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INTERNAL MODEL THEORY FOR DISTRIBUTED PARAMETER SYSTEMS*

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Abstract. In this paper we consider robust output regulation of distributed parameter systems and the internal model principle. The main purpose is to generalize the internal model principle by Francis and Wonham for infinite-dimensional systems and clarify the relationships between different generalizations of the internal model. We also construct a signal generator capable of generating infinite-dimensional polynomially increasing signals.

Key words. infinite-dimensional systems, robust regulation, internal model

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1. Introduction.

Distributed parameter systems are used to model various types of systems including heat and diffusion processes, vibrations, and delay systems. Robustness of a controller is an essential property because of the unavoidable inaccuracy of the mathematical model compared to the real world system. Regulation and robust regulation of distributed parameter systems have been studied extensively during the last 30 years. In his Ph.D. thesis Bhat [2] extended structural stability results of Francis and Wonham [6] mainly to time-delay systems. This theory was later partly generalized by Immonen [11] for distributed parameter systems with infinite-dimensional signal generators. Also the robust regulation theory by Davison [4] was extended to infinite-dimensional systems by Pohjolainen [16]. These results were later extended to more general classes of reference and disturbance signals by Hämäläinen and Pohjolainen [8] and for well-posed systems by Rebarber and Weiss [18]. Regulation theory without the robustness aspect has been studied by Schumacher [19] and Byrnes et al. [3].

The robust regulation problem consists of two problems, which can be studied separately, one of *robust stabilization* and one of *robust regulation*, as defined in this paper. This can be seen directly from the decomposition of the state of the closed-loop system

$$x_e(t) = T_{A_e}(t)(x_{e0} - \Sigma v_0) + \Sigma v(t), \tag{1.1}$$

where $T_{A_e}(t)$ is the semigroup generated by the system operator of the closed-loop system, A_e, $v(t)$ is the state of the exosystem $\dot{v} = Sv$ and Σ is the solution of the associated Sylvester equation

$$\Sigma S = A_e \Sigma + B_e. \tag{1.2}$$

The *robust stabilization* part of the output regulation problem is related to the first term of (1.1). This part consists of choosing controller parameters such that the closed-loop system is stable and this stability is preserved under a suitable class of

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perturbations. Whenever this is achieved, the first term in (1.1) decays with time, and the state asymptotically approaches the behavior of the second term $\Sigma v(t)$. This can be seen as a dynamic steady state for the closed-loop system. The second part of the robust output regulation problem, the *robust regulation* part, is related to this second term $\Sigma v(t)$ of (1.1). It consists of choosing the controller parameters such that the perturbed dynamic steady state still gives the desired output.

In finite-dimensional control theory the famous internal model principle of Francis and Wonham [6] states that a stabilizing feedback controller solves the robust output regulation problem if and only if it contains a *p-copy internal model* of the exosystem. Here p refers to the dimension of the output space and the definition of this p-copy internal model is that *the minimal polynomial of S divides at least p invariant factors of* $\mathcal{G}_1$, where S and $\mathcal{G}_1$ are the system operators of the signal generator and the controller, respectively.

The internal model principle has also been approached using properties of certain Sylvester equations. This is easy to understand, because in the robust regulation part of the robust output regulation problem we want to choose the controller parameters in such a way that the solution of (1.2) has certain properties. Immonen [11] defined *internal model structure* (*IMS*) in such a way that if the controller has IMS, then the Sylvester equation (1.2) with the perturbed system's parameters still leads to the correct output at the dynamic steady state of the closed-loop system.

The definition of the IMS was given in terms of certain Sylvester equations and because of this it is often hard to check whether a controller has this property. Hämäläinen and Pohjolainen [9] later found sufficient conditions for a controller to have IMS. The origin of these sufficient conditions is in the proof of the internal model principle in [7] and they are given in terms of the controller's parameters on the spectrum of the exosystem. Although these conditions were also called the *internal model*, they were used only as sufficient conditions for the IMS and it was not discussed whether or not they are also necessary. We refer to these conditions as $\mathcal{G}$-conditions. In this paper we extend these conditions for more general signal generators.

All of these concepts, the p-copy internal model of Francis and Wonham, the IMS of Immonen, and the $\mathcal{G}$-conditions of Hämäläinen and Pohjolainen, are related to the robust output regulation problem. It is of course natural that in the transition from finite-dimensional to infinite-dimensional systems, also the concept of internal model has been redefined. The main reason for this is that the minimal polynomials and invariant factors used in the original definition by Francis and Wonham are unavailable for infinite-dimensional operators. However, for example, in the case of the three concepts discussed here, it is hard to see how the different definitions are related.

The purpose of this paper is to show that the internal model principle for the p-copy internal model can be formulated and proved for distributed parameter systems with infinite-dimensional exosystems. We also give precise conditions for the equivalence between the p-copy internal model, IMS, and the $\mathcal{G}$-conditions.

We first generalize the p-copy internal model of Francis and Wonham for distributed parameter systems and infinite-dimensional exosystems. This has not been done previously even when the signal generators considered have been finite-dimensional. The original definition based on minimal polynomials and invariant factors cannot be generalized for infinite-dimensional operators, but there exists an equivalent definition using Jordan canonical forms. More precisely, in the finite-dimensional case, a controller contains a p-copy internal model of the signal generator if *whenever $s \in \sigma(S)$ is an eigenvalue of S such that $d(s)$ is the dimension of the* *largest Jordan block associated to s, then $s \in \sigma(\mathcal{G}_1)$ and $\mathcal{G}_1$ has at least p Jordan blocks of dimension greater than or equal to $d(s)$ associated to s* [6]. Even though the Jordan canonical form is unavailable for infinite-dimensional operators, it is still possible to generalize this definition.

We prove the internal model principle for the p-copy internal model by proving that under suitable assumptions the concepts mentioned earlier, the IMS, the $\mathcal{G}$-conditions, and the generalization of the p-copy internal model given in this paper are all equivalent. Since the internal model structure of the controller is equivalent to the robust regulation property, this proves that a controller is robustly regulating if and only if it contains a p-copy internal model of the exosystem.

The extension of the internal model principle is by itself an important extension of finite-dimensional control theory to distributed parameter systems. Furthermore, the equivalence of the concepts of the IMS, the $\mathcal{G}$-conditions, and the p-copy internal model establishes several additional new results.

Perhaps the most important one of these new results is the extension of the results of [17, 9] which state that a controller satisfying the $\mathcal{G}$-conditions is robustly regulating. Our results prove the converse argument, i.e., that a robustly regulating controller necessarily satisfies the $\mathcal{G}$-conditions. This is a new result which shows that although the $\mathcal{G}$-conditions were introduced as purely sufficient conditions for the IMS, they can in fact be considered as an alternative definition of the internal model. The importance of this result comes from the fact that of the three considered definitions of the internal model, the $\mathcal{G}$-conditions have the following advantages over the other two: They are much more concrete than the IMS but require less assumptions than the p-copy internal model.

Most of the theory developed for robust output regulation for distributed parameter systems considers only reference and disturbance signals which are generated by finite-dimensional exosystems. More general classes of signals to be regulated can be achieved if also the exosystem is allowed to be infinite-dimensional. This has been studied recently in [11, 9]. In these references the signal generator is constructed in such a way that it is only possible to generate bounded uniformly continuous signals. These signals are indeed very general in the context of robust regulation, where the properties of the system often dictate the minimum requirements of the signals one can hope to track [12]. Still, this type of exosystem has the drawback that it can only generate bounded signals. In many engineering applications it is necessary to generate signals which have a growth rate of t or t^n for some $n \in \mathbb{N}$. One commonly used signal of this type is the ramp signal. In this paper we extend the signal generator used in [9] so that it can generate polynomially increasing signals. This is done by defining an operator consisting of an infinite number of Jordan blocks.

Since we are using a more general signal generator, the results of this paper also extend the general robust regulation theory presented in [17] and [9], where the signal generator was assumed to be finite-dimensional and infinite-dimensional with a diagonal system operator, respectively.

It turns out that the internal model principle actually depends only on the robust regulation part of the robust output regulation problem. This can also be seen from the statement of the internal model principle, where the controller is *assumed* to be stabilizing. Because of this, we do not consider the stabilization of the closed-loop system. Of course the solution of the robust output regulation problem depends also on the stabilization part, and in the case of an infinite-dimensional signal generator the stabilization of the closed-loop system can be problematic. In [9] it is shown how the closed-loop system can be stabilized if the signal generator has a diagonal system operator.

One important result shown in [9] was that the smoother the reference and disturbance signals are, the weaker the assumptions needed for the solvability of the output regulation problem. In [9] the conditions for the smoothness of the signals were imposed on the operators of the exosystem. In this paper we show how these conditions can be imposed on the initial value of the exosystem instead. This is done by allowing the solution Σ of the regulator equations to be an unbounded operator. This approach has the advantage that it gives more concrete correspondence between the level of smoothness of the signals and the strictness of the conditions for the solvability of the output regulation problem.

We use infinite-dimensional Sylvester equations with unbounded operators. A fair amount of theory exists on the properties and the solvability of this type of equation [20, 15, 1]. However, since one of our unbounded operators is of particular form, conditions for the solvability of these equations are derived directly.

To illustrate the applicability of our results we present a concrete example, where we design an observer-based robust controller for a finite-dimensional system with an exosystem capable of generating infinite-dimensional linearly increasing signals.

In section 2 we introduce the notation, construct the exosystem capable of generating polynomially increasing infinite-dimensional signals, and state the basic assumptions on the system and the controller. In section 3 we present the output regulation problem and show that the solvability of this problem can be characterized by the solvability of certain constrained Sylvester equations. These results are used in section 4, where we formulate the robust output regulation problem and divide it into two parts, the robust stabilization part and the robust regulation part. In this section we also show that the IMS of Immonen is equivalent to the robust regulation property of the controller. In section 5 we show that the IMS is equivalent to the $\mathcal{G}$-conditions. The main result of the section is Theorem 5.2. In section 6 we generalize the p-copy internal model, show that under suitable assumptions this property is equivalent to the $\mathcal{G}$-conditions, and combine the results in the previous sections to prove the extension of the internal model principle. The main results of the section are Theorems 6.2 and 6.9. An example of application of the theory is presented in section 7. Section 8 contains concluding remarks.

2. Notation and definitions.

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$, and $\mathcal{R}(A)$ the domain, kernel, and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$, and $\rho(A)$ denote the spectrum, the point spectrum, and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$.

