Robust Controller Design for Infinite-Dimensional Exosystems

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Abstract

In this paper we consider robust output regulation of distributed parameter systems with infinite-dimensional exosystems capable of generating polynomially growing signals. We design an observer-based error feedback controller solving the control problem. The controller is chosen in such a way that it incorporates an internal model of the infinite-dimensional exosystem. The remaining parameters of the controller are chosen to stabilize the closed-loop system strongly. We also analyze the classes of signals generated by the exosystem. In particular we explore the connection between the smoothness properties of the reference and disturbance signals and the strictness of the conditions required for the existence of a controller solving the robust output regulation problem.

KEY WORDS: Robust output regulation, distributed parameter systems, infinite-dimensional exosystem

1 Introduction

The topic of this paper is robust output regulation problem for linear distributed parameter systems. Mathematical systems of this class include models describing various natural phenomena including heat and diffusion processes, vibrations and delay systems. The study of output regulation and robust output regulation of infinite-dimensional systems has been active since the early 1980's [24, 20, 23, 2, 14, 5, 10, 6]. Recently in a series of papers [21, 18, 6, 16] one of the key results of linear multivariable control theory, the *internal model principle* of Francis and Wonham [4], was generalized for distributed parameter systems. This result can be used to characterize the error feedback controllers achieving robust output tracking and disturbance rejection of a distributed parameter system provided that the controller stabilizes the closed-loop system. More precisely, a stabilizing controller was shown to solve the robust output regulation problem if and only if it *incorporates an internal model* of the exosystem [16].

In [16] it was shown that the robust output regulation problem can be divided into two equally interesting and challenging parts. The first half of the problem consists of building an internal model of the exosystem into the controller, and the second one of strongly stabilizing the closed-loop system consisting of the plant and the controller. These two goals were formulated mathematically in [16], but the authors only solved the first half of the robust output regulation problem. The purpose of the current paper is to complete this study by solving the remaining half of the problem, i.e., the problem of stabilizing the closed-loop system strongly in a situation where the reference and disturbance signals are generated using an infinite-dimensional nondiagonal exosystem.

In addition to addressing the problem of stabilizing the closed-loop system we also present new results concerning the infinite-dimensional block diagonal exosystem introduced in [16]. Signal generators of this type provide a way to consider output tracking and disturbance rejection of very general polynomially growing signals. In applications the necessity of an infinite-dimensional signal generator arises from a need to track or reject, for example, periodic signals that are not continuously differentiable. Such situations are

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often encountered in the control of robot arms and disk drive systems [29, 7, and references therein]. On the other hand, in the same engineering applications it is also essential to be able to track and reject signals that have polynomially increasing components. Generating such signals is not possible with the infinite-dimensional signal generators studied earlier in the literature [11, 6, 7]. In this paper we complete the study of the block diagonal exosystems by characterizing the classes of reference and disturbance signals they generate. The analysis also immediately yields a method for constructing an exosystem generating given signals in the appropriate classes of functions. In particular we show that in the case of continuous periodic signals the exosystem can be chosen in such a way that it is easy to generate signals with a predetermined level of smoothness.

Compared to the current results in output regulation theory the main novelty in this paper is the use of state space methods and infinite-dimensional exosystems in considering robust output tracking and disturbance rejection of nonsmooth polynomially bounded signals. In particular, output regulation of nonsmooth periodic signals has been studied in the frequency domain within an area called *repetitive control* [7, 27, 29]. The state space approach reveals, in particular, the fact that the considered type of closed-loop stability is crucial to the solvability of the control problem. We also see a very clear connection between the properties of the system to be controlled and the minimal allowed level of smoothness of the exogeneous signals.

In order to solve the robust output regulation problem we generalize the observer-based controller used in the connection of the problem for finite-dimensional and infinite-dimensional diagonal exosystems [9, 6]. We show that the parameters of the controller can be chosen in such a way that the general structures of the operators guarantee the controller to incorporate an internal model of the infinite-dimensional exosystem. The remaining parameters of the controller can subsequently be freely chosen in order to achieve strong stability of the closed-loop system. In particular we show that the problem of stabilizing the closed-loop system can in a fairly straightforward manner be reduced to stabilizing the *internal model* in the controller. More precisely, this requires choosing a bounded linear operator K_2 in such a way that the operator

$$S + B_1 K_2 \tag{1}$$

generates a strongly stable semigroup. Here S is the block diagonal system operator of the exosystem and B_1 is a rank one bounded linear operator with a specific structure. In the case where S is a diagonal operator and the pair (S, B_1) is approximately controllable, the strong stabilization of the operator (1) can be achieved with a choice $K_2 = -B_1^*$. However, this choice for the stabilizing feedback provides very little information on the spectrum of the stabilized operator. In particular, the conditions for the solvability of the robust output regulation problem require some knowledge on the behavior of the resolvent operator of the closed-loop system. The behavior of this resolvent operator is, in turn, dependent on the behavior of the resolvent operator of (1). To help us determine whether the controller satisfies these conditions imposed on the closed-loop system, we use a new approach to stabilizing the internal model. In particular, we use the technique of *pole placement* of an infinite spectrum [28, 8, 30, 22] to reassign the imaginary eigenvalues of the operator S. This approach provides us very precise information on the spectrum of the stabilized operator and, even more importantly, on the behavior of the resolvent operator $R(\lambda, S + B_1 K_2)$ on the imaginary axis.

In [16] the authors introduced a new type of exosystem by constructing a block diagonal operator consisting of an infinite number of finite-dimensional Jordan blocks. In this paper we show that the signals generated by such exosystems are in general of form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t), \tag{2}$$

where $y_j(\cdot)$ are *almost periodic functions* for all $j \in \{1, \ldots, n\}$. This result shows that the block diagonal signal generator extends both finite-dimensional exosystems and infinite-dimensional diagonal exosystems [11, 6]. We also pay special attention to the case where the eigenvalues of the exosystem are of form

$$i\omega_k = i\frac{2\pi k}{\tau}, \qquad k \in \mathbb{Z} \tag{3}$$

for some $\tau > 0$. We show that in this situation the functions $y_j(\cdot)$ in (2) are continuous τ-periodic functions for all $j \in \{1, \ldots, n\}$. The smoothness properties of such functions are directly related to the asymptotic behavior of their Fourier coefficients [13]. We use this theory to relate the smoothness properties of the generated reference and disturbance signals to the corresponding choices for the initial states of the exosystem.

Our construction of the exosystem extends results in [11, 12] for polynomially increasing exogeneous signals and corresponding block diagonal signal generators.

We show that the analysis of the classes of signals generated by the infinite-dimensional exosystem also leads to a very concrete connection between the smoothness of the considered exogeneous signals and the conditions for the solvability of the robust output regulation problem. Establishing this link is made possible by the fact that the theory of robust output regulation in [16] allowed the solutions of the regulator equations to be unbounded operators. More precisely, it was shown that there is a link between the strictness of the conditions for the solvability of the robust output regulation problem and the choice of the initial states of the exosystem in a scale space $W_m = (\mathcal{D}(S^m), \|\cdot\|_m)$, where $\|\cdot\|_m$ denotes the graph norm of the operator $(I+S)^m$. In this paper we further show that these scale spaces are related to the smoothness properties of the generated reference and disturbance signals. We also extend the corresponding results for scale spaces W_α of arbitrary real orders $\alpha \geq 0$. The importance of this extension comes from the fact that it allows us to establish sharper bounds and conditions for the solvablity of the robust output regulation problem.

The above relationship becomes even more concrete when considering controllers of a particular type. We show that in the case of the observer-based error feedback controller studied in this paper the conditions for the solvability of the robust regulation problem most notably involves the behavior of the transfer function $P_K(\lambda)$ of the stabilized plant at the frequencies $i\omega_k$ of the signal generator. We show that the asymptotic decay rate of the values $P_K(i\omega_k)$ as $|k| \to \infty$ is directly related to the minimal level of smoothness the reference and disturbance signals must have in order for the conditions for the solvability of the robust output regulation problem to be satisfied. In other words, the behavior of $P_K(i\omega_k)$ can immediately be used to characterize the classes of signals that can be tracked and rejected using a controller of this type.

To illustrate the applicability of our results we present a concrete example in which we design an observer-based robust controller for a finite-dimensional system with an exosystem capable of generating a class of infinite-dimensional linearly growing signals. We derive the expressions for all components of the controller and determine the classes of reference and disturbance signals the controller is guaranteed to be able to regulate. We also discuss the robustness properties of the resulting control law, as well as the effect of the robustness of the controller to the classes of signals that can be tracked and rejected.

In the construction of the controller we restrict our attention to a situation which is a special case in two regards. First of all, we assume our exosystem has at most a finite number of nontrivial Jordan blocks and that all but a finite number of its eigenvalues are simple and uniformly separated. This type of signal generator can be viewed as a composite exosystem consisting of a finite-dimensional part and an infinite-dimensional diagonal part. Therefore, the signals we can under our assumptions consider are in general of form (2), where $y_0(\cdot) = y_{ap}(\cdot)$ is in general an almost periodic function and the functions $y_j(\cdot)$ for $j \in \{1, \ldots, n\}$ are linear combinations of trigonometric functions. This assumption on the structure of the signal generator is restrictive, but the generated signals still include the most important polynomially bounded functions considered in applications. In particular, we can consider any signal of the form (2) where $y_0(\cdot)$ is a continuous periodic function. Indeed, in order to generate a continuous τ-periodic function it is sufficient that the exosystem contains the simple and uniformly separated eigenvalues $(i\omega_k)_{k\in\mathbb{Z}}$ given by (3). The assumption that the infinite part of the spectrum of S has a uniform gap is not crucial to our approach to the stabilization of the closed-loop system. It can be replaced with the requirement that the spectrum of S does not have any finite accumulation points, if we have an asymptotic lower bound for the distances of the neighboring eigenvalues. However, the price of this added generality is that the conditions for the stabilizability of the closed-loop system become more complicated.

In this paper we also only consider the single-input single-output case. This restriction is not essential to the existence of a controller solving the robust output regulation problem. In fact, similar methods are also applicable in the case of a finite-dimensional output space, and even for an inifinite-dimensional output space provided that we replace the strong stability of the closed-loop system with weak stability [6, Sec. 7]. However, although the methods for these more general systems can be used to effectively stabilize the closed-loop system, they provide little information regarding the behavior of the resulting closed-loop system relevant to the additional conditions for the solvability of the robust output regulation problem. We restrict our attention to the single-input single-output case, because in this situation it is possible to stabilize the closed-loop system using a technique that subsequently allows us to derive easily verifiable sufficient conditions for solvability of the robust output regulation problem.

The structure of the rest of the paper is as follows. In Section 2 we introduce notation and state the standing assumptions on the plant, the exosystem, the controller, and the closed-loop system. Section 3 is devoted to the study of the classes of signals generated by the infinite-dimensional exosystem. In Section 4 we formulate the robust output regulation problem. The construction of an observer-based dynamic error feedback controller solving this problem is presented in detail in Section 5. The choices for the parameters of the controller are illustrated with an example in Section 6. Section 7 contains concluding remarks.

2 Mathematical Preliminaries

In this section we introduce the notation used in the paper and state the basic assumptions on the system, the exosystem and the controller. The main problem of this paper, the robust output regulation problem, is formulated in Section 4.

2.1 Notation

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The dual pairing on a Banach space and the inner product on a Hilbert space are both denoted by $\langle \cdot, \cdot \rangle$. If $(x_k)_{k\in\mathbb{Z}}$ is a sequence of complex numbers and $\alpha > 0$, we denote $x_k = \mathcal{O}(|k|^\alpha)$ if there exist constants $M > 0$ and $N \in \mathbb{N}$ such that

$$|x_k| \le M|k|^\alpha$$

for all $k \in \mathbb{Z}$ with $|k| \ge N$.

2.2 The Plant and the Infinite-Dimensional Exosystem

In this paper we consider the control of a linear distributed parameter system of form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w_s(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) + Du(t) + w_m(t) \end{aligned}$$

on a Banach space X. Here $x(t) \in X$ is the state of the system, $u(t) \in U$ the input and $y(t) \in Y$ the output. The input space U and the output space Y are general Hilbert spaces. We assume that A generates a strongly continuous semigroup on X and that the rest of the operators are bounded in such a way that $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$. For $\lambda \in \rho(A)$ the transfer function of the plant is given by $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$. When considering the stabilization of the closed-loop system in Section 5, we only consider single-input single-output systems, i.e., we assume $U = Y = \mathbb{C}$.

The considered reference signals as well as the disturbance signals to the state and the output, $w_s(t)$ and $w_m(t)$, respectively, are assumed to be generated by an infinite-dimensional exosystem

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{4a}$$
$$w_s(t) = E_s v(t), \tag{4b}$$
$$w_m(t) = E_m v(t), \tag{4c}$$
$$y_{ref}(t) = F_r v(t). \tag{4d}$$

The operators $S : \mathcal{D}(S) \subset W \to W$, $E_s \in \mathcal{L}(W, X)$, $E_m \in \mathcal{L}(W, Y)$, and $F_r \in \mathcal{L}(W, Y)$ satisfy the assumptions stated below. In particular, in the following we choose the system operator S to be an infinite-dimensional block diagonal operator consisting of finite-dimensional Jordan blocks.

The state space W of the exosystem is chosen to be a separable Hilbert space with an orthonormal basis

$$\left\{ \phi_k^l \in W \;\middle|\; k \in \mathbb{Z},\ l = 1, \dots, n_k \right\}.$$

By this we mean that

$$W=\overline{\operatorname{span}}\left\{\phi_k^l\right\}_{kl} \quad \text{and} \quad \langle \phi_k^l,\phi_n^m\rangle = \begin{cases} 1 & k=n,\, l=m \\ 0 & \text{otherwise.}\end{cases}$$

The lengths $n_k \in \mathbb{N}$ of the subsequences are assumed to be uniformly bounded. For a given ordered sequence of frequencies $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ the operators $S_k \in \mathcal{L}(W)$ representing the finite-dimensional Jordan blocks are defined as

$$S_k = i\omega_k \langle \cdot, \phi_k^1\rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle \cdot, \phi_k^l\rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1}\right).$$

The system operator S of the infinite-dimensional exosystem (4) on the space W is defined by

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \qquad \mathcal{D}(S) = \left\{ v\in W \;\middle|\; \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty \right\}$$

and the output operators E_s, E_m and F_r are assumed to be Hilbert–Schmidt operators, i.e. they satisfy

$$\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k} \|E_s\phi_k^l\|^2<\infty, \qquad \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k} \|E_m\phi_k^l\|^2<\infty, \quad \text{and} \quad \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k} \|F_r\phi_k^l\|^2<\infty.$$

Defining $E = E_s \in \mathcal{L}(W,X)$ and $F = E_m - F_r \in \mathcal{L}(W,Y)$ we can write the system in a standard form

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+Ev(t), \quad x(0)=x_0\in X \\ e(t) &= Cx(t)+Du(t)+Fv(t) \end{aligned}$$

where $e(t)\in Y$ is the regulation error and $v(t)\in W$ is the state of the exosystem (4). We further assume that $\sigma(A)\cap\sigma(S)=\varnothing$ and that the transfer function of the plant satisfies $P(i\omega_k)\neq 0$ for all $k\in\mathbb{Z}$.

