Reduced Order Controller Design for Robust Output Regulation

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We study robust output regulation for parabolic partial differential equations and other infinite-dimensional linear systems with analytic semigroups. As our main results we show that robust output tracking and disturbance rejection for our class of systems can be achieved using a finite-dimensional controller and present algorithms for construction of two different internal model based robust controllers. The controller parameters are chosen based on a Galerkin approximation of the original PDE system and employ balanced truncation to reduce the orders of the controllers. In the second part of the paper we design controllers for robust output tracking and disturbance rejection for a 1D reaction–diffusion equation with boundary disturbances, a 2D diffusion–convection equation, and a 1D beam equation with Kelvin–Voigt damping.

***Index Terms*—Robust output regulation, partial differential equation, controller design, Galerkin approximation, model reduction.**

I. Introduction

In the *robust output regulation problem* the main objective is to design a dynamic error feedback controller so that the output $y(t)$ of the linear infinite-dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{dist}(t), \quad x(0) = x_0 \in X \tag{1a}$$
$$y(t) = Cx(t) + Du(t) + D_d w_{dist}(t) \tag{1b}$$

on a Hilbert space X converges to a given reference signal $y_{ref}(t)$ despite the external disturbance signal $w_{dist}(t)$, i.e.,

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad \text{as} \quad t \to \infty.$$

In addition, the control is required to be robust in the sense that the designed controller achieves the output tracking and disturbance rejection even under uncertainties and perturbations in the parameters (A, B, B_d, C, D, D_d) of the system (see Section II for the detailed assumptions on (1)). The closed-loop system consisting of (1) and a dynamic error feedback controller is depicted in Figure 1. In particular the controller only uses the knowledge of the regulation error $e(t) = y(t) - y_{ref}(t)$.

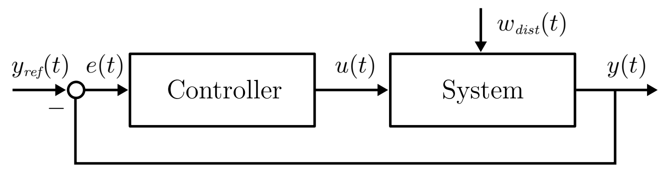

Fig. 1. Dynamic error feedback control scheme.

The design of controllers for robust output regulation of infinite-dimensional linear systems has been studied in several references [13], [14], [18], [24], [25], [29], [31], and many articles also study the controller design for output tracking and disturbance rejection without the robustness requirement [4], [9], [10], [33], [37]. In this paper we concentrate on construction of *finite-dimensional* low-order robust controllers for control systems (1) with distributed inputs and outputs. The motivation for this research arises from the fact that the robust controllers introduced in earlier references [14], [24] are necessarily inifinite-dimensional unless the system (1) is either exponentially stable or stabilizable by static output feedback.

As the main results of this paper we introduce two finite-dimensional controllers that solve the robust output regulation problem for possibly unstable parabolic PDE systems. The controller design is based on the *internal model principle* [8], [12], [26] which characterizes the solvability of the control problem. The general structures of the controllers are based on two infinite-dimensional controllers presented in [14] and [24], respectively. Both of the infinite-dimensional controllers from [14], [24] incorporate an observer-type copy of the original system that is used in stabilizing the closed-loop system. In this paper these observer-type parts are replaced with finite-dimensional low-order systems that are constructed based on a Galerkin approximation of the system (A, B, C, D) and subsequent model reduction using balanced truncation. All controller parameters are computed based on a finite-dimensional approximation of (A, B, C, D) and only involve matrix computations. In particular, when using the Finite Element Method, both the approximation of the system (1) and the model reduction step in the controller construction can be completed efficiently using existing software implementations, and this facilitates straightforward construction of our robust controllers even for complicated PDE systems. The finite-dimensional controllers introduced in this paper can also be preferable to the low-gain robust controllers [13], [18], [31] for exponentially stable systems, since they can typically achieve larger closed-loop stability margins and faster convergence rates of the output.

In the second part of the paper we employ the construction algorithms to design controllers for robust output regulation of selected classes of PDE models — a 1D reaction–diffusion equation, a 2D reaction–diffusion–convection equation, and a 1D beam equation with Kelvin–Voigt damping. The general assumptions on the Galerkin approximation scheme used in the controller design have been verified in the literature for several classes of PDE models and the Finite Element approximation

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schemes used in this paper.

