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Periodic Output Regulation for Distributed Parameter Systems

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Abstract In this paper the output regulation of a linear distributed parameter system with a nonautonomous periodic exosystem is considered. It is shown that the solvability of the output regulation problem can be characterized by the solvability of a certain constrained infinite-dimensional Sylvester differential equation. Conditions are given for the existence of feedforward and feedback controllers solving the regulation problem along with a method for their construction. The theoretical results are applied to output regulation of a controlled delay equation.

Keywords Sylvester differential equation · strongly continuous evolution family · output regulation · time-dependent systems

Mathematics Subject Classification (2000) 39B42 · 47D06 · 93C05 · 93B52

1 Introduction

Research of output regulation of distributed parameter systems has been active during the last thirty years since the problem was first studied by Schumacher [18]. Since then many of the classical results of finite-dimensional control theory have been extended for infinite-dimensional systems, see [16, 6, 17, 3, 1] and references therein. One of the recent approaches to this problem follows the treatment by Byrnes et al. [2] who showed that the solvability of the output regulation problem can be related to

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the solvability of the so-called regulator equations. In particular the approach of using these Sylvester-type equations allows a uniform treatment of the theory of output regulation for different types of controllers.

In the existing theory of output regulation of distributed parameter systems the considered reference and disturbance signals are usually assumed to be of form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t) \tag{1}$$

where the coefficient functions $y_k(\cdot)$ are trigonometric functions. Recently there has been interest in generalizing the classes of reference and disturbance signals considered for output regulation. In particular output regulation of general continuous periodic signals has found applications in control of robot arms, disk drives, magnetic power supplies of synchrotrons [19,8]. Generating these types of signals with autonomous exosystems requires that the signal generator is an infinite-dimensional system on a Hilbert space. Robust output regulation of distributed parameter systems with infinite-dimensional signal generators capable of generating bounded periodic signals have been studied by Immonen and Pohjolainen [11] and by Hämäläinen and Pohjolainen [7]. These results have also been extended in [14] for signal generators capable of generating signals of form (1) where the coefficient functions $y_k(\cdot)$ are continuous periodic signals.

In this paper we will study the theory of output regulation arising from the use of a different approach in generating nonsmooth reference and disturbance signals. We will allow the exosystem to be a finite-dimensional time-dependent linear system of form

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in \mathbb{C}^q, \tag{2a}$$

$$y_{ref}(t) = F(t)v(t) \tag{2b}$$

where $S(\cdot)$ and $F(\cdot)$ are periodic functions with the same period, i.e. there exists $T > 0$ such that $S(t+T) = S(t)$ and $F(t+T) = F(t)$ for all $t \in \mathbb{R}$. These types of exosystems provide a very natural way of generating reference and disturbance signals of form (1), where the coefficient functions $y_k(\cdot)$ are arbitrary continuous T-periodic functions. For example, if we are interested in regulating a continuous periodic function $y_0 : \mathbb{R} \to \mathbb{C}$ we can choose $\mathbb{C}^q = \mathbb{C}$ and use a periodic exosystem with parameters

$$S(t) \equiv 0, \qquad F(t) = y_0(t), \qquad \forall t \geq 0.$$

The signal $y_0(\cdot)$ is then generated with the initial state $v_0 = 1$, since the corresponding output of the exosystem is given by

$$y_{ref}(t) = F(t)e^{0\cdot t}v_0 = F(t)v_0 = y_0(t)$$

for all $t \geq 0$.

Output regulation of finite-dimensional linear systems with these types of exosystems have been considered recently in Zhang and Serrani [20] for periodic exosystems and by Ichikawa and Katayama [10] in the nonperiodic case. In [20] the authors formulated the Periodic Output Regulation Problem related to a time-invariant finite-dimensional system together with a periodic signal generator (2) and showed that in the same way as in the time-invariant case the solvability of this problem can again be characterized by the solvability of certain regulator equations. The main difference arising from the use of a nonautonomous signal generator is that the regulator equations become dependent on time and one of the equations changes from a matrix Sylvester equation to a *Sylvester differential equation* which is a linear matrix differential equation.

Although the infinite-dimensional exosystem and the periodic exosystem (2) can both be used to generate signals of form (1) where $y_k(\cdot)$ are continuous periodic functions, the signal generators have fundamental differences. By definition the controller must solve the output regulation problem for all signals generated by the exosystem with functions $S(\cdot)$ and $F(\cdot)$. One of the main differences is the size of the classes of signals generated by the two types of exosystems. The infinite-dimensional exosystem can be chosen in such a way that it is — for example — possible to generate all absolutely continuous 2π-periodic signals by appropriate choices of the initial state of the exoystem. On the other hand, given a finite set of T-periodic functions the periodic exosystem (2) can be chosen in such a way that it generates only these predetermined functions and their linear combinations. This illustrates the sense in which the classes of signals generated by the infinite-dimensional exosystems are significantly larger than the ones generated by the periodic exosystems. If we are interested in regulating a selected set of signals, then this property of the periodic exosystem can be a considerable advantage. The reason for this is that the exosystem can be chosen to generate only few unnecessary signals. The lack of such unnecessary signals will in turn simplify the controller designed to solve the output regulation problem.

The infinite-dimensional exosystem has the property that the smoothness of the generated signals can be related to the corresponding initial states of the exosystem. This allows a simple classification of the signals based on their smoothness. The usefulness of this classification arises from the fact that the smoothness of the reference and disturbance signals can also be used to weaken the conditions required for the solvability of the output regulation problem [14]. In the case of the periodic exosystem the initial state does not affect the smoothness of the signals and the differentiability properties of the generated signals can be determined directly from those of the functions $S(\cdot)$ and $F(\cdot)$.

In this paper we consider a time-invariant infinite-dimensional system with a finite-dimensional periodic signal generator (2) and look for a time-dependent controller to solve the problem of output regulation and disturbance rejection. This leads to a situation where the state of the closed-loop system consisting of the plant and the controller is determined by an abstract differential equation of form

$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)v(t), \qquad x_e(0) = x_{e0} \in X_e$$

where $(A_e(t), \mathscr{D}(A_e(t)))$ is a family of unbounded operators and $B_e(\cdot)$ is an operator-valued mapping. The well-posedness of equations of this type is a complicated subject but these questions can be answered using the theory of *strongly continuous evolution families* [4, 15] associated to nonautonomous Cauchy problems on Banach spaces.

The first main result of this paper is that the solvability of the periodic output regulation problem can be characterized by the properties of the periodic solution of the infinite-dimensional *Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t), \tag{3}$$

where $A_e(\cdot)$ and $B_e(\cdot)$ are the T-periodic families of operators associated to the closed-loop system and $S(\cdot)$ is the periodic function from the signal generator (2). This follows from a result that if the Sylvester differential equation (3) has a periodic solution $\Sigma_\infty(\cdot)$, then the state $x_e(t)$ of the closed-loop system can be expressed using a formula

$$x_e(t) = U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0) + \Sigma_\infty(t)v(t) \tag{4}$$

where $U_e(t,s)$ is the strongly continuous evolution family related to the closed-loop system. The importance of this formula comes from the fact that it allows us to study the asymptotic behaviour of the regulation error. In fact, if the closed-loop system is stable, then the first term in (4) goes to zero and the state of the closed-loop system behaves as

$$x_e(t) \sim \Sigma_\infty(t)v(t)$$

asymptotically, where $v(t)$ is the state of the periodic exosystem. This can be seen as a *dynamic steady state* of the closed-loop system. These new results generalize the theory of output regulation with time-invariant signal generators where it is known that the solvability of the output regulation problem can be characterized using the solvability of the regulator equations [2,7,14]. To the authors' knowledge the Sylvester differential equation has not been used earlier to study the output regulation of infinite-dimensional systems.

The main generalization presented in this paper compared to the work of Zhang and Serrani [20] is to allow the plant to be a distributed parameter system instead of a linear finite-dimensional system. As was already mentioned, this extension requires a more careful approach to the systems involved to guarantee that these time-dependent distributed parameter systems have well-defined states. Also the Sylvester differential equation (3) changes from a matrix differential equation to an infinite-dimensional operator differential equation. The solvability of these types of equations is a research topic on its own and because of this we will dedicate Section 4 to a detailed investigation of this question. We will first define the classical solution of a Sylvester differential equation and find the formula for this solution under certain assumptions on the plant, the controller and the exogeneous signals. We will use this solution to define the mild solution of the Sylvester differential equation. Finally, we will show that under suitable assumptions equation (3) has a unique periodic mild solution.

Zhang and Serrani assumed that the reference and disturbance signals are generated by a periodic exosystem of form (2) where the matrix-valued functions $S(\cdot)$ and $F(\cdot)$ are smooth. Under this assumptions also the generated reference and disturbance signals are smooth functions. In this paper we extend this exosystem by allowing the function $S(\cdot)$ to be locally integrable and the output function $F(\cdot)$ to be continuous. This type of exosystem is capable of generating signals which are not necessarily continuously differentiable. Because of this extension the results presented in this paper are new even for finite-dimensional systems.

In addition to characterizing the solvability of the periodic output regulation problem, we also present conditions for the existence of two separate types of controllers — a feedforward controller and an observer-based error feedback controller — solving the regulation problem along with methods for their construction. This is done by generalizing controllers used for time-invariant finite-dimensional [9] and infinite-dimensional [2, 11] linear systems to our case.

We will conclude the paper by considering an example of application of the theoretical results presented earlier in the paper. To this end we will consider a controlled scalar system with a delay formulated as a distributed parameter system on a Hilbert space. We choose a periodic exosystem in such a way that it can generate as a reference signal a *triangle signal*. This signal is continuous but not continuously differentiable. Because of this, it would be impossible to generate this signal with a finite-dimensional time-invariant exosystem. We construct an observer-based periodic feedback controller such that the output of the plant regulates the signals of the chosen exosystem.

The paper is organized as follows. In Section 2 we introduce notation, recall the definition of a strongly continuous evolution family and state the basic assumptions on the system, the exosystem and the controllers. In Section 3 we formulate the Periodic Output Regulation Problem for infinite-dimensional systems and characterize its solvability using the properties of infinite-dimensional Sylvester differential equations. Before proving this main result we investigate the solvability of the Sylvester differential equations in Section 4. The theorem characterizing the solvability of the Periodic Output Regulation Problem is then proved in Section 5. In Section 6 we present sufficient conditions for the existence of feedforward and feedback controllers solving the problem and methods for their construction. The theoretical results are applied to a controlled delay system in Section 7. Section 8 contains concluding remarks.

2 Notation and Definitions

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathscr{D}(A)$ and $\mathscr{R}(A)$ the domain and the range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathscr{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda,A) = (\lambda I - A)^{-1}$. The space of continuous functions $f : \mathbb{R} \to X$ is denoted by $C(\mathbb{R},X)$. The space of T-periodic continuous functions is defined as

$$C_T(\mathbb{R},X) = \left\{ f : \mathbb{R} \to X \;\middle|\; f \text{ is continuous}, f(t+T) = f(t) \;\; \forall t \in \mathbb{R} \right\}.$$

Similarly we denote by $C_T^1(\mathbb{R},X)$ the space of T-periodic continuously differentiable functions. We denote by $C_T(\mathbb{R},\mathscr{L}(X,Y))$ and $C_T(\mathbb{R},\mathscr{L}_s(X,Y))$ the spaces of T-periodic functions with values in $\mathscr{L}(X,Y)$ which are continuous in uniform and strong operator topologies, respectively. For $I \subset \mathbb{R}$ we denote by $C^1(I,\mathscr{L}_s(X,Y))$ the space of functions which are continuously differentiable with respect to the strong operator topology of $\mathscr{L}(X,Y)$. For an operator-valued function $A(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(X,Y))$ we denote

$$\|A\|_\infty = \sup_{t\in[0,T]} \|A(t)\|.$$

In dealing with infinite-dimensional nonautonomous systems we use the concept of a strongly continuous evolution family [15, Ch. 5], [4, Sec. VI.9].