The operators S_k in the definition of the infinite-dimensional exosystem satisfy

$$(i\omega_k I - S_k)\phi_k^1 = 0, \qquad (S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1} \quad \forall l\in\{2,\dots,n_k\}$$

and thus they can indeed be viewed as single Jordan blocks of dimensions n_k associated to eigenvalues $i\omega_k$. Since the operator S is an infinite block diagonal operator consisting of operators the S_k, it can be considered to be a generalization of a matrix in a Jordan canonical form. It is straightforward to verify that the spectrum of the operator S satisfies

$$\sigma(S) = \overline{\sigma_p(S)} = \overline{\{i\omega_k\}}_{k\in\mathbb{Z}},$$

where the line denotes the closure of the set in $\mathbb{C}$. Moreover, the operator S generates a C_0-group $T_S(t)$ satisfying

$$T_S(t)v = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\sum_{l=1}^{n_k}\langle v,\phi_k^l\rangle \sum_{j=1}^{l}\frac{t^{l-j}}{(l-j)!}\phi_k^j, \qquad v\in W,\ t\in\mathbb{R}.$$

This group is polynomially bounded forward and backwards in time. More precisely, for any $n\in\mathbb{N}$ such that $n\geq n_k$ for all $k\in\mathbb{Z}$ there exists $M_S\geq 1$ such that

$$\|T_S(t)\| \leq M_S(|t|^n+1), \qquad \forall t\in\mathbb{R}.$$

This implies that the growth bound of the C_0-group is $\omega_0(T_S(t))=0$. For $k\in\mathbb{Z}$ we define

$$d_k = \max\{\, n_l \mid l\in\mathbb{Z},\ \omega_l=\omega_k \,\},$$

which corresponds to the dimension of the largest Jordan block associated to an eigenvalue $i\omega_k \in \sigma_p(S)$. For $k \in \mathbb{Z}$ we denote by P_k the orthogonal projection

$$P_k = \sum_{l=1}^{n_k} \langle \cdot, \phi_k^l \rangle \phi_k^l$$

onto the finite-dimensional subspace $\operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ of W. With this notation the domain of the operator S satisfies

$$\mathcal{D}(S) = \left\{ v \in W \mid \sum_{k\in\mathbb{Z}} \omega_k^2 \|P_k v\|^2 < \infty \right\} = \left\{ v \in W \mid \sum_{k\in\mathbb{Z}} (1+\omega_k^2) \|P_k v\|^2 < \infty \right\}.$$

To analyze the classes of signals generated by the exosystem we define a set of scale spaces $W_\alpha \subset W$ related to the system operator S of the exosystem. They will be used in the classification of the generated signals based on which spaces W_α the corresponding initial states belong to. In the next section we will show that this kind of classification has a close relationship to the smoothness properties of the generated signals.

Definition 1. For $\alpha \geq 0$ we denote by $(W_\alpha, \|\cdot\|_\alpha)$ the space

$$W_\alpha = \left\{ v \in W \mid \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v\|^2 < \infty \right\}$$

with norm $\|\cdot\|_\alpha$ defined by

$$\|v\|_\alpha^2 = \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v\|^2, \qquad v \in W_\alpha.$$

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For all $\alpha \geq 0$ the spaces $(W_\alpha, \|\cdot\|_\alpha)$ are Hilbert spaces, and for $0 \leq \beta \leq \alpha$ we have $W_\alpha \subset W_\beta$ and

$$\|v\|_\beta \leq \|v\|_\alpha \qquad \forall v \in W_\alpha$$

For nonnegative integer values $m \in \mathbb{N}_0$ the spaces W_m coincide with the domains $\mathcal{D}((S+I)^m)$ and the norms $\|\cdot\|_m$ are equivalent to the norms defined by the mappings $v \mapsto \|(S+I)^m v\|$ on W_m. It can also be verified that the spaces W_α are invariant under the group $T_S(t)$, the restrictions $T_S(t)|_{W_\alpha}$ are strongly continuous groups on W_α and the generators of these groups are $S|_{W_\alpha} : \mathcal{D}(S|_{W_\alpha}) \subset W_\alpha \to W_\alpha$ with domains $\mathcal{D}(S|_{W_\alpha}) = W_{\alpha+1}$.

2.3 The Controller and the Closed-Loop System

We consider the dynamic error feedback controller

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z \\ u(t) &= K z(t) \end{aligned}$$

on a Banach-space Z. Here $z(t) \in Z$ is the state of the controller, $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$. The closed-loop system consisting of the plant and the controller on $X_e = X \times Z$ with state $x_e(t) = (x(t), z(t))^T$ is given by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e v(t), \quad x_e(0) = x_{e0} = (x_0, z_0)^T \\ e(t) &= C_e x_e(t) + D_e v(t), \end{aligned}$$

where $C_e = (C \;\; DK)$, $D_e = F$,

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} \quad \text{and} \quad B_e = \begin{pmatrix} E \\ \mathcal{G}_2 F \end{pmatrix}.$$

The operator $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ generates a C_0-semigroup $T_{A_e}(t)$ on X_e. Furthermore, the operator B_e satisfies

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\|B_e\phi_k^l\|^2 &= \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\left(\|E_s\phi_k^l\| + \|\mathcal{G}_2(E_m - F_r)\phi_k^l\|\right)^2 \\ &= \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k} 2\left(\|E_s\phi_k^l\|^2 + 2\|\mathcal{G}_2\|^2(\|E_m\phi_k^l\|^2 + \|F_r\phi_k^l\|^2)\right) < \infty. \end{aligned} \tag{6}$$

Before turning to consider the robust output regulation problem, we will analyze in detail the classes of signals generated by our infinite-dimensional exosystem.

3 The Classes of Reference and Disturbance Signals

Using the formal definition and the properties of the infinite-dimensional exosystem presented in the previous section we can study the generated reference and disturbance. To this end we consider the generation of signals $y_{ref}(\cdot) : \mathbb{R} \to Y$, where Y is a Banach space. We will show that the produced signals are of form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t), \tag{7}$$

where the coefficient functions $y_j(\cdot)$ are *almost periodic functions* [1, Def. 4.5.6]. Such bounded and uniformly continuous functions can be characterized by the fact that they can be uniformly approximated by *trigonometric polynomials*, i.e., linear combinations of functions of form $t \mapsto e^{i\omega t}y$, where $\omega \in \mathbb{R}$ and $y \in Y$.

In the next section we will further concentrate on the case where these coefficient functions are periodic with the same period length. For such reference signals we will in particular study the relationship between the smoothness of the coefficient functions $y_j(\cdot)$ and the choice of the initial state v_0 from the space W_α for some $\alpha \geq 0$.

Theorem 2. *The signals generated by the infinite-dimensional exosystem are of form* (7) *where* $y_j(\cdot) : \mathbb{R} \to Y$ *are almost periodic functions for all* $j \in \{0, \dots, n\}$*, and where* $n = \max_{k\in\mathbb{Z}} n_k - 1$.

Proof. For all initial states $v_0 \in W$ the state of the exosystem is given by $v(t) = T_S(t)v_0$ and thus

$$\begin{aligned} y_{ref}(t) = Fv(t) &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\sum_{l=1}^{n_k}\langle v_0, \phi_k^l\rangle \sum_{j=1}^{l}\frac{t^{l-j}}{(l-j)!}F\phi_k^j = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\sum_{l=1}^{n_k}\langle v_0, \phi_k^l\rangle \sum_{j=0}^{l-1}\frac{t^j}{j!}F\phi_k^{l-j} \\ &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\sum_{j=0}^{n_k-1} t^j \cdot \frac{1}{j!}\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle F\phi_k^{l-j} = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\sum_{j=0}^{n_k-1} a_{jk}t^j, \end{aligned}$$

where we have denoted

$$a_{jk} = \frac{1}{j!}\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle F\phi_k^{l-j} \in Y. \tag{8}$$

Let $n = \max_{k\in\mathbb{Z}} n_k - 1$ and define $a_{jk} = 0 \in Y$ for all $k \in \mathbb{Z}$ and $j \in \{n_k + 1, \dots, n\}$. Then for any $j \in \{0, \dots, n\}$ we have

$$\begin{aligned} j! \cdot \sum_{k\in\mathbb{Z}}\|a_{jk}\| &= \sum_{k\in\mathbb{Z}}\|\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle F\phi_k^{l-j}\| \leq \sum_{k\in\mathbb{Z}}\sum_{l=j+1}^{n_k}|\langle v_0, \phi_k^l\rangle| \cdot \|F\phi_k^{l-j}\| \\ \leq \sum_{k\in\mathbb{Z}}&\left(\sum_{l=j+1}^{n_k}|\langle v_0, \phi_k^l\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{n_k-j}\|F\phi_k^l\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}|\langle v_0, \phi_k^l\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\|F\phi_k^l\|^2\right)^{\frac{1}{2}} < \infty \end{aligned}$$

and thus $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ for all $j \in \{0,\dots,n\}$. This implies that

$$y_{ref}(t) = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{j=0}^{n_k-1} a_{jk} t^j = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{j=0}^{n} a_{jk} t^j = \sum_{j=0}^{n} t^j \sum_{k\in\mathbb{Z}} a_{jk} e^{i\omega_k t} = \sum_{j=0}^{n} t^j y_j(t),$$

where we have in turn denoted

$$y_j(t) = \sum_{k\in\mathbb{Z}} a_{jk} e^{i\omega_k t} \tag{9}$$

for all $j \in \{0,\dots,n\}$. Changing the order or summation is allowed since the series in question are absolutely convergent. To prove the theorem it is now sufficient to show that the functions $y_j(\cdot)$ are almost periodic. This follows directly from the fact that if $j \in \{0,\dots,n\}$ and if $N > 0$, then the functions

$$t \mapsto \sum_{k=-N}^{N} a_{jk} e^{i\omega_k t}$$

are trigonometric polynomials and $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ implies

$$\left\| y_j(t) - \sum_{k=-N}^{N} a_{jk} e^{i\omega_k t} \right\| = \left\| \sum_{|k|>N} a_{jk} e^{i\omega_k t} \right\| \le \sum_{|k|>N} \|a_{jk}\| \longrightarrow 0$$

uniformly in $t \in \mathbb{R}$ as $N \to \infty$. □

The proof of Theorem 2 also presents a method for constructing an exosystem to generate a given signal of form (7), where the coefficient functions can be written as (9). This can be done in a very straightforward manner by choosing the output operator F and the initial state v_0 of the exosystem in such a way that the equations (8) are satisfied for all $k \in \mathbb{Z}$. The next lemma presents suitable choices for the parameters.

Lemma 3. *Let $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ and assume that for $j \in \{0,\dots,n\}$ the coefficient functions $y_j(\cdot)$ can be written in the form* (9) *with $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$. For $k \in \mathbb{Z}$ define*

$$c_k = \begin{cases} \max\left\{\sqrt{\|a_{jk}\|}\right\}_{j=0}^{n} & a_{jk} \neq 0 \text{ for some } j = 0,\dots,n \\ 1 & \text{otherwise.} \end{cases}$$

The signal (7) *can be generated by an infinite-dimensional exosystem with $n_k = n+1$ for all $k \in \mathbb{Z}$ and $F \in \mathcal{L}(W,Y)$ defined by*

$$F\phi_k^1 = \frac{1}{c_k}(n_k-1)!\, a_{n_k-1,k},$$
$$F\phi_k^l = \frac{1}{c_k}\left((n_k-l)!\, a_{n_k-l,k} - (n_k-l+1)!\, a_{n_k-l+1,k}\right), \quad l \in \{2,\dots,n_k\}.$$

The signal (7) *is generated with an initial state $v_0 \in W$ of the exosystem satisfying*

$$\langle v_0, \phi_k^l \rangle = \max\left\{\sqrt{\|a_{jk}\|}\right\}_{j=0}^{n}$$

for all $k \in \mathbb{Z}$ and $l \in \{1,\dots,n_k\}$.

Proof. A straightforward computation shows that these choices of F and v_0 satisfy equations (8) for all $k \in \mathbb{Z}$. It is thus sufficient to show that $v_0 \in W$ and

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

For the initial state we have

$$\sum_{k\in\mathbb{Z}} \|P_k v_0\|^2 = (n+1)\sum_{k\in\mathbb{Z}} \max\left\{\|a_{jk}\|\right\}_{j=0}^n < \infty$$

since $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ for all $j \in \{0,\dots,n\}$, and thus $v_0 \in W$. On the other hand, the output operator F satisfies

$$\|F\phi_k^1\| = \frac{n!}{c_k}\|a_{nk}\| \le n! \max\left\{\sqrt{\|a_{jk}\|}\right\}_{j=0}^n$$

$$\|F\phi_k^l\| \le \frac{n!}{c_k}\left(\|a_{n+1-l,k}\| + \|a_{n+2-l,k}\|\right) \le 2n! \max\left\{\sqrt{\|a_{jk}\|}\right\}_{j=0}^n$$

and thus

$$\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k} \|F\phi_k^l\|^2 \le 4(n+1)(n!)^2 \sum_{k\in\mathbb{Z}} \max\left\{\|a_{jk}\|\right\}_{j=0}^n < \infty$$

again since $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ for all $j \in \{0,\dots,n\}$. This concludes the proof. □

The formula in (8) shows us that the coefficients a_{jk} of the functions $y_j(\cdot)$ are determined equally by both the output operator F of the exosystem and the choice v_0 of the initial state. Because of this it is evident that different initial states of the same exosystem can generate very different types of signals. We will see an illustrative example of this property of the infinite-dimensional exosystem in the next section.