The possibility of using Galerkin approximations in the controller design is based on the theory developed in [1], [2], [15], [16], [21], [23], [36]. Using Galerkin approximations in dynamic stabilization is a well-known and frequently used technique [6], [16], [22], and in this paper we employ the same methodology in constructing finite-dimensional low-order controllers for robust output regulation. In the proofs of our main results we show that the closed-loop systems with our reduced order controllers approximate — in the sense of graph topology — closed-loop systems with infinite-dimensional controllers which can be shown to achieve closed-loop stability, and therefore the controllers achieve robust output regulation provided that the orders of the approximations are sufficiently high. The graph topology was first used for the dynamic stabilization problem with Galerkin approximations in [22], and a detailed theoretic framework for constructing controllers based on balanced truncations was presented in [6]. Our proofs are especially based on the techniques in [21], [22]. Controller construction for robust output regulation using Galerkin approximations was first studied in [28] for a 1D heat equation with constant coefficients. In this paper we improve and extend the controller design method to be applicable for a larger class of control systems, include model reduction as a part of the design procedure, and consider two different controller structures.

The reference signals $y_{ref} : \mathbb{R} \to \mathbb{C}^p$ and the disturbance signals $w_{dist} : \mathbb{R} \to \mathbb{C}^{2q+1}$ we consider are of the form

$$y_{ref}(t) = a_0^1(t) + \sum_{k=1}^{q} (a_k^1(t)\cos(\omega_k t) + b_k^1(t)\sin(\omega_k t)) \quad (2a)$$

$$w_{dist}(t) = a_0^2(t) + \sum_{k=1}^{q} (a_k^2(t)\cos(\omega_k t) + b_k^2(t)\sin(\omega_k t)) \quad (2b)$$

for some known frequencies $\{\omega_k\}_{k=0}^q \subset \mathbb{R}$ with $0 = \omega_0 < \omega_1 < \ldots < \omega_q$ and *unknown* coefficient polynomial vectors $\{a_k^j(t)\}_{k,j}$ and $\{b_k^j(t)\}_{k,j}$ with real or complex coefficients (any of the polynomials are allowed to be zero). We assume the maximum orders of the coefficient polynomial vectors are known, so that $a_k^1(t), b_k^1(t) \in \mathbb{C}^p$ and $a_k^2(t), b_k^2(t) \in \mathbb{C}^{m_d}$ are polynomial of order at most $n_k - 1$ for each k.

Remark I.1. In (2), $a_0^1(t)$ and $a_0^2(t)$ correspond to the frequency $\omega_0 = 0$. The constructions of the controllers are carried out with ω_0 being present, but there are situations where tracking of signals with this frequency component can not be achieved (namely, when the system (1) has an invariant zero at $0 \in \mathbb{C}$). In this situation the construction of the matrices G_1, G_2, and K_1 in Section III can be modified in a straightforward manner to remove this frequency from the controller.

Throughout the paper we consider distributed control and observation, i.e., B and C are bounded linear operators. Also the disturbance input operator B_d is assumed to be bounded, but under this assumption it is also possible to reject boundary disturbances for many classes of PDEs as demonstrated in Section V-A. Indeed, since $w_{dist}(\cdot)$ in (2) is smooth, boundary disturbances can in many situations be written in the form (1) with a bounded operator B_d and a modified disturbance signal including the derivative $\dot{w}_{dist}(\cdot)$ [7, Sec. 3.3]. Since $\dot{w}_{dist}(\cdot)$ is also of the form (2b) with the same frequencies and coefficient polynomial vectors of order at most $n_k - 1$, the modified disturbance signal belongs to the same original class of signals. Moreover, since the operators B_d and D_d are not used in any way in the controller construction in Section III, rejection of boundary disturbances can be done without computing B_d and D_d explicitly — it is sufficient to know such operators exist. This extremely useful property is based on the fact that a robust internal model based controller will achieve disturbance rejection for *any* disturbance input and feedthrough operators B_d and D_d and any signals of the form (2).

The paper is organised as follows. In Section II we state the standing assumptions, formulate the robust output regulation problem, and summarise the Galerkin approximations and the balanced truncation method. In Section III we present our main results including the construction of the two finite-dimensional robust controllers. The main theorems are proved in Section IV. Section V focuses on robust controller design for particular PDE models. Concluding remarks are presented in Section VI. Section A contains helpful lemmata.

A. Notation

The inner product on a Hilbert space X is denoted by $\langle \cdot, \cdot \rangle$. For a linear operator $A : X \to Y$ we denote by $D(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$, and $\rho(A)$ denote the spectrum, the point spectrum, and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1}$. For a fixed $\alpha \in \mathbb{R}$ we denote

$$\mathcal{H}_\infty(\mathbb{C}_\alpha^+) = \left\{ G : \mathbb{C}_\alpha^+ \to \mathbb{C} \;\middle|\; G \text{ is analytic, } \sup_{s \in \mathbb{C}_\alpha^+} |G(s)| < \infty \right\}$$

where $\mathbb{C}_\alpha^+ = \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \alpha \}$. For $\alpha = 0$ we use the notation $\mathcal{H}_\infty = \mathcal{H}_\infty(\mathbb{C}_0^+)$. We denote by $M(\mathcal{H}_\infty)$ the set of matrices with entries in $\mathcal{H}_\infty$.