Let X, Y, U be Banach spaces and let W be a Hilbert space. Let $(i\omega_k)_{k\in\mathbb{Z}} \in i\mathbb{R}$ be a sequence with no finite accumulation points and assume that for all $k \in \mathbb{Z}$ the set $I_k = \left\{ j \in \mathbb{Z} \;\middle|\; \omega_j = \omega_k \right\}$ is finite. Let $\left\{ \phi_k^l \;\middle|\; k \in \mathbb{Z},\, l = 1, \dots, n_k \right\} \subset W$, where $n_k < \infty$ for all $k \in \mathbb{Z}$, be an orthonormal basis of W, i.e., $W = \overline{\operatorname{span}}\{\phi_k^l\}_{kl}$ and $\langle \phi_k^l, \phi_n^m \rangle = \delta_{kn}\delta_{lm}$. Furthermore, assume that there exists $N_d \in \mathbb{N}$ such that $n_k \le N_d$ for all $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$ define an operator $S_k \in \mathcal{L}(W)$ such that

$$S_k = i\omega_k \langle \cdot, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle \cdot, \phi_k^l \rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1} \right).$$

The operator S_k then satisfies $(i\omega_k I - S_k)\phi_k^1 = 0$ and $(S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1}$ for all $l \in \{2, \dots, n_k\}$ and thus corresponds to a single Jordan block associated to an eigenvalue $i\omega_k$. We define the operator $S : \mathcal{D}(S) \subset W \to W$ as

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \qquad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty \right\}.$$

For $k \in \mathbb{Z}$ define $d_k = \max\{ n_l \mid l \in \mathbb{Z},\ \omega_l = \omega_k \}$. Since the operators S_k can be seen as Jordan blocks of S, this value corresponds to the dimension of the largest Jordan block associated to an eigenvalue $i\omega_k \in \sigma(S)$. The spectrum of the operator S satisfies $\sigma(S) = \sigma_p(S) = \{i\omega_k\}_k$ and S generates a C_0-group $T_S(t)$ on W given by

$$T_S(t)v = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v, \phi_k^l \rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!} \phi_k^j, \qquad v \in W,\ t \in \mathbb{R}.$$

For $k \in \mathbb{Z}$ denote by P_k the orthogonal projection $P_k = \sum_{l=1}^{n_k} \langle \cdot, \phi_k^l \rangle \phi_k^l$ onto the finite-dimensional subspace $W_k = \operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ of W.

We consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ev, \quad x(0) = x_0 \in X, \\ e &= Cx + Du + Fv, \end{aligned}$$

where $x(t) \in X$ is the state of the system, $e(t) \in Y$ is the regulation error, and $u(t) \in U$ is the input for $t \geq 0$. We assume that $A : \mathcal{D}(A) \subset X \to X$ generates a C_0-semigroup on X, and the other operators are bounded, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$, $E \in \mathcal{L}(W, X)$, and $F \in \mathcal{L}(W, Y)$. For $\lambda \in \rho(A)$ the transfer function of the plant is $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$ and we assume that $\sigma(A) \cap \sigma(S) = \varnothing$. Here $v(t) \in W$ is the state of the exosystem

$$\dot{v} = Sv, \qquad v(0) = v_0 \in W$$

on W. The dynamic feedback controller on a Banach space Z is of the form

$$\begin{aligned} \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 e, \quad z(0) = z_0 \in Z, \\ u &= Kz, \end{aligned}$$

where $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, and $K \in \mathcal{L}(Z, U)$. The closed-loop system on $X_e = X \times Z$ with state $x_e(t) = (x(t), z(t))^T$ is given by

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e v, \quad x_e(0) = x_{e0} = (x_0, z_0)^T, \\ e &= C_e x_e + D_e v, \end{aligned}$$

where $C_e = [C \quad DK]$, $D_e = F$,

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \quad \text{and} \quad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix}.$$

The operator $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ generates a C_0-semigroup $T_{A_e}(t)$ on X_e.

Since $\sigma(S) = \{i\omega_k\}_k \subset i\mathbb{R}$, we have $1 \in \rho(S)$. For $m \in \mathbb{N}_0$ the Sobolev space of order m associated to S is the Banach space $W^m = (\mathcal{D}(S^m), \|\cdot\|_m)$, with norm

$\|v\|_m = \|(S-I)^m v\|_W$ for $v \in \mathcal{D}(S^m)$ [5, sect. II.5]. With this definition we have $W^0 = W$ and $\|\cdot\|_0 = \|\cdot\|_W$. We also have that

$$\mathcal{D}(S^m) = \left\{ v \in W \mid \sum_{k\in\mathbb{Z}} \omega_k^{2m} \|P_k v\|^2 < \infty \right\}$$

and the function $h_m : W^m \to \mathbb{R}$ defined such that $h_m(v)^2 = \sum_{k\in\mathbb{Z}}(1+\omega_k^2)^m \|P_k v\|^2$ for all $v \in \mathcal{D}(S^m)$ is a norm which is equivalent to $\|\cdot\|_m$.

3. Output regulation. The output regulation problem on W^m (ORPm) is stated as follows.

PROBLEM 1 (Output regulation problem on W^m). *Let $m \in \mathbb{N}_0$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:*

- *The closed-loop system operator A_e generates a strongly stable C_0-semigroup on X_e.*
- *For all initial states $v_0 \in W^m$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.*

The problem statement contains two parts. The first requires the stabilization of the closed-loop system and the second that the regulation error goes to zero asymptotically. In this paper we are concerned only about the regulation part of the problem. To this end, we do not consider if and how the closed-loop system can be stabilized strongly but *assume* that it can be done. In [9] Hämäläinen and Pohjolainen show how and under what assumptions the closed-loop system can be stabilized if the exosystem has a diagonal system operator.

In this section we show that the solution of an associated Sylvester equation describes the asymptotic behavior of the closed-loop system and that the solvability of the output regulation problem can be characterized by the solvability of this equation with an additional *regulation constraint*. Together this Sylvester equation and the regulation constraint are called the *regulator equations*.

Let $m \in \mathbb{N}_0$ be fixed for the rest of the section. The next assumption gives conditions for the solvability of the regulator equations.

ASSUMPTION 1. *Assume that for every $k \in \mathbb{Z}$ and $l \in \{1, \ldots, n_k\}$ we have $B_e\phi_k^l \in \mathcal{R}(i\omega_k I - A_e)^{n_k - l + 1}$ and*

$$\sup_{\|x_e'\|\le 1} \sum_{k\in\mathbb{Z}} \frac{1}{(1+\omega_k^2)^m} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, x_e' \rangle \right|^2 < \infty,$$

where $x_e' \in X_e'$, the dual space of X_e.

Theorem 3.1 gives a characterization for the solvability of the output regulation problem.

THEOREM 3.1. *Assume $(\mathcal{G}_1, \mathcal{G}_2, K)$ are such that A_e generates a strongly stable C_0-semigroup on X_e and that Assumption 1 is satisfied. Then the following are equivalent:*

(a) *The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the output regulation problem on W^m.*

(b) *There exists a unique operator $\Sigma \in \mathcal{L}(W^m, X_e)$ such that $\Sigma(W^{m+1}) \subset \mathcal{D}(A_e)$ and*

$$\Sigma S = A_e \Sigma + B_e, \tag{3.1a}$$

$$0 = C_e \Sigma + D_e, \tag{3.1b}$$

where the equations are considered on W^{m+1}.

Theorem 3.1 shows the important result that the smoothness of the reference and disturbance signals has a direct effect on the conditions for the solvability of the output regulation problem. More precisely, if we want to regulate and reject signals which correspond to the initial states $v \in W^m$ of the exosystem for some m, it is sufficient for the solvability of the output regulation problem that Assumption 1 is satisfied for this m and that the regulator equations (3.1) have a solution which is in $\mathcal{L}(W^m, X_e)$.

The proof of Theorem 3.1 is based on the following two lemmas.

LEMMA 3.2. *If Assumption* 1 *is satisfied, the Sylvester equation* $\Sigma S = A_e\Sigma + B_e$ *on* W^{m+1} *has a unique solution* $\Sigma \in \mathcal{L}(W^m, X_e)$ *such that for all* $v \in W^m$

$$\Sigma v = \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v, \phi_k^l\rangle \sum_{j=1}^{l}(-1)^{l-j}R(i\omega_k, A_e)^{l+1-j}B_e\phi_k^j.$$

Proof. For brevity we denote $R_k = R(i\omega_k, A_e)$. Since the function h_m in section 2 defines a norm equivalent to $\|\cdot\|_m$, there exists $C > 0$ such that $h_m(v) \le C\|v\|_m$. Using the Cauchy–Schwarz inequality twice we see that for all $v \in W^m$

$$\begin{aligned}\|\Sigma v\| &= \sup_{\|x_e'\|\le 1}|\langle \Sigma v, x_e'\rangle| \le \sup_{\|x_e'\|\le 1}\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}|\langle v, \phi_k^l\rangle| \left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle\right| \\ &\le \sup_{\|x_e'\|\le 1}\sum_{k\in\mathbb{Z}}\|P_k v\|\frac{(1+\omega_k^2)^{\frac{m}{2}}}{(1+\omega_k^2)^{\frac{m}{2}}}\cdot\left(\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle\right|^2\right)^{\frac{1}{2}} \\ &\le C\|v\|_m\cdot\left(\sup_{\|x_e'\|\le 1}\sum_{k\in\mathbb{Z}}\frac{1}{(1+\omega_k^2)^m}\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle\right|^2\right)^{\frac{1}{2}}\end{aligned}$$

and thus $\Sigma \in \mathcal{L}(W^m, X_e)$. Let $s \in \rho(A_e)$ and $v \in W^{m+1}$. Denote $R_s = R(s, A_e)$. Now

$$R(s, A)\Sigma(S - sI)v = \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle (S - sI)v, \phi_k^l\rangle \sum_{j=1}^{l}(-1)^{l-j}R_s R_k^{l+1-j}B\phi_k^j.$$

Using the definition of S we see that the terms in the sum over $k \in \mathbb{Z}$ are equal to

$$\begin{aligned}&\sum_{l=1}^{n_k-1}\sum_{j=1}^{l}\left(\langle v, \phi_k^l\rangle(-1)^{l-j}(i\omega_k - s)R_s R_k^{l+1-j}B_e\phi_k^j + \langle v, \phi_k^{l+1}\rangle(-1)^{l-j}R_s R_k^{l+1-j}B_e\phi_k^j\right) \\ &+ \langle v, \phi_{n_k}^{n_k}\rangle\sum_{j=1}^{n_k}(-1)^{n_k-j}(i\omega_k - s)R_s R_k^{n_k+1-j}B_e\phi_k^j.\end{aligned}$$