3.1 Signals in the Sobolev Spaces of Periodic Functions

We conclude the analysis of the infinite-dimensional signal generator by considering signals of form (7) where the coefficient functions $y_j(\cdot)$ are periodic functions with the same period $\tau > 0$. To this end we assume that the frequencies of the exosystem are given by

$$(\omega_k)_{k\in\mathbb{Z}} = \left(\frac{2\pi k}{\tau}\right)_{k\in\mathbb{Z}}.$$

It is well-known that for periodic functions the smoothness properties can be characterized via the asymptotic behavior of their Fourier coefficients. We will now show that taking an advantage of this property we can easily relate the smoothness properties of the generated signals to the choices of the initial states of the exosystem in the spaces W_α. The importance of this result will become clear later in the paper when we further relate the choices of the initial states in the spaces W_α to the strictness of the conditions for the solvability of the robust output regulation problem for the generated signals.

To classify the generated signals we use the *Sobolev spaces of periodic functions* [13, Sec. 3.6] defined below. In the definition the values $\hat{f}(k) \in Y$ denote the Fourier coefficients of a continuous τ-periodic function $f(\cdot)$, i.e.,

$$f(t) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{i\omega_k t}, \qquad \forall t \in \mathbb{R}.$$

Definition 4 (Sobolev spaces of periodic functions)**.** For $\alpha > \frac{1}{2}$ the Hilbert spaces

$$H^\alpha_{per}(0,\tau) = \left\{ f \in C_\tau(\mathbb{R},Y) \;\middle|\; \sum_{k\in\mathbb{Z}}(1+\omega_k^2)^\alpha \|\hat{f}(k)\|^2 < \infty \right\}$$

with norms defined by

$$\|f\|^2_{per,\alpha} = \frac{1}{\tau}\sum_{k\in\mathbb{Z}}(1+\omega_k^2)^\alpha\|\hat{f}(k)\|^2, \qquad f \in H^\alpha_{per}(0,\tau)$$

are called the *Sobolev spaces of periodic functions*. ■

The order $\alpha > \frac{1}{2}$ of the space $H^\alpha_{per}(0,\tau)$ is closely related to the smoothness properties of its functions. For example, if $Y = \mathbb{C}$, then for all $m \in \mathbb{N}$ the space $H^m_{per}(0,\tau)$ contains precisely the τ-periodic functions whose distributional derivatives of orders up to m are in $L^2(0,\tau)$. In particular this implies that if $f \in C_\tau(\mathbb{R}, Y)$ is such that the derivatives $f^{(j)}$ exist and are absolutely continuous on $[0,\tau]$ for all $j \in \{0,\ldots,m-1\}$, then $f \in H^m_{per}(0,\tau)$.

The following theorem shows that for any infinite-dimensional signal generator the choice of the initial state of the exosystem in a space W_α directly translates to the smoothness of the generated signal.

Theorem 5. *If $v_0 \in W_\alpha$ for some $\alpha > \frac{1}{2}$, then the coefficient functions of the signal generated by the infinite-dimensional exosystem satisfy $y_j(\cdot) \in H^\alpha_{per}(0,\tau)$.*

Proof. Let $n = \max_{k\in\mathbb{Z}} n_k - 1$ and let $j \in \{0,\ldots,n\}$ be arbitrary. From the proof of Theorem 2 we have that $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$. Together with the formula in (9) this implies that $(a_{jk})_{k\in\mathbb{Z}}$ are the Fourier coefficients of the function $y_j(\cdot)$. Using (8) we can also see that

$$\begin{aligned}\|a_{jk}\| &\le \frac{1}{j!}\sum_{l=j+1}^{n_k} |\langle v_0, \phi_k^l\rangle| \cdot \|F\phi_k^{l-j}\| \le \|F\| \sum_{l=1}^{n_k} |\langle v_0, \phi_k^l\rangle| \le \sqrt{n}\|F\| \left(\sum_{l=1}^{n_k} |\langle v_0, \phi_k^l\rangle|^2\right)^{\frac{1}{2}} \\ &= \sqrt{n}\|F\| \cdot \|P_k v_0\|\end{aligned}$$

for all $k \in \mathbb{Z}$. The fact that $v_0 \in W_\alpha$ now implies

$$\sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|a_{jk}\|^2 \le n\|F\|^2 \cdot \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v_0\|^2 < \infty,$$

and thus $y_j(\cdot) \in H^\alpha_{per}(0,\tau)$. Since $j \in \{0,\ldots,n\}$ was arbitrary, this concludes the proof. □

The next theorem states a converse result which shows that the signal generator can be chosen in such a way that the smoothness of the reference signal is also translated to the property $v_0 \in W_\alpha$ of the corresponding initial state.

Theorem 6. *Let $\beta > \frac{1}{2}$ and assume $y_j(\cdot) \in H^\beta_{per}(0,\tau)$ for all $j \in \{0,\ldots,n\}$. For any $0 \le \alpha < \beta - \frac{1}{2}$ the infinite-dimensional exosystem can be chosen in such a way that the reference signal* (7) *is generated with a choice $v_0 \in W_\alpha$ of the initial state.*

Proof. The functions $y_j(\cdot)$ are of form

$$y_j(t) = \sum_{k\in\mathbb{Z}} a_{jk} e^{i\omega_k t}, \qquad t \in \mathbb{R}.$$

Let $0 \le \alpha < \beta - \frac{1}{2}$, define

$$c_k = \begin{cases} (1+\omega_k^2)^{\frac{\beta-\alpha}{2}} \max\{\|a_{jk}\|\}_{j=0}^n & a_{jk} \ne 0 \text{ for some } j \in \{0,\ldots,n\} \\ 1 & \text{otherwise} \end{cases}$$

and choose W and $F\phi_k^l$ as in Lemma 3. A direct computation shows that the equations (8) are satisfied if we choose the initial state v_0 of the exosystem in such a way that

$$\langle v_0, \phi_k^l\rangle = (1+\omega_k^2)^{\frac{\beta-\alpha}{2}} \max\{\|a_{jk}\|\}_{j=0}^n.$$

It remains to show that $v_0 \in W_\alpha$ and that

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

As in the proof of Lemma 3 it is easy to see that we have

$$\|F\phi_k^l\| \leq \frac{2n!}{c_k} \max\{\|a_{jk}\|\}_{j=0}^n \leq 2n!(1+\omega_k^2)^{\frac{\alpha-\beta}{2}} < \infty$$

and thus

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 \leq 4(n+1)(n!)^2 \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\alpha-\beta} < \infty$$

since $\omega_k = \frac{2\pi k}{\tau}$ and $\alpha - \beta < -\frac{1}{2}$. Furthermore, for v_0 we have that

$$\sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\alpha} \|P_k v_0\|^2 = (n+1) \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\beta} \max\{\|a_{jk}\|^2\}_{j=0}^n < \infty,$$

since $y_j(\cdot) \in H_{per}^{\beta}(0,\tau)$. This concludes the proof. □

As we mentioned in the end of the previous section, the choice of the initial state of the infinite-dimensional signal generator can have a radical effect on the generated signal. The following example illustrates this property of the infinite-dimensional signal generators.

Example 7. Let $Y = \mathbb{C}$ and $\omega_k = \frac{2\pi k}{\tau}$ and choose the parameters of the infinite-dimensional exosystem in such a way that $n_k = 1$ and

$$F\phi_k = \frac{1}{|k|}$$

for all $k \in \mathbb{Z}$. Since the signals generated by the exosystem satisfy $\hat{y}_{ref}(k) = \langle v_0, \phi_k\rangle F\phi_k$, we can see that any given signal $y_{ref}(\cdot) \in H_{per}^1(0,\tau)$ can be generated with this particular exosystem by choosing $v_0 \in W$ in such a way that

$$\langle v_0, \phi_k\rangle = |k| a_k.$$

We indeed have $v_0 \in W$, since

$$\sum_{k\in\mathbb{Z}} |\langle v_0, \phi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} \frac{1+\omega_k^2}{1+\omega_k^2} \cdot k^2 |a_k|^2 \leq \sup_{k\in\mathbb{Z}} \frac{k^2}{1+\omega_k^2} \sum_{k\in\mathbb{Z}} (1+\omega_k^2)|a_k|^2 < \infty.$$

Thus this exosystem is capable of generating any reference signal from $H_{per}^1(0,\tau)$.

This reasoning can be further generalized to show that if $y_{ref}(\cdot) \in H_{per}^{\gamma}(0,\tau)$ for $\gamma \geq 1$, then it can be generated using this exosystem with a choice $v_0 \in W_{\gamma-1}$ of the initial state. Indeed, if we choose v_0 as above, then $v_0 \in W_{\gamma-1}$ follows directly from

$$\sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\gamma-1} |\langle v_0, \phi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} \frac{\left(1+\omega_k^2\right)^{\gamma}}{1+\omega_k^2} \cdot k^2 |a_k|^2 \leq \sup_{k\in\mathbb{Z}} \frac{k^2}{1+(2\pi k/\tau)^2} \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\gamma} |a_k|^2 < \infty.$$

■

4 The Robust Output Regulation Problem

In this section we outline the main control problem studied in the paper and briefly recall some results concerning the solvability of this problem. On a general level we are interested in choosing the parameters of the error feedback controller in such a way that the following are satisfied.

- The closed-loop system is strongly stable.

- The regulation error decays to zero asymptotically.
- If the parameters of the plant are perturbed in such a way that the closed-loop stability is preserved, then the regulation error goes to zero asymptotically.

It is well-known that the solvability of the first two parts of this problem is connected to the solvability of the so-called *regulator equations* [4, 2, 6]

$$\Sigma S = A_e \Sigma + B_e \tag{10a}$$
$$0 = C_e \Sigma + D_e. \tag{10b}$$

In particular, if the closed-loop system is stable and the Sylvester equation (10a) has a bounded solution, then the regulation error decays asymptotically if and only if the *regulation constraint* (10b) is satisfied. The reason behind this connection is that if (10a) has a solution Σ, then for any initial states x_{e0} and v_0 of the closed-loop system and the exosystem, respectively, the regulation error can be written in the form

$$e(t) = C_e T_{A_e}(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e)v(t). \tag{11}$$

If the semigroup $T_{A_e}(t)$ related to the closed-loop system is strongly stable (i.e. if $T_{A_e}(t)x_e \to 0$ as $t \to \infty$ for all $x_e \in X_e$), then the first part of the above expression decays to zero. Furthermore, the second part of the regulation error will be zero whenever the regulation constraint (10b) is satisfied.

In [16] it was shown that the conditions for the solvability of the output regulation problem can be made less strict if we allow the solution of the Sylvester equation (10a) to be an operator belonging to $\mathcal{L}(W_\alpha, X_e)$ for some $\alpha > 0$. It was further shown that if the Sylvester equation has a solution $\Sigma \in \mathcal{L}(W_\alpha, X_e)$, we can consider tracking and rejection of exogeneous signals corresponding to the initial states in the scale space W_α of the infinite-dimensional exosystem. In the light of Theorems 5 and 6 we can immediately see that in the case of periodic signals this establishes a link between the strictness of the requirement for the solvability of the Sylvester equation (10a) and the level of smoothness of the reference and disturbance signals. In this paper we simplify the associated conditions by considering a slightly relaxed sufficient condition (12) for the solvability of the Sylvester equation equation (10a).

Theorem 8. *Let $\alpha \geq 0$. Assume $(\mathcal{G}_1, \mathcal{G}_2, K)$ are such that A_e generates a strongly stable C_0-semigroup on X_e, $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, and*

$$\sup_{k\in\mathbb{Z}} \; (1+\omega_k^2)^{-\alpha} \|R(i\omega_k, A_e)\| < \infty. \tag{12}$$

Then the Sylvester equation (10a) *considered on $W_{\alpha+1}$ has a unique solution $\Sigma \in \mathcal{L}(W_\alpha, X_e)$ and the following are equivalent.*

1. *The regulation error decays to zero asymptotically for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ of the closed-loop system and the exosystem, respectively.*
2. *The operator Σ satisfies the regulation constraint* (10b).

Proof. The conclusions of the theorem follow from [16, Thm. 3.1] once we verify that [16, Ass. 1] is satisfied for $m = \alpha$. In [16] the scale spaces W_α were only defined for $\alpha = m \in \mathbb{N}_0$. However, the choice of the space only affects the conditions for the operator Σ solving the Sylvester equation (10a) to be in $\mathcal{L}(W_\alpha, X_e)$, and the proofs in [16] can be used as they are once we replace $m \in \mathbb{N}_0$ with $\alpha \geq 0$.

Since $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, we have $\mathcal{R}(i\omega_k I - A_e)^l = X_e$ for all $l \in \{1, \dots, n_k\}$. Choose $n \in \mathbb{N}$ such that $n_k \leq n$ for all $k \in \mathbb{Z}$. Then for any $x_e^* \in X_e^*$ with $\|x_e^*\| \leq 1$ we have

$$\left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right| \leq \sum_{j=1}^{l} \|R(i\omega_k, A_e)\|^{l+1-j} \|B_e \phi_k^j\| \cdot \|x_e^*\|$$

$$\leq \max\{\|R(i\omega_k, A_e)\|, \|R(i\omega_k, A_e)\|^{n_k}\} \cdot \sum_{j=1}^{n_k} \|B_e \phi_k^j\|$$

$$\leq \max\{1, \|R(i\omega_k, A_e)\|^{n_k}\} \cdot \sqrt{n_k} \cdot \left(\sum_{j=1}^{n_k} \|B_e \phi_k^j\|^2 \right)^{\frac{1}{2}}$$

for all $k \in \mathbb{Z}$. Condition (12) implies that there exists $M \geq 0$ such that

$$\frac{\max\{1, \|R(i\omega_k, A_e)\|^{2n_k}\}}{(1+\omega_k^2)^\alpha} \leq M$$

for all $k \in \mathbb{Z}$. Since by (6) the operator B_e satisfies $(B_e\phi_k^l)_{kl} \in \ell^2(X_e)$, we have

$$\sup_{\|x_e^*\|\leq 1} \sum_{k\in\mathbb{Z}} \frac{1}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e\phi_k^j, x_e^* \rangle \right|^2$$
$$\leq \sum_{k\in\mathbb{Z}} \frac{\max\{1, \|R(i\omega_k, A_e)\|^{2n_k}\}}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} n_k \sum_{j=1}^{n_k} \|B_e\phi_k^j\|^2 \leq n^2 M \sum_{k\in\mathbb{Z}} \sum_{j=1}^{n_k} \|B_e\phi_k^j\|^2 < \infty.$$

This concludes the proof. □

If we use the relaxed conditions for the solvability of the Sylvester equation (10a) presented in the above theorem, we can state the robust output regulation problem mathematically in the following way.

