, in practical applications we would also like to be able to determine a lower bound for the size of the internal model based on the class of admissible perturbations. For this purpose, however, the bound given in Theorem 8 turns out to be inconvenient. The reason for this is that the operators K and $P_K^\dagger$ used in defining values p_k depend on the parameters of the controller. The following theorem shows that if the plant has an equal number of inputs and outputs, we obtain a lower bound that is only dependent on the class of perturbations.

Theorem 11. *Assume $U = Y = \mathbb{C}^p$. Let $k \in \{1,\dots,q\}$ and define*

$$\tilde{p}_k = \dim \operatorname{span}\{ \tilde{P}(i\omega_k)^{-1}F\phi_k^1 \mid (\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}_0 \},$$

where $\mathcal{O}_0 \subset \mathcal{O}$ is a given class of perturbations for which the operators $\tilde{P}(i\omega_k)$ are invertible. If the controller is robust with respect to perturbations in $\mathcal{O}_0$, then

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \tilde{p}_k$$

and $\mathcal{G}_1$ has at least $\tilde{p}_k$ independent Jordan chains of lengths greater than or equal to n_k associated to the eigenvalue $i\omega_k$.

Proof. The proof can be carried out as the proof of Theorem 8, but now the unique solution $\boldsymbol{z} = (z_{n_k},\dots,z_1) \in \mathcal{N}(J_{\mathcal{G}_1}(i\omega_k))$ of (7) satisfies

$$\tilde{P}(i\omega_k)Kz_1 = -F\phi_k^1 \quad \Leftrightarrow \quad Kz_1 = -\tilde{P}(i\omega_k)^{-1}F\phi_k^1$$

if the perturbations are such that $\tilde{P}(i\omega_k)$ is invertible.

Again, the number of linearly independent solutions of the above equation must be at least the number of linearly independent elements on the right-hand side for different perturbations. Since Jordan chains originating from linearly independent vectors are independent, this concludes the proof. □

When designing a robust controller for a class $\mathcal{O}_0$ of perturbations, we can simply compute the number of linearly independent vectors $\tilde{P}(i\omega_k)^{-1}F\phi_k^1$ given by different $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}_0$. By Theorem 11 this immediately gives us a lower bound for how many times the largest Jordan block associated to the frequency $i\omega_k$ must be copied in the controller.

VI. Shock Absorber Model

We close the paper by considering an example of application of the theoretic results presented in the earlier sections. To this end, we consider a system consisting of three identical but independent shock absorber models. We begin by building a one-dimensional feedback controller to achieve output tracking of reference signals with a single frequency $i\beta$. As in Example 6, we can then study the robustness properties of this controller using Theorem 4. We will see that the controller is robust with respect to perturbations that affect the transfer functions of the different subsystems at $\lambda = i\beta$ in the same way, or whose effect on the transfer functions vanish at this frequency. We continue the example by extending the controller in such a way that the first subsystem can be perturbed independently of the two other subsystems without destoying the output tracking.

In this example, the behavior of an individual shock absorber is described by an equation

$$\ddot{q}(t) + r\dot{q}(t) + q(t) = F(t).$$

where $r > 0$ is the damping coefficient. If we control the external force $F(t)$ and observe the position, the standard form for a single system becomes

$$\dot{x}_k(t) = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} x_k(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k(t), \quad x_k(0) = \begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix}$$
$$y_k(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x_k(t).$$

We assume that in the nominal situation the damping coefficients have values $r = 1$, and later in the example consider the effects of uncertainty in these parameters. The system to be controlled consists of 3 identical and independent systems, and the plant can be written as

$$\dot{x}(t) = \operatorname{diag}(A_0, A_0, A_0)\, x(t) + \operatorname{diag}(B_0, B_0, B_0)\, u(t)$$
$$y(t) = \operatorname{diag}(C_0, C_0, C_0)\, x(t)$$

where $x(t) = (x_1^1(t), x_1^2(t), x_2^1(t), x_2^2(t), x_3^1(t), x_3^2(t))^T$, $u(t) = (u_1(t), u_2(t), u_3(t))^T$, $y(t) = (y_1(t), y_2(t), y_3(t))^T$, and

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The plant is exponentially stable, and its transfer function is given by

$$P(\lambda) = CR(\lambda, A)B = \operatorname{diag}(C_0R(\lambda, A_0)B_0)$$
$$= \operatorname{diag}\left(\frac{1}{\lambda^2+\lambda+1} \right) = \frac{1}{\lambda^2+\lambda+1} I$$

for all $\lambda \notin \sigma_p(A_0)$.