Definition 1 A family of bounded operators $(U(t,s))_{t\geq s} \subset \mathscr{L}(X)$ is called a *strongly continuous evolution family* if

(a) $U(s,s) = I$ for $s \in \mathbb{R}$;
(b) $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$;
(c) $\left\{ (t,s) \in \mathbb{R}^2 \mid t \geq s \right\} \ni (t,s) \mapsto U(t,s)$ is a strongly continuous mapping.

A strongly continuous evolution family is called *exponentially bounded* if there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}$$

for all $t \geq s$. The evolution family is called *periodic* (with period T) if $U(t+T,s+T) = U(t,s)$ for all $t \geq s$.

Strongly continuous evolution families are related to nonautonomous abstract Cauchy problems. If we consider such an equation

$$\dot{x}(t) = A(t)x(t) + f(t), \tag{5a}$$
$$x(s) = x_s \in X \tag{5b}$$

and if $U(t,s)$ is a strongly continuous evolution family associated to the family $(A(t), \mathscr{D}(A(t)))$ of operators, then if for every $s \in \mathbb{R}$ this equation has a classical solution $x(\cdot) \in C^1([s,\infty),X)$ such that $x(t) \in \mathscr{D}(A(t))$ for all $t \geq s$, this solution is given by

$$x(t) = U(t,s)x_s + \int_s^t U(t,r)f(r)ds \tag{6}$$

for all $t \geq s$. On the other hand, under weaker conditions this function can also be seen as a mild solution of the abstract Cauchy problem [12].

Definition 2 If there exits a strongly continuous evolution family related to the family $(A(t), \mathscr{D}(A(t)))$ of operators and if $f \in C(\mathbb{R},X)$, then for all $s \in \mathbb{R}$ the function $x(\cdot) \in C([s,\infty),X)$ defined by (6) is called the *mild solution* of the nonautonomous abstract Cauchy problem (5).

2.1 The Plant and the Exosystem

Throughout this paper we consider a linear distributed parameter system

$$\dot{x} = Ax + Bu + w_s(t), \qquad x(0) = x_0 \in X \tag{7a}$$
$$y = Cx + Du + w_m(t) \tag{7b}$$

on a Banach space X. We assume that A generates a strongly continuous semigroup $T(t)$ on X and the rest of the operators are bounded, $B \in \mathscr{L}(U,X)$, $C \in \mathscr{L}(X,Y)$, $D \in \mathscr{L}(U,Y)$ where U and Y are Hilbert spaces. The disturbance signals $w_s(t)$ and $w_m(t)$ and the reference signal $y_{ref}(t)$ are generated by a time-periodic exosystem

$$\dot{v} = S(t)v, \qquad v(0) = v_0 \in W \tag{8a}$$
$$w_s(t) = E_d(t)v(t), \tag{8b}$$
$$w_m(t) = F_d(t)v(t), \tag{8c}$$
$$y_{ref}(t) = F_{ref}(t)v(t), \tag{8d}$$

on $W = \mathbb{C}^q$ where $S(\cdot) \in \mathbb{R} \to \mathscr{L}(W)$ is a T-periodic function such that $S(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathscr{L}(W))$, and the rest of the functions satisfy $E_d(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,X))$, $F_d(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,Y))$ and $F_{ref}(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,Y))$. The strongly continuous evolution family related to the exosystem is denoted by $U_S(t,s)$. Since W is a finite-dimensional space, the evolution family $U_S(t,s)$ is sometimes also called the *fundamental matrix* of the differential equation (8). Since we assumed that $S(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathscr{L}(W))$, we have that $(s,t) \mapsto U_S(t,s)$ is a continuous mapping. Since the state of the exosystem (8) is $v(t) = U_S(t,0)v_0$, this immediately implies that the reference and disturbance signals are continuous functions. However, they are not necessarily smooth as is the case with the signals generated by time-invariant finite-dimensional exosystems. The signals need not even be continuously differentiable. In fact, any function $f \in C_T(\mathbb{R}, \mathbb{C})$ can be generated by this exosystem by simply choosing $W = \mathbb{C}$, $S(t) \equiv 0$ and $F_{ref}(t) = f(t)$ for all $t \in \mathbb{R}$.

Our assumptions on the imply certain additional properties for the exosystem. Since the function $S(\cdot)$ is T-periodic, it is easy to see that also the evolution family $U_S(t,s)$ is T-periodic in the sense of Definition 1. It also follows from the fact that the operators $S(t)$ for $t \in \mathbb{R}$ are bounded that the properties of Definition 1 hold for arbitrary values of $s,t,r \in \mathbb{R}$ and that for all $t,s \in R$ we have $U_S(t,s)^{-1} = U_S(s,t)$.

It should be noted that the most important difference between the periodic exosystem (8) and a corresponding finite-dimensional time-invariant exosystem is that $E_d(\cdot)$, $F_d(\cdot)$ and $F_{ref}(\cdot)$ are allowed to be periodic functions. On the contrary, allowing $S(\cdot)$ to depend on time brings no additional generality to the classes of signals generated by exosystems of this form. Indeed, the so-called Floquet Representation Theorem [5, Thm. III.7.1] implies that the periodic exosystem (8) can always be rewritten in such a way that $\tilde{S}(\cdot) \equiv \tilde{S}$ is a constant matrix and $\tilde{E}_d(\cdot)$, $\tilde{F}_d(\cdot)$ and $\tilde{F}_{ref}(\cdot)$ are continuous T-periodic functions. However, we choose to work with an exosystem of form (8), because in certain situations allowing $S(\cdot)$ to depend on time is more illustrative or more natural for a given application. This is the case, for example, when we consider tracking of signals with periodically modulated frequencies.

We consider asymptotic regulation and disturbance rejection and because of this we are not concerned with reference and disturbance signals which decay asymptotically. We can therefore assume without loss of generality that the eigenvalues of the matrix $U_S(T,0)$ associated to the exosystem have magnitude greater than or equal to one.

Assumption 1 *We have* $|\lambda| \geq 1$ *for all* $\lambda \in \sigma(U_S(T,0))$.

Defining $E(t) = E_d(t)$ and $F(t) = F_d(t) - F_{ref}(t)$ for all $t \in [0,T]$ we have $E(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,X))$ and $F(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,Y))$. The system (7) can now be written in a standard form as

$$\dot{x} = Ax + Bu + E(t)v, \qquad x(0) = x_0 \in X \tag{9a}$$

$$e = Cx + Du + F(t)v \tag{9b}$$

with the exosystem (8). Here $e(t) = y(t) - y_{ref}(t) \in Y$ is the regulation error.

2.2 Two Types of Controllers

In this paper we consider two types of T-periodic controllers, a feedforward controller and a dynamic error feedback controller. When using the feedforward controller we assume the full state $x(t)$ of the system (9) is available for feedback. The control signal is constructed using $x(t)$ and the state $v(t)$ of the exosystem by

$$u(t) = Kx(t) + L(t)v(t), \tag{10}$$

where $K \in \mathscr{L}(X,U)$ and $L(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,X))$ are the parameters of the controller.

The dynamic error feedback controller is a dynamical system of form

$$\dot{z} = \mathscr{G}_1(t)z + \mathscr{G}_2(t)e, \qquad z(0) = z_0 \in Z \tag{11a}$$

$$u = K(t)z \tag{11b}$$

on a Banach space Z where $(\mathscr{G}_1(t), \mathscr{D}(\mathscr{G}_1(t)))$ is a family of unbounded operators, $\mathscr{G}_2(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(Y,Z))$ and $K(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(Z,U))$.

2.3 The Closed-Loop System

The closed-loop system consisting of the plant and the chosen controller can be written as

$$\dot{x}_e = A_e(t)x_e + B_e(t)v, \qquad x_e(0) = x_{e0} \in X_e \tag{12a}$$

$$e = C_e(t)x_e + D_e(t)v \tag{12b}$$

on a Banach space X_e. In the case of the feedforward controller we have $X_e = X$, $x_e(t) = x(t)$ and the operator-valued functions of the closed-loop system are given by

$$A_e(t) \equiv A + BK, \qquad B_e(t) = BL(t) + E(t),$$
$$C_e(t) \equiv C + DK, \qquad D_e(t) = DL(t) + F(t).$$

On the other hand, for the error feedback controller we have $X_e = X \times Z$, $x_e(t) = (x(t), z(t))^T \in X_e$ and the parameters of the closed-loop system are given by

$$A_e(t) = \begin{pmatrix} A & BK(t) \\ \mathscr{G}_2(t)C & \mathscr{G}_1(t) + \mathscr{G}_2(t)DK(t) \end{pmatrix}, \qquad B_e(t) = \begin{pmatrix} E(t) \\ \mathscr{G}_2(t)F(t) \end{pmatrix},$$

$C_e(t) = \begin{pmatrix} C & DK(t) \end{pmatrix}$ and $D_e(t) = F(t)$. In both of these cases we clearly have that $B_e(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X_e))$, $C_e(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(X_e, Y))$ and $D_e(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, Y))$.

We make the following standing assumption in order to guarantee that the closed-loop system has a well-defined mild state.

Assumption 2 *There exists an exponentially bounded strongly continuous evolution family $U_e(t,s)$ associated to the closed-loop system.*

Under this assumption we have that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the mild state of the closed-loop system is given by

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)v(s)ds.$$

It should be noted that in the case of the feedforward controller Assumption 2 is always satisfied and for the evolution family $U_e(t,s)$ we have

$$U_e(t,s) = T_{A+BK}(t-s),$$

where $T_{A+BK}(t)$ is the C_0-semigroup generated by the operator $A + BK$.

3 The Periodic Output Regulation Problem

In this section we formulate the problem of output regulation and disturbance rejection of the plant (9) when the reference and disturbance signals are generated by the exosystem (8). This problem consists of choosing the parameters of a feedforward or a feedback controller in such a way that the closed-loop system is stable and the regulation error goes to zero asymptotically.

We will also present the main result of this paper which states that the solvability of the regulation problem can be characterized using the solvability of the constrained periodic Sylvester differential equations

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) \tag{13a}$$
$$0 = C_e(t)\Sigma(t) + D_e(t). \tag{13b}$$

These equations are a generalization of the so-called *regulator equations* familiar from the output regulation of distributed parameter systems with time-invariant exosystems [2, 14]. In the case of a time-invariant exosystem these equations are constrained Sylvester operator equations independent of time.

The reason behind the relationship between the solvability of the constrained Sylvester differential equations (13) and the solvability of the output regulation problem is that if the Sylvester differential equation (13a) has a T-periodic solution $\Sigma_\infty(\cdot)$, then the state $x_e(t)$ of the closed-loop system is given by

$$x_e(t) = U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0) + \Sigma_\infty(t)v(t), \tag{14}$$

where x_{e0} and v_0 are the initial states of the closed-loop systems and the exosystem, respectively, and $v(t) = U_S(t,0)v_0$ is the state of the exosystem. From this formula it is clear that if the controller parameter are chosen in such a way that the closed-loop system is stable, then the first term in (14) goes to zero asymptotically. In other words, the state of the closed-loop system approaches the behaviour of the term

$$x_e(t) \sim \Sigma_\infty(t)v(t)$$

as time goes to infinity. Because of this, the term $\Sigma_\infty(t)v(t)$ can be seen as a *dynamic steady state* of the closed-loop system. The role of equation (13b) is now to guarantee that this dynamic steady state gives the desired output despite the disturbance signals.

We will now state the main problem of this paper, the *Periodic Output Regulation Problem*. The statement of the problem does not depend on the form of the controller, i.e. on whether we use a feedforward of a feedback controller. Similarly the main result of this paper — the characterization of the controllers solving this problem — is given in terms of the parameters of the closed-loop system and is valid for both types of controllers. We consider the different controller types separately in Section 6, where we design feedforward and feedback controllers solving the Periodic Output Regulation Problem.