II. Robust Output Regulation, Galerkin Approximation, and Model Reduction

In this section we state our main assumption on the system (1) and the controller and formulate the robust output regulation problem. We also review selected important background results concerning Galerkin approximations and balanced truncation.

We consider a control system (1) on a Hilbert space X, and we assume $V \subset X$ is another Hilbert space with a continuous and dense injection $\iota : V \to X$. Let $a(\cdot, \cdot) : V \times V \to \mathbb{C}$ be a bounded and coercive sesquilinear form, i.e., there exist $c_1, c_2, \lambda_0 > 0$ such that for all $\phi, \ \psi \in V$ we have

$$|a(\phi, \psi)| \le c_1 \|\phi\|_V \|\psi\|_V$$

$$\operatorname{Re} a(\phi, \phi) + \lambda_0 \|\phi\|_X^2 \ge c_2 \|\phi\|_V^2.$$

We assume A is defined by $a(\cdot, \cdot)$ so that

$$\langle -A\phi, \psi \rangle = a(\phi, \psi), \qquad \forall \phi \in D(A), \psi \in V,$$
$$D(A) = \{ \phi \in V \mid a(\phi, \cdot) \text{ has an extension to } X \}.$$

As shown in [1, Sec. 2], the operator $A : D(A) \subset X \to X$ is such that $A - \lambda_0 I$ generates an analytic semigroup on X.

In (1) B, C, and D are the *input operator*, *output operator* and *feedthrough operator*, respectively, and B_d and D_d are the input operator and feedthrough operator, respectively, for the disturbance input $w_{dist}(t)$. These operators are assumed to be bounded so that $B \in \mathcal{L}(U, X)$, $B_d \in \mathcal{L}(U_d, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$, and $D_d \in \mathcal{L}(U_d, Y)$ where $U = \mathbb{C}^m$ or $U = \mathbb{R}^m$ is the input space, $U_d = \mathbb{C}^{m_d}$ or $U_d = \mathbb{R}^{m_d}$ is the disturbance input space, and $Y = \mathbb{C}^p$ or $Y = \mathbb{R}^p$ is the output space. We assume the pair (A, B) is exponentially stablizable and (C, A) is exponentially detectable. The transfer function of (1) is denoted by

$$P(\lambda) = CR(\lambda, A)B + D, \qquad \lambda \in \rho(A).$$

We make the following standing assumption which is also necessary for the solvability of the robust output regulation problem. The condition means that (A, B, C, D) is not allowed to have invariant zeros at the frequencies $\{i\omega_k\}_{k=0}^q$ in (2).

Assumption II.1. *Let $K \in \mathcal{L}(X, U)$ be such that $A + BK$ generates an exponentially stable semigroup. We assume $P_K(i\omega_k) = (C + DK)R(i\omega_k, A + BK)B + D \in \mathbb{C}^{p\times m}$ is surjective for every $k \in \{0, \dots, q\}$.*

Due to standard operator identities, the surjectivity of $P_K(i\omega_k)$ is independent of the choice of the stabilizing feedback operator K. Moreover, for any $k \in \{0, \dots, q\}$ for which $i\omega_k \in \rho(A)$ the matrix $P_K(i\omega_k)$ is surjective if and only if $P(i\omega_k)$ is surjective.

We consider the design of internal model based error feedback controllers of the form

$$\dot z(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t) \tag{3a}$$
$$u(t) = K z(t) \tag{3b}$$

where $e(t) = y(t) - y_{ref}(t)$ is the *regulation error*, $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, and $K \in \mathcal{L}(Z, U)$. Letting $x_e(t) = (x(t), z(t))^T$ and $w_{ext}(t) = (w_{dist}(t), y_{ref}(t))^T$, the system and the controller can be written together as a *closed-loop system* on the Hilbert space $X_e = X \times Z$ (see [14], [26] for details)

$$\dot x_e(t) = A_e x_e(t) + B_e w_{ext}(t), \qquad x_e(0) = x_{e0}$$
$$e(t) = C_e x_e(t) + D_e w_{ext}(t)$$

where $x_{e0} = (x_0, z_0)^T$ and

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \qquad B_e = \begin{bmatrix} B_d & 0 \\ \mathcal{G}_2 D_d & -\mathcal{G}_2 \end{bmatrix},$$
$$C_e = \begin{bmatrix} C, & DK \end{bmatrix}, \qquad D_e = \begin{bmatrix} D_d, & -I \end{bmatrix}.$$

The operator A_e generates a strongly continuous semigroup $T_e(t)$ on X_e.

The Robust Output Regulation Problem. Choose $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

(a) The semigroup $T_e(t)$ is exponentially stable.

