

A Reduced Order Controller for Output Tracking of a Kelvin–Voigt Beam

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Abstract: We study output tracking and disturbance rejection for an Euler–Bernoulli beam with Kelvin–Voigt damping. The system has distributed control and pointwise observation. As our main result we design a finite-dimensional low-order internal model based controller that is based on a spectral Galerkin method and model reduction by Balanced Truncation. The performance of the designed controller is demonstrated with numerical simulations and compared to the performance of a low-gain internal model based controller.

Keywords: PDE control, Euler–Bernoulli beam, output tracking, disturbance rejection, robust output regulation, distributed parameter system.

1. INTRODUCTION

In this paper we design a finite-dimensional error feedback controller for output tracking and disturbance rejection of an Euler–Bernoulli beam equation with Kelvin–Voigt damping on $\Omega = (-1, 1)$ (Ito and Morris, 1998, Sec. 3),

$$v_{tt}(\xi, t) + (EIv_{\xi\xi} + d_{KV}Iv_{\xi\xi t})_{\xi\xi}(\xi, t) + d_v v_t(\xi, t) \quad (1.1a)$$

$$= b_1(\xi)u_1(t) + b_2(\xi)u_2(t) + B_{d0}(\xi)w_{dist}(t) \quad (1.1b)$$

$$v(-1, t) = v_{\xi}(-1, t) = 0, \quad (1.1c)$$

$$v(1, t) = v_{\xi}(1, t) = 0 \quad (1.1d)$$

$$v(\xi, 0) = v_0(\xi), \quad v_t(\xi, 0) = v_1(\xi), \quad (1.1e)$$

$$y(t) = (v(\xi_1, t), v(\xi_2, t))^T. \quad (1.1f)$$

The parameters $E > 0$ and $I > 0$ are the (constant) elastic modulus and the second moment of area, respectively, and $d_{KV} > 0$ and $d_v \geq 0$ are the coefficient associated to the Kelvin–Voigt damping and viscous damping, respectively. The beam is assumed to have constant density $\rho = 1$. The system has two inputs $u(t) = (u_1(t), u_2(t))^T$ and two outputs $y(t) = (y_1(t), y_2(t))^T$ (see Section 2 for details).

We study *output tracking* and *disturbance rejection*, where the aim is to design a control law in such a way that the output $y(t)$ of (1.1) converges to a given reference signal $y_{ref}(t)$, i.e., $\|y(t) - y_{ref}(t)\| \rightarrow 0$ as $t \rightarrow \infty$, despite the external disturbance signals $w_{dist}(t)$. The considered signals $y_{ref} : [0, \infty) \rightarrow \mathbb{R}^p$ and $w_{dist} : [0, \infty) \rightarrow \mathbb{R}^{n_d}$ are of the form

$$y_{ref}(t) = a_0^1 + \sum_{k=1}^q (a_k^1 \cos(\omega_k t) + b_k^1 \sin(\omega_k t)) \quad (1.2a)$$

$$w_{dist}(t) = a_0^2 + \sum_{k=1}^q (a_k^2 \cos(\omega_k t) + b_k^2 \sin(\omega_k t)) \quad (1.2b)$$

for some known frequencies $\{\omega_k\}_{k=0}^q \subset \mathbb{R}$ with $0 = \omega_0 < \omega_1 < \dots < \omega_q$ and possibly *unknown* constants $\{a_k^j\}_{k,j} \subset$

\mathbb{R} and $\{b_k^j\}_{k,j} \subset \mathbb{R}$ (any of the constants are allowed to be zero). This control problem — typically called *output regulation* — has been studied extensively in the literature for controlled partial differential equations (Xu and Sallet, 2014; Deutscher, 2015; Xu and Dubljevic, 2016; Jin and Guo, 2019) and distributed parameter systems (Pohjola, 1981; Hämäläinen and Pohjola, 2000; Byrnes et al., 2000; Rebarber and Weiss, 2003; Immonen, 2006, 2007; Natarajan et al., 2014; Paunonen, 2016).

In this paper we solve the output tracking and disturbance rejection problem with a finite-dimensional dynamic error feedback controller introduced recently in (Paunonen and Phan, 2020). The controller design is based on Galerkin approximation theory for a class of linear systems (Banks and Kunisch, 1984; Morris, 1994), in particular including controlled parabolic PDEs, and it uses model reduction to reduce the dimension of the controller¹. Output regulation of an Euler–Bernoulli beam with Kelvin–Voigt damping (single-input-single-output with clamped–free boundary conditions) was also considered in (Paunonen and Phan, 2020, Sec. V.C), where the controller was based on the Finite Element Method. As the main novelty of this paper we instead base our controller on a *spectral Galerkin method* with non-local basis functions based on Chebyshev polynomials (Shen, 1995; Shen et al., 2011).