Using the resolvent equation we see that this in turn is equal to

$$\sum_{j=1}^{n_k-1}\left(\sum_{l=j}^{n_k-1}\langle v,\phi_k^l\rangle(-1)^{l-j}R_sR_k^{l-j}B_e\phi_k^j+\sum_{l=j+1}^{n_k}\langle v,\phi_k^l\rangle(-1)^{l-j+1}R_sR_k^{l-j}B_e\phi_k^j\right)$$
$$-\sum_{j=1}^{n_k-1}\sum_{l=j}^{n_k-1}\langle v,\phi_k^l\rangle(-1)^{l-j}R_k^{l+1-j}B_e\phi_k^j-\langle v,\phi_k^{n_k}\rangle\sum_{j=1}^{n_k}(-1)^{n_k-j}R_k^{n_k+1-j}B_e\phi_k^j$$
$$+\langle v,\phi_k^{n_k}\rangle\sum_{j=1}^{n_k}(-1)^{n_k-j}R_sR_k^{n_k-j}B_e\phi_k^j$$
$$=-\sum_{l=1}^{n_k}\sum_{j=1}^{l}\langle v,\phi_k^l\rangle(-1)^{l-j}R_k^{l+1-j}B_e\phi_k^j+\sum_{j=1}^{n_k}\langle v,\phi_k^j\rangle R_sB_e\phi_k^j.$$

This implies

$$R(s,A_e)\Sigma(S-sI)v=\sum_{k\in\mathbb{Z}}\left(-\sum_{l=1}^{n_k}\sum_{j=1}^{l}\langle v,\phi_k^l\rangle(-1)^{l-j}R_k^{l+1-j}B_e\phi_k^j+\sum_{j=1}^{n_k}\langle v,\phi_k^j\rangle R_sB_e\phi_k^j\right)$$
$$=-\Sigma v+R(s,A_e)B_ev$$

or $\Sigma v=R(s,A_e)B_ev-R(s,A_e)\Sigma(S-sI)v$. This shows that $\Sigma(W^{m+1})\subset\mathcal{D}(A_e)$ and

$$(sI-A_e)\Sigma v=B_ev-\Sigma(S-sI)v.$$

This concludes that $\Sigma S=A_e\Sigma+B_e$ on W^{m+1}.

Finally, we will show that Σ is unique. Assume $\Sigma_1\in\mathcal{L}(W^m,X_e)$ such that $\Sigma_1(W^{m+1})\subset\mathcal{D}(A_e)$ and $\Sigma_1S=A_e\Sigma_1+B_e$ on W^{m+1}. Then for all $k\in\mathbb{Z}$ we have

$$B_e\phi_k^1=(i\omega_kI-A_e)\Sigma_1\phi_k^1,\quad B_e\phi_k^2=(i\omega_kI-A_e)\Sigma_1\phi_k^2+\Sigma_1\phi_k^1,\dots,$$
$$B_e\phi_k^{n_k}=(i\omega_kI-A_e)\Sigma_1\phi_k^{n_k}+\Sigma_1\phi_k^{n_k-1}$$

since $S\phi_k^1=i\omega_k\phi_k^1$ and $S\phi_k^l=i\omega_k\phi_k^l+\phi_k^{l-1}$ for $l\in\{2,\dots,n_k\}$. A direct computation shows that for all $l\in\{1,\dots,n_k\}$ we have $\Sigma_1\phi_k^l=\sum_{j=1}^{l}(-1)^{l-j}R(i\omega_k,A_e)^{l+1-j}B_e\phi_k^j$ and thus for all $v\in W^m$

$$\Sigma_1v=\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v,\phi_k^l\rangle\Sigma_1\phi_k^l=\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v,\phi_k^l\rangle\sum_{j=1}^{l}(-1)^{l-j}R(i\omega_k,A_e)^{l+1-j}B_e\phi_k^j=\Sigma v.$$

This concludes that $\Sigma_1=\Sigma$. ◻

The next lemma shows that the solution of the Sylvester equation (3.1a) describes the asymptotic behavior of the closed-loop system. This was proved in [9] for a diagonal operator S and a bounded operator Σ. We extend the proof to our case.

LEMMA 3.3. *Assume $\Sigma\in\mathcal{L}(W^m,X_e)$ is such that $\Sigma(W^{m+1})\subset\mathcal{D}(A_e)$ and*

$$\Sigma S=A_e\Sigma+B_e \tag{3.2}$$

on W^{m+1}. Then for all $t\geq0$ and for all initial values $x_{e0}\in X_e$ and $v_0\in W^m$ the regulation error $e(t)$ is given by

$$e(t)=C_eT_{A_e}(t)(x_{e0}-\Sigma v_0)+(C_e\Sigma+D_e)v(t),$$

where $T_{A_e}(t)$ is the C_0-semigroup generated by A_e on X_e, and $v(t)$ is the state of the exosystem, $v(t) = T_S(t)v_0$. Furthermore, if $T_{A_e}(t)$ is strongly stable, we have for the state of the closed-loop system $x_e(t)$ and the regulation error $e(t)$ that

$$\lim_{t\to\infty} \|x_e(t) - \Sigma v(t)\| = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \|e(t) - (C_e\Sigma + D_e)v(t)\| = 0. \tag{3.3}$$

Proof. We will first show that for $x_{e0} \in X_e$ and $v_0 \in W^m$ the state of the closed-loop system is given by

$$x_e(t) = T_{A_e}(t)(x_{e0} - \Sigma v_0) + \Sigma v(t) \qquad \forall t \geq 0, \tag{3.4}$$

where $v(t) = T_S(t)v_0$ is the state of the exosystem. Let $x_{e0} \in X_e$, $v_0 \in W^m$, and $t > 0$. The state of the closed-loop system is given by

$$x_e(t) = T_{A_e}(t)x_{e0} + \int_0^t T_{A_e}(t-s)B_e v(s)ds \qquad \forall t \geq 0.$$

Using (3.2) we see that for any $w \in W^{m+1}$

$$\begin{aligned} T_{A_e}(t-s)B_e T_S(s)w &= T_{A_e}(t-s)(\Sigma S - A_e\Sigma)T_S(s)w \\ &= -T_{A_e}(t-s)A_e\Sigma T_S(s)w + T_{A_e}(t-s)\Sigma S T_S(s)w = \frac{d}{ds}\left(T_{A_e}(t-s)\Sigma T_S(s)w\right) \end{aligned}$$

and thus

$$\int_0^t T_{A_e}(t-s)B_e T_S(s)w ds = \Sigma T_S(t)w - T_{A_e}(t)\Sigma w. \tag{3.5}$$

Since the operators on both sides of this equation are in $\mathcal{L}(W^m, X_e)$ and since W^{m+1} is dense in W^m, (3.5) also holds for any $w \in W^m$. This implies that

$$x_e(t) = T_{A_e}(t)x_{e0} + \Sigma T_S(t)v_0 - T_{A_e}(t)\Sigma v_0 = T_{A_e}(t)(x_{e0} - \Sigma v_0) + \Sigma v(t).$$

The regulation error is given by $e(t) = C_e x_e(t) + D_e v(t)$, and using (3.4) we get

$$e(t) = C_e x_e(t) + D_e v(t) = C_e T_{A_e}(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(t).$$

If the semigroup $T_{A_e}(t)$ is strongly stable, we also see that the limits in (3.3) are satisfied. ∎

Finally, we will present the proof of Theorem 3.1.

Proof of Theorem 3.1. We will first prove that (b) implies (a). Assume (b) holds and that there exists an operator $\Sigma \in \mathcal{L}(W^m, X_e)$ with $\Sigma(W^{m+1}) \subset \mathcal{D}(A_e)$ satisfying the regulator equations (3.1). Since $T_{A_e}(t)$ is strongly stable we have from Lemma 3.3 that for all initial values $x_{e0} \in X_e$ and $v_0 \in W^m$

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|e(t) - (C_e\Sigma + D_e)v(t)\| = 0$$

since $C_e\Sigma + D_e = 0$. Thus the controller solves the ORPm.

It remains to prove that (a) implies (b). Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the ORPm. From Lemma 3.2 we see that there exists a unique $\Sigma \in \mathcal{L}(W^m, X_e)$ with $\Sigma(W^{m+1}) \subset \mathcal{D}(A_e)$ satisfying (3.1a). Since the controller solves the ORPm, using Lemma 3.3 we have that for all $x_{e0} \in X_e$ and $v_0 \in W^m$

$$\|(C_e\Sigma + D_e)T_S(t)v_0\| \leq \|(C_e\Sigma + D_e)T_S(t)v_0 - e(t)\| + \|e(t)\| \overset{t\to\infty}{\longrightarrow} 0$$

and thus $\lim_{t\to\infty}\|(C_e\Sigma + D_e)T_S(t)v_0\| = 0$ for every $v_0 \in W^m$. Let $k \in \mathbb{Z}$ and $l \in \{1,\dots,n_k\}$. We have $T_S(t)\phi_k^l = e^{i\omega_k t}\sum_{j=1}^l \frac{t^{l-j}}{(l-j)!}\phi_k^j$ for all $t \geq 0$. Because of this,

$$0 = \lim_{t\to\infty}\|(C_e\Sigma + D_e)T_S(t)\phi_k^1\| = \lim_{t\to\infty}\|e^{i\omega_k t}(C_e\Sigma + D_e)\phi_k^1\| = \|(C_e\Sigma + D_e)\phi_k^1\|$$

and thus $(C_e\Sigma + D_e)\phi_k^1 = 0$. Using this we get

$$\begin{aligned}0 &= \lim_{t\to\infty}\|(C_e\Sigma + D_e)T_S(t)\phi_k^2\| = \lim_{t\to\infty}\left\|e^{i\omega_k t}\left(t(C_e\Sigma + D_e)\phi_k^1 + (C_e\Sigma + D_e)\phi_k^2\right)\right\|\\ &= \|(C_e\Sigma + D_e)\phi_k^2\|,\end{aligned}$$

which implies $(C_e\Sigma + D_e)\phi_k^2 = 0$. Continuing this we eventually get

$$\begin{aligned}0 &= \lim_{t\to\infty}\|(C_e\Sigma + D_e)T_S(t)\phi_k^{n_k}\|\\ &= \lim_{t\to\infty}\left\|\sum_{j=1}^{n_k-1}\frac{t^{n_k-j}}{(n_k-j)!}(C_e\Sigma + D_e)\phi_k^j + (C_e\Sigma + D_e)\phi_k^{n_k}\right\| = \|(C_e\Sigma + D_e)\phi_k^{n_k}\|\end{aligned}$$

and thus $(C_e\Sigma + D_e)\phi_k^{n_k} = 0$. This concludes that $(C_e\Sigma + D_e)\phi_k^l = 0$ for every $l \in \{1,\dots,n_k\}$, and since $k \in \mathbb{Z}$ was arbitrary and $\{\phi_k^l\}$ are a basis of W, we have that $C_e\Sigma + D_e = 0$. Thus also (3.1b) is satisfied. ∎

We conclude this section by showing that the convergence of the series in Assumption 1 is in fact necessary for the operator Σ in Lemma 3.2 to be in $\mathcal{L}(W_m, X_e)$.