The Robust Output Regulation Problem on W_α. Let $\alpha \geq 0$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:

- The closed-loop system operator A_e generates a strongly stable C_0-semigroup on X_e, we have $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, and
$$\sup_{k\in\mathbb{Z}} (1+\omega_k^2)^{-\alpha} \|R(i\omega_k, A_e)\| < \infty. \tag{13}$$
- For all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.
- If the parameters (A, B, C, D, E, F) are perturbed to (A', B', C', D', E', F') in such a way that the new closed-loop system (A'_e, B'_e, C'_e, D'_e) is strongly stable and it satisfies $\sigma(A'_e) \cap \sigma_p(S) = \varnothing$, and (13), then $\lim_{t\to\infty} e(t) = 0$ for all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$.

■

The robust output regulation problem is formulated in such a way that we only consider perturbations under which the stability of the closed-loop system is preserved. The main reason for studying this type of 'conditional' robustness is that the problem of determining perturbations preserving the strong stability of a semigroup is a difficult and largely open problem. In the special case where the semigroup is generated by a Riesz-spectral operator some classes of finite-rank perturbations preserving the strong and polynomial stability types of the semigroup have been presented in [17].

Presenting a solution to the above control problem is the main topic of the rest of this paper. The results presented in Sections 4 and 5 of [16] imply that in order to solve the robust output regulation problem the dynamic error feedback controller must incorporate *an internal model* of the infinite-dimensional exosystem (4). In particular, for this it is sufficient to choose the parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following $\mathcal{G}$*-conditions* [16, Def. 5.1] are satisfied.

Definition 9 (The $\mathcal{G}$-conditions)**.** A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to satisfy the $\mathcal{G}$*-conditions* related to the infinite-dimensional exosystem in Section 2.2 if

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \forall k \in \mathbb{Z}, \tag{14a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{14b}$$

and

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1) \qquad \forall k \in \mathbb{Z}. \tag{14c}$$

■

Roughly stated, a controller incorporating an internal model of the exosystem solves the robust output regulation problem if it also stabilizes the closed-loop system. In the following section we outline a procedure for choosing the controller parameters in such a way that these goals are achieved.

5 Controller Design for Robust Output Regulation

In this section we consider designing an error feedback controller to solve the robust output regulation problem formulated in the previous section. The results presented in [16] show us that it suffices to choose a controller which

1. satisfies the $\mathcal{G}$-conditions,
2. strongly stabilizes the closed-loop system,
3. satisfies conditions of Theorem 8 for α.

We consider single-input single-output systems, i.e. $U = Y = \mathbb{C}$, and construct a controller in the case where the system operator S of the exosystem has at most a finite number of nontrivial Jordan blocks and the asymptotic part of its spectrum consists of simple and uniformly separated eigenvalues. In particular the assumptions on the exosystem allow tracking and rejection of continuous τ-periodic functions.

Assumption 10. *Assume the following are satisfied.*

1. *All but a finite number of the eigenvalues $\sigma(S) = \{i\omega_k\}_{k\in\mathbb{Z}}$ of S are simple and have a uniform gap, i.e., there exists $N \in \mathbb{N}$ such that*

$$\inf_{k\neq l} |\omega_k - \omega_l| > 0,$$

   where $|k|, |l| \geq N$.

2. *The pair (A, B) is exponentially stabilizable and the pair (C, A) exponentially detectable.*

Unfortunately, these standing assumptions do not yet guarantee the solvability of the robust output regulation problem. In particular, in order to stabilize the internal model of the exosystem in the controller, we need conditions not only on the structure of the controller, but also on the choices of its individual parameters. These additional assumptions are stated in Theorems 13 and 15.

In the course of this section we will also very clearly see that in general the best we can hope for is the strong stability of the closed-loop system. The reason for this is that internal model containing the copy of exosystem in the controller must be stabilized with a bounded feedback. If the exosystem has an infinite number of eigenvalues on the imaginary axis, then the exponential stabilization of the closed-loop system is in general impossible even if these eigenvalues are all simple [15, Cor. 3.58].

5.1 An Observer Based Controller Satisfying the $\mathcal{G}$-Conditions

In this section we introduce the general structure of the feedback controller we use to solve the robust output regulation problem. We show that the forms of the parameters guarantee that the controller satisfies the $\mathcal{G}$-conditions in Definition 9. The remaining parameters of the controller are fixed in Sections 5.2 and 5.3 to stabilize the closed-loop system and ensure that the conditions of Theorem 8 are satisfied.

The structure of our observer-based feedback controller is specified below.

Definition 11. The parameters of the error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ on the space $Z = X \times W$ are chosen to be of form

$$\mathcal{G}_1 = \begin{pmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & S \end{pmatrix}, \quad \mathcal{G}_2 = \begin{pmatrix} -L \\ G_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & K_2 \end{pmatrix},$$

where $G_2 = g_2 \in W$ is such that $\langle g_2, \phi_k^{n_k} \rangle \neq 0$ for all $k \in \mathbb{Z}$, and where $K_1 \in \mathcal{L}(X, \mathbb{C})$, $K_2 \in \mathcal{L}(W, \mathbb{C})$, and $L \in \mathcal{L}(\mathbb{C}, X)$. ■

The copy of the operator S in $\mathcal{G}_1$ is loosely called the *internal model* of the exosystem in the controller. As stated by the internal model principle studied in [16], the dimension of the output space determines the number of copies of the dynamics of the signal generator the controller must contain in order for it to satisfy the $\mathcal{G}$-conditions. We are only considering systems with a single output, and because of this one copy of the operator S is sufficient. In the case of a p-dimensional output space we would have to replace the operator S in $\mathcal{G}_1$ with an operator G_1 copying the dynamics of the exosystem p times [6].

We will now show that if the operators K_1, K_2 and L are chosen in such a way that the spectra of the closed-loop system and the exosystem are disjoint, then the controller satisfies the $\mathcal{G}$-conditions in Definition 9. Using the results presented in [16] we can then immediately conclude that with these choices of parameters the controller solves the robust output regulation problem if the closed-loop system is stabilized in such a way that $\sigma(A_e) \cap \sigma(S) = \varnothing$ and the condition (13) is satisfied. The problem of choosing the remaining operators to strongly stabilize the closed-loop system is the topic of the next two sections.

Theorem 12. *If $\sigma(A_e) \cap \sigma(S) = \varnothing$, the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions.*

Proof. Since $g_2 \neq 0$ we have that $\mathcal{G}_2 y \neq 0$ for all $y \in \mathbb{C}$ and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$.

Let $k \in \mathbb{Z}$ and assume $(x, v)^T \in \mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Definition 11 then implies that there exist $x_1 \in \mathcal{D}(A)$, $v_1 \in \mathcal{D}(S)$ and $y \in \mathbb{C}$ such that

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} i\omega_k I - A - BK_1 - L(C + DK_1) & -(B + LD)K_2 \\ 0 & i\omega_k I - S \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -L \\ G_2 \end{pmatrix} y.$$

The second line of this equation shows that $(i\omega_k I - S)v_1 = G_2 y$, and using the structure of the operator S further implies

$$\langle g_2, \phi_k^{n_k} \rangle y = \langle G_2 y, \phi_k^{n_k} \rangle = \langle (i\omega_k I - S)v_1, \phi_k^{n_k} \rangle = (i\omega_k - i\omega_k)\langle v_1, \phi_k^{n_k} \rangle = 0.$$

Since $\langle g_2, \phi_k^{n_k} \rangle \neq 0$ by definition, we must have $y = 0$. This also concludes $(x, v)^T = \mathcal{G}_2 y = 0$, and thus $\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$.

Let $k \in \mathbb{Z}$ be such that $n_k = d_k$. Let $(x, v)^T \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1}$. The triangular structure of the operator $\mathcal{G}_1$ clearly also implies $v \in \mathcal{N}(i\omega_k I - S)^{d_k - 1}$. Since $\sigma(A_e) \cap \sigma(S) = \varnothing$, we have from [16, Lem. 5.7] that $\mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2) = Z$ and thus there exist $x_1 \in \mathcal{D}(A)$, $v_1 \in \mathcal{D}(S)$, and $y \in \mathbb{C}$ such that

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} i\omega_k I - A - BK_1 - L(C + DK_1) & -(B + LD)K_2 \\ 0 & i\omega_k I - S \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} -L \\ G_2 \end{pmatrix} y.$$

The second line of this equation further implies $v = (i\omega_k I - S)v_1 + G_2 y$. The structure of the operator S_k implies

$$\begin{aligned}
(i\omega_k I - S_k) &= -\sum_{l=2}^{d_k} \langle \cdot, \phi_k^l \rangle \phi_k^{l-1}, \\
(i\omega_k I - S_k)^2 &= \sum_{j=2}^{d_k} \langle \sum_{l=2}^{d_k} \langle \cdot, \phi_k^l \rangle \phi_k^{l-1}, \phi_k^j \rangle \phi_k^{j-1} = \sum_{l=3}^{d_k} \langle \cdot, \phi_k^l \rangle \phi_k^{l-2} \\
&\vdots \\
(i\omega_k I - S_k)^{d_k - 1} &= (-1)^{d_k - 1} \langle \cdot, \phi_k^{d_k} \rangle \phi_k^1,
\end{aligned}$$

and finally $(i\omega_k I - S_k)^{d_k} = 0$. Since we have by assumption that $v \in \mathcal{N}(i\omega_k I - S)^{d_k - 1}$, the properties of the projection P_k can be used to further show that

$$\begin{aligned}
0 &= P_k (i\omega_k I - S)^{d_k - 1} v = (i\omega_k I - S_k)^{d_k - 1} v \\
&= (i\omega_k I - S_k)^{d_k - 1} \left((i\omega_k I - S)v_1 + G_2 y \right) \\
&= (i\omega_k I - S_k)^{d_k} v_1 + (i\omega_k I - S_k)^{d_k - 1} G_2 y = (-1)^{d_k - 1} y \langle g_2, \phi_k^{d_k} \rangle \phi_k^1.
\end{aligned}$$

Since $\langle g_2, \phi_k^{d_k} \rangle \neq 0$, we must have $y = 0$. This immediately implies

$$\begin{pmatrix} x \\ v \end{pmatrix} = (i\omega_k I - \mathcal{G}_1) \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} \in \mathcal{R}(i\omega_k I - \mathcal{G}_1),$$

and thus $\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$. This concludes that the controller satisfies the $\mathcal{G}$-conditions. □

5.2 Stabilization of the Closed-Loop System

We now turn to the problem of choosing the parameters K_1, K_2 and L of the observer-based controller in Definition 11 in such a way that the closed-loop is strongly stable and $\sigma(A_e) \cap \sigma(S) = \varnothing$. We will first show that the problem can be reduced to the feedback stabilization of the internal model, which is then considered separately in Section 5.3. The main result of this section is presented in the next theorem, which also lists the appropriate choices for the parameters of the controller.

Theorem 13. *Choose $K_{11} \in \mathcal{L}(X, \mathbb{C})$ and $L \in \mathcal{L}(\mathbb{C}, X)$ such that $A + BK_{11}$ and $A + LC$ are exponentially stable. Then the Sylvester equation*

$$SH_{e1} = H_{e1}(A + BK_{11}) + G_2(C + DK_{11}) \tag{15}$$

on $\mathcal{D}(A)$ has a unique solution $H_{e1} \in \mathcal{L}(X, W)$ satisfying $H_{e1}(\mathcal{D}(A)) \subset \mathcal{D}(S)$.

Denote $B_1 = H_{e1}B + G_2 D$ and assume $K_2 \in \mathcal{L}(W, \mathbb{C})$ can be chosen in such a way that the semigroup generated by the operator $S + B_1 K_2$ is strongly stable and $\sigma(S + B_1 K_2) \cap \sigma(S) = \varnothing$. Then for the choice $K_1 = K_{11} + K_2 H_{e1}$ the closed-loop system is strongly stable and $\sigma(A_e) \cap \sigma(S) = \varnothing$.

We will first consider the solvability of the Sylvester equation in the theorem. We will also need a similar result later in Section 5.3. For this reason, the following lemma is presented for more general operators $\tilde{A}$ and $\tilde{G}$ in place of $A + BK_{11}$ and $G_2(C + DK_{11})$, respectively.

Lemma 14. *Assume that $\tilde{A} : \mathcal{D}(\tilde{A}) \subset \tilde{X} \to \tilde{X}$ generates an exponentially stable semigroup on a Banach space $\tilde{X}$ and that $\tilde{G} \in \mathcal{L}(\tilde{X}, W)$. Then the Sylvester equation $SH = H\tilde{A} + \tilde{G}$ has a unique solution $H \in \mathcal{L}(\tilde{X}, W)$ satisfying $H(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(S)$. The operator H is given by*

$$H = \left(\ldots, H_{-1}^T, H_0^T, H_1^T, \ldots\right)^T, \qquad H_k = \sum_{l=1}^{n_k} (-1)^{l-1} J_{n_k}^{l-1} P_k \tilde{G} R(i\omega_k, \tilde{A})^l,$$

where $J_{n_k} \in \mathcal{L}(\operatorname{span}\{\phi_k^l\}_{l=1}^{n_k})$ is an operator corresponding to a single $n_k \times n_k$ Jordan block with eigenvalue 0.

Proof. Since $\tilde{A}$ generates an exponentially stable semigroup and since the growth bound of the semigroup generated by $-S$ polynomially bounded, we have from [19] that the Sylvester equation has a unique solution $H \in \mathcal{L}(\tilde{X}, W)$ and $H(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(S)$. It remains to show that the given operator H is such that $H(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(S)$ and that it satisfies the Sylvester equation.