The Periodic Output Regulation Problem *Choose the parameters of a T-periodic controller in such a way that the following are satisfied:*

1. *The evolution family $U_e(t,s)$ is exponentially stable, i.e. there exist constants $M_e \geq 1$ and $\omega_e > 0$ such that $\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$.*
2. *For all initial values $x_{e0} \in X_e$ and $v_0 \in W$ the closed-loop system* (12) *and the exosystem* (8)*, respectively, the regulation error satisfies*

$$\lim_{t\to\infty} e(t) = 0.$$

The following theorem is the first main result of this paper. It characterizes the controllers solving the Periodic Output Regulation Problem in terms of the behaviour of the periodic solution of an infinite-dimensional Sylvester differential equation.

Theorem 1 *Assume the controller stabilizes the closed-loop system exponentially. Then the periodic Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) \tag{15}$$

has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X_e))$ such that

$$\Sigma_\infty(t)v = \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v\,ds$$

for all $v \in W$. The controller solves the Periodic Output Regulation Problem if and only if this solution satisfies

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0 \tag{16}$$

for all $t \in [0,T]$.

The proof of the theorem is divided into parts. We will first study the periodic Sylvester differential equation separately in the next section. We will consider the solvability of this equation and define in detail what is meant by the concept of unique periodic mild solution used in the statement of Theorem 1. We will further show — towards proving the above result — that under the assumptions of Theorem 1 the Sylvester differential equation (15) has a unique periodic mild solution. Subsequently in Section 5 we will finish the proof of Theorem 1 by showing that the solution of the Sylvester differential equation can be used to write the state of the closed-loop system in the form (14) and that the asymptotic behaviour of the closed-loop system leads to zero regulation error if and only if the regulation constraint (16) is satisfied.

4 The Infinite-Dimensional Sylvester Differential Equation

In this section we consider the infinite-dimensional Sylvester differential equation

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t), \qquad \Sigma(0) = \Sigma_0 \tag{17}$$

on the interval $[0,T]$. The equation is considered in the strong sense for $v \in W$. As we saw in the previous section, this equation is closely related to the Periodic Output Regulation Problem. In this section we will define the classical and mild solutions of this equation and prove the first part of Theorem 1, i.e. that under the assumptions stated the Sylvester differential equation (15) has a unique periodic mild solution $\Sigma_\infty(\cdot)$.

If the space X_e is finite-dimensional and if the operator-valued functions appearing in (17) are continuous — as is the situation in [20] — the solvability of the Sylvester differential equation is a fairly straightforward matter. In the infinite-dimensional case, however, the question on the solvability of the equation becomes much more complicated. The first sign of this is that in a strict sense equation (17) only makes sense if $\mathscr{R}(\Sigma(t)) \subset \mathscr{D}(A_e(t))$ for all $t \in [0,T]$. We also need stronger and more complicated conditions for the application of the Leibniz integral rule, which is the core of the proof of the solvability of the equation. The use of this result in the finite-dimensional case requires continuity of the functions $A_e(\cdot)$, $S(\cdot)$ and $B_e(\cdot)$. For matrix-valued functions continuity is a straight-forward matter, unlike for families of unbounded operators. We will see later that the solvability of the Sylvester differential equation (17) in this strict sense requires very restrictive assumptions on the evolution families and functions involved. Fortunately it turns out that for the purposes of Periodic Output Regulation Problem it is sufficient to consider a weaker type of solution of this equation, more precisely the mild solution defined later in this section.

We begin by defining the classical solution of the Sylvester differential equation. Our main intention is to consider the periodic solutions of the equation. For this it is sufficient to start by considering the solution of the equation on the interval $[0,T]$.

Definition 3 A function $\Sigma(\cdot) \in C^1([0,T], \mathscr{L}_s(W,X_e))$ such that $\mathscr{R}(\Sigma(t)) \subset \mathscr{D}(A_e(t))$ for all $t \in [0,T]$ is called the *classical solution* of the Sylvester differential equation (17) if it satisfies the equation on $[0,T]$.

The next theorem states sufficient conditions for the solvability of the problem and shows the form of the solution under these conditions. These conditions are very restrictive for the purposes of Periodic Output Regulation. However, we only use the theorem to obtain the form of the solution in order to define the mild solution of the Sylvester differential equation. The *parabolic conditions* [15, Sec 5.6] which the family $(A_e(t), \mathscr{D}(A_e(t)))$ is assumed to satisfy in essence require that the operators $A_e(t)$ for $t \in [0,T]$ are generators of analytic semigroups on X_e.

Theorem 2 *Assume the following are satisfied.*

1. *There exists $\mu \in \mathbb{R}$ such that $U_e(t,s)$ satisfies the* parabolic conditions:
   (P$_1$) *The domain $\mathscr{D}(A_e(t)) =: \mathscr{D}(A_e)$ is independent of $t \in [0,T]$ and dense in X_e.*
   (P$_2$) *We have $\{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \mu \} \subset \rho(A_e(t))$ for every $t \in [0,T]$ and there exists a constant $M \geq 1$ such that*
   $$\|R(\lambda, A_e(t))\| \leq \frac{M}{|\lambda - \mu| + 1}, \qquad \operatorname{Re}\lambda \geq \mu,\ t \in [0,T].$$
   (P$_3$) *There exists a constant $L \geq 0$ such that for $t,s,r \in [0,T]$*
   $$\|(A_e(t) - A_e(s))R(\mu, A_e(r))\| \leq L|t-s|.$$
2. *The domain $\mathscr{D}(A_e(t)^*) =: \mathscr{D}(A_e^*)$ is independent of $t \in [0,T]$ and dense in X_e^*. For all $x \in X_e$ and $x^* \in \mathscr{D}(A_e^*)$ the mapping*
   $$t \mapsto \langle x, A_e(t)^* x^* \rangle$$
   is continuous on $[0,T]$.
3. *The function $S(\cdot) : [0,T] \to \mathscr{L}(W)$ is strongly continuous.*
4. *The function $B_e(\cdot)$ is strongly continuous on $[0,T]$ and strongly continuously differentiable on $(0,T)$.*
5. $\mathscr{R}(\Sigma_0) \subset \mathscr{D}(A_e)$.

Then the infinite-dimensional Sylvester differential equation (17) *has a unique classical solution $\Sigma(\cdot)$ on $[0,T]$ such that*

$$\Sigma(t)v = U_e(t,0)\Sigma_0 U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds \tag{18}$$

for all $v \in W$.

Proof Since $U_e(t,s)$ satisfies the parabolic conditions, we have from [15, Sec 5.6] that for all $x \in X_e$, $y \in \mathscr{D}(A_e)$, and $t > s$

$$\frac{\partial}{\partial t} U_e(t,s)x = A_e(t)U_e(t,s)x, \qquad \frac{\partial}{\partial s} U_e(t,s)y = -U_e(t,s)A_e(s)y.$$

Also, since $S(\cdot) : [0,T] \to \mathscr{L}(W)$ is a strongly continuous function the evolution family $U_S(t,s)$ satisfies

$$\frac{\partial}{\partial s} U_S(t,s)v = -U_S(t,s)S(s)v, \qquad \frac{\partial}{\partial t} U_S(t,s)v = S(t)U_S(t,s)v$$

for all $v \in W$ and $t < s$.

Let $v \in W$, $x^* \in \mathscr{D}(A_e^*)$, and $s \in [0,T]$. Using the above differentiation rules we see that for any $t \in (s,T]$

$$\begin{aligned}
&\frac{\partial}{\partial t} \langle U_e(t,s)B_e(s)U_S(s,t)v, x^* \rangle \\
&= \langle A_e(t)U_e(t,s)B_e(s)U_S(s,t)v, x^* \rangle - \langle U_e(t,s)B_e(s)U_S(s,t)S(t)v, x^* \rangle \\
&= \langle U_e(t,s)B_e(s)U_S(s,t)v, A_e(t)^* x^* \rangle - \langle U_e(t,s)B_e(s)U_S(s,t)S(t)v, x^* \rangle \\
&\frac{\partial}{\partial t} \langle U_e(t,0)\Sigma_0 U_S(0,t)v, x^* \rangle \\
&= \langle A_e(t)U_e(t,0)\Sigma_0 U_S(0,t)v, x^* \rangle - \langle U_e(t,0)\Sigma_0 U_S(0,t)S(t)v, x^* \rangle \\
&= \langle U_e(t,0)\Sigma_0 U_S(0,t)v, A_e(t)^* x^* \rangle - \langle U_e(t,0)\Sigma_0 U_S(0,t)S(t)v, x^* \rangle
\end{aligned}$$

To show that (18) is a solution of the Sylvester differential equation we will use the Leibniz integral rule. This result states that if $f : \{ (t,s) \mid 0 \le s \le t \le T \} \to \mathbb{C}$ is continuous in t and s, if $\frac{\partial}{\partial t} f(t,s)$ exists and is integrable with respect to s and if there exists a function $f_1 \in L^1(0,T)$ such that

$$\left| \frac{\partial}{\partial t} f(t,s) \right| \le |f_1(s)|,$$

then the mapping $t \mapsto \int_0^t f(t,s)ds$ is differentiable on $(0,T)$ and

$$\frac{d}{dt} \int_0^t f(t,s)ds = f(t,t) + \int_0^t \frac{\partial}{\partial t} f(t,s)ds.$$

Our assumptions imply that the function

$$(t,s) \to f(t,s) = \langle U_e(t,s)B_e(s)U_S(s,t)v, x^* \rangle$$

is continuous for $0 \le s \le t \le T$ and the computation above shows that it is continuously differentiable with respect to t. Since the mappings $(t,s) \to U_e(t,s)$ and $(t,s) \to U_S(s,t)$ are strongly continuous, there exist constants $M_e, M_S > 0$ such that

$$\max_{0 \le s \le t \le T} \|U_e(t,s)\| \le M_e, \qquad \max_{0 \le s \le t \le T} \|U_S(s,t)\| \le M_S.$$

Using these estimates we see that

$$\begin{aligned}\left|\frac{\partial}{\partial t}f(t)\right| &\le \|U_e(t,s)B_e(s)U_S(s,t)v\|\cdot\|A_e(t)^*x^*\| + \|U_e(t,s)B_e(s)U_S(s,t)S(t)v\|\cdot\|x^*\|\\ &\le \|U_e(t,s)\|\cdot\|B_e(s)\|\cdot\|U_S(s,t)\|\left(\|v\|\cdot\|A_e(t)^*x^*\| + \|S(t)\|\cdot\|v\|\cdot\|x^*\|\right)\\ &\le M_eM_S\|v\|\cdot\|B_e\|_\infty\left(\max_{r\in[0,T]}\|A_e(r)^*x^*\| + \|x^*\|\cdot\|S\|_\infty\right) =: f_1(s).\end{aligned}$$

Since f_1 is a constant function we have $f_1 \in L^1(0,T)$. This concludes that we can use the Leibniz integral rule.