LEMMA 3.4. *If the operator Σ defined in Lemma 3.2 is in $\mathcal{L}(W_m, X_e)$, then Assumption 1 is satisfied.*

Proof. Assume $\Sigma \in \mathcal{L}(W_m, X_e)$ and denote $R_k = R(i\omega_k, A_e)$ for brevity. There exists $M \geq 0$ such that for all $v \in W_m$ we have

$$\sup_{\|x_e'\|\leq 1}\left|\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v,\phi_k^l\rangle\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle\right| = \|\Sigma v\| \leq M\|v\|_m. \tag{3.6}$$

Let $x_e' \in X_e'$ be such that $\|x_e'\| \leq 1$ and let $N_1, N_2 \in \mathbb{N}$. Choose $v \in W_m$ such that if $-N_1 \leq k \leq N_2$, then

$$\langle v,\phi_k^l\rangle = \frac{1}{(1+\omega_k^2)^m}\sum_{j=1}^{l}(-1)^{l-j}\overline{\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle}$$

and $\langle v,\phi_k^l\rangle = 0$ otherwise. Since the function h_m in section 2 defines a norm equivalent to $\|\cdot\|_m$, there exists $C > 0$ such that $\|v\|_m \leq Ch_m(v)$. We have from (3.6) that

$$\begin{aligned}&\sum_{k=-N_1}^{N_2}\frac{1}{(1+\omega_k^2)^m}\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle\right|^2 \leq \|\Sigma v\| \leq M\|v\|_m\\ &\leq CMh_m(v) = CM\left(\sum_{k=-N_1}^{N_2}\frac{(1+\omega_k^2)^m}{(1+\omega_k^2)^{2m}}\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e'\rangle\right|^2\right)^{\frac{1}{2}}\end{aligned}$$

and thus

$$\sum_{k=-N_1}^{N_2} \frac{1}{(1+\omega_k^2)^m} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e' \rangle \right|^2 \leq C^2 M^2.$$

Since this holds for all N_1, N_2, we see by letting $N_1, N_2 \to \infty$ that

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1+\omega_k^2)^m} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e' \rangle \right|^2 \leq C^2 M^2,$$

and since $x_e' \in X_e'$ with $\|x_e'\| \leq 1$ was arbitrary, we see that the supremum of the left-hand side of the previous inequality over all $x_e' \in X_e'$ with $\|x_e'\| \leq 1$ must be less than or equal to $C^2 M^2$. This concludes the proof. □

4. Robust output regulation and internal model structure. In this section we consider robust output regulation. The robust output regulation problem on W^m (RORPm) is stated as follows.

PROBLEM 2 (Robust output regulation problem on W^m). *Let $m \in \mathbb{N}_0$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:*

- *The closed-loop system operator A_e generates a strongly stable C_0-semigroup on X_e.*
- *For all initial states $v_0 \in W^m$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.*
- *If the parameters (A, B, C, D, E, F) are perturbed to (A', B', C', D', E', F') in such a way that the new closed-loop system operator A_e' generates a strongly stable C_0-semigroup and Assumption 1 is satisfied, then $\lim_{t\to\infty} e(t) = 0$ for all initial states $v_0 \in W^m$ and $x_{e0} \in X_e$.*

The formula

$$e(t) = C_e T_{A_e}(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(t) \tag{4.1}$$

in Lemma 3.3 gives us valuable insight into the behavior of the regulation error. It shows that the regulation error $e(t)$ consists of two somewhat independent parts. The first term depends only on the behavior of the closed-loop system and not of the exosystem. This part goes to zero for all initial values x_{e0} and v_0 if the closed-loop system is strongly stable. On the other hand, the second term depends only on the behavior of the exosystem and not of the closed-loop system. This part goes to zero for all initial states v_0 of the exosystem if the regulation constraint $C_e\Sigma + D_e = 0$ is satisfied. The formula (4.1) is also independent of the operators A_e, B_e, C_e, and D_e in the sense that it holds for all such operators whenever Σ is a solution of the Sylvester equation $\Sigma S = A_e \Sigma + B_e$. This observation allows us to consider the robust output regulation as a problem consisting of two parts. For this we denote by A_e', B_e', C_e', and D_e' the operators of the perturbed closed-loop system, i.e., the closed-loop system consisting of the perturbed system and the controller.

If the operators of the system are perturbed, we first encounter the problem of *robust stabilization* related to the first term in (4.1). If the strong stability of the closed-loop system is preserved and Assumption 1 is satisfied for the perturbed operators, we know that the Sylvester equation

$$\Sigma' S = A_e' \Sigma' + B_e' \tag{4.2}$$

has a unique bounded solution, and the formula (4.1) describes the behavior of the regulation error of the perturbed system. Since the perturbed closed-loop system is strongly stable, the first term in (4.1) still approaches zero asymptotically.

The second part of the robust output regulation problem is the problem of *robust regulation* related to the second term in (4.1). This problem consists of choosing the controller parameters in such a way that the regulation error at the dynamic steady state of the closed-loop system is zero. This steady state error is precisely the second term in (4.1). This term must approach zero for all initial values v_0, which is achieved if and only if $C_e'\Sigma' + D_e' = 0$, where Σ' is the solution of the perturbed Sylvester equation (4.2). We can express this requirement more mathematically by stating that the perturbations preserving the closed-loop stability need to satisfy

$$\Sigma' S = A_e'\Sigma' + B_e' \quad \Rightarrow \quad C_e'\Sigma' + D_e' = 0. \tag{4.3}$$

In the above terminology a controller solves the robust output regulation problem if it solves both robust stabilization and robust regulation parts. In this paper we are interested only in the robust regulation part and do not consider the problem of stabilizing the closed-loop system. To this end, motivated by the previous observations, we define a *robustly regulating* controller, i.e., a controller solving the robust regulation part of the robust output regulation problem, in the following way.

DEFINITION 4.1. *A controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is called* robustly regulating *if the condition* (4.3) *is satisfied for all operators* A_e', B_e', C_e', D_e' *of the closed-loop system and* $\Sigma' \in \mathcal{L}(W^m, X_e)$ *with* $\Sigma'(W^{m+1}) \subset \mathcal{D}(A_e')$.

We will now give the definition of the IMS of Immonen [11]. We extend the original definition in which the operator Γ was assumed to be bounded.

DEFINITION 4.2 (Internal model structure (IMS)). *A controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is said to have* IMS *if*

$$\forall\ \Gamma, \Delta: \qquad \Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2\Delta \quad \Rightarrow \quad \Delta = 0, \tag{4.4}$$

where $\Gamma \in \mathcal{L}(W^m, Z)$ *with* $\Gamma(W^{m+1}) \subset \mathcal{D}(\mathcal{G}_1)$ *and* $\Delta \in \mathcal{L}(W, Y)$.

Theorem 4.3 shows that the robust regulation property of a controller is equivalent to the IMS.

THEOREM 4.3. *A controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is robustly regulating on* W^m *if and only if it has IMS on* W^m.

Proof. Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is robustly regulating. Let $\Gamma \in \mathcal{L}(W^m, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ be such that $\Gamma(W^{m+1}) \subset \mathcal{D}(\mathcal{G}_1)$ and $\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2\Delta$. We need to show that $\Delta = 0$. Let A, B, C, and D be any operators and choose $E = -BK\Gamma$ and $F = \Delta - DK\Gamma$. Then $\Sigma = \begin{bmatrix}0, \ \Gamma\end{bmatrix}^T \in \mathcal{L}(W^m, X_e)$ is an operator such that $\Sigma(W^{m+1}) \subset \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) = \mathcal{D}(A_e)$ and for all $v \in W^{m+1}$ we have

$$\Sigma S v = \begin{bmatrix} 0Sv \\ \Gamma S v \end{bmatrix} = \begin{bmatrix} A0v + BK\Gamma v \\ \mathcal{G}_2 C0v + \mathcal{G}_1\Gamma v + \mathcal{G}_2 DK\Gamma v \end{bmatrix} + \begin{bmatrix} -BK\Gamma v \\ \mathcal{G}_2(\Delta v - DK\Gamma v) \end{bmatrix} = A_e\Sigma v + B_e v$$

and thus $\Sigma S = A_e\Sigma + B_e$ on W^{m+1}. Condition (4.3) now implies that for all $v \in W^m$

$$0 = C_e\Sigma v + D_e v = C0v + DK\Gamma v + (\Delta - DK\Gamma)v = \Delta v.$$

Since W^m is dense in W, this concludes that $\Delta = 0$ and thus the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS.

Assume now that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS and A_e, B_e, and $\Sigma \in \mathcal{L}(W^m, X_e)$ are operators such that $\Sigma(W^{m+1}) \subset \mathcal{D}(A_e)$ and $\Sigma S = A_e\Sigma + B_e$. We will now show that $C_e\Sigma + D_e = 0$.

Since $X_e = X \times Z$ and $\mathcal{D}(A_e) = \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1)$, the operator Σ is of form $\Sigma = \begin{bmatrix}\Pi, \ \Gamma\end{bmatrix}^T$, where $\Pi \in \mathcal{L}(W^m, X)$ and $\Gamma \in \mathcal{L}(W^m, Z)$ are operators such that $\Pi(W^{m+1}) \subset \mathcal{D}(A)$ and $\Gamma(W^{m+1}) \subset \mathcal{D}(\mathcal{G}_1)$. Now $\Sigma S = A_e\Sigma + B_e$ implies that for all $v \in W^{m+1}$ we have

$$\begin{bmatrix}\Pi S v\\ \Gamma S v\end{bmatrix} = \begin{bmatrix}A & BK\\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK\end{bmatrix}\begin{bmatrix}\Pi v\\ \Gamma v\end{bmatrix} + \begin{bmatrix}Ev\\ \mathcal{G}_2 F v\end{bmatrix} = \begin{bmatrix}A\Pi v + BK\Gamma v + Ev\\ \mathcal{G}_1\Gamma v + \mathcal{G}_2(C\Pi + DK\Gamma + F)v\end{bmatrix}.$$

The second line implies that we have $\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F)$ on W^{m+1}, and thus we have from (4.4) that $0 = C\Pi + DK\Gamma + F = C_e\Sigma + D_e$. This concludes the proof. ☐

The IMS is independent of the operator K. This operator is needed only in the robust stabilization part of the robust output regulation problem.