(b) There exists $M_e, \omega_e > 0$ such that for all initial states $x_0 \in X$ and $z_0 \in Z$ and for all signals $w_{dist}(t)$ and $y_{ref}(t)$ of the form (2) we have

$$\|y(t) - y_{ref}(t)\| \le M_e e^{-\omega_e t}(\|x_{e0}\| + \|\Lambda\|). \tag{4}$$

where Λ is a vector containing the coefficients of the polynomials $\{a_k^j(t)\}_{k,j}$ and $\{b_k^j(t)\}_{k,j}$ in (2).

(c) When (A, B, B_d, C, D, D_d) are perturbed to $(\tilde A, \tilde B, \tilde B_d, \tilde C, \tilde D, \tilde D_d)$ in such a way that the perturbed closed-loop system remains exponentially stable, then for all $x_0 \in X$ and $z_0 \in Z$ and for all signals $w_{dist}(t)$ and $y_{ref}(t)$ of the form (2) the regulation error satisfies (4) for some modified constants $\tilde M_e, \tilde\omega_e > 0$.

The *internal model principle* [26, Thm. 6.9] implies that in order to achieve robust output tracking of the reference signal $y_{ref}(t)$, it is both necessary and sufficient that the following are satisfied.

- The controller (3) incorporates *an internal model* of the reference and disturbance signals in (2).
- The semigroup $T_e(t)$ generated by A_e is exponentially stable.

As shown in Section III, the internal model property of the controller can be guaranteed by choosing a suitable structure for the operator $\mathcal{G}_1$. The rest of structure and parameters of the controller are then chosen so that the closed-loop system becomes exponentially stable.

A. Background on Galerkin Approximations

Let V^N be a sequence of finite dimensional subspaces of V and let P^N be the orthogonal projection of X onto V^N. Throughout the paper we assume the approximating subspaces (V^N) have the property that any element $\phi \in V$ can be approximated by elements in V^N in the norm on V, i.e.,

$$\forall \phi \in V\, \exists (\phi^N)_N,\ \phi^N \in V^N : \quad \|\phi^N - \phi\|_V \xrightarrow{N\to\infty} 0. \tag{5}$$

We define the approximations $A^N : V^N \to V^N$ of A by

$$\langle -A^N \phi, \psi\rangle = a(\phi, \psi) \quad \text{for all} \quad \phi, \psi \in V^N,$$

that is, A^N is defined via the restriction of $a(\cdot,\cdot)$ to $V^N \times V^N$. For $B \in \mathcal{L}(U, X)$ we define $B^N \in \mathcal{L}(U, V^N)$ by

$$\langle B^N u, \psi\rangle = \langle u, B^*\psi\rangle \quad \text{for all} \quad \psi \in V^N,$$

and $C^N \in \mathcal{L}(V^N, Y)$ is defined as the restriction of $C \in \mathcal{L}(X, Y)$ onto V^N. Note that computing the Galerkin approximation of $B_d \in \mathcal{L}(U_d, X)$ is not necessary.

Lemma II.2. *Under the standing assumptions on A and the approximating finite-dimensional subspaces V^N, the following hold.*

(a) *If $\tilde B \in \mathcal{L}(\mathbb{C}^{m_0}, X)$ and $\tilde C \in \mathcal{L}(X, \mathbb{C}^{m_0})$, then*

$$\|P^N R(\lambda, A + \tilde B\tilde C)x - R(\lambda, A^N + \tilde B^N \tilde C^N)P^N x\| \xrightarrow{N\to\infty} 0$$

for all $\lambda \in \rho(A + \tilde B\tilde C)$ and $x \in X$.

(b) *Let $\tilde B \in \mathcal{L}(\mathbb{C}^{m_0}, X)$ and $\tilde C \in \mathcal{L}(X, \mathbb{C}^{p_0})$ be such that $(A, \tilde B, \tilde C)$ is exponentially stabilizable and detectable. If*

$(\tilde{B}_0^N)_N$ and $(\tilde{C}_0^N)_N$ are two sequences such that $\tilde{B}_0^N \in \mathcal{L}(\mathbb{C}^{m_0}, V^N)$ and $\tilde{C}_0^N \in \mathcal{L}(V^N, \mathbb{C}^{p_0})$ for all N and

$$\|\tilde{B}_0^N - \tilde{B}\|_{\mathcal{L}(U,X)} \to 0 \quad \text{and} \quad \|\tilde{C}_0^N P^N - \tilde{C}\|_{\mathcal{L}(X,Y)} \to 0$$

as $N \to \infty$, then $\tilde{C}_0^N R(\cdot, A^N)\tilde{B}_0^N$ converge to the transfer function $\tilde{C}R(\cdot, A)\tilde{B}$ in the graph topology of $M(\mathcal{H}_\infty)$ as $N \to \infty$.