The motivation for this study comes from the fact that spectral methods are powerful numerical approximation tools — typically achieving great accuracy with low numbers of basis functions — but to our knowledge they have not been used previously in controller design for output

¹ Note that the Internal Model Principle (Paunonen and Pohjola, 2010) requires a controller solving the control problem for all $y_{ref}(t)$ and $w_{dist}(t)$ in (1.2) necessarily has dimension of at least “number of outputs \times number of (complex) frequencies”, i.e., $2(2q+1)$. Thus in our setting the controller having “low order” means that the dimension is not much higher than $2(2q+1)$.

regulation of PDEs. In addition, in the case of Chebyshev functions the **Chebfun** MATLAB library (available at <https://www.chebfun.org/>) (Driscoll et al., 2014; Trefethen, 2013) can be employed in computing the Galerkin approximation.

Finally, we compare the performance of our controller to the so-called “simple” internal model based controller, and demonstrate that our new controller achieves a considerably improved rate of convergence of the output tracking.

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2. THE OUTPUT REGULATION PROBLEM

In (1.1) the functions $b_1(\cdot), b_2(\cdot) \in C^1(-1, 1; \mathbb{R})$ are fixed input profiles satisfying $b_j(\pm 1) = b'_j(\pm 1) = 0$, $j = 1, 2$, and the disturbance input term has the form $B_{d0}(\xi)w_{dist}(t) = \sum_{k=1}^{n_d} b_{dk}(\xi)w_{dist}^k(t)$ for $w_{dist}(t) = (w_{dist}^k(t))_{k=1}^{n_d}$ and for some fixed but unknown profile functions $b_{dk}(\cdot) \in C^1(-1, 1; \mathbb{R})$ with $b_{dk}(\pm 1) = b'_{dk}(\pm 1) = 0$ for $k \in \{1, \dots, n_d\}$ ².

The Output Regulation Problem. *Design a dynamic error feedback controller such that the following hold.*

- (1) *The closed-loop system is exponentially stable.*
- (2) *For any initial values of the system (1.1) and the controller and for any $\{a_k^j\}_{k,j} \subset \mathbb{R}$ and $\{b_k^j\}_{k,j} \subset \mathbb{R}$*

$$\|y(t) - y_{ref}(t)\| \rightarrow 0$$

at a uniform exponential rate as $t \rightarrow \infty$.

As shown in (Paunonen and Phan, 2020), the controller constructed in this paper is also *robust* in the sense that it tolerates changes and uncertainty in some of the parameters of the system (1.1) (see (Paunonen and Phan, 2020) for details).

3. REDUCED ORDER INTERNAL MODEL BASED CONTROLLER DESIGN

The controller we design is the finite-dimensional “Observer-based robust controller” introduced in (Paunonen and Phan, 2020, Sec. III.A). It has the general form

$$\dot{z}_1(t) = G_1 z_1(t) + G_2 e(t) \quad (3.1a)$$

$$\dot{z}_2(t) = (A_L^r + B_L^r K_2^r) z_2(t) + B_L^r K_1^N z_1(t) - L^r e(t) \quad (3.1b)$$

$$u(t) = K_1^N z_1(t) + K_2^r z_2(t) \quad (3.1c)$$

with state $(z_1(t), z_2(t))^T \in Z := Z_0 \times \mathbb{C}^r$ and input $e(t) = y(t) - y_{ref}(t)$. The matrices $(G_1, G_2, A_L^r, B_L^r, K_1^N, K_2^r, L^r)$ are constructed using the algorithm in Section 3.2. Theorem III.1 in (Paunonen and Phan, 2020) states that if the

² Theory would allow weaker assumptions on b_j and b_{dk} , but spectral methods do not work well for discontinuous functions.

order $N \in \mathbb{N}$ of the Galerkin approximation in **Step 2** of the algorithm and the order $r \leq N$ of the Balanced Truncation model reduction in **Step 4** are sufficiently high, then the controller (3.1) solves the output regulation problem.

3.1 The Galerkin Approximation of the Beam Model

The controller design uses a Galerkin approximation (A^N, B^N, C^N) of (1.1) written as first order system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_d w_{dist}(t) \\ y(t) &= Cx(t) \end{aligned}$$

on a Hilbert space X . We review the construction of (A^N, B^N, C^N) in this section. As described in (Paunonen and Phan, 2020), the controller design algorithm requires that the operator A is associated to a coercive sesquilinear $a(\cdot, \cdot)$ form defined on another Hilbert space $V \subset X$. In order to utilise the spectral Galerkin method based on Chebyshev polynomials, we formulate the beam system (1.1) as a first order system with state $x(t) = (v(\cdot, t), \dot{v}(\cdot, t))^T$ on the space $X = V_0 \times L_\omega^2(-1, 1)$ where $L_\omega^2(-1, 1)$ is the L^2 -space with the weight $\omega(\xi) = (1 - \xi^2)^{-1/2}$ and $V_0 = \{f \in H_\omega^2(-1, 1) \mid f(\pm 1) = f'(\pm 1) = 0\}$. Here also the Sobolev space $H_\omega^2(-1, 1)$ is defined with the weight $\omega(\cdot)$. The motivation for the use of the weighted spaces is that the Chebyshev polynomials $T_k(\cdot)$ are orthogonal with respect to the inner product of $L_\omega^2(-1, 1)$, and this property can be leveraged in computing the Galerkin approximation (Shen, 1995).