We saw in the beginning of this section that in order to solve the regulation part of the problem, we must choose the controller parameter such that (4.3) holds for all perturbations preserving the closed-loop stability. This way it would seem that the solution to the regulation part of the problem depends on the stabilization part, since the stabilization part determines the perturbations for which we need to consider the condition (4.3). On the other hand, in Definition 4.1 we required that (4.3) is satisfied for *all* possible perturbations of the operators. This choice was made because it turns out that a controller actually satisfies (4.3) for all perturbations preserving the closed-loop stability if and only if it satisfies it for arbitrary perturbations. This is shown by the next lemma.

PROPOSITION 4.4. *Let $(\mathcal{G}_1, \mathcal{G}_2, K)$ be the controller. If there exist operators (A, B, C, D, E, F) such that the closed-loop system is strongly stable, then the following are equivalent:*

(a) *Condition* (4.3) *is satisfied for all A_e, B_e, C_e, D_e, and $\Sigma \in \mathcal{L}(W^m, X_e)$ with $\Sigma(W^{m+1}) \subset \mathcal{D}(A_e)$ such that the closed-loop system is stable.*

(b) *Condition* (4.3) *is satisfied for all A_e, B_e, C_e, D_e, and $\Sigma \in \mathcal{L}(W^m, X_e)$ with $\Sigma(W^{m+1}) \subset \mathcal{D}(A_e)$.*

Proof. It is sufficient to show that if (a) is satisfied, then the controller has IMS. This can be seen directly from the first part of the proof of Lemma 4.3 if we choose operators A, B, C, and D such that the closed-loop system is stable. Because the operators E and F do not appear in the operator A_e, they do not affect the closed-loop stability, and the rest of the proof can be used as it is. ☐

5. The $\mathcal{G}$-conditions.

In this section we compare the IMS defined in the previous section to the $\mathcal{G}$-conditions by Hämäläinen and Pohjolainen [9, 17]. In the previous references the signal generator was assumed to be diagonal or finite-dimensional, respectively. We first extend the definition of the $\mathcal{G}$-conditions for an infinite-dimensional signal generator with nontrivial Jordan block structure.

DEFINITION 5.1 ($\mathcal{G}$-conditions). *A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to satisfy* the $\mathcal{G}$-conditions *related to the exosystem S if*

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \forall k \in \mathbb{Z}, \tag{5.1a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{5.1b}$$

and

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1) \qquad \forall k \in \mathbb{Z}. \tag{5.1c}$$

The $\mathcal{G}$-conditions depend on the exosystem S. Its contribution to the conditions are the eigenvalues $i\omega_k \in \sigma(S)$ and the dimensions d_k of the largest Jordan blocks associated to them.

Let $m \in \mathbb{N}_0$ be arbitrary and fixed. The following theorem is the main result of this section. It states that under suitable assumptions the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS if and only if it satisfies the $\mathcal{G}$-conditions.

THEOREM 5.2. *Let $(\mathcal{G}_1, \mathcal{G}_2, K)$ be a controller such that $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$. Then it has IMS if and only if it satisfies the $\mathcal{G}$-conditions.*

It is worthwhile to note that the definition of the IMS depends on the value m and the space W^m on which the robust output regulation problem is considered, but the $\mathcal{G}$-conditions are independent of this. Thus Theorem 5.2 implies that if the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS for some value of m, then it has it for all $m \in \mathbb{N}_0$.

Theorem 5.2 extends the results of [17, 9, 14]. In the first two references it was shown that a controller satisfying the $\mathcal{G}$-conditions has IMS, but the signal generator was assumed to be finite-dimensional and infinite-dimensional with a diagonal system operator, respectively. For an exosystem with a diagonal system operator the condition (5.1c) becomes redundant. In [14] it was shown that if the exosystem has a diagonal system operator, then the controller has IMS if and only if it satisfies the $\mathcal{G}$-conditions.

We prove the theorem in parts. Lemmas 5.3, 5.4, and 5.5 prove that the IMS of the controller implies that it satisfies the $\mathcal{G}$-conditions. Lemma 5.6 shows that also the converse holds.

LEMMA 5.3. *If the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS, then* (5.1a) *is satisfied.*

Proof. Let $k \in \mathbb{Z}$ and let $w \in \mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $z \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that $w = (i\omega_k I - \mathcal{G}_1)z = \mathcal{G}_2 y$. Choose $\Gamma = \langle \cdot, \phi_k^{n_k} \rangle z \in \mathcal{L}(W^m, Z)$ and $\Delta = \langle \cdot, \phi_k^{n_k} \rangle y \in \mathcal{L}(W, Y)$. Now $\mathcal{R}(\Gamma) \subset \mathcal{D}(\mathcal{G}_1)$. For any $v \in W^{m+1}$

$$
\begin{aligned}
(\Gamma S - \mathcal{G}_1 \Gamma)v &= \langle Sv, \phi_k^{n_k} \rangle z - \langle v, \phi_k^{n_k} \rangle \mathcal{G}_1 z = \langle S_k v, \phi_k^{n_k} \rangle z - \langle v, \phi_k^{n_k} \rangle \mathcal{G}_1 z \\
&= \left\langle i\omega_k \langle v, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle v, \phi_k^l \rangle (i\omega_k \phi_k^l + \phi_k^{l-1}), \phi_k^{n_k} \right\rangle z - \langle v, \phi_k^{n_k} \rangle \mathcal{G}_1 z \\
&= \langle v, \phi_k^{n_k} \rangle (i\omega_k I - \mathcal{G}_1) z = \langle v, \phi_k^{n_k} \rangle \mathcal{G}_2 y = \mathcal{G}_2 (\langle v, \phi_k^{n_k} \rangle y) = \mathcal{G}_2 \Delta v.
\end{aligned}
$$

Thus we have $\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta$ and our assumption implies that $\Delta = 0$. Now $0 = \Delta \phi_k^{n_k} = \langle \phi_k^{n_k}, \phi_k^{n_k} \rangle y = y$ and thus also $w = \mathcal{G}_2 y = 0$. ∎

LEMMA 5.4. *If the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS, then* (5.1b) *is satisfied.*

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and let $\phi \in \mathcal{D}(S)$ be such that $\|\phi\| = 1$. Choose the operators $\Gamma = 0 \in \mathcal{L}(W^m, Z)$ and $\Delta = \langle \cdot, \phi \rangle y$. Then $\mathcal{R}(\Gamma) = \{0\} \subset \mathcal{D}(\mathcal{G}_1)$ and for all $v \in W^{m+1}$ we have $\Gamma S v = 0$ and $\mathcal{G}_1 \Gamma v + \mathcal{G}_2 \Delta v = 0 + \langle v, \phi \rangle \mathcal{G}_2 y = 0$. Thus we have $\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta$ and our assumption implies $\Delta = 0$. Now $0 = \Delta \phi = \langle \phi, \phi \rangle y = y$ and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$. ∎

To prove the condition (5.1c) we need to assume that $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$.

LEMMA 5.5. *If $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$, and if the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has IMS, then* (5.1c) *is satisfied.*

Proof. Since $d_k = \max\{ n_l \mid l \in \mathbb{Z}, \, \omega_l = \omega_k \}$, it is sufficient to prove that for all $k \in \mathbb{Z}$ we have $\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$. Let $k \in \mathbb{Z}$ and $z \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k - 1}$. Since $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$, there exist $z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$z = (i\omega_k I - \mathcal{G}_1) z_1 + \mathcal{G}_2 y. \tag{5.2}$$

Choose $\Gamma \in \mathcal{L}(W^m, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ such that $\Delta = (-1)^{n_k}\langle \cdot, \phi_k^{n_k}\rangle y$ and

$$\Gamma = \left(\sum_{l=1}^{n_k-1} (-1)^{l-1}\langle \cdot, \phi_k^l\rangle (i\omega_k I - \mathcal{G}_1)^{n_k-1-l} z \right) + (-1)^{n_k-1}\langle \cdot, \phi_k^{n_k}\rangle z_1.$$

Since $z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $(i\omega_k I - \mathcal{G}_1)^l z \in \mathcal{D}(\mathcal{G}_1)$ for all $l \in \{0, \ldots, n_k - 2\}$, we have $\mathcal{R}(\Gamma) \subset \mathcal{D}(\mathcal{G}_1)$. Now for $l \in \{2, \ldots, n_k - 1\}$ we have

$$\begin{aligned}
(\Gamma S - \mathcal{G}_1\Gamma)\phi_k^1 &= (i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^1 = (i\omega_k I - \mathcal{G}_1)^{n_k-1} z = 0 = \mathcal{G}_2\Delta\phi_k^1,\\
(\Gamma S - \mathcal{G}_1\Gamma)\phi_k^l &= (i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^l + \Gamma\phi_k^{l-1} = (-1)^{l-1}(i\omega_k I - \mathcal{G}_1)(i\omega_k I - \mathcal{G}_1)^{n_k-1-l} z\\
&\quad + (-1)^{l-2}(i\omega_k I - \mathcal{G}_1)^{n_k-1-(l-1)} z = 0 = \mathcal{G}_2\Delta\phi_k^l,
\end{aligned}$$

and finally using (5.2)

$$\begin{aligned}
(\Gamma S - \mathcal{G}_1\Gamma)\phi_k^{n_k} &= (i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^{n_k} + \Gamma\phi_k^{n_k-1} = (-1)^{n_k-1}(i\omega_k I - \mathcal{G}_1)z_1 + (-1)^{n_k-2} z\\
&= (-1)^{n_k-1}\left((i\omega_k I - \mathcal{G}_1)z_1 - z\right) = (-1)^{n_k-1}(-\mathcal{G}_2 y)\\
&= \mathcal{G}_2\left((-1)^{n_k}\langle \phi_k^{n_k}, \phi_k^{n_k}\rangle y\right) = \mathcal{G}_2\Delta\phi_k^{n_k}.
\end{aligned}$$

This concludes that $\Gamma S v = \mathcal{G}_1\Gamma v + \mathcal{G}_2\Delta v$ for all $v \in \operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$. Since clearly $\Gamma\phi_j^l = 0$ and $\Delta\phi_j^l = 0$ for all $j \neq k$ and $l \in \{1, \ldots, n_j\}$, we have that for all $v \in W^{m+1}$

$$\Gamma S v = \Gamma P_k S v = \Gamma S P_k v = \mathcal{G}_1\Gamma P_k v + \mathcal{G}_2\Delta P_k v = \mathcal{G}_1\Gamma v + \mathcal{G}_2\Delta v$$

and thus $\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2\Delta$ on W^{m+1}. Now our assumption implies that $\Delta = 0$ and thus $0 = (-1)^{n_k-1}\Delta\phi_k^{n_k} = \|\phi_k^{n_k}\|^2 y = y$. Substituting this into (5.2) we get $z = (i\omega_k I - \mathcal{G}_1)z_1$, which concludes that $z \in \mathcal{R}(i\omega_k I - \mathcal{G}_1)$. ∎

Finally, Lemma 5.6 proves that if the controller satisfies the $\mathcal{G}$-conditions, then it has IMS.