We consider output tracking of signals with a single frequency component $\beta \in \mathbb{R}$. To this end, we choose an exosystem described by the equations

$$\dot{v}(t) = i\beta, \qquad v(0) = v_0$$
$$y_{ref}(t) = \left(1,1,1\right)^T v(t) = \mathbf{1}v_0$$

on $W = \mathbb{C}$. We now have $S = i\beta \in \mathcal{L}(\mathbb{C})$, $i\omega_0 = i\beta \in i\mathbb{R}$, and $\phi = 1 \in \mathcal{N}(i\omega_0 - \mathcal{G}) = \mathbb{C}$. Furthermore, with this choice we have $F = -\mathbf{1}$ in the standard form, and the transfer function of the plant

$$P(i\omega_0) = P(i\beta) = \frac{1}{(i\beta)^2 + i\beta + 1} I = \frac{1}{1-\beta^2+i\beta} I$$

at $i\omega_0$ is invertible.

As the first controller we consider

$$\dot{z}(t) = i\beta \cdot z(t) + \left(1,0,0\right)e(t)$$
$$u(t) = Kz(t) = \left(k_1,k_2,k_3\right)^T z(t),$$

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where we choose the elements of K in such a way that the controller solves the output regulation problem for the nominal plant. This can be done by ensuring that the closed-loop system is stable and the regulator equations have a solution. In particular, since $\mathcal{G}_1 = i\beta$, Theorem 4 implies that the regulator equations are satisfied if

$$0 = P(i\beta)K\Gamma + F$$

for some $\Gamma \in \mathcal{N}(i\omega_0 - \mathcal{G}_1) = \mathbb{C}$.

Since the closed-loop system is finite-dimensional, its stability can be determined based on the locations of the eigenvalues of A_e. The plant is stable, and thus for all $\lambda \in \overline{\mathbb{C}^+}$ the inverse of $\lambda - A_e$ can be computed using the inverse of $\lambda - A$ and that of its Schur complement $S_A(\lambda) = \lambda - \mathcal{G}_1 - \mathcal{G}_2 P(\lambda)K$. In particular, the closed-loop system will become exponentially stable if we choose K in such a way that $\lambda - \mathcal{G}_1 - \mathcal{G}_2 P(\lambda)K$ has no roots in $\overline{\mathbb{C}^+}$. A direct computation yields

$$\begin{aligned} S_A(\lambda) &= \lambda - \mathcal{G}_1 - \mathcal{G}_2 P(\lambda)K = \lambda - \frac{k_1}{\lambda^2+\lambda+1} \\ &= \frac{\lambda^3+\lambda^2+\lambda-k_1}{\lambda^2+\lambda+1}. \end{aligned}$$

If we choose $k_1 = -1/2$, then $S_A(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{C}^+}$, and the closed-loop system is exponentially stable. The regulator equations have a solution if the regulation constraint is satisfied, i.e., if

$$\begin{aligned} & P(i\beta)K\Gamma + F = 0 \\ \Leftrightarrow\quad & \frac{1}{1-\beta^2+i\beta}\begin{pmatrix} -1/2 \\ k_2 \\ k_3 \end{pmatrix}\Gamma + (-\mathbf{1}) = 0 \\ \Leftrightarrow\quad & \frac{1}{1-\beta^2+i\beta}\begin{pmatrix} -1/2 \\ k_2 \\ k_3 \end{pmatrix}\Gamma = \mathbf{1}. \end{aligned}$$

If we choose $\Gamma = -2(1-\beta^2+i\beta)$ and $k_2 = k_3 = -1/2$, then the regulator equations have a solution. This concludes that with the choice $K = -1/2 \cdot \mathbf{1}$ the controller solves the output regulation problem.