We now have for the function $\Sigma(\cdot)$ in (18) that

$$\begin{aligned}&\frac{d}{dt}\langle\Sigma(t)v,x^*\rangle = \frac{\partial}{\partial t}\langle U_e(t,0)\Sigma_0U_S(0,t)v,x^*\rangle + \frac{\partial}{\partial t}\int_0^t\langle U_e(t,s)B_e(s)U_S(s,t)v,x^*\rangle ds\\ &= \langle U_e(t,0)\Sigma_0U_S(0,t)v,A_e(t)^*x^*\rangle - \langle U_e(t,0)\Sigma_0U_S(0,t)S(t)v,x^*\rangle\\ &\quad + \int_0^t\left(\langle U_e(t,s)B_e(s)U_S(s,t)v,A_e(t)^*x^*\rangle - \langle U_e(t,s)B_e(s)U_S(s,t)S(t)v,x^*\rangle\right)ds\\ &\quad + \langle U_e(t,t)B_e(t)U_S(t,t)v,x^*\rangle\\ &= \langle\Sigma(t)v,A_e(t)^*x^*\rangle - \langle\Sigma(t)S(t)v,x^*\rangle + \langle B_e(t)v,x^*\rangle \qquad (19)\end{aligned}$$

We will next show that the mapping $t\mapsto\Sigma(t)v$ is continuously differentiable on $(0,T)$ and that $\Sigma(t)v\in\mathscr{D}(A_e)$ for all $t\in[0,T]$. We will do this by first considering the nonautonomous Cauchy problem

$$\dot{x}(t) = A_e(t)x(t) + B_e(t)U_S(t,0)w, \qquad x(0) = \Sigma_0 w,$$

where $w\in W$. Since $x(0)\in\mathscr{D}(A_e)$ and since $t\mapsto B_e(t)U_S(t,0)w$ is continuously differentiable we have from [15, Thm 5.7.1] that this equation has a unique classical solution given by

$$x(t) = U_e(t,0)\Sigma_0 w + \int_0^t U_e(t,s)B_e(s)U_S(s,0)w\,ds$$

such that $x(\cdot)$ is continuously differentiable on $(0,T)$ and $x(t)\in\mathscr{D}(A_e(t))$ for all $t\in[0,T]$. If we denote by $H(\cdot):[0,T]\to\mathscr{L}(W,X_e)$ the mapping $x(t)=H(t)w$, then $t\mapsto H(t)$ is strongly continuously differentiable on $(0,T)$ and $\mathscr{R}(H(t))\subset\mathscr{D}(A_e)$. Since $t\mapsto U_S(0,t)$ is strongly continuously differentible, by choosing $w=U_S(0,t)v$ we finally obtain that the mapping

$$\begin{aligned}t\mapsto H(t)U_S(0,t)v &= U_e(t,0)\Sigma_0U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,0)U_S(0,t)v\,ds\\ &= U_e(t,0)\Sigma_0U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds\\ &= \Sigma(t)v\end{aligned}$$

is continuously differentiable on $(0,T)$ and $\Sigma(t)v \in \mathscr{D}(A_e)$ for all $[0,T]$. Equation (19) now becomes

$$\langle \frac{d}{dt}\Sigma(t)v, x^*\rangle + \langle \Sigma(t)S(t)v, x^*\rangle = \langle A_e(t)\Sigma(t)v, x^*\rangle + \langle B_e(t)v, x^*\rangle$$

Since $x^* \in \mathscr{D}(A_e^*)$ was arbitrary and since $\mathscr{D}(A_e^*)$ is dense in X_e^*, this implies

$$\frac{d}{dt}\Sigma(t)v + \Sigma(t)S(t)v = A_e(t)\Sigma(t)v + B_e(t)v.$$

Since $v \in W$ was arbitrary, this concludes that $\Sigma(\cdot)$ satisfies the Sylvester differential equation.

To prove the uniqueness of the solution, let $\Sigma_1(\cdot) \in C^1([0,T], \mathscr{L}_s(W,X))$ be a classical solution of the Sylvester differential equation (18). Letting $v \in W$ and applying both sides of the equation to $U_S(s,t)v \in W$ we obtain

$$\begin{aligned}
&\dot{\Sigma}_1(s)U_S(s,t)v + \Sigma_1(s)S(s)U_S(s,t)v = A_e(s)\Sigma_1(s)U_S(s,t)v + B_e(s)U_S(s,t)v \\
\Rightarrow\quad &U_e(t,s)\dot{\Sigma}_1(s)U_S(s,t)v + U_e(t,s)\Sigma_1(s)S(s)U_S(s,t)v \\
&= U_e(t,s)A_e(s)\Sigma_1(s)U_S(s,t)v + U_e(t,s)B_e(s)U_S(s,t)v \\
\Rightarrow\quad &\frac{d}{ds}\left(U_e(t,s)\Sigma_1(s)U_S(s,t)v\right) = U_e(t,s)B_e(s)U_S(s,t)v
\end{aligned}$$

Integrating both sides of the equation from 0 to t and using $\Sigma_1(0) = \Sigma_0$ gives

$$\begin{aligned}
\int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds &= U_e(t,t)\Sigma_1(t)U_S(t,t)v - U_e(t,0)\Sigma_1(0)U_S(0,t)v \\
&= \Sigma_1(t)v - U_e(t,0)\Sigma_0 U_S(0,t)v
\end{aligned}$$

and thus $\Sigma_1(\cdot) = \Sigma(\cdot)$. □

As was already mentioned, the conditions imposed on the evolution family $U_e(t,s)$ in Theorem 2 essentially require that for $t \in [0,T]$ the operators $A_e(t)$ generate analytic semigroups on X_e. An immediate consequence of this is that in particular also the semigroup generated by A on X must be analytic. In this paper, however, we do not need to restrict ourselves to the cases where the Sylvester differential equations have classical solutions. It turns out it is sufficient to consider their *mild solutions*, as given in the following definition.

Definition 4 Let $\Sigma_0 \in \mathscr{L}(W,X_e)$. Under Assumption 2 the operator-valued function $\Sigma(\cdot) \in C([0,T], \mathscr{L}_s(W,X_e))$ defined in (18) is called the *mild solution* of the Sylvester differential equation (17) on $[0,T]$.

We can finally turn our attention to the periodic solutions of the Sylvester differential equation. By the periodic Sylvester differential equation we mean the equation

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) \tag{20}$$

on $\mathbb{R}$. The mild solution of this equation is a mild solution $\Sigma(\cdot)$ of the Sylvester differential equation (17) on $[0,T]$ corresponding to an initial condition $\Sigma(0) = \Sigma_0 \in$

$\mathscr{L}(W,X_e)$ for which $\Sigma(\cdot)$ is a periodic function. The following theorem states that if the closed-loop system is exponentially stable, then under the standing assumptions the periodic Sylvester differential equation (20) has a unique periodic mild solution and that this solution has period T.

Theorem 3 *If $U_e(t,s)$ is exponentially stable, then the periodic Sylvester differential equation* (20) *has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,X_e))$ such that*

$$\Sigma_\infty(t)v = \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)vds$$

for all $v \in W$.

Proof We will first show that $\Sigma_\infty(\cdot)$ is a mild solution of the Sylvester differential equation (17). Since for every $v \in W$ we have

$$\Sigma_\infty(t)v = U_e(t,0)\int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)U_S(0,t)vds + \int_0^t U_e(t,s)B_e(s)U_S(s,t)vds,$$

it suffices to show that the linear operator $\Sigma_\infty(0)$ defined by

$$\Sigma_\infty(0)v = \int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)vds, \qquad v \in W$$

is bounded. Before we can do this we need to show that Assumption 1 implies that there exist constants $n_S \in \mathbb{N}_0$ and $M_S > 0$ such that $\|U_S(t,s)\| \le M_S((s-t)^{n_S}+1)$ for all $t \le s$. To see this, let $R \in \mathscr{L}(W) = \mathbb{C}^{q\times q}$ be such that

$$U_S(0,T) = U_S(T,0)^{-1} = RJR^{-1},$$

where J is the Jordan canonical form of $U_S(T,0)^{-1}$. Let $t \le s$ and let $t_0, s_0 \in [0,T)$ and $m_t, m_s \in \mathbb{Z}$ be such that $t = m_tT + t_0$ and $s = m_sT + s_0$. We have $m_t \le m_s$ and taking advantage of the periodicity of the evolution family $U_S(t,s)$

$$\begin{aligned}
\|U_S(t,s)\| &= \|U_S(m_tT+t_0, m_sT+s_0)\| \\
&= \|U_S(m_tT+t_0,(m_t+1)T)U_S((m_t+1)T, m_sT)U_S(m_sT, m_sT+s_0)\| \\
&= \|U_S(t_0,T)U_S(0,T)^{m_s-m_t-1}U_S(0,s_0)\| \\
&\le \|R\|\|R^{-1}\|\|J^{m_s-m_t-1}\| \max_{t_0\in[0,T]}\|U_S(t_0,0)\| \cdot \max_{s_0\in[0,T]}\|U_S(0,s_0)\|
\end{aligned}$$

If $J_k \in \mathbb{C}^{n\times n}$ is a single Jordan block in J associated to an eigenvalue $\lambda_k \in \mathbb{C}$, we then have from Assumption 1 that $|\lambda_k| \le 1$. If we denote $H_k = J_k - \lambda_k I$, then $\|H_k\| = 1$ and $H_k^n = 0$. For brevity denote $m_d = m_s - m_t - 1$. Whenever $m_d \ge n$ we have $m_dT \le s-t$ and thus

$$\begin{aligned}
\|J_k^{m_d}\| = \|(\lambda_k I + H_k)^{m_d}\| &= \left\|\sum_{l=0}^{m_d}\binom{m_d}{l}\lambda_k^{m_d-l}H_k^l\right\| \le \sum_{l=0}^{n-1}\frac{m_d!}{l!(m_d-l)!}|\lambda_k|^{m_d-l}\|H_k\|^l \\
&\le \sum_{l=0}^{n-1} m_d(m_d-1)\cdots(m_d-l+1) \le nm_d^{n-1} = \frac{n}{T^{n-1}}(m_dT)^{n-1} \le \frac{n}{T^{n-1}}(s-t)^{n-1}
\end{aligned}$$

Since $\|J\| = \max_k \|J_k\|$, this implies that there exist constants $n_S \in \mathbb{N}_0$ and $\tilde{M}_S > 0$ such that $\|U_S(t,s)\| \leq \tilde{M}_S (s-t)^{n_S}$ for all $t \leq s$ for which $s-t$ is sufficiently large. Combining this with the fact that $\|U_S(t,s)\|$ is uniformly bounded on finite intervals, we can conclude that there exists a constant $M_S > 0$ such that for all $t \leq s$ we have $\|U_S(t,s)\| \leq M_S((s-t)^{n_S}+1)$.

Since the closed-loop system is stable there exist constants $M_e \geq 1$ and $\omega_e > 0$ such that for all $t \geq s$ we have $\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$. Now for all $v \in W$ we have

$$\begin{aligned}
&\| \int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)v\,ds \| \leq \int_{-\infty}^{0} \|U_e(0,s)B_e(s)U_S(s,0)v\| ds \\
&\leq M_e M_S \|B_e\|_\infty \int_{-\infty}^{0} ((-s)^{n_S}+1)e^{\omega_e s} ds \cdot \|v\| =: M\|v\|,
\end{aligned}$$

where $M < \infty$ and thus $\Sigma_\infty(0) \in \mathscr{L}(W,X_e)$.

To prove the periodicity of $\Sigma_\infty(\cdot)$, let $t \in \mathbb{R}$. We then have for all $v \in W$

$$\begin{aligned}
\Sigma_\infty(t+T)v &= \int_{-\infty}^{t+T} U_e(t+T,s)B_e(s)U_S(s,t+T)v\,ds \\
&= \int_{-\infty}^{t} U_e(t+T,s+T)B_e(s+T)U_S(s+T,t+T)v\,ds \\
&= \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v\,ds = \Sigma_\infty(t)v.
\end{aligned}$$

This shows that $\Sigma_\infty(\cdot)$ is T-periodic.