Similarly as in (Ito and Morris, 1998, Sec. 3), letting $w_{dist}(t) \equiv 0$, taking an inner product $\langle \cdot, \cdot \rangle_\omega$ (the inner product of $L_\omega^2(-1, 1)$) of both sides of (1.1) with a $\psi_1 \in V_0$ and integrating by parts leads to the weak form

$$\langle \ddot{v}, \psi_1 \rangle_\omega + \langle EIV'' + d_{KV}I\dot{v}'', (\omega\psi_1)'' \rangle_{L^2} \quad (3.2a)$$

$$+ d_v \langle \dot{v}, \psi_1 \rangle_\omega = \langle b_1 u_1 + b_2 u_2, \psi_1 \rangle_\omega. \quad (3.2b)$$

Defining the state of the first order system as $x(t) = (v(\cdot, t), \dot{v}(\cdot, t))$, the above weak form can be written as

$$\langle \dot{x}(t), \psi \rangle_X + a(x(t), \psi) = \langle Bu(t), \psi \rangle_X, \quad \psi \in V$$

on $V = V_0 \times V_0$ if we define $Bu = [b_1 u_1 + b_2 u_2]^0$ for $u = (u_1, u_2)^T \in \mathbb{C}^2$ and

$$\begin{aligned} a(\phi, \psi) &= -\langle \phi_2, \psi_1 \rangle_{V_0} + \langle EIV\phi_1'' + d_{KV}I\dot{\phi}_2'', (\omega\psi_2)'' \rangle_{L^2} \\ &\quad + d_v \langle \phi_2, \psi_2 \rangle_\omega \end{aligned}$$

for $\phi = (\phi_1, \phi_2)^T \in V$, $\psi = (\psi_1, \psi_2)^T \in V$. By (Maday, 1990, Lem. 5.1) we can define an inner product on V_0 by

$$\langle \phi_1, \psi_1 \rangle_{V_0} = EI \langle \phi_1'', (\omega\psi_1)'' \rangle_{L^2}, \quad \phi_1, \psi_1 \in V_0,$$

and the norm $\|\cdot\|_{V_0}$ induced by $\langle \cdot, \cdot \rangle_{V_0}$ is equivalent to the norm on $H_\omega^2(-1, 1)$. The following lemma shows that if $X = V_0 \times L_\omega^2(-1, 1)$ and $V = V_0 \times V_0$ are equipped with the norms $\|(\phi_1, \phi_2)^T\|_X^2 = \|\phi_1\|_{V_0}^2 + \|\phi_2\|_\omega^2$ and $\|(\phi_1, \phi_2)^T\|_V^2 = \|\phi_1\|_{V_0}^2 + \|\phi_2\|_{V_0}^2$, then the form $a(\cdot, \cdot)$ is bounded and coercive, and therefore the standing assumptions in (Paunonen and Phan, 2020) are satisfied. The proof of Lemma 3.1 using (Maday, 1990) was given to the authors by the anonymous referee.

Lemma 3.1. The sesquilinear form $a(\cdot, \cdot)$ is bounded and coercive, i.e., there exist $q_1, q_2 > 0$ and $\lambda_0 \in \mathbb{R}$ such that

$$\begin{aligned} |a(\phi, \psi)| &\leq q_1 \|\phi\|_V \|\psi\|_V, & \forall \phi, \psi \in V \\ \operatorname{Re} a(\phi, \phi) &\geq q_2 \|\phi\|_V^2 - \lambda_0 \|\phi\|_X^2 & \forall \phi \in V. \end{aligned}$$

Proof. By (Maday, 1990, Lem. 5.1) there exists $\beta > 0$ such that $\|f\|_\omega \leq \beta \|f\|_{V_0}$ for all $f \in V_0$. Let $\phi = (\phi_1, \phi_2)^T \in V$ and $\psi = (\psi_1, \psi_2)^T \in V$ be arbitrary. Since $|\langle \phi_k, \psi_j \rangle_{V_0}| \leq \|\phi_k\|_{V_0} \|\psi_j\|_{V_0} \leq \|\phi\|_V \|\psi\|_V$ for $k, j \in \{1, 2\}$, we have

$$\begin{aligned} |a(\phi, \psi)| &= |-\langle \phi_2, \psi_1 \rangle_{V_0} + \langle \phi_1, \psi_2 \rangle_{V_0} + \frac{d_{KV}}{E} \langle \phi_2, \psi_2 \rangle_{V_0} \\ &\quad + d_v \langle \phi_2, \psi_2 \rangle_\omega| \\ &\leq \left(2 + \frac{d_{KV}}{E}\right) \|\phi\|_V \|\psi\|_V + d_v \|\phi_2\|_\omega \|\psi_2\|_\omega \\ &\leq \left(2 + \frac{d_{KV}}{E} + d_v \beta^2\right) \|\phi\|_V \|\psi\|_V. \end{aligned}$$