Proof. It is shown in [21, Thm. 5.2] that

$$\|P^N R(\lambda, A)x - R(\lambda, A^N)P^N x\| \overset{N\to\infty}{\longrightarrow} 0 \qquad \forall x \in X$$

for some $\lambda \in \rho(A)$. Since $\tilde{B}^N \tilde{C}^N P^N \to \tilde{B}\tilde{C}$ strongly as $N \to \infty$, the resolvent identity and standard perturbation formulas imply that part (a) is true.

To prove part (b), let $K \in \mathcal{L}(X, U)$ be such that $A + \tilde{B}K$ is exponentially stable. Then by [21, Thm. 5.2–5.3] and standard perturbation theory $A^N + \tilde{B}_0^N K P^N$ are uniformly exponentially stable for large N. The functions $\tilde{C}_0^N R(\cdot, A^N)\tilde{B}_0^N$ and $\tilde{C}R(\cdot, A)\tilde{B}$ have right coprime factorizations in $M(\mathcal{H}_\infty)$ given by

$$\begin{aligned}\tilde{C}_0^N R(\cdot, A^N)\tilde{B}_0^N &= \tilde{C}_0^N R(\cdot, A^N + \tilde{B}_0^N K P^N)\tilde{B}_0^N \\ &\quad \times (I + K P^N R(\cdot, A^N + \tilde{B}_0^N K P^N)\tilde{B}_0^N)^{-1} \\ \tilde{C}R(\cdot, A)\tilde{B} &= \tilde{C}R(\cdot, A + \tilde{B}K)\tilde{B}(I + KR(\cdot, A + \tilde{B}K)\tilde{B})^{-1}.\end{aligned}$$

To conclude that $\tilde{C}_0^N R(\cdot, A^N)\tilde{B}_0^N$ converges to $\tilde{C}R(\cdot, A)\tilde{B}$ in the graph topology, it suffices to show that $\tilde{C}_0^N R(\cdot, A^N + \tilde{B}_0^N K P^N)\tilde{B}_0^N$ and $K P^N R(\cdot, A^N + \tilde{B}_0^N K P^N)\tilde{B}_0^N$ converge to $\tilde{C}R(\cdot, A + \tilde{B}K)\tilde{B}$ and $KR(\cdot, A + \tilde{B}K)\tilde{B}$ in $M(\mathcal{H}_\infty)$, respectively. We will only show the convergence of $\tilde{C}_0^N R(\cdot, A^N + \tilde{B}_0^N K P^N)\tilde{B}_0^N$ since the second convergence can be shown analogously.

By [21, Thm. 4.2 & Cor. 4.3] the transfer functions $\tilde{C}P^N R(\cdot, A^N + P^N \tilde{B} K P^N)P^N \tilde{B}$ converge to $\tilde{C}R(\cdot, A + \tilde{B}K)\tilde{B}$ in $M(\mathcal{H}_\infty(\mathbb{C}^+_{-\varepsilon}))$ for some $\varepsilon > 0$. Standard perturbation theory implies that for small $\varepsilon > 0$ we also have $\sup_{\mathrm{Re}\,\lambda < -\varepsilon} \|R(\lambda, A^N + \tilde{B}_0^N K P^N) - R(\lambda, A^N + P^N \tilde{B} K P^N)\| \to 0$ as $N \to \infty$. Together with the convergences of $\tilde{B}_0^N$ and $\tilde{C}_0^N$ and the triangle inequality it is easy to show that $\tilde{C}_0^N R(\cdot, A^N + \tilde{B}_0^N K P^N)\tilde{B}_0^N$ converges to $\tilde{C}R(\cdot, A + \tilde{B}K)\tilde{B}$ in $M(\mathcal{H}_\infty)$. This completes the proof. □

B. Model Reduction via Balanced Truncation

We use balanced truncation [20], [27] to reduce the order of our controllers. For a general minimal and stable finite-dimensional system (A, B, C) on $\mathbb{C}^N$ the reduced order model (A^r, B^r, C^r) on $\mathbb{C}^r$ is computed as follows [3, Sec. 2.1].

(1) Find a minimal "internally balanced realization" (A_b, B_b, C_b) of (A, B, C) as described in [3, Sec. 2.1].

(2) The controllability Gramian $\Sigma_B \geq 0$ and the observability Gramian $\Sigma_C \geq 0$ of (A_b, B_b, C_b), defined as the solutions of

$$\begin{cases} A_b \Sigma_B + \Sigma_B A_b^* = -B_b B_b^* \\ A_b^* \Sigma_C + \Sigma_C A_b = -C_b^* C_b, \end{cases}$$

have the property $\Sigma_B = \Sigma_C = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N)$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N > 0$ are the *Hankel singular values* of (A, B, C).