A. Robustness Properties of the Control Law

The theory presented earlier in this paper, and in particular Theorem 4, states that the control law is robust with respect to any perturbations for which the closed-loop system stability is preserved and for which the equations

$$\tilde{P}(i\beta)Kz = -F \tag{14a}$$
$$(i\beta - \mathcal{G}_1)z = 0 \tag{14b}$$

have a solution $z \in \mathbb{C}$. Since $i\beta - \mathcal{G}_1 = 0$, the second equation is in particular satisfied for all $z \in \mathbb{C}$. If the parameters of the subsystems are perturbed, then the perturbed transfer function $\tilde{P}(\lambda)$ is of the form

$$\begin{aligned} \tilde{P}(\lambda) &= \mathrm{diag}\left(\tilde{P}_1(\lambda), \tilde{P}_2(\lambda), \tilde{P}_3(\lambda)\right) \\ &= P(\lambda) + \mathrm{diag}\left(\delta_1(\lambda), \delta_2(\lambda), \delta_3(\lambda)\right), \end{aligned}$$

where $\delta_j(\lambda)$ are functions that are analytic in $\overline{\mathbb{C}^+}$. If we denote $\alpha = 1/z \in \mathbb{C}$ and use $P(i\beta) = \frac{1}{1-\beta^2+i\beta}I$, then the first equation in (14) becomes

$$\begin{aligned} & (P(i\beta) + \Delta(i\beta))K = -\alpha F \\ \Leftrightarrow\quad & \Delta(i\beta)K = -\alpha F - \frac{1}{1-\beta^2+i\beta}K \\ \Leftrightarrow\quad & -1/2 \cdot \begin{pmatrix} \delta_1(i\beta) \\ \delta_2(i\beta) \\ \delta_3(i\beta) \end{pmatrix} = \left(\alpha + \frac{1}{2(1-\beta^2+i\beta)}\right)\cdot \mathbf{1} \\ \Leftrightarrow\quad & \delta_j(i\beta) = -\frac{1}{1-\beta^2+i\beta} - 2\alpha, \quad j = 1,2,3. \end{aligned}$$

Since α may be an arbitrary nonzero complex number, this shows that the control law is robust with respect to any perturbations that affect the transfer functions $P_j(\lambda)$ of the subsystems at $\lambda = i\beta$ in the same way. In particular, this includes any perturbations that are not visible in the transfer functions of the subsystems at this frequency.

In the case where $\beta = 0$, the considered reference signals are constant functions, and the control law is in particular robust with respect to small changes in the damping coefficients of the subsystems. Indeed, if the damping coefficients r of the different subsystems are perturbed independently of each others, then the perturbed system operator of the plant is given by

$$\tilde{A}_0^j = \begin{pmatrix} 0 & 1 \\ -1 & -r_j \end{pmatrix}, \qquad A = \mathrm{diag}\left(\tilde{A}_0^1, \tilde{A}_0^2, \tilde{A}_0^3\right)$$

and the perturbed transfer function becomes

$$\begin{aligned} \tilde{P}(\lambda) &= CR(\lambda, \tilde{A})B \\ &= \mathrm{diag}(C_0 R(\lambda, \tilde{A}_0^1)B_0, C_0 R(\lambda, \tilde{A}_0^2)B_0, C_0 R(\lambda, \tilde{A}_0^3)B_0) \\ &= \mathrm{diag}\left(\frac{1}{\lambda^2+r_1\lambda+1}, \frac{1}{\lambda^2+r_2\lambda+1}, \frac{1}{\lambda^2+r_3\lambda+1}\right) \end{aligned}$$

for all $\lambda \notin \sigma_p(\tilde{A})$. Since changing the values of the damping coefficients can be written as an additive perturbation to A, we know that for sufficiently small changes (i.e., $r_j \approx 1$) the closed-loop system remains exponentially stable and $i\omega_0 = 0 \in \rho(\tilde{A})$. However, if we for any such perturbed values r_j compute the value of the perturbed transfer function $\tilde{P}(\lambda)$ at the frequency $\lambda = i\beta = 0$, we can see that

$$\begin{aligned} \tilde{P}(i\beta) &= \mathrm{diag}\left(\frac{1}{0^2+r_1 0+1}, \frac{1}{0^2+r_2 0+1}, \frac{1}{0^2+r_3 0+1}\right) \\ &= I = P(i\beta). \end{aligned}$$

This means that uncertainties in the damping coefficients do not affect the transfer function of the plant at $\lambda = 0$. In particular this concludes that the one-dimensional controller is robust with respect to any changes in the values r_j for which the closed-loop system is exponentially stable and $0 \in \rho(\tilde{A})$.

B. Extending the Internal Model for Given Perturbations

In this section we will extend the controller in such a way that the control law is robust with respect to perturbations that affect the subsystems in such a way that $\tilde{P}_2(i\beta) = \tilde{P}_3(i\beta)$.