Thus the first claim holds with $q_1 = 2 + d_{KV}/E + d_v \beta^2$. Moreover, the definitions of $\|\cdot\|_V$ and $\|\cdot\|_X$ imply

$$\begin{aligned} \operatorname{Re} a(\phi, \phi) &= \frac{d_{KV}}{E} \langle \phi_2, \phi_2 \rangle_{V_0} + d_v \langle \phi_2, \phi_2 \rangle_\omega \\ &= \frac{d_{KV}}{E} \|\phi\|_V^2 - \frac{d_{KV}}{E} \|\phi_1\|_{V_0}^2 + d_v \|\phi_2\|_\omega^2 \\ &\geq \frac{d_{KV}}{E} \|\phi\|_V^2 - \frac{d_{KV}}{E} (\|\phi_1\|_{V_0}^2 + \|\phi_2\|_\omega^2) \end{aligned}$$

and thus the second claim holds with $q_2 = \lambda_0 = \frac{d_{KV}}{E}$. \square

The Galerkin approximation is defined by constructing a sequence of approximating finite-dimensional subspaces V^N of V . The approximating subspaces are required to have the property that (see (Morris, 1994, Sec. 5.2)) *any element $\phi \in V$ can be approximated by elements in V^N in the norm on V , i.e.,*

$$\forall \phi \in V \exists (\phi^N)_N, \phi^N \in V^N : \|\phi^N - \phi\|_V \xrightarrow{N \rightarrow \infty} 0.$$

The matrix $A^N : V^N \rightarrow V^N$ is defined by restricting $a(\cdot, \cdot)$ to $V^N \times V^N$, i.e.,

$$\langle -A^N \phi, \psi \rangle = a(\phi, \psi) \text{ for all } \phi, \psi \in V^N.$$

The input matrix $B^N \in \mathcal{L}(U, V^N)$ is defined by

$$\langle B^N u, \psi \rangle = \langle u, B^* \psi \rangle \text{ for all } \psi \in V^N,$$

and $C^N \in \mathcal{L}(V^N, Y)$ is the restriction of $C \in \mathcal{L}(X, Y)$ onto V^N . Approximating (or even knowing!) $B_d \in \mathcal{L}(U_d, X)$ is not necessary in our controller design.

In this paper we define the approximating subspaces $V^N = V_0^N \times V_0^N$ according to the *spectral Galerkin method* in (Shen, 1995). The basis functions ϕ_k of V^N are defined to be linear combinations of the Chebyshev polynomials $T_k(\xi)$ so that ϕ_k satisfy the boundary conditions of $V = V_0 \times V_0$. As shown in (Shen, 1995, Sec. 3.1), we can choose

$$V_0^N = \operatorname{span}\{\phi_0(\cdot), \dots, \phi_{N-4}(\cdot)\}$$

where for $k = \{0, \dots, N-4\}$ we have

$$\phi_k(\xi) = T_k(\xi) - \frac{2(k+2)}{k+3} T_{k+2}(\xi) + \frac{k+1}{k+3} T_{k+4}(\xi).$$

Then $V_0^N = \{f \in \operatorname{span}\{T_0(\cdot), \dots, T_N(\cdot)\} \mid f(\pm 1) = f'(\pm 1) = 0\}$. In the Galerkin approximation the solution $v(\xi, t)$ of (1.1) is approximated with

$$v_N(\xi, t) = \sum_{k=0}^{N-4} \alpha_k(t) \phi_k(\xi).$$

The matrices of the approximate system (A_N, B_N, C_N) are then derived from the system of ordinary differential equations which are obtained from the second order weak

form (3.2) with $\psi_2 = \phi_l$ for $l \in \{0, \dots, N-4\}$. The resulting system is

$$\begin{aligned} &\sum_{k=0}^{N-4} \langle \phi_k, \phi_l \rangle_\omega (\ddot{\alpha}_k(t) + d_v \dot{\alpha}_k(t)) \\ &\quad + \sum_{k=0}^{N-4} \langle \phi_k'', (\omega \phi_l)'' \rangle_{L^2} (EI \alpha_k(t) + d_{KV} I \dot{\alpha}_k(t)) \\ &= \langle b_1, \phi_l \rangle_\omega u_1(t) + \langle b_2, \phi_l \rangle_\omega u_2(t) \end{aligned}$$

for $l \in \{0, \dots, N-4\}$. The output matrix C_N defined by

$$y(t) = \begin{bmatrix} v_N(\xi_1, t) \\ v_N(\xi_2, t) \end{bmatrix} = \sum_{k=0}^{N-4} \begin{bmatrix} \phi_k(\xi_1) \\ \phi_k(\xi_2) \end{bmatrix} \alpha_k(t).$$

Setting $\alpha(t) = (\alpha_0(t), \dots, \alpha_{N-4}(t))^T$, we have

$$\begin{aligned} M \ddot{\alpha}(t) + EI \cdot F \alpha(t) + (d_{KV} F + d_v M) \dot{\alpha}(t) &= B_0^N u(t) \\ y(t) &= C_0^N \alpha(t), \end{aligned}$$