(3) If we write

$$A_b = \begin{bmatrix} A^r & A_b^{12} \\ A_b^{12} & A_b^{22} \end{bmatrix}, \quad B_b = \begin{bmatrix} B^r \\ B_b^2 \end{bmatrix}, \quad C_b = \begin{bmatrix} C^r, & C_b^2 \end{bmatrix}$$

where $A^r \in \mathbb{C}^{r\times r}$, $B^r \in \mathbb{C}^{r\times m}$ and $C^r \in \mathbb{R}^{p\times r}$, then (A^r, B^r, C^r) is the desired reduced order model.

Lemma II.3. *The distance in the graph topology between the stable system (A, B, C) on $\mathbb{C}^N$ and its balanced truncation (A^r, B^r, C^r) satisfies*

$$d(CR(\cdot, A)B, C^r R(\cdot, A^r)B^r) \leq M \sum_{k=r+1}^{N} \sigma_k$$

for some constant $M > 0$ independent of $r \in \{1, \ldots, N\}$.

Proof. The convergence in the graph topology follows from the corresponding $M(\mathcal{H}_\infty)$-error bound [11] and the fact that for stable systems the distance in the graph topology and $M(\mathcal{H}_\infty)$-norm are equivalent. □

Remark II.4. Improved numerical stability of the model reduction algorithm can be achieved by omitting the explicit computation of the balanced realization and instead using a "balancing-free" method such those in [34] (`balred` in Matlab) or [32] (`hankelmr` in Matlab). Both of these methods produce reduced order models which satisfy the estimate in Lemma II.3. As demonstrated by the proofs in Section IV, the balanced truncation can be replaced by any other model reduction method that approximates a stable finite-dimensional system in the $M(\mathcal{H}_\infty)$-norm.

III. Finite-Dimensional Robust Controller Design

In this section we present algorithms for constructing two finite-dimensional reduced order controllers that solve the robust output regulation problem. The constructions use the following data:

- Frequencies $\{\omega_k\}_{k=1}^q$ of the reference and disturbance signals (2).
- Maximal orders $n_k - 1$ of the coefficient polynomials $a_k^1(t), a_k^2(t), b_k^1(t)$, and $b_k^2(t)$ associated to each ω_k in (2).
- The dimension of the output space $\dim Y = p$
- Galerkin approximations (A^N, B^N, C^N) of (1).
- The values $P(i\omega_k)$ of the transfer function through the invertibility condition of $P(i\omega_k)K_1^{1k}$ (only for the dual observer-based controller when $\dim Y < \dim U$).

The construction does not use any information on the disturbance operators B_d and D_d or knowledge of the phases and amplitudes of $y_{ref}(\cdot)$ and $w_{dist}(\cdot)$. Indeed, robustness guarantess that the same controller will achieve output tracking and disturbance rejection for *any* operators B_d and D_d, and for all coefficient polynomials $a_k^1(t), a_k^2(t), b_k^1(t)$, and $b_k^2(t)$ of orders at most $n_k - 1$.

In the constructions, the role of the component G_1 of the system matrix $\mathcal{G}_1$ is to guarantee that the controller contains a suitable internal model of the signals (2). Expressed in terms of spectral properties, the internal model requires that $i\omega_k \in \sigma_p(\mathcal{G}_1)$ *for all* $k \in \{0, \ldots, q\}$ *and $\mathcal{G}_1$ has at least* $p = \dim Y$ *independent Jordan chains of length greater than* *or equal to n_k associated to each eigenvalue $i\omega_k$* (see [24, Def. 4]). The steps following the choice of G_1 fix the remaining parameters of the controllers in such a way that the closed-loop system becomes exponentially stable. The choices of the parameters are based on solutions of *finite-dimensional* algebraic Riccati equations involving the Galerkin approximation of (1). Increasing the sizes of the parameters $\alpha_1, \alpha_2 \geq 0$ improves the stability margin of the closed-loop system and leads to faster convergence rate for the output, but choosing too large values often causes numerical issues in solving the Riccati equations. In the final part of the algorithms the order of the finite-dimensional controller is reduced using balanced truncation.

The construction does not give precise bounds for the sizes of the Galerkin approximation or the model reduction, but instead only guarantees that robust output regulation is achieved for approximations of sufficiently high orders. As seen in Section IV, the key requirement on the orders of these approximations is the ability of the reduced order controller to approximate the behaviour of a full infinite-dimensional observer-based robust controller. As Lemma II.3 indicates, the validity of the reduced order approximation in the graph topology depends on the decay of the Hankel singular values. While for some particular finite-dimensional systems reduction may be impossible (i.e., only the choice $r = N$ is possible for achieving a given accuracy), the Hankel singular values of Galerkin approximations of parabolic PDE systems typically decay fairly rapidly and because of this reduction is usually possible.