It is easy to see that $H \in \mathcal{L}(\tilde{X}, W)$. We will now show that if we denote $H_k = P_k H$, then for all $k \in \mathbb{Z}$ the operator H satisfies

$$S_k H_k = H_k \tilde{A} + \tilde{G}_k \tag{16}$$

To this end, let $k \in \mathbb{Z}$ and $x \in \mathcal{D}(\tilde{A})$. Since $S_k = i\omega_k P_k + J_{n_k}$, $R(i\omega_k, \tilde{A})\tilde{A}x = -x + i\omega_k R(i\omega_k, \tilde{A})x$, and since $J_{n_k}^{n_k} = 0$, a straightforward computation shows

$$\begin{aligned} S_k H_k x - H_k \tilde{A} x = & \sum_{l=1}^{n_k} (-1)^{l-1} S_k J_{n_k}^{l-1} P_k \tilde{G} R(i\omega_k, \tilde{A})^l x - \sum_{l=1}^{n_k} (-1)^{l-1} J_{n_k}^{l-1} P_k \tilde{G} R(i\omega_k, \tilde{A})^l \tilde{A} x \\ = & \sum_{l=1}^{n_k} (-1)^{l-1} (i\omega_k J_{n_k}^{l-1} + J_{n_k}^l) P_k \tilde{G} R(i\omega_k, \tilde{A})^l x \\ & - \sum_{l=1}^{n_k} (-1)^{l-1} J_{n_k}^{l-1} P_k \tilde{G} \left(-R(i\omega_k, \tilde{A})^{l-1} + i\omega_k R(i\omega_k, \tilde{A})^l \right) x \\ = & \sum_{l=1}^{n_k-1} (-1)^{l-1} J_{n_k}^l P_k \tilde{G} R(i\omega_k, \tilde{A})^l x + \sum_{l=1}^{n_k} (-1)^{l-1} J_{n_k}^{l-1} P_k \tilde{G} R(i\omega_k, \tilde{A})^{l-1} x = P_k \tilde{G} x. \end{aligned}$$

This concludes that the given operator H satisfies (16) for all $k \in \mathbb{Z}$.

If $x \in \mathcal{D}(\tilde{A})$, then we can use (16) to show that

$$\sum_{k\in\mathbb{Z}} \|S_k H x\|^2 = \sum_{k\in\mathbb{Z}} \|S_k H_k x\|^2 = \sum_{k\in\mathbb{Z}} \|H_k \tilde{A} x + P_k \tilde{G} x\|^2 \le 2 \sum_{k\in\mathbb{Z}} \|P_k H \tilde{A} x\|^2 + 2 \sum_{k\in\mathbb{Z}} \|P_k \tilde{G} x\|^2 < \infty$$

since $H\tilde{A}x \in W$ and $\tilde{G}x \in W$. By definition this means that $Hx \in \mathcal{D}(S)$. Since $x \in \mathcal{D}(\tilde{A})$ was arbitrary, this concludes $H(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(S)$.

Finally, the equations (16) and $H(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(S)$ together with the properties of the operator S imply that for all $x \in \mathcal{D}(\tilde{A})$ we have

$$SHx = \sum_{k\in\mathbb{Z}} S_k P_k H x = \sum_{k\in\mathbb{Z}} \left(P_k H \tilde{A} x + P_k \tilde{G} x \right) = H\tilde{A}x + \tilde{G}x.$$

This concludes the proof. □

We can now complete the proof of Theorem 13 by showing that if the parameters of the controller are chosen as suggested, then the closed-loop system is strongly stable and $\sigma(A_e) \cap \sigma(S) = \varnothing$.

Proof of Theorem 13. The solvability of the Sylvester equation in the theorem follows directly from Lemma 14.

If the feedback controller has the structure described in Definition 11, the system operator of the closed-loop system is given by

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} = \begin{pmatrix} A & BK_1 & BK_2 \\ -LC & A + BK_1 + LC & BK_2 \\ G_2 C & G_2 DK_1 & S + G_2 DK_2 \end{pmatrix}.$$

If we choose a similarity transform $Q_e \in \mathcal{L}(X \times X \times W, X \times W \times X)$ in such a way that

$$Q_e = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{pmatrix} \qquad \text{and} \qquad Q_e^{-1} = \begin{pmatrix} I & 0 & 0 \\ I & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

we can then define an operator $\tilde{A}_e$ on the space $X \times W \times X$ by

$$\tilde{A}_e = Q_e A_e Q_e^{-1} = \begin{pmatrix} A + BK_1 & BK_2 & BK_1 \\ G_2(C + DK_1) & S + G_2 DK_2 & G_2 DK_1 \\ 0 & 0 & A + LC \end{pmatrix}.$$

It is well-known that $\tilde{A}_e$ generates a semigroup, and that this semigroup is strongly stable if and only if the semigroup $T_e(t)$ generated by A_e is. The triangular structure of $\tilde{A}_e$ further implies that since $A + LC$ is exponentially stable, this operator generates a strongly stable semigroup if the operator

$$\tilde{A}_{e1} = \begin{pmatrix} A + BK_1 & BK_2 \\ G_2(C + DK_1) & S + G_2 DK_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ G_2 C & S \end{pmatrix} + \begin{pmatrix} B \\ G_2 D \end{pmatrix} \begin{pmatrix} K_1 & K_2 \end{pmatrix}$$

is strongly stable [6, Lem. 20]. Using $K_1 = K_{11} + K_2 H_{e1}$ shows that

$$\tilde{A}_{e1} = \begin{pmatrix} A + BK_{11} & 0 \\ G_2(C + DK_{11}) & S \end{pmatrix} + \begin{pmatrix} B \\ G_2 D \end{pmatrix} \begin{pmatrix} K_2 H_{e1} & K_2 \end{pmatrix}.$$

We can now use the solution H_{e1} of the Sylvester equation in the theorem to choose a similarity transform $Q_{e1} \in \mathcal{L}(X \times W)$ satisfying

$$Q_{e1} = \begin{pmatrix} I & 0 \\ H_{e1} & I \end{pmatrix}, \qquad Q_{e1}^{-1} = \begin{pmatrix} I & 0 \\ -H_{e1} & I \end{pmatrix}.$$

A direct computation shows that since H_{e1} is the solution of the Sylvester equation, we have

$$\begin{aligned} Q_{e1} \begin{pmatrix} A + BK_{11} & 0 \\ G_2(C + DK_{11}) & S \end{pmatrix} Q_{e1}^{-1} &= \begin{pmatrix} A + BK_{11} & 0 \\ H_{e1}(A + BK_{11}) + G_2(C + DK_{11}) - SH_{e1} & S \end{pmatrix} \\ &= \begin{pmatrix} A + BK_{11} & 0 \\ 0 & S \end{pmatrix}. \end{aligned}$$

Therefore, if we denote $B_1 = H_{e1}B + G_2 D$ and define $A_{e1} = Q_{e1}\tilde{A}_{e1}Q_{e1}^{-1}$, we then have

$$\begin{aligned} A_{e1} = Q_{e1}\tilde{A}_{e1}Q_{e1}^{-1} &= \begin{pmatrix} A + BK_{11} & 0 \\ 0 & S \end{pmatrix} + \begin{pmatrix} B \\ H_{e1}B + G_2 D \end{pmatrix} \begin{pmatrix} K_2 H_{e1} - K_2 H_{e1} & K_2 \end{pmatrix} \\ &= \begin{pmatrix} A + BK_{11} & BK_2 \\ 0 & S + B_1 K_2 \end{pmatrix} \end{aligned}$$

Since the operators $A + BK_{11}$ and $S + B_1 K_2$ generate exponentially and strongly stable semigroups, respectively, the operator A_{e1} generates a strongly stable semigroup [6, Lem. 20]. Using this and the earlier arguments we can conclude that the closed-loop system is strongly stable.

The operator K_2 was chosen in such a way that $\sigma(S + B_1 K_2) \cap \sigma(S) = \varnothing$, and since $A + BK_{11}$ and $A + LC$ are generators of exponentially stable semigroups, we also have

$$\sigma(A + BK_{11}) \cap \sigma(S) = \varnothing \qquad \text{and} \qquad \sigma(A + LC) \cap \sigma(S) = \varnothing.$$

We can now use properties of triangular block operators and the similarities between the operators to deduce that

$$\sigma(A_{e1}) \cap \sigma(S) = \varnothing \quad \Rightarrow \quad \sigma(\tilde{A}_{e1}) \cap \sigma(S) = \varnothing \quad \Rightarrow \quad \sigma(\tilde{A}_e) \cap \sigma(S) = \varnothing \quad \Rightarrow \quad \sigma(A_e) \cap \sigma(S) = \varnothing.$$

This concludes the proof. □

5.3 Stabilization of the Internal Model

In this section we complete the construction of the controller by stabilizing the internal model in the operator $\mathcal{G}_1$. Our main goal is to choose a feedback $K_2 \in \mathcal{L}(W, \mathbb{C})$ in such a way that the operator

$$S + B_1 K_2$$

generates a strongly stable semigroup on W. Here $B_1 = H_{e1}B + G_2 D$, as was defined in the proof of Theorem 13, and H_{e1} is the solution of the Sylvester equation (15). For choosing a suitable operator K_2 we use pole placement of an infinite spectrum [28, 8, 30, 22]. This approach allows us to directly verify that the condition $\sigma(S + B_1 K_2) \cap \sigma(S) = \varnothing$ is satisfied, and subsequently to derive asymptotic estimates for the behavior of the resolvent of the closed-loop system. We will see in Section 5.4 that such asymptotic estimates concerning the closed-loop system are essential to determining on which of the scale spaces W_α the controller solves the robust output regulation problem.

The first one of the conditions in Assumption 10 means that there exists a finite set $I_S \subset \mathbb{Z}$ of indices and a constant $d > 0$ such that $n_k = 1$ for all $k \in \mathbb{Z} \setminus I_S$ and

$$|\omega_k - \omega_l| \geq d > 0$$

for $k,l \in \mathbb{Z}\setminus I_S$ such that $k \neq l$. The operators S, B_1 and K_2 can therefore be decomposed into

$$S = \begin{pmatrix} S_f & 0 \\ 0 & S_i \end{pmatrix}, \qquad B_1 = \begin{pmatrix} B_f \\ B_i \end{pmatrix}, \qquad K_2 = \begin{pmatrix} K_f & K_i \end{pmatrix} \tag{17}$$

according to the decomposition $W = W^f \times W^i$ of the state space of the exosystem, where

$$W^f = \operatorname{span}\left\{ \phi_k^l \;\middle|\; k \in I_S,\ l = 1,\ldots,n_k \right\}, \qquad W^i = \overline{\operatorname{span}}\{\phi_k^1\}_{k\in\mathbb{Z}\setminus I_S}.$$

The labels 'f' and 'i' stand for 'finite' and 'infinite' parts of the spaces and operators. The parts S_f, B_f and K_f are operators on finite-dimensional spaces and S_i is an infinite-dimensional diagonal operator

$$S_i v = \sum_{k\in\mathbb{Z}\setminus I_S} i\omega_k \langle v, \phi_k^1\rangle \phi_k^1, \qquad \mathcal{D}(S_i) = \left\{ v \in W^i \;\middle|\; \sum_{k\in\mathbb{Z}\setminus I_S} \omega_k^2 |\langle v,\phi_k^1\rangle|^2 < \infty \right\}.$$

Our standing assumptions concerning the spectrum of the exosystem also imply that we can assume the frequencies to be ordered in such a way that $\omega_k \leq \omega_l$ for all $k,l \in \mathbb{Z}\setminus I_S$ such that $k \leq l$. For notational convinience we also assume $0 \in I_S$.

Theorem 15 below shows us the final conditions required to strongly stabilize the closed-loop system in such a way that $\sigma(A_e)\cap\sigma(S) = \varnothing$. To express these assumptions we need the complex-valued function $P_K(\cdot)$ defined by

$$P_K(\lambda) = (C + DK_{11})R(\lambda, A + BK_{11})B + D \tag{18}$$

for all $\lambda \in \rho(A + BK_{11})$. This function is the transfer function of the original plant after being stabilized with an input $u = K_{11}x + \tilde{u}$. It is well-known that the invertibility of a transfer function is preserved under this type of feedback. Because of this, our assumption on the invertibility of the operators $P(i\omega_k)$ made in Section 2 also implies that we have $P_K(i\omega_k) \neq 0$ for all $k \in \mathbb{Z}$.

Theorem 15. *Assume there exist $\beta, c > 0$ such that*

$$|P_K(i\omega_k)| \cdot |\langle g_2, \phi_k^1\rangle| \geq \frac{c}{|k|^\beta} \tag{19}$$

for large enough $|k|$. Then the operator $K_2 \in \mathcal{L}(W,\mathbb{C})$ can be chosen in such a way that the semigroup generated by the operator $S + B_1K_2$ is strongly stable and $\sigma(S + B_1K_2)\cap\sigma(S) = \varnothing$. Furthermore, for any $\gamma > \beta + \frac{1}{2}$ the operator K_2 can be chosen in such a way that the asymptotic behavior of the resolvent of the closed-loop system satisfies

$$\|R(i\omega_k, A_e)\| = \mathcal{O}\left(|k|^\gamma\right).$$

Proof. We begin by showing that the pair (S_f, B_f) of finite-dimensional operators is controllable. Since the matrix S_f consists of Jordan blocks, it is sufficient to show that $\langle B_f, \phi_k^{n_k}\rangle \neq 0$ for all $k \in I_S$. Using the formula $B_1 = H_{e1}B + G_2D$, Lemma 14 and (18) we have

$$\begin{aligned}
\langle B_1, \phi_k^{n_k}\rangle &= \langle H_kB + G_2D, \phi_k^{n_k}\rangle \\
&= \left\langle \sum_{l=1}^{n_k} (-1)^{l-j} J_{n_k}^{l-1} P_k G_2 (C + DK_{11}) R(i\omega_k, A + BK_{11})^l B, \phi_k^{n_k} \right\rangle + \langle g_2, \phi_k^{n_k}\rangle D \\
&= \sum_{l=1}^{n_k} \left[(-1)^{l-j} \langle J_{n_k}^{l-1} g_2, \phi_k^{n_k}\rangle (C + DK_{11}) R(i\omega_k, A + BK_{11})^l B \right] + \langle g_2, \phi_k^{n_k}\rangle D \\
&= \langle g_2, \phi_k^{n_k}\rangle (C + DK_{11}) R(i\omega_k, A + BK_{11}) B + \langle g_2, \phi_k^{n_k}\rangle D \\
&= \langle g_2, \phi_k^{n_k}\rangle P_K(i\omega_k) \neq 0
\end{aligned}$$

for all $k \in \mathbb{Z}$. This concludes that the pair (S_f, B_f) is controllable. Since these are finite-dimensional operators, we can now choose an operator $K_{f1} \in \mathcal{L}(W^f, \mathbb{C})$ in such a way that $S_f + B_f K_{f1}$ is exponentially stable.