It remains to prove that the periodic Sylvester differential equation (20) has no other periodic solutions. To this end, let $\Sigma(\cdot)$ be any periodic mild solution of the equation corresponding to an arbitrary initial condition $\Sigma(0) = \Sigma_0 \in \mathscr{L}(W,X_e)$, i.e.

$$\Sigma(t)v = U_e(t,0)\Sigma_0 U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds$$

for $v \in W$. The difference $\Delta(t)v = \Sigma_\infty(t)v - \Sigma(t)v$ satisfies

$$\begin{aligned}
\Delta(t)v &= \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v\,ds - U_e(t,0)\Sigma_0 U_S(0,t)v - \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds \\
&= \int_{-\infty}^{0} U_e(t,s)B_e(s)U_S(s,t)v\,ds - U_e(t,0)\Sigma_0 U_S(0,t)v \\
&= U_e(t,0)\Sigma_\infty(0)U_S(0,t) - U_e(t,0)\Sigma_0 U_S(0,t)v = U_e(t,0)\Delta(0)U_S(0,t)v.
\end{aligned}$$

Thus

$$\|\Delta(t)\| \leq M_e M_S (t^{n_S}+1)e^{-\omega_e t}\|\Delta(0)\|$$

and the assumption $\omega_e > 0$ implies $\lim_{t\to\infty} \Delta(t) = 0$. Since $\Sigma(\cdot)$ is periodic and since $\lim_{t\to\infty} \|\Sigma(t) - \Sigma_\infty(t)\| = 0$, we must have $\Sigma(t) \equiv \Sigma_\infty(t)$. This concludes that no other periodic solutions than $\Sigma_\infty(\cdot)$ may exist. □

This concludes the treatment of the solvability of the Sylvester differential equation in this paper. In the next section we will use these results to prove Theorem 1.

5 The Dynamic Steady State of the Closed-Loop System

The idea behind the proof of Theorem 1 is as was outlined in the beginning of Section 3. We will first prove that the state of the closed-loop system can be expressed using the unique periodic mild solution $\Sigma_\infty(\cdot)$ of the Sylvester differential equation (15). It turns out that this representation of the state of the closed-loop system is actually equivalent to the Sylvester differential equation. This is shown in Theorem 4, which is a generalization of the corresponding result for distributed parameter systems with time-invariant exosystems [7, 14]. This formula will allow us to investigate the asymptotic behaviour of the regulation error and to complete the proof of Theorem 1 by showing that the regulation error decays to zero asymptotically if and only if the regulation constraint (16) is satisfied.

Theorem 4 *Let $\Sigma(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X_e))$. Then the following are equivalent.*

1. *The function $\Sigma(\cdot)$ is a periodic mild solution of the Sylvester differential equation* (15)*.*
2. *For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system and the exosystem the state of the closed-loop system can be written as*

$$x_e(t) = U_e(t,0)(x_{e0} - \Sigma(0)v_0) + \Sigma(t)v(t), \qquad t \geq 0. \tag{21}$$

If these conditions are satisfied, then the regulation error is given by

$$e(t) = C_e(t)U_e(t,0)(x_{e0} - \Sigma(0)v_0) + (C_e(t)\Sigma(t) + D_e(t))v(t), \qquad t \geq 0 \tag{22}$$

for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system and the exosystem.

Proof For any initial conditions $x_{e0} \in X_e$ and $v_0 \in W$ and for any $t \geq 0$ the state of the closed-loop system (12) is given by

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0 ds.$$

Using this and $v(t) = U_S(t,0)v_0$ we immediately see that the state of the closed-loop having representation (21) for some $\Sigma(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X_e))$ and for all initial states is equivalent to

$$\int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0 ds = -U_e(t,0)\Sigma(0)v_0 + \Sigma(t)U_S(t,0)v_0, \qquad \forall v_0 \in W.$$

Since $U_S(0,t)$ is invertible, every $v_0 \in W$ can be written as $v_0 = U_S(0,t)w_0$ for some $w_0 \in W$. Thus the above condition is equivalent the fact that

$$\begin{aligned} \int_0^t U_e(t,s)B_e(s)U_S(s,0)U_S(0,t)w_0 ds = &- U_e(t,0)\Sigma(0)U_S(0,t)w_0 \\ &+ \Sigma(t)U_S(t,0)U_S(0,t)w_0 \\ \Leftrightarrow \qquad \int_0^t U_e(t,s)B_e(s)U_S(s,t)w_0 ds = &- U_e(t,0)\Sigma(0)U_S(0,t)w_0 + \Sigma(t)w_0, \end{aligned}$$

for all $w_0 \in W$. This is precisely the condition that $\Sigma(\cdot)$ is a mild solution of the Sylvester differential equation (15) corresponding to an initial condition $\Sigma(0)$. Since $\Sigma(\cdot)$ is a periodic function, this is finally equivalent to the fact that it is a periodic mild solution of the Sylvester differential equation (15).

If the state of the closed-loop system has the representation (21), then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error is given by

$$e(t) = C_e(t)x_e(t) + D_e v(t) = C_e(t)U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0) + (C_e(t)\Sigma_\infty(t) + D_e(t))\, v(t).$$

This concludes the proof. □

As was stated before, if the closed-loop system is stable then the formula (21) allows us to investigate the asymptotic behaviour of the regulation error in the case. This is shown in the next lemma.

Lemma 1 *Assume the closed-loop system is exponentially stable and the Sylvester differential equation* (15) *has a periodic mild solution* $\Sigma(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X_e))$. *Then for all initial states* $x_{e0} \in X_e$ *and* $v_0 \in W$ *of the closed-loop system and the signal generator the state of the closed-loop system and the regulation error satisfy*

$$\lim_{t\to\infty} \|x_e(t) - \Sigma(t)v(t)\| = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|e(t) - (C_e(t)\Sigma(t) + D_e(t))v(t)\| = 0. \quad (23)$$

Proof Since the closed-loop system is exponentially stable, there exist constants $M_e \geq 1$ and $\omega_e > 0$ such that $\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$ for all $t \geq s$.

Formulas (21) and (22) in Theorem 4 implies that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ we have

$$\|x_e(t) - \Sigma(t)v(t)\| = \|U_e(t,0)(x_{e0} - \Sigma(0)v_0)\| \leq M_e e^{-\omega_e t} \|x_{e0} - \Sigma(0)v_0\| \longrightarrow 0$$

and

$$\begin{aligned} \|e(t) - (C_e(t)\Sigma(t) + D_e(t))v(t)\| &= \|C_e(t)U_e(t,0)(x_{e0} - \Sigma(0)v_0)\| \\ &\leq M_e e^{-\omega_e t} \|C_e\|_\infty \|x_{e0} - \Sigma(0)v_0\| \longrightarrow 0 \end{aligned}$$

as $t \to \infty$, since $\omega_e > 0$. □

We can finally use the previous results to present the proof of Theorem 1.

Proof (Proof of Theorem 1*)* Since $U_e(t,s)$ is exponentially stable we have from Theorem 3 that the periodic Sylvester differential equation (15) has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X_e))$ given by the appropriate formula.

Assume first that the periodic solution $\Sigma_\infty(\cdot)$ of the Sylvester differential equation satisfies the regulation constraint

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0$$

for all $t \in [0,T]$. Since the functions are T-periodic, this is satisfied for all $t \in \mathbb{R}$. Using this and Lemma 1 we have that for all initial values $x_{e0} \in X_e$ and $v_0 \in W$

$$\|e(t)\| = \|e(t) - (C_e(t)\Sigma_\infty(t) + D_e(t))v(t)\| \longrightarrow 0$$

as $t \to \infty$ and thus the controller solves the Periodic Output Regulation Problem.

It remains to show the converse implication. To this end, we assume that the controller solves the Periodic Output Regulation Problem and show that the regulation constraint (16) is satisfied. Let $t_0 \in [0,T)$ be arbitrary and denote $t = t_0 + nT$ for $n \in \mathbb{N}$. Since the controller solves the Periodic Output Regulation Problem we have from Lemma 1 that for all $v_0 \in W$ and arbitrary $x_{e0} \in X_e$

$$\begin{aligned} &\|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t,0)v_0\| = \|(C_e(t)\Sigma(t) + D_e(t))U_S(t,0)v_0\| \\ \le &\|(C_e(t)\Sigma_\infty(t) + D_e(t))\, U_S(t,0)v_0 - e(t)\| + \|e(t)\| \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. Let $\lambda \in \sigma(U_S(T,0))$ and let $\{\phi_k\}_{k=1}^m$ be a Jordan chain associated to this eigenvalue. By Assumption 1 we have $|\lambda| \ge 1$. Now $U_S(T,0)\phi_1 = \lambda\phi_1$ and

$$U_S(T,0)\phi_k = \lambda\phi_k + \phi_{k-1}, \quad k \in \{2,\dots,m\}. \tag{24}$$

Using the periodicity of the evolution family $U_S(t,s)$ we get

$$\begin{aligned} U_S(t,0) &= U_S(t_0+nT,0) = U_S(t_0+nT,nT)U_S(nT,(n-1)T)\cdots U_S(T,0) \\ &= U_S(t_0,0)U_S(T,0)^n \end{aligned}$$

and thus

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t,0)\phi_1\| \\ &= \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t_0,0)\phi_1\| \cdot \left(\lim_{n\to\infty} |\lambda|^n\right), \end{aligned}$$

which implies $(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t_0,0)\phi_1 = 0$ since $|\lambda| \ge 1$. Using this and (24) we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t,0)\phi_2\| \\ &= \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t_0,0)\phi_2\| \cdot \left(\lim_{n\to\infty} |\lambda|^n\right) \end{aligned}$$

and thus also $(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t_0,0)\phi_2 = 0$. Continuing this we finally obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t,0)\phi_m\| \\ &= \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t_0,0)\phi_m\| \cdot \left(\lim_{n\to\infty} |\lambda|^n\right) \end{aligned}$$

which implies $(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, U_S(t_0,0)\phi_m = 0$. Since $\lambda \in \sigma(U_S(T,0))$ and the associated Jordan chain were arbitrary, we have that $(C_e(t_0)\Sigma(t_0) + D_e(t_0))U_S(t_0,0) = 0$. The invertibility of $U_S(t_0,0)$ further concludes that $C_e(t_0)\Sigma(t_0) + D_e(t_0) = 0$. Since $t_0 \in [0,T)$ was arbitrary, this finally shows that $C_e(t)\Sigma(t) + D_e(t) = 0$ for every $t \in [0,T]$. This concludes the proof. □

6 Controller Design

In this section we show how to construct two types of controllers to solve the Periodic Output Regulation Problem. We will consider a feedforward controller and an observer-based error feedback controller. These controllers generalize the corresponding well-known controllers for time-invariant finite-dimensional [9] and infinite-dimensional [11] systems.

6.1 A Feedforward Controller

The following theorem presents sufficient conditions for the solvability of the Periodic Output Regulation Problem using a feedforward controller and the appropriate choices of the parameters. Using this type of controller is possible when the state of the plant is available for feedback. In particular this covers the case where the original plant is already exponentially stable.