The main results, Theorems III.1 and III.2, confirm that the constructed controllers solve the robust output regulation problem. The proofs of the theorems are presented in Section IV. The proofs also show that the Riccati equations in Step 3 can be solved approximately in order to improve computational efficiency, as long as the approximation scheme is such that the approximation errors of K^N and L^N are small.

A. Observer-Based Finite-Dimensional Controller

Our first finite-dimensional robust controller is of the form

$$\dot{z}_1(t) = G_1 z_1(t) + G_2 e(t) \tag{6a}$$
$$\dot{z}_2(t) = (A_L^r + B_L^r K_2^r) z_2(t) + B_L^r K_1^N z_1(t) - L^r e(t) \tag{6b}$$
$$u(t) = K_1^N z_1(t) + K_2^r z_2(t) \tag{6c}$$

with state $(z_1(t), z_2(t))^T \in Z := Z_0 \times \mathbb{C}^r$ and input $e(t) = y(t) - y_{ref}(t)$. The matrices $(G_1, G_2, A_L^r, B_L^r, K_1^N, K_2^r, L^r)$ are chosen using the algorithm below. More precisely, (G_1, G_2) are as in Step 1, K_1^N is as in Step 3, and $(A_L^r, B_L^r, L^r, K_2^N)$ are as in Step 4. The parts G_1, G_2, K_1^N are *the internal model* in the controller. The terminology "observer-based controller" arises from the property that the finite-dimensional subsystem (6b) approximates (in a certain sense) a full infinite-dimensional observer for (1).

PART I. The Internal Model

Step 1: We choose $Z_0 = Y^{n_0} \times Y^{2n_1} \times \ldots \times Y^{2n_q}$, $G_1 = \operatorname{diag}(J_0^Y, \ldots, J_q^Y) \in \mathcal{L}(Z_0)$, and $G_2 = (G_2^k)_{k=0}^q \in \mathcal{L}(Y, Z_0)$. The parts of G_1 and G_2 are chosen as follows. For $k = 0$, let

$$J_0^Y = \begin{bmatrix} 0_p & I_p & & \\ & 0_p & \ddots & \\ & & \ddots & I_p \\ & & & 0_p \end{bmatrix}, \qquad G_2^0 = \begin{bmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{bmatrix}$$

where 0_p and I_p are the $p \times p$ zero and identity matrices, respectively. For $k \in \{1, \ldots, q\}$ we choose

$$J_k^Y = \begin{bmatrix} \Omega_k & I_{2p} & & \\ & \Omega_k & \ddots & \\ & & \ddots & I_{2p} \\ & & & \Omega_k \end{bmatrix}, \qquad G_2^k = \begin{bmatrix} 0_{2p} \\ \vdots \\ 0_{2p} \\ I_p \\ 0_p \end{bmatrix}$$

where $\Omega_k = \begin{bmatrix} 0_p & \omega_k I_p \\ -\omega_k I_p & 0_p \end{bmatrix}$. The pair (G_1, G_2) is controllable by construction.

PART II. The Galerkin Approximation and Stabilization.

Step 2: For a fixed and sufficiently large $N \in \mathbb{N}$, apply the Galerkin approximation described in Section II-A to the system (A, B, C) to arrive at the finite-dimensional system (A^N, B^N, C^N) on V^N.

Step 3: Choose the parameters $\alpha_1, \alpha_2 \geq 0$, $Q_1 \in \mathcal{L}(U_0, X)$, and $Q_2 \in \mathcal{L}(X, Y_0)$ with U_0, Y_0 Hilbert in such a way that the systems $(A + \alpha_1 I, Q_1, C)$ and $(A + \alpha_2 I, B, Q_2)$ are both exponentially stabilizable and detectable. Let Q_1^N and Q_2^N be the approximations of Q_1 and Q_2, respectively, according to the approximation V^N of V. Let $Q_0 \in \mathcal{L}(Z_0, \mathbb{C}^{p_0})$ be such that (Q_0, G_1) is observable, and let $R_1 \in \mathcal{L}(Y)$ and $R_2 \in \mathcal{L}(U)$ be positive definite matrices. Denote

$$A_s^N = \begin{bmatrix} G_1 & G_2 C^N \\ 0 & A^N \end{bmatrix}, \; B_s^N = \begin{bmatrix} G_2 D \\ B^N \end{bmatrix}, \; Q_s^N = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_2^N \end{bmatrix}.$$