Since S_i is a diagonal operator, we can clearly use Lemma 14 to show that the Sylvester equation

$$S_i H = H(S_f + B_f K_{f1}) + B_i K_{f1}$$

has a unique solution $H \in \mathcal{L}(W^f, W^i)$ satisfying

$$Hv = \sum_{k \in \mathbb{Z} \setminus I_S} \langle B_i K_{f1} R(i\omega_k, S_f + B_f K_{f1})v, \phi_k^1 \rangle \phi_k^1$$

for all $v \in W^f$. We now choose $K_f = K_{f1} + K_i H$. In this notation we have

$$S + B_1 K_2 = \begin{pmatrix} S_f + B_f K_{f1} & 0 \\ B_i K_{f1} & S_f \end{pmatrix} + \begin{pmatrix} B_f \\ B_i \end{pmatrix} \begin{pmatrix} K_i H & K_i \end{pmatrix},$$

and as in the proof of Theorem 13 we can use the fact that the operator H is the solution of the Sylvester equation to show

$$\begin{aligned} \begin{pmatrix} I & 0 \\ H & I \end{pmatrix} (S + B_1 K_2) \begin{pmatrix} I & 0 \\ -H & I \end{pmatrix} &= \begin{pmatrix} S_f + B_f K_{f1} & 0 \\ 0 & S_i \end{pmatrix} + \begin{pmatrix} B_f \\ HB_f + B_i \end{pmatrix} \begin{pmatrix} K_i H - K_i H & K_i \end{pmatrix} \\ &= \begin{pmatrix} S_f + B_f K_{f1} & B_f K_i \\ 0 & S_i + B_2 K_i \end{pmatrix}. \end{aligned} \tag{20}$$

Here we have denoted $B_2 = HB_f + B_i \in \mathcal{L}(\mathbb{C}, W^i)$. For any $u \in \mathbb{C}$ we have

$$\begin{aligned} B_2 u = (HB_f + B_i)u &= \sum_{k \in \mathbb{Z} \setminus I_S} \left\langle B_i (K_{f1} R(i\omega_k, S_f + B_f K_{f1}) B_f u + u), \phi_k^1 \right\rangle \phi_k^1 \\ &= \sum_{k \in \mathbb{Z} \setminus I_f} (K_{f1} R(i\omega_k, S_f + B_f K_{f1}) B_f u + u) \langle B_i, \phi_k^1 \rangle \phi_k^1. \end{aligned} \tag{21}$$

The above formulas imply that since $S_f + B_f K_{f1}$ is exponentially stable, the operator $S + B_1 K_2$ can be stabilized by choosing $K_i \in \mathcal{L}(W^i, \mathbb{C})$ in such a way that $S_i + B_2 K_i$ generates a strongly stable semigroup on W^i [6, Lem. 20].

We will choose operator K_i using pole placement of an infinite spectrum [28, 25]. Let $\gamma > \beta + \frac{1}{2}$ and choose

$$\mu_k = -\frac{1}{|k|^\gamma} + i\omega_k$$

for $k \in \mathbb{Z} \setminus I_S$ (recall that we assumed $0 \in I_S$). In particular we will show that we can choose $K_i \in \mathcal{L}(W^i, \mathbb{C})$ in such a way that $\sigma(S_i + B_2 K_i) = \{\mu_k\}_k$ and the operator $S_i + B_2 K_i$ is a strongly stable Riesz-spectral operator with at most finite number of nonsimple eigenvalues. Denote

$$d = \inf_{k \neq l} |\omega_k - \omega_l| > 0,$$

where $k, l \in \mathbb{Z} \setminus I_S$. For all $\lambda \in \mathbb{C}$ such that $\operatorname{dist}(\lambda, i\omega_k) > \frac{1}{3}d$ we have

$$\sum_{k \in \mathbb{Z} \setminus I_S} \left| \frac{\langle B_2, \phi_k^1 \rangle}{\lambda - i\omega_k} \right|^2 \leq \frac{3}{d} \sum_{k \in \mathbb{Z} \setminus I_S} |\langle B_2, \phi_k^1 \rangle|^2 \leq \frac{3}{d} \|B_2\|^2 < \infty, \tag{22a}$$

$$\sum_{\substack{k \in \mathbb{Z} \setminus I_S \\ k \neq l}} \left| \frac{\langle B_2, \phi_k^1 \rangle}{i\omega_l - i\omega_k} \right|^2 \leq \frac{1}{d} \sum_{\substack{k \in \mathbb{Z} \setminus I_S \\ k \neq l}} |\langle B_2, \phi_k^1 \rangle|^2 \leq \frac{1}{d} \|B_2\|^2 < \infty. \tag{22b}$$

Our next step is to derive an asymptotic lower bound for the behavior of the terms $|\langle B_2, \phi_k^1\rangle|$. We can first observe that we must have $K_{f1}R(i\omega_k, S_f)B_f \neq 1$ for all $k \in \mathbb{Z} \setminus I_S$, since otherwise we would have

$$(i\omega_k I - S_f - B_f K_{f1})R(i\omega_k, S_f)B_f = (i\omega_k I - S_f)R(i\omega_k, S_f)B_f - B_f K_{f1}R(i\omega_k, S_f)B_f = 0,$$

i.e., $i\omega_k \in \sigma(S_f + B_f K_{f1})$. This, however, is impossible since $S_f + B_f K_{f1}$ is exponentially stable. The application of the well-known Sherman-Morrison formula therefore implies

$$K_f R(i\omega_k, S_f + B_f K_{f1})B_f + 1 = \frac{1}{1 - K_{f1}R(i\omega_k, S_f)B_f} \neq 0$$

for all $k \in \mathbb{Z} \setminus I_S$. On the other hand, using the form of the operator $B_1 = H_{e1}B + G_2 D$ and Lemma 14 we can see that for $k \in \mathbb{Z} \setminus I_S$

$$\langle B_i, \phi_k^1\rangle = \langle B_1, \phi_k^1\rangle = \langle G_2(C + DK_{11})R(i\omega_k, A + BK_{11})B + G_2 D, \phi_k^1\rangle = \langle g_2, \phi_k^1\rangle P_K(i\omega_k).$$

By Definition 11 and the property $P_K(i\omega_k) \neq 0$ we also see that these terms must be nonzero for all $k \in \mathbb{Z} \setminus I_S$. These together with the formula (21) for the operator B_2 imply that for all $k \in \mathbb{Z} \setminus I_S$

$$\langle B_2, \phi_k^1\rangle = (K_{f1}R(i\omega_k, S_f + B_f K_{f1})B_f + 1)\langle B_i, \phi_k^1\rangle \neq 0. \tag{23}$$

Furthermore, since the norms $\|R(i\omega_k, S_f)\|$ decay to zero as $|k| \to \infty$, have an estimate

$$\begin{aligned}|\langle B_2, \phi_k^1\rangle| &= |(K_{f1}R(i\omega_k, S_f + B_f K_{f1})B_f + 1)| \cdot |\langle g_2, \phi_k^1\rangle P_K(i\omega_k)| \\ &\geq \frac{|\langle g_2, \phi_k^1\rangle| \cdot |P_K(i\omega_k)|}{1 + |K_{f1}R(i\omega_k, S_f)B_f|} \geq \frac{|\langle g_2, \phi_k^1\rangle| \cdot |P_K(i\omega_k)|}{1 + \|K_{f1}\| \cdot \|R(i\omega_k, S_f)\| \cdot \|B_f\|} \geq \frac{1}{2}|\langle g_2, \phi_k^1\rangle| \cdot |P_K(i\omega_k)|\end{aligned}$$

for all $k \in \mathbb{Z} \setminus I_S$ with a large enough $|k|$. Our assumption (19) finally implies that there exist a constant $c > 0$ such that for all $k \in \mathbb{Z} \setminus I_S$ with a large enough $|k|$

$$|\langle B_2, \phi_k^1\rangle| \geq \frac{1}{2}|\langle g_2, \phi_k^1\rangle| \cdot |P_K(i\omega_k)| \geq c|k|^{-\beta}$$

and thus for a large enough $N \in \mathbb{N}$ we also have

$$\sum_{|k| \geq N} \left| \frac{\mu_k - i\omega_k}{\langle B_2, \phi_k^1\rangle} \right|^2 \leq \frac{1}{c^2} \sum_{|k| \geq N} \frac{|k|^{2\beta}}{|k|^{2\gamma}} \leq \frac{1}{c^2} \sum_{|k| \geq N} \frac{1}{|k|^{2(\gamma - \beta)}} < \infty, \tag{24}$$

since $2(\gamma - \beta) > 1$.

Since the conditions (22), (23), and (24) are satisfied, we have from [28, Thm. 1] that there exists an operator $K_i \in \mathcal{L}(W^i, \mathbb{C})$ such that $S_i + B_2 K_i$ is a strongly stable Riesz-spectral operator with eigenvalues $\{\mu_k\}_{k \in \mathbb{Z} \setminus I_S}$ and at most finite number of these eigenvalues are nonsimple. Since

$$\sigma(S + B_1 K_2) \subset \sigma(S_f + B_f K_{f1}) \cup \sigma(S_i + B_2 K_i) = \sigma(S_f + B_f K_{f1}) \cup \{\mu_k\}_{k \in \mathbb{Z} \setminus I_S}$$

where $S_f + B_f K_{f1}$ is exponentially stable, we have that $\sigma(S + B_1 K_2) \cap \sigma(S) = \varnothing$. This concludes that the internal model can be stabilized using a bounded feedback K_2. The infinite part of this operator is obtained by choosing $K_i = \langle \cdot, h\rangle$, where $h \in W^i$ is given by

$$\begin{aligned}h &= \sum_{k \in \mathbb{Z} \setminus I_S} h_k \phi_k^1, \\ \overline{h_k} &= \frac{\mu_k - i\omega_k}{\langle B_2, \phi_k^1\rangle} \prod_{\substack{l \in \mathbb{Z} \setminus I_S \\ l \neq k}} \frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} = \frac{1}{|k|^\gamma \langle B_2, \phi_k^1\rangle} \prod_{\substack{l \in \mathbb{Z} \setminus I_S \\ l \neq k}} \left(1 + i\frac{1}{|l|^\gamma(\omega_l - \omega_k)}\right).\end{aligned}$$

In the remaining part of the proof we derive the estimate for the asymptotic behavior of the resolvent of the closed-loop system. To estimate the resolvent operators of the various composite operators we will use the fact that if X_1 and X_2 are Banach spaces, and if $A_{11} \in \mathcal{L}(X_1)$, $A_{12} \in \mathcal{L}(X_2, X_1)$ and $A_{22} \in \mathcal{L}(X_2)$, we then have

$$\left\| \begin{pmatrix} A_{11} & A_{11}A_{12}A_{22} \\ 0 & A_{22} \end{pmatrix} \right\| \leq (\|A_{11}\| + 1)(\|A_{12}\| + 1)(\|A_{22}\| + 1). \tag{25}$$

This follows directly from the estimate

$$\begin{aligned} \left\| \begin{pmatrix} A_{11} & A_{11}A_{12}A_{22} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| &\leq \|A_{11}x_1 + A_{11}A_{12}A_{22}x_2\| + \|A_{22}x_2\| \\ &\leq (\|x_1\| + \|x_2\|)\left(\|A_{11}\| + \|A_{11}\|\|A_{12}\|\|A_{22}\| + \|A_{22}\|\right) \\ &\leq \|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\| \max\{\|A_{12}\|, 1\}(\|A_{11}\|(1 + \|A_{22}\|) + \|A_{22}\|) \\ &\leq \|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|(\|A_{11}\| + 1)(\|A_{12}\| + 1)(\|A_{22}\| + 1). \end{aligned}$$

Here we used the norm $\|(x_1, x_2)^T\| = \|x_1\| + \|x_2\|$ on the composite space $X_1 \times X_2$. A different choice for the norm would have only resulted in a constant $M > 0$ on the right-hand side of the estimate in (25).

We will start with the asymptotic behavior of $R(i\omega_k, S_i + B_2K_i)$. The fact that $S_i + B_2K_i$ is a Riesz-spectral operator and all but a finite number of its eigenvalues are simple implies that there exists an isomorphism $Q_i \in \mathcal{L}(W^i)$ such that

$$S_i + B_2K_i = Q_i \begin{pmatrix} S_i^{fin} & 0 \\ 0 & S_i^{inf} \end{pmatrix} Q_i^{-1},$$

where S_i^{fin} is a finite-dimensional exponentially stable operator and $S_i^{inf} = \operatorname{diag}(\mu_k)_{|k| \geq N}$ for some $N \in \mathbb{N}$. This means that the resolvent operator of $S_i + B_2K_i$ satisfies

$$\begin{aligned} \|R(i\omega_k, S_i + B_2K_i)\| &= \left\| Q_i \begin{pmatrix} R(i\omega_k, S_i^{fin}) & 0 \\ 0 & R(i\omega_k, S_i^{inf}) \end{pmatrix} Q_i^{-1} \right\| \\ &\leq \|Q_i\|\|Q_i^{-1}\| \cdot \max\left\{ \|R(i\omega_k, S_i^{fin})\|, \|R(i\omega_k, S_i^{inf})\| \right\} \end{aligned}$$

for all $k \in \mathbb{Z} \setminus I_S$. For $k \in \mathbb{Z}$ with $|k| \geq N$ the norm of $R(i\omega_k, S_i^{fin})$ is uniformly bounded and

$$\|R(i\omega_k, S_i^{inf})\| = \frac{1}{|i\omega_k - \mu_k|} = |k|^\gamma.$$

This immediately implies

$$\|R(i\omega_k, S_i + B_2K_i)\| = \mathcal{O}(|k|^\gamma).$$

We can now turn to considering the asymptotic behavior of the resolvent operator of $S + B_1K_2$. Similarly as in the derivation of the estimate (25) we can easily see that

$$\left\| \begin{pmatrix} I & 0 \\ H & I \end{pmatrix} \right\| \leq \|H\| + \|I\| + \|I\| = \|H\| + 2.$$