where $M = (\langle \phi_k, \phi_l \rangle_\omega)_{lk} \in \mathbb{R}^{(N-3) \times (N-3)}$ and $F = (\langle \phi_k'', (\omega \phi_l)'' \rangle_{L^2})_{lk} \in \mathbb{R}^{(N-3) \times (N-3)}$. The exact values of the inner products $M_{lk} := \langle \phi_k, \phi_l \rangle_\omega$ and $F_{lk} := \langle \phi_k'', (\omega \phi_l)'' \rangle_{L^2}$ are given in Lemma 3.2 below. The values $\langle b_1, \phi_l \rangle_\omega$ and $\langle b_2, \phi_l \rangle_\omega$ can be computed based on the (truncated) Chebyshev series expansions of b_1 and b_2 . Indeed, if $b_j(\xi) = \sum_{k=0}^{\infty} q_k^j T_k(\xi)$, then the orthogonality of the basis $\{T_k\}_{k=0}^{\infty}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\omega$ implies that for $j = 1, 2$ and $l \in \{0, \dots, N-4\}$ we have

$$\begin{aligned} \langle b_j, \phi_l \rangle_\omega &= \langle b_j, T_l \rangle_\omega - \frac{2(l+2)}{l+3} \langle b_j, T_{l+2} \rangle_\omega + \frac{l+1}{l+3} \langle b_j, T_{l+4} \rangle_\omega \\ &= \|T_l\|_\omega^2 q_l^j - \frac{2(l+2)}{l+3} \|T_{l+2}\|_\omega^2 q_{l+2}^j + \frac{l+1}{l+3} \|T_{l+4}\|_\omega^2 q_{l+4}^j. \end{aligned}$$

Here $\|T_0\|_\omega^2 = \pi$ and $\|T_l\|_\omega^2 = \pi/2$ for $l \geq 1$. In Matlab the required Chebyshev coefficients $\{q_l^j\}_{l=0}^{N-4}$, $j = 1, 2$, of b_1 and b_2 are readily available using **Chebfun**. Finally, based on the formula for $\phi_k(\cdot)$, the matrix C_0^N has the form $C_0^N = [c_0, \dots, c_{N-4}]$ where

$$c_k = \begin{bmatrix} T_k(\xi_1) \\ T_k(\xi_2) \end{bmatrix} - \frac{2(k+2)}{k+3} \begin{bmatrix} T_{k+2}(\xi_1) \\ T_{k+2}(\xi_2) \end{bmatrix} + \frac{k+1}{k+3} \begin{bmatrix} T_{k+4}(\xi_1) \\ T_{k+4}(\xi_2) \end{bmatrix}.$$

Lemma 3.2. (Shen, 1995, Lem. 3.1) The nonzero components M_{lk} , $l, k \in \{0, \dots, N-4\}$, of $M \in \mathbb{R}^{(N-3) \times (N-3)}$ are $M_{00} = 35\pi/18$,

$$M_{ll} = \frac{\pi [(l+1)^2 + 4(l+2)^2 + (l+3)^2]}{2(l+3)^2}, \quad l \geq 1$$

$$M_{l,l+2} = M_{l+2,l} = -\frac{\pi [(l+2)(l+5) + (l+1)(l+4)]}{(l+3)(l+5)}$$

$$M_{l,l+4} = M_{l+4,l} = \frac{\pi(l+1)}{2(l+3)}.$$

The nonzero elements F_{lk} , $l, k \in \{0, \dots, N-4\}$, of $F \in \mathbb{R}^{(N-3) \times (N-3)}$ are

$$F_{ll} = 8(l+1)^2(l+2)(l+4)\pi$$

$$F_{lk} = \frac{8\pi(l+1)(l+2) [l(l+4) + 3(k+2)^2]}{k+3},$$

for $k = l+2, l+4, \dots$

In summary, the matrices (A_N, B_N, C_N) of the Galerkin approximation are given by $C_N = [C_0^N, 0]$ and

$$A_N = \begin{bmatrix} 0 & I_{N \times N} \\ -EIM^{-1}F & -d_{KV}M^{-1}F - d_v I \cdot I_{N \times N} \end{bmatrix},$$

$$B_N = \begin{bmatrix} 0 \\ M^{-1}B_0^N \end{bmatrix}.$$

3.2 The Reduced Order Controller Design Algorithm

The following algorithm from (Paunonen and Phan, 2020, Sec. III.A) determines the parameters of the controller (3.1) so that (G_1, G_2) are as in **Step 1**, K_1^N is as in **Step 3**, and $(A_L^r, B_L^r, L^r, K_2^r)$ are as in **Step 4**. The parts G_1, G_2, K_1^N constitute *the internal model* of the controller, and its construction is based on the fact that the system (1.1) has two outputs, i.e., $Y = \mathbb{C}^2$.

PART I. The Internal Model

Step 1: Choose $Z_0 = Y^{2q+1} = \mathbb{C}^{4q+2}$,

$$G_1 = \text{diag}(0_2, \Omega_1, \dots, \Omega_q) \in \mathcal{L}(Z_0)$$

$$G_2 = [I_2, I_2, 0_2, \dots, I_2, 0_2]^T \in \mathcal{L}(Y, Z_0),$$

where $\Omega_k = \begin{bmatrix} 0_2 & \omega_k I_2 \\ -\omega_k I_2 & 0_2 \end{bmatrix}$ for all $k \in \{1, \dots, q\}$ and 0_2 and I_2 are the 2×2 zero and identity matrices. The pair (G_1, G_2) is controllable by construction.

PART II. The Galerkin Approximation and Stabilization.