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For such perturbations, the perturbed transfer function is of the form

$$\tilde{P}(\lambda) = P(\lambda) + \Delta(\lambda) = P(\lambda) + \operatorname{diag}\left(\delta_1(\lambda), \delta_2(\lambda), \delta_3(\lambda)\right),$$

where the functions $\delta_j(\cdot)$ satisfy $\delta_2(i\beta) = \delta_3(i\beta)$. In other words, we would like to achieve robustness with respect to the perturbations that affect the last two subsystems at the frequency $\lambda = i\beta$ in the same way. For brevity, we denote $\delta_0^1 = \delta_1(i\beta)$ and $\delta_0^2 = \delta_2(i\beta) = \delta_3(i\beta)$.

We can first of all use Theorem 11 to compute a lower bound for $\dim \mathcal{N}(i\beta - \mathcal{G}_1)$ for a control law that is robust with respect to such perturbations. For any perturbations of small enough norm we now have (for brevity, denote $\alpha = 1 - \beta^2 + i\beta \in \mathbb{C}$)

$$\begin{aligned}\tilde{y}_n &= \tilde{P}(i\beta)^{-1} F\phi = \left(\frac{1}{\alpha} I + \Delta(i\beta)\right)^{-1} F \\ &= \alpha \left(I + \alpha\Delta(i\beta)\right)^{-1} F \\ &= \alpha \operatorname{diag}\left(\frac{1}{\alpha\delta_1(0)+1}, \frac{1}{\alpha\delta_2(0)+1}, \frac{1}{\alpha\delta_3(0)+1}\right) \mathbf{1} \\ &= \left(\frac{\alpha}{\alpha\delta_0^1+1}, \frac{\alpha}{\alpha\delta_0^2+1}, \frac{\alpha}{\alpha\delta_0^2+1}\right)^T .\end{aligned}$$

It is clear that perturbations with different δ_0^1 and δ_0^2 can produce at most 2 linearly independent elements $\tilde{y}_n$. We can therefore use Theorem 11 to conclude that if the controller is robust with respect to all perturbations satisfying $\delta_2(i\beta) = \delta_3(i\beta)$, then we necessarily have $\dim \mathcal{N}(i\beta - \mathcal{G}_1) \geq 2$.

In the following we will show that we can achieve robustness with respect to the appropriate perturbations with a proper choice of a controller incorporating a two-dimensional internal model of the exosystem. Such a controller can be constructed by choosing $\mathcal{G}_1 = i\beta \in \mathbb{C}^{2\times 2}$, and correspondingly extending the matrices $\mathcal{G}_2$ and K,

$$\mathcal{G}_2 = \begin{pmatrix} 1 & 0 & 0 \\ g_1 & g_2 & g_3 \end{pmatrix}, \qquad K = \begin{pmatrix} -1/2 & k_1 \\ -1/2 & k_2 \\ -1/2 & k_3 \end{pmatrix}.$$

We will fix the new parameters $\{g_j\}$ and $\{k_j\}$ in such a way that the closed-loop system is stable, the controller solves the output regulation problem, and the control law is robust with respect to the desired class of perturbations.

Regardless of the choices of the free parameters, the regulator equations for the nominal plant have a solution. Indeed, we have $i\beta - \mathcal{G}_1 = 0 \in \mathbb{C}^{2\times 2}$, and choosing $\Gamma = (-2(1-\beta^2+i\beta), 0)^T \in \mathbb{C}^2$ we get

$$\begin{aligned}P(i\beta)K\Gamma &= \frac{1}{1-\beta^2+i\beta} \begin{pmatrix} -1/2 & k_1 \\ -1/2 & k_2 \\ -1/2 & k_3 \end{pmatrix} \begin{pmatrix} -2(1-\beta^2+i\beta) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{1} = -F.\end{aligned}$$