Using this and (20) we see that the resolvent of the stabilized internal model $S + B_1K_2$ satisfies

$$\begin{aligned} &\|R(i\omega_k, S + B_1K_2)\| \\ \leq\ &(\|H\| + 2)^2 \left\| \begin{pmatrix} R(i\omega_k, S_f + B_fK_{f1}) & R(i\omega_k, S_f + B_f)B_fK_iR(i\omega_k, S_i + B_2K_i) \\ & R(i\omega_k, S_i + B_2K_i) \end{pmatrix} \right\| \\ \leq\ &(\|H\| + 2)^2(\|B_fK_i\| + 1)(\|R(i\omega_k, S_f + B_fK_{f1})\| + 1)(\|R(i\omega_k, S_i + B_2K_i)\| + 1) \end{aligned}$$

Since $\|R(i\omega_k, S_f + B_f K_{f1})\|$ is uniformly bounded with respect to $k \in \mathbb{Z} \setminus I_S$, this estimate implies

$$\|R(i\omega_k, S + B_1 K_2)\| = \mathcal{O}(|k|^\gamma).$$

This, in turn, can be used to estimate the behavior of the resolvent $R(i\omega_k, \tilde{A}_{e1})$. Using the definition of the operator A_{e1} in the proof of Theorem 13 and the estimate (25), we obtain

$$\begin{aligned}
\|R(i\omega_k, \tilde{A}_{e1})\| &= \|Q_{e1}^{-1} R(i\omega_k, A_{e1}) Q_{e1}\| \\
&\le \|Q_{e1}^{-1}\| \|Q_{e1}\| \left\| \begin{pmatrix} R(i\omega_k, A + BK_{11}) & R(i\omega_k, A + BK_{11}) BK_2 R(i\omega_k, S + B_1 K_2) \\ 0 & R(i\omega_k, S + B_1 K_2) \end{pmatrix} \right\| \\
&\le \|Q_{e1}^{-1}\| \|Q_{e1}\| (\|R(i\omega_k, A + BK_{11})\| + 1)(\|BK_2\| + 1)(\|R(i\omega_k, S + B_1 K_2)\| + 1).
\end{aligned}$$

Since $A + BK_{11}$ generates an exponentially stable semigroup, the terms $\|R(i\omega_k, A + BK_{11})\|$ are uniformly bounded with respect to k and thus

$$\|R(i\omega_k, \tilde{A}_{e1})\| = \mathcal{O}(|k|^\gamma).$$

Finally, we can estimate the behavior of the resolvent operators $R(i\omega_k, A_e)$ of the closed-loop system. Similarly as above, we can use the definition of the operator $\tilde{A}_e$ in the proof of Theorem 13 and the estimate (25) to show

$$\begin{aligned}
\|R(i\omega_k, A_e)\| &= \|Q_e^{-1} R(i\omega_k, \tilde{A}_e) Q_e\| \\
&\le \|Q_e^{-1}\| \|Q_e\| \left\| \begin{pmatrix} R(i\omega_k, \tilde{A}_{e1}) & R(i\omega_k, \tilde{A}_{e1}) \begin{pmatrix} B \\ G_2 D \end{pmatrix} K_1 R(i\omega_k, A + LC) \\ 0 & R(i\omega_k, A + LC) \end{pmatrix} \right\| \\
&\le \|Q_e^{-1}\| \|Q_e\| (\|R(i\omega_k, \tilde{A}_{e1})\| + 1) \left(\left\| \begin{pmatrix} B \\ G_2 D \end{pmatrix} K_1 \right\| + 1 \right) (\|R(i\omega_k, A + LC)\| + 1).
\end{aligned}$$

Since $A + LC$ generates an exponentially stable semigroup, the norms $\|R(i\omega_k, A + LC)\|$ are uniformly bounded with respect to $k \in \mathbb{Z}$. Because of this, the previous estimate implies

$$\|R(i\omega_k, A_e)\| = \mathcal{O}(|k|^\gamma).$$

This concludes the proof. □

Remark 16. Since there is some freedom in choosing the parameter $G_2 \in \mathcal{L}(\mathbb{C}, W)$ of the controller, Theorem 15 concludes that the stabilization of the internal model can be achieved using bounded feedback whenever the values $P_K(i\omega_k)$ of the transfer function of the stabilized plant decay to zero at a rate that is at most polynomial. Moreover, it also shows that this rate is reflected in the behavior of the resolvent of the stabilized closed-loop system at the eigenvalues $i\omega_k$ of the exosystem. For finite-dimensional systems the assumption on the polynomial decay of the transfer function is always satisfied. However, in the case of infinite-dimensional systems the situation is more complicated, and in particular the values of the transfer function can approach zero at a faster rate even in the case of well-behaved systems.

5.4 The Solvability of the Robust Output Regulation Problem

We conclude the study of our observer-based controller by determining on which of the scale spaces W_α it solves the robust output regulation problem. Theorem 17 in particular shows that, provided the values $P_K(i\omega_k)$ considered in the previous section approach zero at a rate that is at most polynomial, such a scale space always exitsts. We also show a concrete connection between the rate of this decay and the smoothness of the exogeneous signals the controller is capable of tracking and rejecting. We conclude the section by discussing possibilities of relaxing our standing assumptions on the spectrum of our exosystem.

Theorem 17. *Assume there exist constants $\beta, c > 0$ such that*

$$|P_K(i\omega_k)| \cdot |\langle g_2, \phi_k^1 \rangle| \geq \frac{c}{|k|^\beta}$$

for large enough $|k|$. If the parameters of the controller are chosen as described earlier in this section for some $\gamma > \beta + \frac{1}{2}$ in Theorem 15, then the controller solves the robust output regulation problem on W_γ.

Proof. We will first verify that condition (13) is satisfied for $\alpha = \gamma$. Theorem 15 and our assumption that all but a finite number of the eigenvalues of S have a uniform gap imply that there exist constants $N \in \mathbb{N}$ and $c, M > 0$ such that

$$\|R(i\omega_k, A_e)\| \leq M|k|^\alpha \qquad \text{and} \qquad |\omega_k| \geq c|k|$$

for all $k \in \mathbb{Z}$ with $|k| \geq N$. For all such $k \in \mathbb{Z}$ we thus have

$$\frac{\|R(i\omega_k, A_e)\|^2}{(1 + \omega_k^2)^\alpha} \leq \frac{M^2|k|^{2\alpha}}{(1 + c^2k^2)^\alpha} \leq \frac{M^2}{c^{2\alpha}} \frac{|k|^{2\alpha}}{|k|^{2\alpha}} = \frac{M^2}{c^{2\alpha}} < \infty,$$

which concludes that condition (13) is satisfied.

Since $\sigma(A_e) \cap \sigma(S) = \varnothing$ and since by Theorem 12 the controller satisfies the $\mathcal{G}$-conditions, Theorem 8 and the results in [16] conclude that the controller solves the robust output regulation problem on W_γ. □

The above theorem also illustrates a close connection between the asymptotic behavior of the values $P_K(i\omega_k)$ of the transfer function of the stabilized plant and the minimal smoothness of the signals our controller is capable of tracking and rejecting. In particular this is visible in the case of τ-periodic reference and disturbance signals generated by a diagonal exosystem with frequencies

$$(\omega_k)_{k\in\mathbb{Z}} = \left(\frac{2\pi k}{\tau}\right)_{k\in\mathbb{Z}}.$$

Combining Theorems 17 and 5 shows us that for such exosystems and the above choices of parameters our controller is guaranteed to track and reject τ-periodic signals belonging to the space $H^\gamma_{per}(0, \tau)$.

The applicability of Theorem 17 has an evident limitation arising from the fact that since the parameter G_2 of the controller is a bounded operator, we necessarily have

$$\sum_{k\in\mathbb{Z}} |\langle g_2, \phi_k^1 \rangle|^2 < \infty. \tag{26}$$

This immediately implies that regardless of the behaviour of the transfer function $P_K(\cdot)$ of the stabilized plant, the constant β in the theorem is always larger than $\frac{1}{2}$. Therefore it is impossible to use our results to guarantee the existence of a controller solving the robust output regulation problem on the space W_α for any $\alpha \leq 1$. In principle this restriction could be overcome by allowing the input operator G_2 of the controller to be unbounded, e.g. $G_2 = g_2 \in W_{-1}$, where the Hilbert space W_{-1} is defined analogously to W_α for $\alpha \geq 0$ (see [3, Sec. II.5] for details). For such operators it is possible that $|\langle g_2, \phi_k^1 \rangle| \geq c > 0$ for some $c > 0$ and for large $|k|$. Then, if in addition $|P_K(i\omega_k)|$ were bounded away from zero at infinity, we could choose $\beta = 0$ in Theorem 17, and the robust output regulation problem on W_γ would be solvable for any $\gamma > \frac{1}{2}$.

However, unbounded input operators G_2 of the error feedback controller are not covered by the present theory. Such classes of controllers would in particular require additional standing assumptions to ensure that the controller and the closed-loop systems have well-defined states [26].

We have assumed that all but a finite number of the eigenvalues of S have a uniform gap. However, this particular assumption is not crucial to the construction of the observer-based controller, and can be relaxed at the cost of added complexity in the results. In particular the same methods can be applied if the infinite part of the spectrum of S consists of simple eigenvalues having no finite accumulation points, and if we have a polynomial bound for the rate at which the neighboring eigenvalues approach each other as they approach infinity. In the stabilization of the internal model $S + B_1K_2$ we would only need to modify the conditions affecting the convergence of the series in (22). If the eigenvalues of S do not have a uniform gap, it is clear that we need to add a requirement that the terms $|\langle B_2, \phi_k^1\rangle|$ approach zero fast enough as $|k| \to \infty$. Since we have seen in the proof of Theorem 15 that for large $|k|$ these terms behave like $|\langle g_2, \phi_k^1\rangle| \cdot |P_K(i\omega_k)|$, the condition could be expressed as a requirement that the asymptotic decay of these terms is sufficiently fast.

On the other hand, if we have knowledge that the eigenvalues of S approach infinity faster than at a constant rate as $|k| \to \infty$, we can use any known bound for this rate to improve the results in Theorem 17. More precisely, if there exist constants $\eta > 1$ and $c > 0$ such that for large $|k|$ we have

$$|\omega_k| \geq c|k|^\eta, \tag{27}$$

it is then easy to show that our controller in fact solves the robust output regulation problem on W_α for $\alpha = \gamma/\eta$. In this situation the infinite part of the spectrum of S still has a uniform gap, and the controller can constructed exactly as described in the earlier sections. Therefore we only need to verify that condition (13) is satisfied. This, however, is easily done using (27), since for all $k \in \mathbb{Z}$ with large enough $|k|$ we now have

$$\frac{\|R(i\omega_k, A_e)\|^2}{(1+\omega_k^2)^\alpha} \leq \frac{M^2|k|^{2\gamma}}{(1+c^2|k|^{2\eta})^\alpha} \leq \frac{M^2}{c^{2\alpha}}\frac{|k|^{2\gamma}}{|k|^{2\alpha\eta}} = \frac{M^2}{c^{2\alpha}} < \infty.$$

In the next section we will study a concrete example on choosing the parameters of the controller to achieve robust output regulation and disturbance rejection of signals generated by an infinite-dimensional exosystem.

6 Robust Controller for a Scalar System

We conclude the paper with a detailed example concerning the construction of robust controllers. To illustrate the use of the theoretic results we consider the problem of steering the output of a scalar system to the reference signals generated by an infinite-dimensional exosystem. We use the methods presented in the earlier sections to strongly stabilize the closed-loop system. We will then see that the use of these particular methods allows us to easily determine the values of $\alpha \geq 0$ for which the controller solves the robust output regulation problem on W_α.

We consider the robust control of a scalar system of form

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t), \qquad x(0) = x_0 \in \mathbb{C} \\ y(t) &= cx(t) + du(t) \end{aligned}$$

on the space $X = \mathbb{C}$. We consider the single-input single-output case and assume $b \neq 0$ and $c \neq 0$. Since the infinite-dimensional exosystem want to consider has $0 \in \sigma_p(S)$, we must also require $a \neq 0$.

As the signal generator we choose an infinite-dimensional exosystem

$$\begin{aligned} \dot{v}(t) &= Sv(t), \qquad v(0) = v_0 \in W \\ y_{ref}(t) &= F_r v(t) \end{aligned}$$

capable of generating the step signal depicted in Figure 1. This signal is of form

$$y_{ref}(t) = t + y_0(t),$$

where $y_0(\cdot)$ is the triangle signal from Figure 2.

On the interval $[0, 2\pi]$ the function $y_0(\cdot)$ can be defined as

$$y_0(t) = \begin{cases} t & 0 \leq t < \pi \\ -t + \pi & \pi \leq t < 2\pi \end{cases}$$

and its Fourier series representation is given by

$$y_0(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt}, \qquad a_k = \frac{1}{2\pi}\int_0^{2\pi} y_0(t)e^{ikt}dt = \begin{cases} \frac{\pi}{2} & k = 0 \\ -\frac{(e^{i\pi k}-1)^2}{2\pi k^2} & k \neq 0. \end{cases}$$

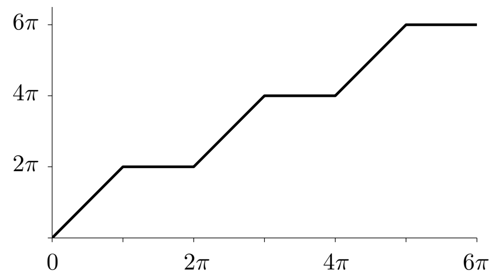

Figure 1: The signal $y_0(t) + t$.

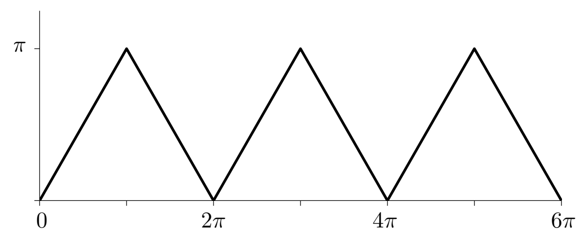

Figure 2: The triangle signal.

This shows that $y_0(\cdot)$ can be generated by a diagonal exosystem with frequencies $i\omega_k = ik$ for $k \in \mathbb{Z}$. Also, since $a_k = 0$ for all even $k \neq 0$, we could have also decided to leave out the corresponding frequencies. We have

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\beta |a_k|^2 = \sum_{k\in\mathbb{Z}} (1+k^2)^\beta \frac{|e^{i\pi k}-1|^4}{4\pi^2 k^4} < \infty$$

if and only if $\beta < \frac{3}{2}$, and thus $y_0(\cdot) \in H^\beta_{per}(0, 2\pi)$ precisely if $\frac{1}{2} < \beta < \frac{3}{2}$.