Step 2: For a sufficiently large $N \in \mathbb{N}$, form a Galerkin approximation (A^N, B^N, C^N) on V^N of the control system (1.1) as described in Section 3.1.

Step 3: Choose parameters $\alpha_1, \alpha_2 \geq 0$, $Q_1 \in \mathcal{L}(U_0, X)$, and $Q_2 \in \mathcal{L}(X, Y_0)$ with U_0, Y_0 Hilbert spaces in such a way that the systems $(A + \alpha_1 I, Q_1, C)$ and $(A + \alpha_2 I, B, Q_2)$ are exponentially stabilizable and detectable. Let Q_1^N and Q_2^N be the approximations of Q_1 and Q_2 , respectively, according to the approximation V^N of V . Let $Q_0 \in \mathcal{L}(Z_0, \mathbb{C}^{p_0})$ be such that (Q_0, G_1) is observable, and let $R_1 \in \mathcal{L}(Y)$ and $R_2 \in \mathcal{L}(U)$ be positive definite matrices. Denote

$$A_s^N = \begin{bmatrix} G_1 & G_2 C^N \\ 0 & A^N \end{bmatrix}, \quad B_s^N = \begin{bmatrix} G_2 D \\ B^N \end{bmatrix}, \quad Q_s^N = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_2^N \end{bmatrix}.$$

Define $L^N = -\Sigma_N C^N R_1^{-1} \in \mathcal{L}(Y, V^N)$ and $K^N := [K_1^N, K_2^N] = -R_2^{-1} (B_s^N)^* \Pi_N \in \mathcal{L}(Z_0 \times V^N, U)$ where Σ_N and Π_N are the non-negative solutions of the finite-dimensional Riccati equations

$$\begin{aligned} (A^N + \alpha_1 I) \Sigma_N + \Sigma_N (A^N + \alpha_1 I)^* \\ - \Sigma_N (C^N)^* R_1^{-1} C^N \Sigma_N &= -Q_1^N (Q_1^N)^* \\ (A_s^N + \alpha_2 I)^* \Pi_N + \Pi_N (A_s^N + \alpha_2 I) \\ - \Pi_N B_s^N R_2^{-1} (B_s^N)^* \Pi_N &= -(Q_s^N)^* Q_s^N. \end{aligned}$$

With these choices $A_s^N + B_s^N K^N$ and $A^N + L^N C^N$ are Hurwitz if N is sufficiently large (Banks and Ito, 1997, Thm. 4.8).

PART III. The Model Reduction

Step 4: For a fixed and suitably large $r \in \mathbb{N}$, $r \leq N$, apply the Balanced Truncation method to the stable finite-dimensional system

$$(A^N + L^N C^N, [B^N + L^N D, L^N], K_2^N)$$

to obtain a stable r -dimensional reduced order system

$$(A_L^r, [B_L^r, L^r], K_2^r).$$

4. NUMERICAL SIMULATIONS

The simulation codes can be downloaded at the address github.com/lassipau/MTNS20-Matlab-simulations.

The controller is designed for a beam (1.1) with parameters

$$E = 10, \quad I = 1, \quad d_{KV} = 0.01, \quad d_v = 0.4.$$

The pointwise measurements of the deflection are at $\xi_1 = -0.6$ and $\xi_2 = 0.3$. The profile functions for the control $u(t) = (u_1(t), u_2(t))^T$ and disturbance $w_{dist}(t) \in \mathbb{R}$ are

$$b_1(\xi) = \frac{1}{3}(\xi + 1)^2(1 - \xi)^6, \quad b_2(\xi) = \frac{1}{3}(\xi + 1)^6(1 - \xi)^2$$

$$b_d(\xi) = \frac{1}{3}(\xi + 1)^2(1 - \xi)^2.$$

Our goal is to design a controller which is capable of tracking and rejecting continuous 2-periodic signals $y_{ref}(t)$ and $w_{dist}(t)$. To this end, we will choose the frequencies $0 = \omega_0 < \omega_1 < \dots < \omega_q$ to be $\omega_k = k\pi$ for $k \in \{0, \dots, q\}$ for some suitable value $q \in \mathbb{N}$, and we use $q = 10$ in our simulations. Due to the robustness of the controller, for any continuous 2-periodic reference signal $y_{ref}^{per}(t)$ the controller will track the truncated part $y_{ref}(t)$ of the form (1.2) with perfect accuracy, and the remaining part $y_{ref}(t) - y_{ref}^{per}(t)$ will appear as an additional external disturbance in the control loop. However, due to the stability of the closed-loop system, the effect of this difference signal on the asymptotic regulation error $e(t)$ is guaranteed to be small as long as the quantity $\max_{t \in [0, 2\pi]} \|y_{ref}(t) - y_{ref}^{per}(t)\|$ is sufficiently small. The smallness of the truncation error can be guaranteed for the most important continuous 2-periodic functions $y_{ref}^{per}(\cdot)$ (outside pathological situations) by choosing a suitably large $q \in \mathbb{N}$. In the simulations we demonstrate the above property of the controller in the tracking of a triangle signal which contains an infinite number of frequency components.