LEMMA 5.6. *If a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions, then it has IMS.*

Proof. Let $\Gamma \in \mathcal{L}(W^m, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ be such that $\Gamma(W^{m+1}) \subset \mathcal{D}(\mathcal{G}_1)$ and

$$\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2\Delta. \tag{5.3}$$

Let $k \in \mathbb{Z}$. Applying both sides of (5.3) to ϕ_k^1 we obtain $(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^1 = \mathcal{G}_2\Delta\phi_k^1$. Now (5.1a) and (5.1b) imply that $\Delta\phi_k^1 = 0$ and $(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^1 = 0$, and if $n_k \geq 2$, we see using the condition (5.1c) that

$$\Gamma\phi_k^1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1) \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Applying both sides of (5.3) to ϕ_k^2 we obtain $(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^2 + \Gamma\phi_k^1 = \mathcal{G}_2\Delta\phi_k^2$. Since $\Gamma\phi_k^1 \in \mathcal{R}(i\omega_k I - \mathcal{G}_1)$, the conditions (5.1a) and (5.1b) imply that we have $\Delta\phi_k^2 = 0$ and $(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^2 + \Gamma\phi_k^1 = 0$. Since $\Gamma\phi_k^1 \in \mathcal{D}(\mathcal{G}_1)$, we have $\Gamma\phi_k^2 \in \mathcal{D}(i\omega_k I - \mathcal{G}_1)^2$. Applying $(i\omega_k I - \mathcal{G}_1)$ to both sides of the latter equation and using $\Gamma\phi_k^1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$, we obtain $(i\omega_k I - \mathcal{G}_1)^2\Gamma\phi_k^2 = 0$. If $n_k \geq 3$, the condition (5.1c) implies

$$\Gamma\phi_k^2 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^2 \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Continuing the same procedure we see that $\Delta\phi_k^l = 0$ and

$$\Gamma\phi_k^l \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^l \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$$

for all $l \in \{1, \dots, n_k - 1\}$. Applying both sides of (5.3) to $\phi_k^{n_k}$ we obtain

$$(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^{n_k} + \Gamma\phi_k^{n_k-1} = \mathcal{G}_2\Delta\phi_k^{n_k},$$

and the conditions (5.1a) and (5.1b) imply that $\Delta\phi_k^{n_k} = 0$. Since $k \in \mathbb{Z}$ was arbitrary, we have shown that $\Delta\phi_k^l = 0$ for all $k \in \mathbb{Z}$ and $l \in \{1, \dots, n_k\}$. Since $\{\phi_k^l\}$ is a basis of W we have $\Delta = 0$. ∎

This concludes the proof of Theorem 5.2. We conclude this section by looking more closely at the assumption $\mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2) = Z$ for all $k \in \mathbb{Z}$. In finite-dimensional control theory this is precisely the condition that all the modes of the exosystem S in the controller $\mathcal{G}_1$ are controllable by $\mathcal{G}_2$. Since $i\omega_k \in i\mathbb{R}$ for all $k \in \mathbb{Z}$, this assumption is always satisfied in the finite-dimensional case if the controller stabilizes the closed-loop system. In the infinite-dimensional case the situation is more complicated. Lemma 5.7 gives a sufficient condition for the assumption to hold.

LEMMA 5.7. *If $\sigma(A_e) \cap \sigma(S) = \varnothing$, then $\mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2) = Z$ for all $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and $z \in Z$. We need to show that there exist $z_0 \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that $z = (i\omega_k I - \mathcal{G}_1)z_0 + \mathcal{G}_2 y$. Since $\sigma(A_e) \cap \sigma(S) = \varnothing$, we have $i\omega_k \in \rho(A_e)$ and $i\omega_k I - A_e$ is surjective. Thus there exist $x_1 \in \mathcal{D}(A)$ and $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{bmatrix} 0 \\ z \end{bmatrix} = (i\omega_k I - A_e)\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} (i\omega_k I - A)x_1 - BKz_1 \\ -\mathcal{G}_2 Cx_1 + (i\omega_k I - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 \end{bmatrix}.$$

The second equation shows that $z = (i\omega_k I - \mathcal{G}_1)z_1 + \mathcal{G}_2(-Cx_1 - DKz_1)$ and thus we can choose $z_0 = z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $y = -Cx_1 - DKz_1 \in Y$. ∎

6. The p-copy internal model. In this section we generalize the definition of the p-copy internal model of Francis and Wonham [6] for distributed parameter systems. We also show that under certain assumptions this property is equivalent to the $\mathcal{G}$-conditions presented in section 5. In the view of the results presented in the previous sections this extends the internal model principle of classical finite-dimensional control theory for distributed parameter systems by concluding that under suitable assumptions a controller is robustly regulating if and only if it contains a p-copy internal model of the exosystem. This is the main result of the paper and is presented in Theorem 6.9.

The classical definition states that if $\dim Y = p$, a controller incorporates a p-copy internal model of the signal generator if *the minimal polynomial of S divides at least p invariant factors of $\mathcal{G}_1$*. This definition cannot be generalized for infinite-dimensional operators $\mathcal{G}_1$ and S, but the p-copy internal model has the following equivalent definition [6]: A controller contains a p-copy internal model of the exosystem if *whenever $s \in \sigma(S)$ is an eigenvalue of S such that $d(s)$ is the dimension of the largest Jordan block associated to s, then $s \in \sigma(\mathcal{G}_1)$ and $\mathcal{G}_1$ has at least p Jordan blocks of dimension greater than or equal to $d(s)$ associated to s.* Since the operators S_k can be seen as Jordan blocks of operator S, this definition can be directly generalized for our exosystem and an infinite-dimensional controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in the following way.

DEFINITION 6.1 (p-Copy internal model). *A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to* incorporate a p-copy internal model *of the exosystem S if for all $k \in \mathbb{Z}$ we have*

$$\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) \geq \dim Y$$

and $\mathcal{G}_1$ has at least $\dim Y$ *independent Jordan chains of length greater than or equal to d_k associated to the eigenvalue $i\omega_k$.*

We will now show that under certain assumptions the controller incorporates a p-copy internal model if and only if it satisfies the $\mathcal{G}$-conditions. Theorem 6.2 is the first main result of this section.

THEOREM 6.2. *Let $\sigma(A_e) \cap \sigma(S) = \varnothing$ and $\dim Y < \infty$. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ contains a p-copy internal model of the exosystem if and only if it satisfies the $\mathcal{G}$-conditions.*

To prove the theorem we need a series of lemmas. The proof of Theorem 6.2 is presented at the end of the section. Since the required lemmas are also useful results considered separately, we will prove them using weaker assumptions whenever possible. In particular, it is interesting to see that the $\mathcal{G}$-conditions imply that the controller incorporates a p-copy internal model even if the space Y is infinite-dimensional. Because of this, the $\mathcal{G}$-conditions are a more suitable choice for the definition of an internal model when generalizing the theory to allow infinite-dimensional input and output spaces.

The proofs make use of the interesting result that under our assumptions for any $k \in \mathbb{Z}$ the operator $P(i\omega_k)K$ restricted to the eigenspace $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ is an isomorphism between the eigenspace and the output space. This relation establishes the fact that every eigenvalue $i\omega_k$ of the exosystem is an eigenvalue of the controller $\mathcal{G}_1$ with a geometric multiplicity equal to the dimension of the output space.

We will start by presenting the following lemma used in the proofs of Lemmas 6.4–6.7.

LEMMA 6.3. *If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is injective for every $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. Let $z \in \mathcal{N}(sI - \mathcal{G}_1)$ be such that $P(s)Kz = 0$. Choose $x = R(s, A)BKz \in \mathcal{D}(A)$. Now

$$(sI - A_e)\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} (sI - A)x - BKz \\ -\mathcal{G}_2 Cx + (sI - \mathcal{G}_1)z - \mathcal{G}_2 DKz \end{bmatrix}$$
$$= \begin{bmatrix} BKz - BKz \\ -\mathcal{G}_2(CR(s, A)B + D)Kz + (sI - \mathcal{G}_1)z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $s \in \sigma(S)$, we know that $s \notin \sigma_p(A_e)$ and thus $sI - A_e$ is injective. This implies that $z = 0$, which concludes that the restriction of $P(s)K$ to $\mathcal{N}(sI - \mathcal{G}_1)$ is an injection. ∎

The following lemma states that if the $\mathcal{G}$-conditions are satisfied, then for all $k \in \mathbb{Z}$ the space $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ is isomorphic to Y, and $\mathcal{G}_1$ has $\dim Y$ independent Jordan chains of length greater than or equal to d_k associated to the eigenvalue $i\omega_k$. This proves a part of Theorem 6.2, but the result is more general in the sense that it does not require Y to be finite-dimensional.

LEMMA 6.4. *If $\sigma(A_e) \cap \sigma(S) = \varnothing$ and the $\mathcal{G}$-conditions are satisfied, then for all $k \in \mathbb{Z}$ the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is an isomorphism between $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ and Y, and $\mathcal{G}_1$ has $\dim Y$ independent Jordan chains of length greater than or equal to d_k associated to the eigenvalue $i\omega_k$.*

Proof. Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. From Lemma 6.3 we see that the operator $(P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)}$ is injective and thus it is sufficient to prove that it is also surjective.