By Theorem 4 the control law is robust respect to any perturbations $\Delta(\lambda)$ satisfying $\Delta(i\beta) = \operatorname{diag}\left(\delta_0^1, \delta_0^2, \delta_0^2\right)$ if the equations

$$\tilde{P}(i\beta)K \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -F, \qquad \text{and} \qquad (i\beta - \mathcal{G}_1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

have a solution $z = (z_1, z_2)^T \in \mathbb{C}^2$. Since again $i\beta - \mathcal{G}_1 = 0$, the second equation is satisfied for all $z \in \mathbb{C}^2$. The first equation becomes (for brevity denote $\alpha = 1 - \beta^2 + i\beta \in \mathbb{C}$)

$$\begin{aligned}&(P(i\beta) + \Delta(i\beta))Kz = -F \\ \Leftrightarrow\ & \left(\frac{1}{\alpha} I + \operatorname{diag}\left(\delta_0^1, \delta_0^2, \delta_0^2\right)\right) \begin{pmatrix} -1/2 & k_1 \\ -1/2 & k_2 \\ -1/2 & k_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbf{1} \\ \Leftrightarrow\ & \begin{pmatrix} (1/\alpha + \delta_0^1)(-z_1/2 + k_1 z_2) \\ (1/\alpha + \delta_0^2)(-z_1/2 + k_2 z_2) \\ (1/\alpha + \delta_0^2)(-z_1/2 + k_3 z_2) \end{pmatrix} = \mathbf{1}\end{aligned}$$

If we choose $k_2 = k_3$, and if the perturbations satisfy $\delta_0^1, \delta_0^2 \neq 1/\alpha$ (which is in particular true for small $|\delta_0^1|$ and $|\delta_0^2|$) then the equations reduce to

$$\begin{aligned}&\begin{cases} (1/\alpha + \delta_0^1)(-z_1/2 + k_1 z_2) = 1 \\ (1/\alpha + \delta_0^2)(-z_1/2 + k_2 z_2) = 1 \end{cases} \\ \Leftrightarrow\ & \begin{pmatrix} -1/2 & k_1 \\ -1/2 & k_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (1/\alpha + \delta_0^1)^{-1} \\ (1/\alpha + \delta_0^2)^{-1} \end{pmatrix}.\end{aligned}$$

This matrix equation has a solution for all perturbations whenever $k_1 \neq k_2$. If we choose $k_1 = 0$ and $k_2 = k_3 = -1/2$, then for all appropriate perturbations the equations (7) have a solution

$$\begin{aligned}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} -1/2 & 0 \\ -1/2 & -1/2 \end{pmatrix}^{-1} \begin{pmatrix} (1/\alpha + \delta_0^1)^{-1} \\ (1/\alpha + \delta_0^2)^{-1} \end{pmatrix} \\ &= 2 \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (1/\alpha + \delta_0^1)^{-1} \\ (1/\alpha + \delta_0^2)^{-1} \end{pmatrix},\end{aligned}$$

and due to Theorem 4 the control law is robust with any perturbations of the appropriate form provided that the closed-loop system is stable.

The stability of the closed-loop system can again be achieved by choosing K and the remaining parameters of $\mathcal{G}_2$ in such a way that the Schur complement $S_A(\lambda) = \lambda - \mathcal{G}_1 - \mathcal{G}_2 P(\lambda) K$ of $\lambda - A$ in $\lambda - A_e$ is invertible for all $\lambda \in \overline{\mathbb{C}^+}$. For our choices $k_1 = 0$ and $k_2 = k_3 = -1/2$ we have

$$\begin{aligned}&S_A(\lambda) = \lambda - \mathcal{G}_1 - \mathcal{G}_2 P(\lambda) K \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \frac{1}{\lambda^2+\lambda+1} \begin{pmatrix} 1 & 0 & 0 \\ g_1 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} -1/2 & 0 \\ -1/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \frac{1/2}{\lambda^2+\lambda+1} \begin{pmatrix} 1 & 0 \\ g_1+g_2+g_3 & g_2+g_3 \end{pmatrix} \\ &= \frac{1}{\lambda^2+\lambda+1} \times \\ &\quad \begin{pmatrix} \lambda^3+\lambda^2+\lambda+1/2 & 0 \\ (g_1+g_2+g_3)/2 & \lambda^3+\lambda^2+\lambda+(g_2+g_3)/2 \end{pmatrix}.\end{aligned}$$

The matrix $S_A(\lambda)$ is nonsingular for all $\lambda \in \overline{\mathbb{C}^+}$ if its diagonal elements have no roots in the closed right half-plane. Since the roots of the polynomial $\lambda^3 + \lambda^2 + \lambda + 1/2$ lie in $\mathbb{C}^-$, the stability depends only on the choices of g_2 and g_3. If we choose $g_2 = 1$ and $g_1 = g_3 = 0$, then $S_A(\lambda)$ is invertible for all $\lambda \in \overline{\mathbb{C}^+}$, and thus the closed-loop system is exponentially stable.