Theorem 5 *Assume the pair (A,B) is exponentially stabilizable and that the constrained Sylvester differential equation*

$$\dot{\Pi}(t)+\Pi(t)S(t) = A\Pi(t)+B\Gamma(t)+E(t) \tag{25a}$$

$$0 = C\Pi(t)+D\Gamma(t)+F(t) \tag{25b}$$

has a periodic mild solution $\Pi(\cdot) \in C_T(\mathbb{R},\mathscr{L}_s(W,X))$ and $\Gamma(\cdot) \in C_T(\mathbb{R},\mathscr{L}_s(W,U))$, then the Periodic Output Regulation Problem is solved by a feedforward controller with parameters $K \in \mathscr{L}(X,U)$ and $L(\cdot) \in C_T(\mathbb{R},\mathscr{L}_s(W,U))$ where K is chosen in such a way that $A+BK$ generates an exponentially stable C_0-semigroup and $L(t) = \Gamma(t)-K\Pi(t)$ for all $t \in [0,T]$, i.e.

$$u(t) = Kx(t)+(\Gamma(t)-K\Pi(t))v(t).$$

Proof Since $A_e(t) \equiv A+BK$, we have from the choice of the operator $K \in \mathscr{L}(X,U)$ that the closed-loop system is exponentially stable. We thus have from Theorem 1 that the feedforward controller solves the Peridic Output Regulation Problem if the unique periodic mild solution $\Sigma_\infty(\cdot)$ of the Sylvester differential equation (15) satisfies $C_e(t)\Sigma_\infty(t)+D_e(t) = 0$ for all $t \in [0,T]$. For the feedforward controller the Sylvester differential equation (15) becomes (writing $L(t) = \Gamma(t)-K\Pi(t)$)

$$\begin{aligned}\dot{\Sigma}(t)+\Sigma(t)S(t) &= A_e(t)\Sigma(t)+B_e(t) = (A+BK)\Sigma(t)+BL(t)+E(t)\\ &= A\Sigma(t)+B\Gamma(t)+E(t)+BK(\Sigma(t)-\Pi(t))\end{aligned}$$

An operator-valued function $\Sigma(\cdot) \in C([0,T],\mathscr{L}_s(W,X))$ is the mild solution of this equation on $[0,T]$ corresponding to an initial condition $\Sigma(0)$ if it satisfies the integral equation

$$\begin{aligned}\Sigma(t)v = T_A(t)\Sigma(0)U_S(0,t)v &+ \int_0^t T_A(t-s)(B\Gamma(s)+E(s))U_S(s,t)v\,ds\\ &+ \int_0^t T_A(t-s)BK(\Sigma(s)-\Pi(s))U_S(s,t)v\,ds.\end{aligned}$$

for all $v \in W$. This shows that $\Sigma(\cdot) = \Pi(\cdot)$ is a mild solution of this equation, because in this case the last integral vanishes and $\Pi(\cdot)$ satisfies the remaining equation as it is the mild solution of the Sylvester differential equation (25a). Since $\Pi(\cdot)$ is periodic and since the closed-loop system is stable, $\Pi(\cdot)$ is the unique periodic mild solution of the Sylvester differential equation (15), i.e. $\Sigma_\infty(\cdot) = \Pi(\cdot)$.

We now have using (25b) that for all $t \in [0,T]$

$$\begin{aligned} C_e(t)\Sigma_\infty(t) + D_e(t) &= (C+DK)\Pi(t) + DL(t) + F(t) \\ &= C\Pi(t) + D(K\Pi(t) + L(t)) + F(t) \\ &= C\Pi(t) + D\Gamma(t) + F(t) = 0. \end{aligned}$$

This concludes the proof. □

6.2 A Feedback Controller

In the case where the state of the plant is not available for feedback we need to use an observer-based controller. The next theorem shows how to construct this type of controller solving the Periodic Output Regulation Problem.

Theorem 6 *Assume that the pair (A,B) is exponentially stabilizable, that there exists a periodic output injection $L(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(Y,X_e))$ such that the evolution family associated to the family*

$$\left(\begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + L(t) \begin{pmatrix} C & F(t) \end{pmatrix}, \mathscr{D}(A) \times W \right) \tag{26}$$

of operators is exponentially stable and assume that the constrained Sylvester differential equation

$$\dot{\Pi}(t) + \Pi(t)S(t) = A\Pi(t) + B\Gamma(t) + E(t) \tag{27a}$$

$$0 = C\Pi(t) + D\Gamma(t) + F(t) \tag{27b}$$

has a periodic mild solution $\Pi(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,X))$ and $\Gamma(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,U))$. Under these assumptions the Periodic Output Regulation Problem is solved by an error feedback controller with parameters

$$\mathscr{G}_1(t) = \begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} K_1 & K_2(t) \end{pmatrix} + L(t) \left(\begin{pmatrix} C & F(t) \end{pmatrix} + DK(t) \right)$$

$$\mathscr{D}(\mathscr{G}_1(t)) =: \mathscr{D}(\mathscr{G}_1) = \mathscr{D}(A) \times W$$

and $\mathscr{G}_2(t) = -L(t)$ on $Z = X \times W$ where $K_1 \in \mathscr{L}(X,U)$ is chosen in such a way that $A + BK_1$ generates an exponentially stable semigroup, $K_2(t) = \Gamma(t) - K_1\Pi(t)$ and $K(t) = \begin{pmatrix} K_1 & K_2(t) \end{pmatrix}$.

Proof Since A generates a semigroup on X and since $S(\cdot)$ is a matrix-valued locally integrable function, it is clear that there exists a strongly continuous evolution family associated to the family $(\mathscr{G}_1(t), \mathscr{D}(\mathscr{G}_1))$ of operators and thus the error feedback controller (11) has a well-defined mild state.

We will first show that there exists a strongly continuous evolution family associated to the closed-loop system and that this evolution family is exponentially stable. For our choice of the feedback controller we have that $X_e = X \times X \times W$ and for the family $(A_e(t), \mathscr{D}(A_e(t)))$ of operators we have $\mathscr{D}(A_e(t)) = \mathscr{D}(A) \times \mathscr{D}(A) \times W$ and

$$A_e(t) = \begin{pmatrix} A & BK_1 & BK_2(t) \\ -L_1(t)C & A + BK_1 + L_1(t)C & E(t) + BK_2(t) + L_1(t)F(t) \\ -L_2(t)C & L_2(t)C & S(t) + L_2(t)F(t) \end{pmatrix}$$

for all $t \geq 0$. Here we have denoted $L(t) = (L_1(t), L_2(t))^T$. Applying a time-invariant similarity transform $T \in \mathscr{L}(X_e)$ such that

$$T = \begin{pmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

we can define

$$\tilde{A}_e(t) = TA_e(t)T^{-1} = \begin{pmatrix} A + BK_1 & BK_1 & BK_2(t) \\ 0 & A + L_1(t)C & E(t) + L_1(t)F(t) \\ 0 & L_2(t)C & S(t) + L_2(t)F(t) \end{pmatrix}$$

and $\mathscr{D}(\tilde{A}_e(t)) = \mathscr{D}(A_e(t))$ for all $t \geq 0$. Since A generates a C_0-semigroup on X, since $S(\cdot)$ is a matrix-valued locally integrable function and since the other operator-valued functions are continuous and uniformly bounded it is straight-forward to show that there exists a strongly continuous evolution family associated to the family $(\tilde{A}_e(t), \mathscr{D}(\tilde{A}_e(t)))$ of operators. Furthermore, this evolution family is exponentially stable since $K_2(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W, X))$, since $A + BK_1$ generates an exponentially stable semigroup and since the evolution family associated to the system (26) of operators is exponentially stable. Clearly the same conclusions now also apply to the family $(A_e(t), \mathscr{D}(A_e(t)))$ of operators of the closed-loop system.

By Theorem 1 it remains to show that the unique periodic solution of the Sylvester differential equation (15) satisfies $C_e(t)\Sigma(t) + D_e(t) = 0$ for all $t \in [0, T]$. Writing $\Sigma(t) = (\Sigma_1(t), \Sigma_2(t))^T$ we have that for a dynamic error feedback controller this equation can be written as a pair of equations

$$\dot{\Sigma}_1(t) + \Sigma_1(t)S(t) = A\Sigma_1(t) + BK(t)\Sigma_2(t) + E(t) \tag{28a}$$

$$\dot{\Sigma}_2(t) + \Sigma_2(t)S(t) = \mathscr{G}_1(t)\Sigma_2(t) + \mathscr{G}_2(t)(C\Sigma_1(t) + DK\Sigma_2(t) + F(t)) \tag{28b}$$

Let $\Pi(\cdot)$ and $\Gamma(\cdot)$ be the solution of equations (27). We will show that the unique periodic mild solution $\Sigma_\infty(\cdot)$ of the Sylvester differential equation (15) is given by $\Sigma_1(\cdot) = \Pi(\cdot)$ and $\Sigma_2(\cdot) = (\Pi(\cdot), I)^T$. If this is the case, then using (27b) and the fact

$$\Gamma(t) = K_1\Pi(t) + K_2(t) = K(t)\begin{pmatrix} \Pi(t) \\ I \end{pmatrix}$$

we have that

$$C_e(t)\Sigma_\infty(t)+D_e(t)=C\Pi(t)+DK(t)\begin{pmatrix}\Pi(t)\\ I\end{pmatrix}+F(t)=C\Pi(t)+D\Gamma(t)+F(t)=0.$$

Theorem 1 then concludes that the dynamic error feedback controller solves the Periodic Output Regulation Problem.

If $\Sigma_2(\cdot)=(\Pi(\cdot),I)^T$, equation (28a) is equal to equation (27a) and thus in this situation the function $\Sigma_1(\cdot)=\Pi(\cdot)$ is a mild solution of (28a) corresponding to the initial value $\Pi(0)\in\mathscr{L}(W,X)$.

It remains to show that if $\Sigma_1(\cdot)=\Pi(\cdot)$, then $\Sigma_2(\cdot)=(\Pi(\cdot),I)^T$ is a mild solution of (28b). Assume $\Sigma_1(\cdot)=\Pi(\cdot)$. The solution of (28b) is of form $\Sigma_2(\cdot)=(\Sigma_{21}(\cdot),\Sigma_{22}(\cdot))^T$. If $\Sigma_{22}(\cdot)=I$, the left-hand side of the equation can be written formally as

$$\dot\Sigma(t)+\Sigma(t)S(t)=\begin{pmatrix}\dot\Sigma_{21}(t)+\Sigma_{21}(t)S(t)\\ S(t)\end{pmatrix}.$$

On the other hand, the right-hand side of the equation becomes

$$\begin{aligned}
&\mathscr{G}_1(t)\Sigma_2(t)+\mathscr{G}_2(t)(C\Pi(t)+DK(t)\Sigma_2(t)+F(t))\\
=&\begin{pmatrix}(A+BK_1)\Sigma_{21}(t)+E(t)+BK_2(t)\\ S(t)\end{pmatrix}+L(t)\left[\begin{pmatrix}C & F(t)\end{pmatrix}+D\begin{pmatrix}K_1 & K_2(t)\end{pmatrix}\right]\begin{pmatrix}\Sigma_{21}(t)\\ I\end{pmatrix}\\
&-L(t)\left(C\Pi(t)+DK(t)\begin{pmatrix}\Sigma_{21}(t)\\ I\end{pmatrix}+F(t)\right)\\
=&\begin{pmatrix}(A+BK_1)\Sigma_{21}(t)+BK_2(t)+E(t)\\ S(t)\end{pmatrix}+L(t)C\left(\Sigma_{21}(t)-\Pi(t)\right)\\
=&\begin{pmatrix}A\Sigma_{21}(t)+B\Gamma(t)+E(t)\\ S(t)\end{pmatrix}+\left(\begin{pmatrix}BK_1\\ 0\end{pmatrix}+L(t)C\right)\left(\Sigma_{21}(t)-\Pi(t)\right).
\end{aligned}$$

Here we have used $K_2(t)=\Gamma(t)-K_1\Pi(t)$. Thus equation (28b) is equivalent to the pair

$$\begin{aligned}
\dot\Sigma_{21}(t)+\Sigma_{21}(t)S(t)&=A\Sigma_{21}(t)+B\Gamma(t)+E(t)+(BK_1+L_1(t)C)(\Sigma_{21}(t)-\Pi(t))\\
S(t)&=S(t)+L_2(t)(\Sigma_{21}(t)-\Pi(t))
\end{aligned}$$

of equations. The second equation is clearly satisfied if $\Sigma_{21}(\cdot)=\Pi(\cdot)$. Furthermore, an operator-valued function $\Sigma_{21}(\cdot)\in C([0,T],\mathscr{L}_s(W,X))$ is a mild solution of the first equation corresponding to an initial value $\Sigma_{21}(0)$ on $[0,T]$ if it satisfies the integral equation

$$\begin{aligned}
\Sigma_{21}(t)v=T_A(t)\Sigma_{21}(0)U_S(0,t)&+\int_0^t T_A(t-s)(B\Gamma(s)+E(s))U_S(s,t)v\,ds\\
&+\int_0^t T_A(t-s)(BK_1+L_1(s)C)\left(\Sigma_{21}(s)-\Pi(s)\right)U_S(s,t)v\,ds
\end{aligned}$$

for all $v \in W$. This shows that $\Sigma_{21}(t) = \Pi(t)$ is a mild solution of this equation, since in this case the last integral vanishes and $\Pi(\cdot)$ satisfies the remaining equation as it is the mild solution of (27a).