The spectral Galerkin approximation used in the controller design is completed with $\dim V_0^N = 39$, in which case the dimension of the approximate first order system (A^N, B^N, C^N) is 78. The uncontrolled system is exponentially stable and it has a stability margin which is approximately 0.35. In **Step 3** of controller design we choose the parameters as $\alpha_1 = 2$, $\alpha_2 = 0.8$ and $R_1 = R_2 = I$, $Q_0 = I$, $Q_1 = I$ and $Q_2 = I$. The order $r \leq N$ of the model reduction in **Step 4** of the controller design can be chosen to be as low as $r = 4$ while still achieving exponential closed-loop stability. Using such a low-dimensional ‘‘observer-part’’ in the controller is made possible by the fact that the uncontrolled system already has quite strong stability properties. The resulting closed-loop system has a stability margin ≈ 1.01 .

Since the internal model in the controller has dimension $2(2q + 1) = 42$, the full size of the controller is $\dim Z = 46$. For simulations we approximate the controlled beam system (1.1) with a separate higher-dimensional spectral Galerkin approximation with $\dim V_0^N = 69$. Figure 1 plots some of the eigenvalues of the closed-loop system.

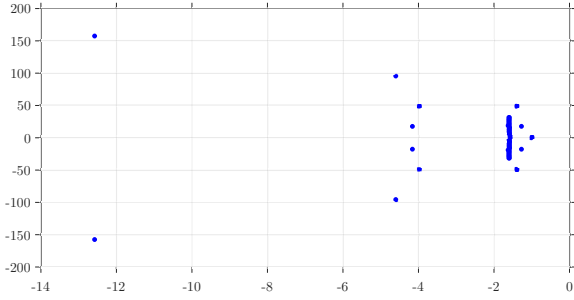


Fig. 1. Eigenvalues of the closed-loop system.

The controller design algorithm in Section 3.2 requires that the system does not have transmission zeros at the (complex) frequencies $\{i\omega_k\}_{k=0}^q \subset i\mathbb{R}$ of the reference and disturbance signals. This property can be tested numerically using the Galerkin approximation of (1.1) and it is satisfied in our simulations.

The designed internal model based controller is capable of tracking any reference signal $y_{ref}(t)$ and any disturbance signal $w_{dist}(t)$ of the form (1.2) with frequencies $(\omega_k)_{k=0}^{10}$ with arbitrary unknown amplitudes and phases. As explained above, the controller will also *approximately* track and reject any continuous 2-periodic reference and disturbance signals. Figure 2 shows the output $y(t)$ and the tracking error in the case where the first component of $y_{ref}(t)$ is a 2-periodic triangle signal, the second component of $y_{ref}(t)$ is identically zero, and the disturbance signal is $w_{dist}(t) = \sin(\pi t) + 0.4 \cos(3\pi t)$. The initial deflection $v_0(\xi)$, the initial velocity $v_1(\xi)$ and the initial state of the controller are all zero.

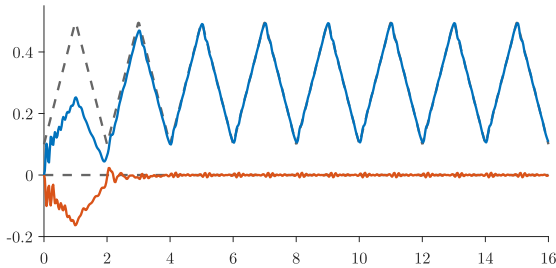


Fig. 2. The output $y(t)$ and the reference signal $y_{ref}(t)$.

Figure 3 plots the norm of the tracking error $\|e(t)\| = \|y(t) - y_{ref}(t)\|_{\mathbb{R}^2}$. The asymptotic residual error is due to the fact that the internal model contains only a finite number of frequencies and is therefore not capable of tracking the nonsmooth triangle signal with perfect accuracy.

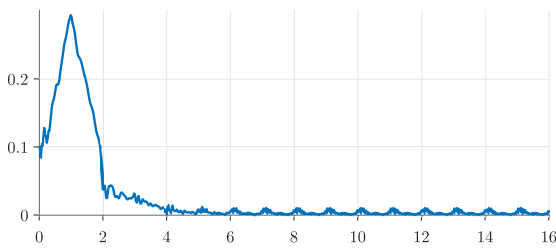


Fig. 3. The norm $\|e(t)\|$ of the tracking error.

Figure 4 depicts the control actions $u_1(t)$ and $u_2(t)$ in the simulation. Finally, Figure 5 plots the deflection $v(\xi, t)$ of the controlled beam.

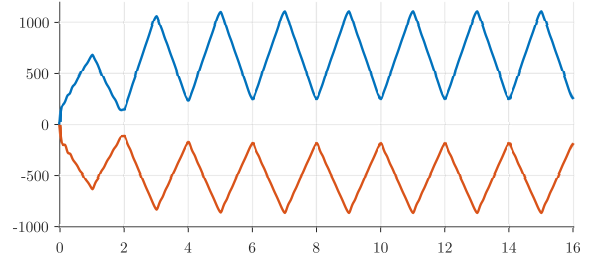


Fig. 4. The control inputs $u_1(t)$ (blue) and $u_2(t)$ (red).