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This concludes the procedure for extending the internal model to accomodate for the perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ for which the perturbed transfer function satisfies $\tilde{P}(i\beta) = P(i\beta) + \operatorname{diag}(\delta_0^1,\delta_0^2,\delta_0^2)$. As the results we see that the controller on $Z=\mathbb{C}^2$ with parameters

$$\mathcal{G}_1 = \begin{pmatrix} i\beta & 0\\ 0 & i\beta\end{pmatrix}, \quad \mathcal{G}_2 = \begin{pmatrix} 1&0&0\\0&1&0\end{pmatrix}, \quad K = -\frac{1}{2}\begin{pmatrix}1&0\\1&1\\1&1\end{pmatrix},$$

solves the output regulation problem and the control law is robust with respect to any small enough perturbations of the appropriate form.

It is quite easy to see from the computations above, that in order to accomodate for perturbations that affect the three different subsystems independently, we need a controller where the frequency $i\beta$ of the exosystem is copied three times. However, such uncertainties still constitute a fairly restricted class of perturbations if we compare them to all the possible perturbations to the parameters (A,B,C,D) of the plant. In particular, such uncertainties still assume that the three subsystems do not interact with each other. On the other hand, the three-dimensional controller required to handle the independent perturbations in the subsystems would also be robust with respect to introduction of arbitrary interconnections between the subsystems (as long as they preserve the closed-loop stability). This follows directly from the internal model principle, since in this example we have $p=\dim Y = 3$.

VII. Conclusions

In this paper we have considered the robust output regulation problem for infinite-dimensional linear systems. In particular, we have studied a situation where the control law is not required to be robust with respect to all perturbations to the parameters of the plant. As the main result, we introduced a simple method for testing the robustness of a feedback control law with respect to given perturbations. The test in particular shows that a full internal model is not always necessary for robustness, and that the perturbations in the parameters of the system affect the output tracking only through the change of the transfer function $P(\lambda)$ at the frequencies $i\omega_k$ of the exosystem.

Furthermore, we have also aimed at identifying the minimal classes of admissible perturbations that necessitate the full p-copy internal model in the controller. We proved that the full internal model is necessary if we require robustness with respect to arbitrarily small rank one perturbations in any one of the operators A, B, C, and D of the plant. In particular, this concludes that even if we do not require robustness with respect to uncertainties in the output operator F of the exosystem, the order of the internal model in the controller may not be reduced. For systems with equal number of inputs and outputs we presented a convenient way of computing a lower bound for the number of copies of the exosystems frequencies that must be included in a controller that is robust with respect to a given class of uncertainties.

The most important topics for future research include using the presented theoretical results to develope a general method for constructing a feedback controller to solve the robust output regulation problem for a given class of perturbations. Moreover, theory similar to the one presented in this paper should also be developed for asymptotic rejection of disturbance signals, which is often considered in connection with asymptotic output tracking.

In this paper we have studied finite-dimensional exosystems and assumed that the input and output operators of the plant are bounded. Extending the results for infinite-dimensional exosystems would enlarge the class of reference signals and would in particular allow tracking of nonsmooth periodic signals. The main difficulty in extending the theory for infinite-dimensional signal generators is that in this situation the closed-loop system may no longer be exponentially stabilizable, and the solvability of the Sylvester equation in the regulator equations requires additional conditions [10], [11]. The class of systems considered in this paper should also be extended to include systems with unbounded operators B and C. Distributed parameter systems with such control and measurement operators are encountered in a wide variety of practical applications.

Appendix

Proof of Lemma 5. Assume first that $\Sigma = (\Pi,\Gamma)^T \in \mathcal{L}(W,X_e)$ is a solution of the Sylvester equation $\Sigma S = \tilde{A}_e\Sigma + B_e$ and let $k\in\{1,\dots,q\}$. Then for all $l\in\{2,\dots,n_k\}$ we have (using $S\phi_k^1 = i\omega_k\phi_k^1$ and $S\phi_k^l = i\omega_k\phi_k^l + \phi_k^{l-1}$)