The signal t can be generated using a single Jordan block associated to an eigenvalue $i\omega_0 = 0$. We therefore choose the state space of our exosystem as

$$W = \overline{\operatorname{span}}\{\phi_0^1, \phi_0^2, \{\phi_k\}_{k\in\mathbb{Z}\setminus\{0\}}\}$$

and the system operator S as

$$S = \langle \cdot, \phi_0^2 \rangle \phi_0^1 + \sum_{k\in\mathbb{Z}\setminus\{0\}} ik \langle \cdot, \phi_k \rangle \phi_k, \qquad \mathcal{D}(S) = \Big\{ v \in W \;\Big|\; \sum_{k\in\mathbb{Z}\setminus\{0\}} k^2 |\langle v, \phi_k \rangle|^2 < \infty \Big\}.$$

A direct computation shows that the reference signals generated by the exosystem are of the form

$$F_r v(t) = \langle v_0, \phi_0^1 \rangle F_r \phi_0^1 + \langle v_0, \phi_0^2 \rangle (t F_r \phi_0^1 + F_r \phi_0^2) + \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{ikt} \langle v_0, \phi_k \rangle F_r \phi_k,$$

and that the reference signal in Figure 1 can be generated by choosing $F_r \in \mathcal{L}(W, \mathbb{C})$ and $v_0 \in W$ as

$$F_r\phi_0^1 = 1, \quad F_r\phi_0^2 = 0, \qquad F_r\phi_k = \frac{1}{k}, \quad \forall k \neq 0$$
$$\langle v_0, \phi_0^1 \rangle = \frac{\pi}{2}, \quad \langle v_0, \phi_0^2 \rangle = 1, \quad \langle v_0, \phi_k \rangle = -\frac{(e^{ik\pi}-1)^2}{2\pi k} \quad \forall k \neq 0.$$

For this choice of the initial state we clearly have $v_0 \in W_\alpha$ for all $\alpha < \frac{1}{2}$. Using Theorem 6 we see that since $y_0(\cdot) \in H^\beta_{per}(0, 2\pi)$ for $\beta < \frac{3}{2}$, it would have been possible to choose the parameters of the exosystem in such a way that $v_0 \in W_\alpha$ for $\alpha < 1$. In this example it would not have been possible to achieve higher α without losing the property

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} |F_r \phi_k^l|^2 < \infty.$$

Similarly as in Example 7 it is easy to see that if $\gamma \geq 1$ then this exosystem is capable of generating all reference and disturbance signals of the form

$$y_{ref}(t) = y_1 t + y_0(t),$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^\gamma_{per}(0, 2\pi)$ with appropriate choices of the initial states $v_0 \in W_{\gamma-1}$.

The operator S has eigenvalues $i\omega_k = ik$ for $k \in \mathbb{Z}$ and the only nontrivial Jordan block in the exosystem is the 2×2-block associated to $i\omega_0 = 0$. The transfer function of our plant is given by

$$P(\lambda) = \frac{cb}{\lambda - a} + d$$

for all $\lambda \neq a$. We assume the parameters of the plant are such that $P(i\omega_k) \neq 0$ for all $k \in \mathbb{Z}$. In particular this requires that $ad \neq bc$.

For these choices of parameters the system and the signal generator satisfy the conditions stated in Assumption 10. We can therefore use the method presented in Section 5 to construct a controller solving the robust output regulation problem.

6.1 Choosing the Parameters of the Controller

We can now choose the first parameters in the operators $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the controller. As the stabilizing feedback and output injection of the pairs (A, B) and (C, A), respectively, we choose

$$\begin{aligned} K_{11} &= -\frac{a+1}{b}, &\quad \Leftrightarrow \quad & A + BK_{11} = -1 \\ L &= -\frac{a+1}{c}, &\quad \Leftrightarrow \quad & A + LC = -1. \end{aligned}$$

The values of the transfer function $P_K(\cdot)$ of the stabilized plant at the eigenvalues $i\omega_k = ik$ of the exosystem are then given by

$$P_K(i\omega_k) = \frac{bc - (a+1)d}{ik+1} + d = \frac{bc + (ik - a)d}{ik+1}.$$

We choose the parameter $G_2 = g_2 \in W$ of the controller as

$$g_2 = \phi_0^2 + \sum_{k\neq 0} \frac{1}{|k|^{2/3}} \phi_k.$$

This operator clearly satisfies the requirement that $\langle g_2, \phi_k^{n_k} \rangle \neq 0$ for all $k \in \mathbb{Z}$.

By Lemma 14 the Sylvester equation $SH = H(A + BK_{11}) + G_2(C + DK_{11})$ has a unique solution $H_{e1} \in \mathcal{L}(\mathbb{C}, W)$ given by

$$H_{e1} = H_0^1 \phi_0^1 + H_0^2 \phi_0^2 + \sum_{k\neq 0} H_k \phi_k,$$

where

$$\begin{aligned} H_0 = \begin{pmatrix} H_0^1 \\ H_0^2 \end{pmatrix} &= \sum_{l=1}^{2} (-1)^{l-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{l-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (C + DK_{11}) R(0, A + BK_{11})^l \\ &= (C + DK_{11}) \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{0 - (-1)} + (-1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\frac{1}{0 - (-1)} \right)^2 \right] \\ &= \left(c - \frac{(a+1)d}{b} \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

and

$$H_k = \langle g_2, \phi_k \rangle (C + DK_{11}) R(i\omega_k, A + BK_{11}) = \left(c - \frac{(a+1)d}{b} \right) \frac{1}{|k|^{2/3}} \cdot \frac{1}{ik+1}$$

for all $k \neq 0$. As was also implied by Lemma 14, we clearly have $H_{e1} \in \mathcal{D}(S)$.

6.2 Solvability of the Robust Output Regulation Problem

We will first estimate the asymptotic behavior of $\langle g_2, \phi_k \rangle$ and $P_K(ik)$ to be able to determine the scale spaces W_α on which it is possible to solve the robust output regulation problem. Using the formulas for these terms we see that for all $k \neq 0$ with a large enough $|k|$ we have

$$|\langle g_2, \phi_k \rangle| \cdot |P_K(ik)| = \frac{1}{|k|^{2/3}} \left| \frac{bc-(a+1)d}{ik+1} + d \right| \geq \frac{1}{|k|^{2/3}} \left(|d| - \frac{|bc-(a+1)d|}{|k|} \right).$$

This immediately implies that if $d \neq 0$, then the conditions of Theorem 17 can be satisfied for $\beta = 2/3$ and if $d = 0$, we can choose $\beta = 5/3$. The results presented in Section 5 thus show for any $\alpha > \beta + \frac{1}{2}$ the parameters of the controller can be chosen in such a way that the robust output regulation problem is solved on W_α. In particular this suggests that, even if $d \neq 0$ (in which case $\alpha > \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$) Theorem 17 does not guarantee that we can choose the parameters of the controller to asymptotically track the step signal in Figure 1, since this reference signal is generated with an initial state $v_0 \notin W_{1/2}$ of the exosystem.

In the following we will assume $d \neq 0$ and stabilize the closed-loop system in such a way that the robust output regulation problem is solved on W_α for $\alpha = 4/3$. Due to the properties of the exosystem the controller will then be able to steer the output of the scalar system to any reference signal

$$y_{ref}(t) = y_1 t + y_0(t),$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^{7/3}_{per}(0, 2\pi)$.

6.3 Stabilization of the Internal Model

The decomposition of the internal model in the controller into exponentially stabilizable and diagonal parts can be done by choosing $I_S = \{0\}$. We then have

$$S_f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B_f = \left(c - \frac{(a+1)d}{b} \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} b + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d = \begin{pmatrix} ad-bc+d \\ -(ad-bc) \end{pmatrix},$$

and pair (S_f, B_f) is stabilizable, since we have previously assumed $ad \neq bc$. The infinite-dimensional diagonal part of the internal model is given by

$$S_i = \sum_{k \neq 0} ik \langle \cdot, \phi_k \rangle \phi_k, \qquad \mathcal{D}(S_i) = \left\{ v \in W \;\middle|\; \sum_{k \neq 0} k^2 |\langle v, \phi_k \rangle|^2 < \infty \right\}$$

$$B_i = \sum_{k \neq 0} \langle g_2, \phi_k \rangle P_K(i\omega_k) \phi_k = \sum_{k \neq 0} \frac{bc + (ik-a)d}{ik+1} \cdot \frac{1}{|k|^{2/3}} \cdot \phi_k.$$

We will first stabilize the pair (S_f, B_f) with a feedback K_{f1}. For this purpose we will choose

$$K_{f1} = \frac{1}{(ad-bc)^2} \big(2(ad-bc),\, 5(ad-bc) + 2d \big).$$

It is well-known that since $\sigma(S_f) = \{0\}$, a value $\lambda \neq 0$ is an eigenvalue of $S_f + B_f K_{f1}$ if and only if $K_{f1} R(\lambda, S_f) B_f = 1$. A direct computation shows that

$$1 - K_{f1} R(\lambda, S_f) B_f = 1 - \frac{1}{\lambda^2} K_{f1} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} B_f = 1 + \frac{3\lambda + 2}{\lambda^2} = \frac{\lambda^2 + 3\lambda + 2}{\lambda^2} = \frac{(\lambda+2)(\lambda+1)}{\lambda^2},$$

and therefore $\sigma(S_f + B_f K_{f1}) = \{-1, -2\}$. This concludes that $S_f + B_f K_{f1}$ is exponentially stable.

From the proof of Theorem 15 we have that for $k \neq 0$

$$\langle B_2, \phi_k \rangle = (K_{f1} R(ik, S_f + B_f K_{f1}) B_f + 1) \langle B_i, \phi_k \rangle = \frac{\langle g_2, \phi_k \rangle P_K(ik)}{1 - K_{f1} R(ik, S_f) B_f}$$

$$= -\frac{1}{|k|^{2/3}} \cdot \frac{bc + (ik-a)d}{ik+1} \cdot \frac{k^2}{(ik+2)(ik+1)} = -\frac{|k|^{4/3}(bc + (ik-a)d)}{(ik+1)^2(ik+2)}$$

As in the proof of Theorem 15, we choose the feedback operator $K_i = \langle \cdot, h \rangle$ as

$$h = \sum_{k \neq 0} h_k \phi_k,$$

where we choose the parameter $\gamma = 4/3$ and

$$\overline{h_k} = \frac{1}{|k|^{4/3} \langle B_2, \phi_k \rangle} \prod_{l \neq 0,k} \left(1 + i \frac{1}{|l|^{4/3}(\omega_l - \omega_k)} \right) = -\frac{(ik+1)^2(ik+2)}{|k|^{8/3}(bc + (ik-a)d)} \prod_{l \neq 0,k} \left(1 + i \frac{1}{|l|^{4/3}(l-k)} \right).$$

For these choices of the parameters Theorem 15 shows us that the resolvent operator of the closed-loop system has asymptotic behavior

$$\|R(ik, A_e)\| = \mathcal{O}(|k|^{4/3}), \tag{28}$$

and by Theorem 17 the dynamic error feedback controller solves the robust output regulation problem on W_α with $\alpha = 4/3$.

6.4 Robustness Properties of the Controller

The results presented in Section 5 now imply that the feedback control law we have constructed is capable of tracking reference signals and rejecting disturbance signals despite perturbations to the parameters A, B, C, D, E, and F of the system as long as the strong stability of the closed-loop system as well as the solvability of the associated Sylvester equation are preserved. In particular this allows such changes and uncertainties in the parameters a, b, c, and d of the plant for which the stability of the closed-loop system is preserved, the condition $\sigma(A_e) \cap \sigma(S) = \varnothing$ remains valid, and for which the resolvent operator of the closed-loop system still has the asymptotic behavior (28).

As we already mentioned, the chosen controller is capable of steering the output of the scalar system to any reference signal of the form

$$y_{ref}(t) = y_1 t + y_0(t),$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^{7/3}_{per}(0, 2\pi)$. The robustness of the controller with respect to perturbations to the operator F_r further enlarges this class of signals. Since this operator (and the operator E) do not appear in the system operator A_e of the closed-loop system, they do not affect the stability of the closed-loop system or the condition (13) guaranteeing the solvability of the Sylvester equation. This means that the operator F_r can be replaced with an arbitrary operator F_r' as long as it satisfies the condition $(F_r' \phi_k) \in \ell^2(\mathbb{C})$ imposed on the parameters of the infinite-dimensional exosystem. Therefore a similar reasoning as in Example 7 shows us that our control law is actually capable of tracking any reference signal of the above form with $y_0(\cdot) \in H^\gamma_{per}(0, 2\pi)$ for $\gamma > \frac{7}{3} - \frac{1}{2} = \frac{11}{6}$.

The same conclusion also applies to the perturbations to the operator $E = 0$. This means that even though we were initially not interested in rejecting disturbance signals affecting the state of the plant, our controller is still capable of handling any such signals generated by the infinite-dimensional exosystem with an output operator $E \in \mathcal{L}(W, \mathbb{C})$ satisfying $(E\phi_k)_k \in \ell^2(\mathbb{C})$.

7 Conclusions

In this paper we have studied the robust output regulation problem for distributed parameter systems and infinite-dimensional exosystems capable of generating polynomially increasing signals. In particular we solved the problem of strongly stabilizing the closed-loop consisting of the plant and the robust observer-based error feedback controller. This problem had been left open in a recent paper [16], where the internal model principle characterizing the controllers solving the robust output regulation problem was extended to distributed parameter systems.

We also analyzed in detail the classes of signals generated by an infinite-dimensional nondiagonal exosystem. In particular we showed that for periodic signals there is a direct connection between the choice of the initial state of the exosystem in a scale space W_α and the level of smoothness of the generated signals. This connection could be further extended to show that knowledge on the smoothness properties of the exogeneous signals can be used to weaken the conditions required for the solvability of the robust output regulation problem.

Further research topics include allowing the input and output operators of the systems to be unbounded and considering signal generators that do not necessarily have pure point spectrum. In particular, unbounded input operator $\mathcal{G}_2$ in the dynamic error feedback controller could potentially allow tracking and rejection of signals with lower levels of smoothness, as was discussed in Section 5.4.

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