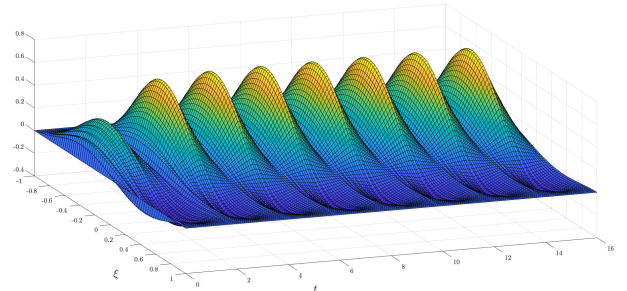


Fig. 5. The deflection of the controlled beam.

4.1 Comparison with a Low-Gain Controller

Because the beam system is exponentially stable, the output tracking problem can alternatively be solved with a “simple” internal model based controller structure whose dynamics only contain the internal model, i.e.,

$$\begin{aligned} \dot{z}(t) &= G_1 z(t) + G_2 e(t), & z(0) &\in \mathbb{R}^{n_c}, \\ u(t) &= K z(t), \end{aligned}$$

where G_1 and G_2 are as in Section 3.2, and thus $n_c = 2(2q+1) = 42$. As shown in (Hämäläinen and Pohjolainen, 2000; Rebarber and Weiss, 2003) or (Paunonen, 2016, Rem. 10), the matrix $K \in \mathbb{R}^{2 \times n_c}$ can be chosen as

$$K = \varepsilon [P(0)^{-1}, \operatorname{Re} P(i\omega_1)^{-1}, \operatorname{Im} P(i\omega_1)^{-1}, \dots, \operatorname{Re} P(i\omega_q)^{-1}, \operatorname{Im} P(i\omega_q)^{-1}],$$

where $P(\lambda) \in \mathbb{C}^{2 \times 2}$ is the transfer function of the system (1.1). For this choice of K , there exists $\varepsilon^* > 0$ such that for any $0 < \varepsilon \leq \varepsilon^*$ the controller achieves output tracking and disturbance rejection for the beam system. In the controller design the values $P(i\omega_k)$ can be reliably computed using a Galerkin approximation of the system due to the results in (Morris, 1994).

The low-gain parameter $\varepsilon > 0$ should be designed so that the closed-loop has the best possible stability margin, and this can be achieved using root locus type analysis. For the beam system (1.1) with the chosen parameters and the internal model (G_1, G_2) with frequencies $\omega_k = k\pi$ for $k \in \{0, \dots, 10\}$, the stabilization of the closed-loop system arising from this controller structure is very difficult due to the high number of unstable frequencies in the internal model. If the number of frequencies is reduced to $\omega_k = k\pi$ for $k \in \{0, \dots, 5\}$ (potentially resulting in lower accuracy of the tracking), the best achievable closed-loop stability margin is approximately 0.0382 with parameter $\varepsilon = 0.076$. The margin can be compared to the stability margin ≈ 1.01 achieved with the reduced order controller in Section 3.2. This smaller stability margin leads to a significantly slower rate of convergence of the

output. Figure 6 depicts the output of the controlled system with the low-gain controller with $\varepsilon = 0.076$.

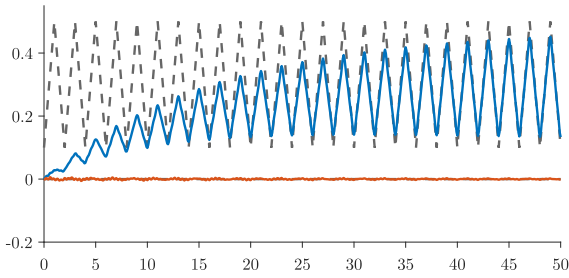


Fig. 6. Output of the system with the low-gain controller.

Note that since in our main controller we were able to choose the size of the reduced order model in **Step 4** as $r = 4$, the dimension of our reduced order controller is only higher by 4 compared to the low-gain controller, but this additional structure of the controller allows the design of a significantly larger stability margin. Since the second controller is based on a low-gain design, it is natural to ask if using this controller leads to reduced control action. However, in the case of a stable linear system with the same number of inputs and outputs the control action required to produce the desired asymptotic output $y_{ref}(t)$ does not depend on the type of the internal model based controller (see (Paunonen, 2017, Lem. 3.10 and proofs of Thm. 3.3 and Lem. 3.4)). Therefore in our simulation the low-gain controller does not achieve output regulation with smaller control gains and the controller requires less control action only for small values of $t \geq 0$. The control inputs produced by the low-gain controller are depicted in Figure 7.

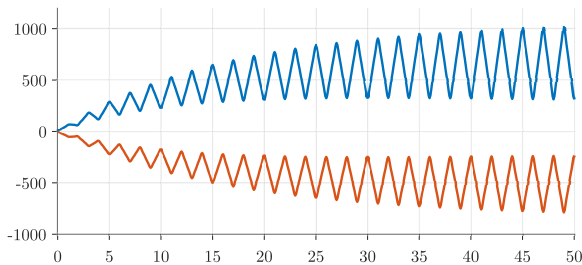


Fig. 7. Control inputs $u_1(t)$ (blue) and $u_2(t)$ (red) produced by the low-gain controller.

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