$$\begin{aligned}
\begin{pmatrix}0\\ \mathcal{G}_2F\phi_k^1\end{pmatrix} &= B_e\phi_k^1 = \Sigma S\phi_k^1 - \tilde{A}_e\Sigma\phi_k^1 && \text{(15a)}\\
&= (i\omega_k - \tilde{A}_e)\Sigma\phi_k^1 && \text{(15b)}\\
&= \begin{pmatrix} (i\omega_k-\tilde{A})\Pi\phi_k^1 - \tilde{B}K\Gamma\phi_k^1\\ (i\omega_k-\mathcal{G}_1)\Gamma\phi_k^1 - \mathcal{G}_2(\tilde{C}\Pi+\tilde{D}K\Gamma)\phi_k^1\end{pmatrix} && \text{(15c)}\\
\begin{pmatrix}0\\ \mathcal{G}_2F\phi_k^l\end{pmatrix} &= B_e\phi_k^l = \Sigma S\phi_k^l - \tilde{A}_e\Sigma\phi_k^l && \text{(15d)}\\
&= (i\omega_k - \tilde{A}_e)\Sigma\phi_k^l + \Sigma\phi_k^{l-1} && \text{(15e)}\\
&= \begin{pmatrix} (i\omega_k-\tilde{A})\Pi\phi_k^l - \tilde{B}K\Gamma\phi_k^l + \Pi\phi_k^{l-1}\\ (i\omega_k-\mathcal{G}_1)\Gamma\phi_k^l - \mathcal{G}_2(\tilde{C}\Pi+\tilde{D}K\Gamma)\phi_k^l + \Gamma\phi_k^{l-1}\end{pmatrix}. && \text{(15f)}
\end{aligned}$$

Since $i\omega_k\in\rho(\tilde{A})$, the first lines of the equations (15) recursively imply that for $l\in\{2,\dots,n_k\}$ we have

$$\begin{aligned}
\Pi\phi_k^1 &= R(i\omega_k,\tilde{A})\tilde{B}K\Gamma\phi_k^1\\
\Pi\phi_k^l &= R(i\omega_k,\tilde{A})\left(\tilde{B}K\Gamma\phi_k^l - \Pi\phi_k^{l-1}\right)\\
&= R(i\omega_k,\tilde{A})\tilde{B}K\Gamma\phi_k^l - R(i\omega_k,\tilde{A})^2\left(\tilde{B}K\Gamma\phi_k^{l-1} - \Pi\phi_k^{l-2}\right)\\
&= \cdots = \sum_{j=0}^{l-1}(-1)^j R(i\omega_k,\tilde{A})^{j+1}\tilde{B}K\Gamma\phi_k^{l-j}.
\end{aligned}$$

Using the definition of Φ_k in (5) we can see that in vector notation the above equations are precisely the equation (8b). Substituting $\Pi\phi_k^l$ into the second lines of the equations (15) we see that for all $l \in \{2, \dots, n_k\}$ we have

$$\begin{aligned}
&(i\omega_k - \mathcal{G}_1)\Gamma\phi_k^1 = \mathcal{G}_2(\tilde{C}\Pi\phi_k^1 + \tilde{D}K\Gamma\phi_k^1 + F\phi_k^1) \\
&\quad = \mathcal{G}_2\left[(\tilde{C}R(i\omega_k, \tilde{A})\tilde{B} + \tilde{D})K\Gamma\phi_k^1 + F\phi_k^1\right] \\
&\quad = \mathcal{G}_2\left(\tilde{P}(i\omega_k)K\Gamma\phi_k^1 + F\phi_k^1\right) \\
&(i\omega_k - \mathcal{G}_1)\Gamma\phi_k^2 + \Gamma\phi_k^1 = \mathcal{G}_2(\tilde{C}\Pi\phi_k^2 + \tilde{D}K\Gamma\phi_k^2 + F\phi_k^2) \\
&\quad = \mathcal{G}_2\left(\tilde{P}(i\omega_k)K\Gamma\phi_k^2 - \tilde{C}R(i\omega_k, \tilde{A})^2\tilde{B}K\Gamma\phi_k^1 + F\phi_k^2\right) \\
&(i\omega_k - \mathcal{G}_1)\Gamma\phi_k^l + \Gamma\phi_k^{l-1} = \mathcal{G}_2(\tilde{C}\Pi\phi_k^l + \tilde{D}K\Gamma\phi_k^l + F\phi_k^l) \\
&\quad = \mathcal{G}_2\Big[\sum_{j=0}^{l-1}(-1)^j\tilde{C}R(i\omega_k, \tilde{A})^{j+1}\tilde{B}K\Gamma\phi_k^{l-j} + \tilde{D}K\Gamma\phi_k^l + F\phi_k^l\Big] \\
&\quad = \mathcal{G}_2\Big[\tilde{P}(i\omega_k)K\Gamma\phi_k^l + \sum_{j=1}^{l-1}(-1)^j\tilde{C}R(i\omega_k, \tilde{A})^{j+1}\tilde{B}K\Gamma\phi_k^{l-j} \\
&\qquad + F\phi_k^l\Big].
\end{aligned}$$

In vector notation this is exactly (8a). Since $k \in \{1, \dots, q\}$ was arbitrary, this concludes the first part of the proof.