Since $\sigma(A_e) \cap \sigma(S) = \varnothing$, we have $s \in \rho(A_e)$ and the operator $sI - A_e$ is surjective. This implies that for all $z \in Z$ there exist $x_1 \in \mathcal{D}(A)$ and $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{bmatrix} 0 \\ z \end{bmatrix} = (sI - A_e)\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} (sI - A)x_1 - BKz_1 \\ -\mathcal{G}_2 Cx_1 + (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 \end{bmatrix}.$$

Since $\sigma(A) \cap \sigma(S) = \varnothing$, we have $s \in \rho(A)$ and we get from the first equation that $x_1 = R(s,A)BKz_1$. Using this we have from the second equation that

$$z = -\mathcal{G}_2 CR(s,A)BKz_1 + (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 = (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 P(s)Kz_1. \tag{6.1}$$

Let $y \in Y$. Then $z = -\mathcal{G}_2 y \in \mathcal{R}(\mathcal{G}_2) \subset Z$ and we can choose $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that (6.1) holds. Now

$$-\mathcal{G}_2 y = (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 P(s)Kz_1 \quad \Leftrightarrow \quad \underbrace{-\mathcal{G}_2 y + \mathcal{G}_2 P(s)Kz_1}_{\in \mathcal{R}(\mathcal{G}_2)} = \underbrace{(sI - \mathcal{G}_1)z_1}_{\in \mathcal{R}(sI - \mathcal{G}_1)}$$
$$\Leftrightarrow \quad \begin{cases} \mathcal{G}_2 y = \mathcal{G}_2 P(s)Kz_1 \\ 0 = (sI - \mathcal{G}_1)z_1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} y = P(s)Kz_1 \\ 0 = (sI - \mathcal{G}_1)z_1 \end{cases}$$

because $\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ and $\mathcal{N}(\mathcal{G}_2) = \{0\}$. This means that for every $y \in Y$ there exists $z_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $y = P(s)Kz_1$ and thus the operator $(P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)}$ is surjective.

This also concludes that $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$. Since Jordan chains related to linearly independent eigenvectors are independent, it remains to show that there exists a Jordan chain of length greater than or equal to d_k related to every vector in $\mathcal{N}(sI - \mathcal{G}_1)$.

We can assume $d_k \geq 2$, because otherwise the proof is complete. Since for all $l \in \mathbb{N}$ we have $\mathcal{N}(sI - \mathcal{G}_1)^l \subset \mathcal{N}(sI - \mathcal{G}_1)^{l+1}$, the condition (5.1c) implies

$$\mathcal{N}(sI - \mathcal{G}_1) \subset \mathcal{N}(sI - \mathcal{G}_1)^2 \subset \cdots \subset \mathcal{N}(sI - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(sI - \mathcal{G}_1). \tag{6.2}$$

Choose $\psi_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ and define $\{\psi_l\}_{l=2}^{d_k}$ recursively as follows: Let $l \in \{2, \ldots, d_k\}$. Assume $\psi_{l-1} \in \mathcal{N}(sI - \mathcal{G}_1)^{l-1}$. We have from (6.2) that there exists $\psi_l \in \mathcal{D}(\mathcal{G}_1)$ such that

$$(\mathcal{G}_1 - sI)\psi_l = \psi_{l-1} \in \mathcal{N}(sI - \mathcal{G}_1)^{l-1} \subset \mathcal{D}(sI - \mathcal{G}_1)^{l-1}.$$

Thus we have $\psi_l \in \mathcal{D}(sI - \mathcal{G}_1)^l$ and $(\mathcal{G}_1 - sI)^l \psi_l = (\mathcal{G}_1 - sI)^{l-1}\psi_{l-1} = 0$. This implies that $\psi_l \in \mathcal{N}(sI - \mathcal{G}_1)^l$.

The set $\{\psi_l\}_{l=1}^{d_k}$ satisfies $(sI - \mathcal{G}_1)\psi_1 = 0$ and $(\mathcal{G}_1 - sI)\psi_l = \psi_{l-1}$ for every $l \in \{2, \ldots, d_k\}$ and thus by possibly adding elements to this set we obtain a Jordan chain $\{\psi_l\}_{l=1}^{m}$ with length $m \geq d_k$. ∎

In the previous lemma we saw that the $\mathcal{G}$-conditions imply that $\mathcal{G}_1$ has *exactly* $\dim Y$ independent Jordan chains. This actually follows from our assumption that $\sigma_p(A_e) \cap \sigma(S) = \varnothing$ as is shown in the next lemma. This is a controllability type result whose basic idea is that if $\mathcal{G}_1$ has an eigenvalue with multiplicity larger than $p = \dim Y$ on the imaginary axis, then this eigenvalue of the pair $(\mathcal{G}_1, \mathcal{G}_2)$ cannot be moved by feedback since $\mathcal{G}_2$ is a rank p operator.

LEMMA 6.5. *If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, then $\mathcal{N}(i\omega_k I - \mathcal{G}_1) \leq \dim Y$ for all $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. We have from Lemma 6.3 that the operator $(P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)} \in \mathcal{L}(\mathcal{N}(sI - \mathcal{G}_1), Y)$ is injective. Using the rank-nullity theorem [13, Thm. 4.7.7] we can conclude that

$$\begin{aligned} \dim \mathcal{N}(sI - \mathcal{G}_1) &= \dim \mathcal{R}\left((P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)}\right) + \dim \mathcal{N}\left((P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)}\right) \\ &= \dim \mathcal{R}\left((P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)}\right) \leq \dim Y. \end{aligned}$$ ∎

The following three lemmas are used to show that if $\dim Y < \infty$, and if the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem, then this controller satisfies the $\mathcal{G}$-conditions. For this it is sufficient to assume that $\sigma_p(A_e) \cap \sigma(S) = \varnothing$. This is satisfied whenever the operator A_e generates a strongly stable C_0-semigroup, because then $\sigma_p(A_e) \subset \mathbb{C}^-$ [10].

LEMMA 6.6. *If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, $\dim Y < \infty$, and the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem, then $\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. Lemma 6.5 shows that we must have $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$. Let $v \in \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $y \in Y$ and $z \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 y = (sI - \mathcal{G}_1)z$. We will first show that there exists $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 P(s) K z_1 = (sI - \mathcal{G}_1)z_1$. From Lemma 6.3 we get that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective, and since $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$ we have that it is invertible. Because of this we can choose $z_0 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that

$$P(s)Kz_0 = y - P(s)Kz \in Y \qquad \Leftrightarrow \qquad y = P(s)K(z + z_0).$$

We then have

$$\mathcal{G}_2 P(s)K(z + z_0) = \mathcal{G}_2 y = v = (sI - \mathcal{G}_1)z = (sI - \mathcal{G}_1)(z + z_0)$$

and thus we can choose $z_1 = z + z_0$.

Choose $x_1 = R(s, A)BKz_1 \in \mathcal{D}(A)$. As in the proof of Lemma 6.3, we see that

$$(sI - A_e)\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2 P(s)Kz_1 + (sI - \mathcal{G}_1)z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $s \in \sigma(S)$ and $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, we have that $sI - A_e$ is injective and thus $z_1 = 0$. This concludes that $v = (sI - \mathcal{G}_1)z_1 = 0$. ∎

LEMMA 6.7. *If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, $\dim Y < \infty$, and the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem, then $\mathcal{N}(\mathcal{G}_2) = \{0\}$.*

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and $k \in \mathbb{Z}$ and denote $s = i\omega_k$. Lemma 6.5 shows that we must have $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$. From Lemma 6.3 we get that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective, and since $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$, it is invertible. This implies that there exists $z_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $y = P(s)Kz_1$ and thus $\mathcal{G}_2 P(s)Kz_1 = 0$. Choose $x_1 = R(s, A)BKz_1 \in \mathcal{D}(A)$. As in the proof of Lemma 6.3, we see that

$$(sI - A_e)\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2 P(s)Kz_1 + (sI - \mathcal{G}_1)z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $s \in \sigma(S)$ and $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, we have that $sI - A_e$ is injective and thus $z_1 = 0$. This also implies $y = P(s)Kz_1 = 0$ and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$. ∎

LEMMA 6.8. *If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, $\dim Y < \infty$, and the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem, then for all $k \in \mathbb{Z}$ we have $\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$.*

Proof. Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$ and $N = \dim Y$. Lemma 6.5 shows that we must have $\dim \mathcal{N}(sI - \mathcal{G}_1) = N$.

By our assumption $\mathcal{G}_1$ has N independent Jordan chains $\{\psi_n^l\}_{l=1}^{m_n}$ with $m_n \geq d_k$ associated to s. Because by the definition of the Jordan chain we have $\psi_n^k \in \mathcal{R}(sI - \mathcal{G}_1)$ for all $n \in \{1, \dots, N\}$ and $k \in \{1, \dots, d_k - 1\}$, it is sufficient to show that

$$\mathcal{N}(sI - \mathcal{G}_1)^m \subset \operatorname{span}\{\, \psi_n^l \mid n = 1, \dots, N,\ l = 1, \dots, m \,\} \tag{6.3}$$

for $m \in \{1, \ldots, d_k - 1\}$. We will do this using induction. Since $\{\psi_n^1\}_{n=1}^N$ is linearly independent and $\psi_n^1 \in \mathcal{N}(sI - \mathcal{G}_1)$ for all $n \in \{1, \ldots, N\}$, we have

$$\mathcal{N}(sI - \mathcal{G}_1) = \operatorname{span}\{\psi_n^1\}_{n=1}^N \tag{6.4}$$

and thus (6.3) holds for $m = 1$.

Assume (6.3) holds for $m = j \in \{1, \ldots, d_k - 2\}$ and let $z \in \mathcal{N}(sI - \mathcal{G}_1)^{j+1}$. Then $z \in \mathcal{D}(\mathcal{G}_1)$ and $(sI - \mathcal{G}_1)z \in \mathcal{N}(sI - \mathcal{G}_1)^j$. Since we assumed (6.3) holds for $m = j$, there exist constants $\left\{ \alpha_n^l \;\middle|\; n = 1, \ldots, N,\, l = 1, \ldots, j \right\}$ such that

$$(sI - \mathcal{G}_1)z = \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^l = \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l (\mathcal{G}_1 - sI)\psi_n^{l+1},$$

where the second equality follows from the fact that $\{\psi_n^l\}_l$ are Jordan chains associated to $\mathcal{G}_1$. This implies

$$(sI - \mathcal{G}_1)\left(z + \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^{l+1} \right) = 0 \qquad \Rightarrow \qquad z + \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^{l+1} \in \mathcal{N}(sI - \mathcal{G}_1).$$

We now have from (6.4) that there exist constants $\{\alpha_n^0\}_{n=1}^N$ such that

$$z + \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^{l+1} = \sum_{n=1}^N \alpha_n^0 \psi_n^1.$$

This concludes that (6.3) holds for $m = j + 1$ and thus completes the proof. □

We can finally present the proof of the first main result of this section.