Since $\Sigma(\cdot) = (\Pi(\cdot), (\Pi(\cdot), I)^T)^T$ is a periodic function and since the closed-loop system is exponentially stable, we have that this function is the unique periodic mild solution of the Sylvester differential equation, i.e. $\Sigma_\infty(\cdot) = (\Pi(\cdot), (\Pi(\cdot), I)^T)^T$. This concludes the proof. □

7 Periodic Output Regulation of a Delay System

In this section we consider output regulation of a controlled scalar system with a delay. Our aim is to asymptotically regulate the output of this system to a triangle signal. We formulate the delay equation as a distributed parameter system on a Hilbert space and construct a periodic exosystem generating the desired reference signal. We will then use Theorems 5 and 6 to construct feedforward and dynamic error feedback controllers solving the Periodic Output Regulation Problem for this exosystem. The feedforward controller can be used if the state of the plant is available for feedback and if we know the initial state of the exosystem. If this is not the case, we need to use the observer-based dynamic error feedback controller constructed in Theorem 6 to estimate the states of the plant and the exosystem.

This example also illustrates the choice of the stabilizing output injection $L(\cdot)$ in the family (26) of operators in a special case where the original plant is exponentially stable and we do not have any disturbance signals to reject. In a more general case we can apply the results presented in [13] to find an appropriate stabilizing function $L(\cdot)$.

7.1 The System

We consider a scalar plant with delay

$$\begin{aligned}
\dot{x}(t) &= -2x(t) + x(t-1) + u(t) && \text{(29a)}\\
y(t) &= x(t) + u(t), && \text{(29b)}\\
x(0) &= \alpha && \text{(29c)}\\
x(\theta) &= f(\theta), \qquad \theta \in [-1,0) && \text{(29d)}
\end{aligned}$$

where $\alpha \in \mathbb{C}$ and $f \in L^2(-1,0)$. Denote $\mathbf{M}_2(-1,0) = \mathbb{C} \times L^2(-1,0)$. This is a Hilbert space with inner product [3, Sec 2.4]

$$\left\langle \begin{pmatrix} \alpha_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ f_2 \end{pmatrix} \right\rangle = \alpha_1 \overline{\alpha}_2 + \langle f_1, f_2 \rangle_{L^2}, \qquad \left\| \begin{pmatrix} \alpha \\ f \end{pmatrix} \right\|^2 = |\alpha|^2 + \|f\|_{L^2}^2.$$

The plant can be given as a linear system of form (9) on $X = \mathbf{M}_2(-1,0)$ by choosing $Y = U = \mathbb{C}$ and

$$A \begin{pmatrix} \alpha \\ f \end{pmatrix} = \begin{pmatrix} -2\alpha + f(-1) \\ \frac{df}{d\theta} \end{pmatrix},$$

$$\mathscr{D}(A) = \left\{ \begin{pmatrix} \alpha \\ f \end{pmatrix} \in \mathbf{M}_2(-1,0) \;\middle|\; f \text{ abs. cont., } \frac{df}{d\theta} \in L^2(-1,0),\ f(0) = \alpha \right\}$$

$B = (1,0)^T$, $C = (1,0)$ and $D = 1$. If $(\alpha, f)^T \in X$, then for the semigroup $T_A(t)$ generated by A we have [3, Thm 2.4.6]

$$T_A(t) \begin{pmatrix} \alpha \\ f \end{pmatrix} = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix} \in \mathbf{M}_2(-1,0),$$

for all $t \geq 0$ where $x(t)$ is the solution of the delay system (29) with input $u(t) \equiv 0$. We know from the theory of ordinary differential equations that $x(t)$ is determined by the integral equation

$$x(t) = e^{-2t}\alpha + \int_0^t e^{-2(t-r)} x(r-1) dr. \tag{30}$$

The function $x(t)$ can be solved from this expression by computing the right-hand side sequentially on intervals $[n, n+1]$ and using the history $x(\theta) = f(\theta)$ for $\theta \in [-1,0)$ on the interval $[0,1]$.

The stability of the plant can be deduced from the location of the roots of the function

$$\Delta(\lambda) = \lambda - (-2) - e^{-\lambda}.$$

More precisely, the plant is exponentially stable if and only if $\Delta(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ [3, Thm 5.1.7]. This is indeed the case since if $\lambda = a + ib$ with $a \geq 0$, then the roots are determined by

$$\Delta(a+ib) = (a + 2 - e^{-a}\cos(b)) + i(b + e^{-a}\sin(b)) = 0$$

The imaginary part of the equation implies that $b = 0$, since for $b \neq 0$ we would have $\operatorname{sinc}(b) = -e^a \leq -1$. This is impossible, since $\operatorname{sinc}(b) > -1$ for all $b \in \mathbb{R}$. If $b = 0$, the real part of the equation implies $2 = e^{-a} - a$. This is impossible, since $a \geq 0$ and thus the right-hand side is less than or equal to 1. This concludes that $\Delta(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$.

7.2 The Exosystem

As an exosystem we choose a one-dimensional periodic exosystem capable of generating the triangle signal depicted in Figure 1.

We choose $W = \mathbb{C}$, $S(t) \equiv 0$ and $F_{ref}(\cdot) \in C_T(\mathbb{R}, \mathscr{L}_s(W,Y)) = C_T(\mathbb{R}, \mathbb{C})$ to be a periodic function with period $T = 2$ and with

$$F_{ref}(t) = \begin{cases} t+1 & 0 \leq t < 1 \\ -t+3 & 1 \leq t < 2. \end{cases} \tag{31}$$

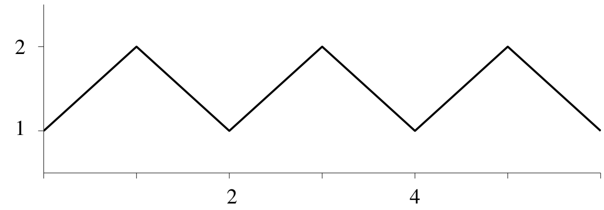

Fig. 1 The triangle signal.

Then the reference signal is given by $y_{ref}(t) = F_{ref}(t)v(t) = F_{ref}(t)v_0$ and the triangle signal in Figure 1 is generated with the initial value $v_0 = 1$ of the exosystem. This also shows that any signal generated by this exosystem is a scalar multiple of the triangle signal.

We assume there are no disturbance signals to the state or to the output of the plant, i.e. $E_d(t) \equiv 0$ and $F_d(t) \equiv 0$. We then have for the standard form that $E(t) \equiv 0$ and $F(t) = -F_{ref}(t)$ for all $t \in \mathbb{R}$.

It is worthwile to remark that the triangle signal in Figure 1 has an infinite number of frequency components. Because of this, generating such a signal with a time-invariant signal generator would require the exosystem to be infinite-dimensional. By construction this exosystem would also be capable of regulating a large class of other nonsmooth 2-periodic signals.

7.3 The Solution of the Constrained Sylvester Differential Equations

In the following we will use Theorems 5 and 6 to construct feedforward and feedback controllers solving the Periodic Output Regulation Problem for the plant and the periodic controller defined above. To this end we will first solve the constrained Sylvester differential equations appearing in these theorems. We will first only present the solution in an abstract way. The numerical estimates required for simulation of the systems are derived later in this section.

With $E(t) \equiv 0$ and $S(t) \equiv 0$ the constrained Sylvester differential equations become

$$\begin{aligned} \dot{\Pi}(t) &= A\Pi(t) + B\Gamma(t) \\ 0 &= C\Pi(t) + D\Gamma(t) + F(t) \end{aligned}$$

Since $D = 1 \neq 0$, we can solve $\Gamma(t) = -C\Pi(t) - F(t)$ from the second equation. Substituting this into the first equation we get

$$\dot{\Pi}(t) = (A - BC)\Pi(t) - BF(t)$$

Since $t \mapsto -F(t)$ is a continuous periodic function this equation has a unique periodic mild solution

$$\Pi(t) = -\int_{-\infty}^{t} T_{A-BC}(t-s)BF(s)ds$$

if the semigroup $T_{A-BC}(t)$ generated by $A-BC$ is exponentially stable (see Theorem 3). Since for all $(\alpha, f)^T \in \mathscr{D}(A)$ we have

$$(A-BC)\begin{pmatrix}\alpha \\ f\end{pmatrix} = \begin{pmatrix}-2\alpha + f(-1) \\ \frac{df}{d\theta}\end{pmatrix} - \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ f\end{pmatrix} = \begin{pmatrix}(-2-1)\alpha + f(-1) \\ \frac{df}{d\theta}\end{pmatrix}$$

it is easy to see that the semigroup $T_{A-BC}(t)$ is of the same form as $T_A(t)$ but the constant -2 has been replaced by -3 in the formula (30). Likewise it is easy to verify that this semigroup is exponentially stable. Substituting $\Pi(\cdot)$ into the formula for $\Gamma(\cdot)$ we obtain

$$\Gamma(t) = -C\Pi(t) - F(t) = -F(t) + \int_{-\infty}^{t} CT_{A-BC}(t-s)BF(s)ds. \tag{32}$$

7.4 The Feedforward Controller

Since the operator A generates an exponentially stable semigroup, we can choose $K = 0 \in \mathscr{L}(W,X)$ in Theorem 6. The theorem states that the feedforward controller solving the Periodic Output Regulation Problem is obtained by choosing

$$L(\cdot) = \Gamma(\cdot) - K\Pi(\cdot) = \Gamma(\cdot) \in C_T(\mathbb{R},\mathbb{C}).$$

Since for any initial state $v_0 \in \mathbb{C}$ the state of the exosystem is given by $v(t) \equiv v_0$, the appropriate feedforward control law is given by

$$u(t) = \Gamma(t)v(t) = \Gamma(t)v_0 = -F(t)v_0 + \int_{-\infty}^{t} CT_{A-BC}(t-s)BF(s)v_0 ds.$$

7.5 The Dynamic Error-Feedback Controller

Since the original system is exponentially stable, we can choose $K_1 = 0$ in Theorem 6 and accordingly

$$K_2(\cdot) = \Gamma(\cdot) - K_1\Pi(\cdot) = \Gamma(\cdot) \in C_T(\mathbb{R},\mathbb{C}).$$

It remains to choose the exponentially stabilizing function $L(\cdot)$ in the family (26) of operators. Since $F(t) = -F_{ref}(t) \leq -1$ for all $t \in \mathbb{R}$, the function $t \mapsto F(t)^{-1}$ is a continuous T-periodic function. If we choose $L(t) = (0, -F(t)^{-1})^T$ for all $t \in \mathbb{R}$, we have $L(\cdot) \in C_T(\mathbb{R}, \mathscr{L}(Y, X \times W))$ and

$$\begin{pmatrix}A & E(t) \\ 0 & S(t)\end{pmatrix} + L(t)\begin{pmatrix}C & F(t)\end{pmatrix} = \begin{pmatrix}A & 0 \\ -F(t)^{-1}C & -F(t)^{-1}F(t)\end{pmatrix} = \begin{pmatrix}A & 0 \\ -F(t)^{-1}C & -1\end{pmatrix}.$$

Since A generates an exponentially stable semigroup and since we have $-F(\cdot)^{-1}C \in C_T(\mathbb{R}, \mathscr{L}(X,W))$, it is straightforward to verify that the evolution family associated to the system (26) of operators is exponentially stable.