Conversely, assume $\Sigma = (\Pi, \Gamma)^T \in \mathcal{L}(W, X_e)$ is such that $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e)$ and (8a) and (8b) are satisfied, and let $k \in \{1, \dots, q\}$. As above, we can see that the equations (8a) and (8b) imply

$$\begin{aligned}
&\Sigma S\phi_k^1 - \tilde{A}_e\Sigma\phi_k^1 = i\omega_k\Sigma\phi_k^1 - \tilde{A}_e\Sigma\phi_k^1 \\
&\quad = \begin{pmatrix} (i\omega_k - \tilde{A})\Pi\phi_k^1 - \tilde{B}K\Gamma\phi_k^1 \\ (i\omega_k - \mathcal{G}_1)\Gamma\phi_k^1 - \mathcal{G}_2(\tilde{C}\Pi + \tilde{D}K\Gamma)\phi_k^1 \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 \\ \mathcal{G}_2F\phi_k^1 \end{pmatrix} = B_e\phi_k^1 \\
&\Sigma S\phi_k^l - \tilde{A}_e\Sigma\phi_k^l = (i\omega_k - \tilde{A}_e)\Sigma\phi_k^l + \Sigma\phi_k^{l-1} \\
&\quad = \begin{pmatrix} (i\omega_k - \tilde{A})\Pi\phi_k^l - \tilde{B}K\Gamma\phi_k^l + \Pi\phi_k^{l-1} \\ (i\omega_k - \mathcal{G}_1)\Gamma\phi_k^l - \mathcal{G}_2(\tilde{C}\Pi + \tilde{D}K\Gamma)\phi_k^l + \Gamma\phi_k^{l-1} \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 \\ \mathcal{G}_2F\phi_k^l \end{pmatrix} = B_e\phi_k^l.
\end{aligned}$$

Since $k \in \{1, \dots, q\}$ was arbitrary, and since the set of vectors $\{\, \phi_k^l \mid k = 1, \dots, q, \ l = 1, \dots, n_k \,\}$ is a basis of W, this concludes that $\Sigma S = \tilde{A}_e\Sigma + B_e$.

If $\Sigma = (\Pi, \Gamma)^T$ is the solution of the Sylvester equation, then (8b) implies that for all $k \in \{1, \dots, q\}$ we have

$$\begin{aligned}
&\tilde{C}_e\Sigma\phi_k^1 + D_e\phi_k^1 = \tilde{C}\Pi\phi_k^1 + \tilde{D}K\Gamma\phi_k^1 + F\phi_k^1 \\
&\quad = \tilde{C}R(i\omega_k, \tilde{A})\tilde{B}K\Gamma\phi_k^1 + \tilde{D}K\Gamma\phi_k^1 + F\phi_k^1 \\
&\quad = \left(\tilde{C}R(i\omega_k, \tilde{A})\tilde{B} + \tilde{D}\right)K\Gamma\phi_k^1 + F\phi_k^1 \\
&\quad = \tilde{P}(i\omega_k)K\Gamma\phi_k^1 + F\phi_k^1 \\
&\tilde{C}_e\Sigma\phi_k^l + D_e\phi_k^l = \tilde{C}\Pi\phi_k^l + \tilde{D}K\Gamma\phi_k^l + F\phi_k^l \\
&\quad = \sum_{j=0}^{l-1}(-1)^j\tilde{C}R(i\omega_k, \tilde{A})^{j+1}\tilde{B}K\Gamma\phi_k^{l-j} + \tilde{D}K\Gamma\phi_k^l + F\phi_k^l \\
&\quad = \tilde{P}(i\omega_k)K\Gamma\phi_k^l + \sum_{j=1}^{l-1}(-1)^j\tilde{C}R(i\omega_k, \tilde{A})^{j+1}\tilde{B}K\Gamma\phi_k^{l-j} + F\phi_k^l
\end{aligned}$$

In vector notation, these equations are exactly (8c). This concludes the proof. □

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