Proof of Theorem 5.2. Lemmas 6.4 and 6.5 show that if the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions, then it incorporates a p-copy internal model of the exosystem. Lemmas 6.6, 6.7, and 6.8 conclude that also the converse holds. □

We will conclude the section by presenting the main result of the paper. This is the extension of the internal model principle for distributed parameter systems with infinite-dimensional exosystems.

THEOREM 6.9. *Assume Y is finite-dimensional and $\sigma(A_e) \cap \sigma(S) = \varnothing$. The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is robustly regulating if and only if it incorporates a p-copy internal model of the exosystem.*

Proof. Theorem 4.3 states that a controller is robustly regulating if and only if it has IMS. Theorem 5.2 and Lemma 5.7 together imply that under our assumptions the controller has IMS if and only if it satisfies the $\mathcal{G}$-conditions. Finally, Theorem 6.2 states that under our assumptions the controller satisfies the $\mathcal{G}$-conditions if and only if it incorporates a p-copy internal model of the exosystem. □

7. Example. In this section we design an observer-based robust controller for a finite-dimensional stable system with an exosystem capable of generating infinite-dimensional linearly increasing signals. The purpose of this example is to illustrate the use of the internal model principle and how to verify the convergence of the series in Assumption 1. Consider a system with operators $A = \operatorname{diag}(-2, -4)$, $B = (1, 1)^T$, $C = (1, 1)$, and $D = 1$ on $X = \mathbb{C}^2$. For the exosystem choose

$$S = \langle \cdot, \phi_0^2 \rangle \phi_0^1 + \sum_{k \neq 0} ik \langle \cdot, \phi_k \rangle \phi_k, \qquad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k \neq 0} k^2 |\langle v, \phi_k \rangle|^2 < \infty \right\}$$

on a Hilbert space $W = \overline{\operatorname{span}}\left\{\phi_0^0, \phi_0^1, \{\phi_k\}_{k\in\mathbb{Z}\setminus\{0\}}\right\}$ consisting of a 2×2 Jordan block related to eigenvalue $i\omega_0 = 0$ and an infinite-dimensional diagonal part with eigenvalues $i\omega_k = ik$. The signals generated by this kind of exosystem are in general of the form $y(t) = \alpha t + y_b(t)$, where $\alpha \in \mathbb{C}$ and y_b is a bounded and uniformly continuous function.

In order to consider robust regulation of a single signal without disturbances, choose $E = 0$ and $F = -F_r$, where $F_r \in \mathcal{L}(W, \mathbb{C})$. In this example we consider output operators F_r for which there exists a constant $M_r \geq 0$ such that $\|F_r\phi_k\| \leq M_r k^{-1}$ for $k \neq 0$.

Choose an observer-based controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that $Z = X \times W$ and

$$\mathcal{G}_1 = \begin{bmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & S \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} -L \\ G_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix},$$

where $G_2 = g_2 \in W$ is such that $\langle g_2, \phi_0^1\rangle = \langle g_2, \phi_0^2\rangle = 1$ and $\langle g_2, \phi_k\rangle = \frac{1}{k}$ for $k \neq 0$. Since $\dim Y = 1$ and since $A + BK_1 + L(C + DK_1)$ is a linear operator on $\mathbb{C}^2$, it is straightforward to verify that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem. Theorem 6.9 thus states that if we can choose the controller parameters K and L in such a way that $\sigma(A_e) \cap \sigma(S) = \varnothing$, then the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is robustly regulating. If in addition the closed-loop system is stable and the series in Assumption 1 converges for some m, then the controller solves the robust output regulation problem on W^m. The output regulation property is then robust with respect to perturbations preserving the strong stability of the closed-loop system and Assumption 1.

We will stabilize the closed-loop system using a procedure similar to the one presented in [9]. Since the exosystem we are considering has a nontrivial Jordan block structure, parts of the method must be generalized to our case. We omit the lengthy but straightforward computations and instead present a list of required steps along with the appropriate modifications. The procedure is as follows.

Step 1. Show that the closed-loop system operator A_e is similar to a block triangular operator with diagonal blocks $A_{e1} : X \times W \to X \times W$ and $A + LC$. Choose the operator L in such a way that $A + LC$ is exponentially stable. The closed-loop system is then strongly stable if the operator A_{e1} generates a strongly stable C_0-semigroup.

Step 2. Show that the operator K_1 can be chosen such that A_{e1} is strongly stable if the operator $S + B_1K_2$ is strongly stable, where $B_1 \in \mathcal{L}(\mathbb{C}, W)$.

Step 3. Denote $W_0 = \operatorname{span}\{\phi_0^1, \phi_0^2\}$, $W_1 = \operatorname{span}\{\phi_k\}_{k\neq 0}$, and write $K_2 = \left[K_2^0, K_2^1\right]$ according to the decomposition $W = W_0 \times W_1$. Show that using a method similar to the one used in Step 2 the part K_2^0 can be chosen such that $S + B_1K_2$ is strongly stable if $S_1 + B_1^1K_2^1$ is strongly stable, where $B_1^1 \in \mathcal{L}(\mathbb{C}, W_1)$ and $S_1 : W_1 \to W_1$ is the diagonal part of S,

$$S_1 = \sum_{k\neq 0} ik\langle\cdot, \phi_k\rangle\phi_k, \qquad \mathcal{D}(S_1) = \left\{ v \in W \,\middle|\, \sum_{k\neq 0} k^2 |\langle v, \phi_k\rangle|^2 < \infty \right\}.$$

Step 4. Using pole placement [21], show that K_2^1 can be chosen such that $S_1 + B_1^1K_2^1$ is strongly stable, has compact resolvent, $\sigma(S_1 + B_1^1K_2^1) = \{-\frac{1}{k^2} + ik\}_{k\neq 0}$, all but a finite number of these eigenvalues are simple, and the generalized eigenvectors of $S_1 + B_1^1K_2^1$ form a Riesz basis of W_1.

Step 5. Using the previous steps conclude that the closed-loop system is strongly stable and show that $\sigma(A_e) \cap \sigma(S) = \varnothing$.

Step 6. Recall the known result that if A, B, and C are linear operators such that one of the operators A and B is bounded and the other one generates a C_0-group, C is bounded, and $\sigma(A) \cap \sigma(B) = \varnothing$, then there exists a bounded linear operator T with a bounded inverse such that $\left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right) = T \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) T^{-1}$ [1, 15]. Use this result repeatedly to show that there exists a bounded linear operator T_e with a bounded inverse such that

$$A_e = T_e \begin{bmatrix} A_b & \\ & A_u \end{bmatrix} T_e^{-1},$$

where A_b is a finite-dimensional operator with spectrum $\sigma(A_b) \subset \mathbb{C}^-$ and A_u is a diagonal operator with eigenvalues $\sigma(A_u) = \{ -\frac{1}{k^2} + ik \mid |k| \geq N_e \geq 1 \}$.

We can now consider the convergence of the series in Assumption 1. Since $i\omega_k \in \rho(A_e)$ for all $k \in \mathbb{Z}$, all the terms are finite and it suffices to consider the tails $|k| \geq N$ of the series. For every $k \in \mathbb{Z}$ with $|k| \geq N_e$ we have $\|R(ik, A_u)\| = k^2$.

Since A_b is a finite-dimensional operator we have that for a large enough $N \geq N_e$

$$\begin{aligned} &\sup_{\|x_e'\| \leq 1} \sum_{|k| \geq N} \frac{1}{(1+\omega_k^2)^m} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, x_e' \rangle \right|^2 \\ &\quad \leq \sum_{|k| \geq N} \frac{\|R(ik, A_e)\|^2 \cdot \|B_e \phi_k\|^2}{(1+k^2)^m} \;\leq\; M^2 \sum_{|k| \geq N} \frac{\max\{k^4, \|R(ik, A_b)\|^2\} \cdot \frac{1}{k^2}}{(1+k^2)^m} \\ &\quad = \; M^2 \sum_{|k| \geq N} \frac{k^2}{(1+k^2)^m}, \end{aligned}$$

where $M = M_r \|T_e\| \cdot \|T_e^{-1}\| \cdot \|G_2\|$. This shows that the series in Assumption 1 converges if $m = 2$.

As already stated, the results in the previous sections conclude that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the robust output regulation problem on W^2; i.e., for any initial states of the system and the controller and for any initial state $v_0 \in W^2$ of the exosystem the regulation error $e(t)$ decays as $t \to \infty$. Furthermore, the regulation property is robust with respect to perturbations of the system's operators preserving the closed-loop stability and Assumption 1. In particular the perturbations to the output operator F_r do not affect the closed-loop stability and the estimate on the convergence of the series in Assumption 1 is valid for any output operator satisfying $\|F_r \phi_k\| \leq M_r k^{-1}$ for some M_r. Because of this we can conclude that the regulation error goes to zero whenever the perturbed output operator satisfies this kind of bound.

8. Conclusions. In this paper the p-copy internal model of Francis and Wonham was generalized for distributed parameter systems. This definition was compared to two other generalizations of internal model found in the literature and the definitions were shown to be equivalent under suitable assumptions. Using this equivalence it was also proved that a controller is robustly regulating if and only if it contains a p-copy internal model of the exosystem. This is an extension of the internal model principle for infinite-dimensional systems.

We constructed a signal generator capable of generating infinite-dimensional polynomially increasing signals. This was done by defining an operator with a possibly infinite number of nontrivial Jordan blocks.

The solution Σ of the regulator equations was allowed to be unbounded. This showed a direct connection between the smoothness of the reference and disturbance signals considered and the assumptions needed for the solvability of the output regulation problem.

Further research topics include the robust stabilization of the closed-loop system, which was not considered in this paper. There exists conditions for the strong and weak stabilization of the exosystem if the signal generator has a diagonal system operator [9]. However, there are very few results on the robustness properties of these stabilizing controllers.

Other further research topics include allowing the operators B and C to be unbounded and considering signal generators which do not necessarily have pure point spectrum.

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