Now Theorem 6 concludes that the Output Regulation Problem is solved by an error feedback controller with parameters satisfying

$$\begin{aligned}\mathscr{G}_1(t) &= \begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} K_1 & K_2(t) \end{pmatrix} + L(t)\left(\begin{pmatrix} C & F(t) \end{pmatrix} + DK(t)\right) \\ &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \Gamma(t) \end{pmatrix} - \begin{pmatrix} 0 \\ F(t)^{-1} \end{pmatrix}\left(\begin{pmatrix} C & F(t) \end{pmatrix} + \begin{pmatrix} 0 & \Gamma(t) \end{pmatrix}\right) \\ &= \begin{pmatrix} A & B\Gamma(t) \\ -F(t)^{-1}C & -1 - F(t)^{-1}\Gamma(t) \end{pmatrix}\end{aligned}$$

and $\mathscr{G}_2(t) = -L(t) = (0, F(t)^{-1})^T$ for all $t \in \mathbb{R}$. The controller consists of a delay system and a one-dimensional ordinary differential equation. Because of this, the initial state of the controller is of form

$$z_0 = \begin{pmatrix} z_0^1 \\ z_0^2 \end{pmatrix}, \qquad z_0^1 = \begin{pmatrix} \alpha \\ f \end{pmatrix},$$

where α and f are the initial value and the history of the delay part of the system, respectively, and z_0^2 is the initial value of the ordinary differential equation part of the system.

7.6 Approximation of the Controller Parameters

To simulate the behaviour of the controlled system, we need to find an approximation for the part $\Gamma(\cdot)$ of the solution of the constrained Sylvester differential equations appearing in the controllers. Since this is a periodic function with period $T = 2$ it is sufficient to compute $\Gamma(t)$ for $t \in [0,2]$. Since we know the term $F(t)$, it further suffices to consider the integral term in (32). This term can be divided into two parts

$$\Gamma_\infty(t) + \Gamma_0(t) = \int_{-\infty}^{0} CT_{A-BC}(t-s)BF(s)ds + \int_0^t CT_{A-BC}(t-s)BF(s)ds \qquad (33)$$

where $\Gamma_\infty(\cdot), \Gamma_0(\cdot) \in ([0,T], \mathbb{C})$. We will show that the function $\Gamma_0(\cdot)$ can be computed explicitly and that we can write $\Gamma_\infty(\cdot)$ in such a way that it is easy approximate numerically with any given finite accuracy.

We will begin by considering $\Gamma_\infty(\cdot)$ in (33). Since $F(\cdot)$ is periodic and even (i.e. $F(-t) = F(t)$), for $T = 2$ and any $t \in [0,2]$ we have

$$\begin{aligned}\Gamma_\infty(t) &= \int_{-\infty}^{0} CT_{A-BC}(t-s)BF(s)ds = \sum_{n=0}^{\infty} \int_{-(n+1)T}^{-nT} CT_{A-BC}(t-s)BF(s)ds \\ &= \sum_{n=0}^{\infty} -CT_{A-BC}(t) \int_T^0 T_{A-BC}(r+nT)BF(-r-nT)dr \\ &= \sum_{n=0}^{\infty} CT_{A-BC}(t+nT) \int_0^T T_{A-BC}(r)BF(r)dr\end{aligned}$$

For the integrand in the last expression we have

$$T_{A-BC}(r)BF(r) = T_{A-BC}(r)\begin{pmatrix}1\\0\end{pmatrix}F(r) = \begin{pmatrix}g(r)\\g(r+\cdot)\end{pmatrix}F(r),$$

where $g(\theta)=0$ for $\theta\in[-1,0)$, $g(r)=e^{-3r}$ for $r\in[0,1)$ and for $r\in[1,2)$

$$g(r) = e^{-3r} + \int_1^r e^{-3(r-s)}e^{-3(s-1)}ds = e^{-3r} + e^{-3(r-1)}(r-1).$$

We can compute the value of this integral using $F(t) = -F_{ref}(t)$ and the expression (31). For $\theta\in[-1,0)$ we then have

$$\begin{aligned}
&\int_0^T T_{A-BC}(r)BF(r)dr = \int_0^T \begin{pmatrix}g(r)\\g(r+\theta)\end{pmatrix}F(r)dr\\
=\ &\int_0^{-\theta}\begin{pmatrix}e^{-3r}\\0\end{pmatrix}(-r-1)dr + \int_{-\theta}^1\begin{pmatrix}e^{-3r}\\e^{-3(r+\theta)}\end{pmatrix}(-r-1)dr\\
&+\int_1^{1-\theta}\begin{pmatrix}e^{-3r}(1+(r-1)e^3)\\e^{-3(r+\theta)}\end{pmatrix}(r-3)dr\\
&+\int_{1-\theta}^2\begin{pmatrix}e^{-3r}(1+(r-1)e^3)\\e^{-3(r+\theta)}(1+(r+\theta-1)e^3)\end{pmatrix}(r-3)dr\\
=\ &\frac{1}{27}\begin{pmatrix}-16+13e^{-3}+6e^{-6}\\16-6\theta-13e^{-3-3\theta}-6e^{-6-3\theta}-6e^{-3-3\theta}\theta\end{pmatrix} =: \begin{pmatrix}\alpha_1\\f_1(\theta)\end{pmatrix}.
\end{aligned}$$

Using this and the formula for the semigroup generated by the operator $A-BC$ we see that

$$\begin{aligned}
\int_{-\infty}^0 CT_{A-BC}(t-s)BF(-s)ds &= \sum_{n=0}^\infty CT_{A-BC}(t+nT)\int_0^T T_{A-BC}(r)BF(r)dr\\
&= \sum_{n=0}^\infty h_\infty(t+nT),
\end{aligned}$$

where the function $h_\infty(\cdot)$ is such that

$$h_\infty(t) = e^{-3t}\alpha_1 + \int_0^t e^{-3(t-s)}h_\infty(s-1)ds$$

and $h_\infty(\theta) = f_1(\theta)$ for $\theta\in[-1,0)$. This function can be evalutated sequentially and we can thus obtain a good approximation for $\Gamma_\infty(\cdot)$ in (33).

It remains to derive an expression for the function $\Gamma_0(\cdot)$. Similarly as above we can see that for all $t\in[0,2]$ we have

$$\Gamma_0(t) = \int_0^t CT_{A-BC}(t-s)BF(s)ds = \int_0^t h_0(t-s)F(s)ds,$$

where $h_0(t) = e^{-3t}$ for $t \in [0,1)$ and for $t \in [1,2)$

$$h_0(t) = e^{-3t} + \int_1^t e^{-3(t-s)} e^{-3(s-1)} ds = e^{-3t} + e^{-3(t-1)}(t-1).$$

This implies that for $t \in [0,1)$ (since $t - s < 1$) we have

$$\int_0^t CT_{A-BC}(t-s)BF(s)ds = \int_0^t e^{-3(t-s)}(-s-1)ds = -\frac{2}{9}(1-e^{-3t}) - \frac{1}{3}t$$

and for $t \in [1,2)$

$$\begin{aligned}
&\int_0^t CT_{A-BC}(t-s)BF(s)ds \\
= &\int_0^{t-1} h_0(\overbrace{t-s}^{>1})F(s)ds + \int_{t-1}^1 h_0(\overbrace{t-s}^{<1})F(s)ds + \int_1^t h_0(\overbrace{t-s}^{<1})F(s)ds \\
= &\int_0^{t-1} (e^{-3(t-s)} + e^{-3(t-s-1)}(t-s-1))(-s-1)ds + \int_{t-1}^1 e^{-3(t-s)}(-s-1)ds \\
&+ \int_1^t e^{-3(t-s)}(s-3)ds \\
= &-\frac{1}{27}\left[6e^{-3t} + (6t+1)^{-3(t-1)} + (6t-28)\right].
\end{aligned}$$

This concludes that we have

$$\Gamma_0(t) = \begin{cases} e^{-3t} + e^{-3(t-1)}(t-1) & t \in [0,1) \\ -\frac{1}{27}\left[6e^{-3t} + (6t+1)^{-3(t-1)} + (6t-28)\right] & t \in [1,2) \end{cases}$$

This explicit expression for $\Gamma_0(\cdot)$ and the series representation obtained for $\Gamma_\infty(\cdot)$ conclude that we can easily obtain good approximations for the function $\Gamma(\cdot)$ and the parameters of the controller.

7.7 Simulation

With the aid of the approximations above we can simulate the output of the system with the two types of controllers. We consider the regulation of the triangle signal depicted in Figure 1 and thus simulate the systems for the intial value $v_0 = 1$ of the exosystem.

Figure 2 shows the simulated output of the system and the feedforward controller for the initial state $x_0 = (0, \sin(2\pi\cdot))^T$ of the system. In the figure the dashed line is the reference signal and the solid line is the output of the system with the controller.

In the case where the states of the system and the exosystem are unavailable, the dynamics of the observer-based controller estimate these states to produce the

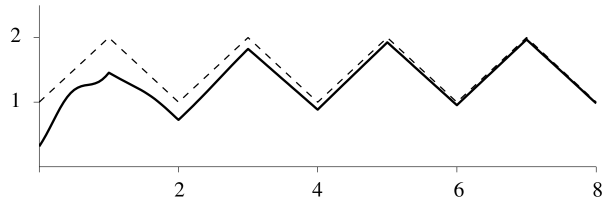

Fig. 2 The system with the feedforward controller.

control signal. Figure 3 shows the simulated output of the system with a dynamic error feedback controller for initial states

$$x_0 = \begin{pmatrix} 0 \\ \sin(2\pi\cdot) \end{pmatrix}, \qquad z_0 = \begin{pmatrix} z_0^1 \\ 1 \end{pmatrix}, \qquad z_0^1 = \begin{pmatrix} 1 \\ \cos(4\pi\cdot) \end{pmatrix}$$

of the system and the controller.

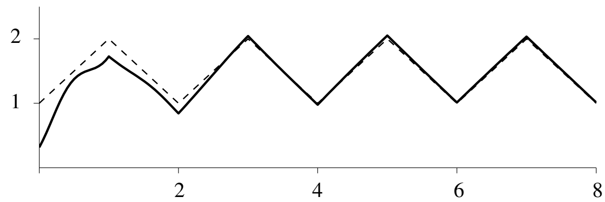

Fig. 3 The system with the feedback controller.

8 Conclusions

In this paper we have considered the infinite-dimensional Periodic Output Regulation Problem consisting of output regulation and disturbance rejection of a time-invariant distributed parameter system together with a periodic nonautonomous finite-dimensional exosystem. We have shown that the periodic exosystem is a very natural way of generating reference and disturbance signals previously available for consideration only when using an infinite-dimensional exosystem.

We used the properties of the solution of an infinite-dimensional Sylvester differential equation to characterize the solvability of the output regulation problem. Our results show that methods similar to the ones familiar from the time-invariant situation can be successfully applied also for distributed parameter systems with periodic exosystems. In particular this applies to the fact that the state of the closed-loop system and its asymptotic behaviour can be described using the periodic solution of the Sylvester differential equation.

Using the characterization of the solvability of the output regulation problem we were also able to construct concrete controllers achieving output regulation and disturbance rejection. Construction of the controllers was also illustrated with an example where we regulated the output of a scalar delay system to a triangle signal.

We have considered an error feedback controller solving the Periodic Output Regulation Problem. In the case of a time-invariant exosystem this type of controllers are known to have good robustness properties. On the other hand, for time-dependent exosystems it remains an open question whether the parameters of the plant can be pertubed without destroying the regulation property of the controller.

Another important research topic is the stablization of the closed-loop system. The stabilization of a time-dependent system is more complicated than the stabilization problem in the time-invariant case, but this problem has been studied also in infinite-dimensional spaces [13].

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