Define $L^N = -\Sigma_N C^N R_1^{-1} \in \mathcal{L}(Y, V^N)$ and $K^N := \begin{bmatrix} K_1^N, \; K_2^N \end{bmatrix} = -R_2^{-1} (B_s^N)^* \Pi_N \in \mathcal{L}(Z_0 \times V^N, U)$ where Σ_N and Π_N are the non-negative solutions of the finite-dimensional Riccati equations

$$\begin{aligned} &(A^N + \alpha_1 I)\Sigma_N + \Sigma_N (A^N + \alpha_1 I)^* \\ &\qquad - \Sigma_N \left(C^N\right)^* R_1^{-1} C^N \Sigma_N = -Q_1^N (Q_1^N)^* \\ &(A_s^N + \alpha_2 I)^* \Pi_N + \Pi_N (A_s^N + \alpha_2 I) \\ &\qquad - \Pi_N B_s^N R_2^{-1} \left(B_s^N\right)^* \Pi_N = -\left(Q_s^N\right)^* Q_s^N. \end{aligned}$$

The exponential stabilizability of the pair $(A_s^N + \alpha_2 I, B_s^N)$ for large N follows from [21, Sec. 5.2] and Lemma A.2. With the above choices the matrices $A_s^N + B_s^N K^N$ and $A^N + L^N C^N$ are Hurwitz if N is sufficiently large [1, Thm. 4.8].

PART III. The Model Reduction

Step 4: For a fixed and suitably large $r \in \mathbb{N}$, $r \leq N$, apply the balanced truncation method in Section II-B to the stable finite-dimensional system

$$(A^N + L^N C^N, [B^N + L^N D, \; L^N], K_2^N)$$

to obtain a stable r-dimensional reduced order system

$$(A_L^r, [B_L^r,\ L^r], K_2^r)\,.$$

Theorem III.1. *Let Assumption* II.1 *be satisfied. The finite-dimensional controller* (6) *solves the Robust Output Regulation Problem provided that the order N of the Galerkin approximation and the order r of the model reduction are sufficiently high.*

If $\alpha_1, \alpha_2 > 0$, then the controller achieves a uniform stability margin in the sense that for any fixed $0 < \alpha < \min\{\alpha_1, \alpha_2\}$ the operator $A_e + \alpha I$ will generate an exponentially stable semigroup if N and $r \le N$ are sufficiently large.

B. Dual Observer-Based Finite-Dimensional Controller

The second controller we construct is of the form

$$\dot z_1(t) = G_1 z_1(t) + G_2^N C_K^r z_2(t) + G_2^N e(t) \tag{7a}$$
$$\dot z_2(t) = (A_K^r + L^r C_K^r) z_2(t) + L^r e(t) \tag{7b}$$
$$u(t) = K_1 z_1(t) - K_2^r z_2(t) \tag{7c}$$

with state $(z_1(t), z_2(t)) \in Z := Z_0 \times \mathbb{C}^r$, and the matrices $(G_1, G_2^N, A_K^r, C_K^r, K_1, K_2^r, L^r)$ are chosen using the algorithm below. More precisely, (G_1, K_1) are as in Step 1, G_2^N is as in Step 3, and $(A_K^r, C_K^r, K_2^r, L^r)$ are as in Step 4. The terminology "dual observer-based controller" is motivated by the property that the dual system of (7) will in fact achieve closed-loop stability with the dual (A^*, C^*, B^*, D^*) of the original system (1). Since X_e is a Hilbert space, we can use this property in proving closed-loop stability in Section IV.

PART I. The Internal Model

Step 1: We choose $Z_0 = Y^{n_0} \times Y^{2n_1} \times \ldots \times Y^{2n_q}$, $G_1 = \mathrm{diag}(J_0^Y, \ldots, J_q^Y) \in \mathcal{L}(Z_0)$, and $K_1 = [K_1^0, \ldots, K_1^q] \in \mathcal{L}(Z_0, U)$. The parts of G_1 and K_1 are chosen as follows. For $k = 0$, let

$$J_0^Y = \begin{bmatrix} 0_p & I_p & & \\ & 0_p & \ddots & \\ & & \ddots & I_p \\ & & & 0_p \end{bmatrix}$$

and $K_1^0 = [K_1^{01},\ 0_p,\ \ldots,\ 0_p]$, where 0_p and I_p are the $p \times p$ zero and identity matrices, respectively. For $k \in \{1, \ldots, q\}$ we choose

$$J_k^Y = \begin{bmatrix} \Omega_k & I_{2p} & & \\ & \Omega_k & \ddots & \\ & & \ddots & I_{2p} \\ & & & \Omega_k \end{bmatrix}, \qquad \Omega_k = \begin{bmatrix} 0_p & \omega_k I_p \\ -\omega_k I_p & 0_p \end{bmatrix}$$

and $K_1^k = [K_1^{k1}, 0_p, 0_{2p}, \ldots, 0_{2p}]$. For each $k \in \{0, \ldots, q\}$ the matrices $K_1^{k1} \in \mathcal{L}(Y, U)$ are chosen[1] so that $P(i\omega_k)K_1^{k1} \in \mathcal{L}(Y)$ are boundedly invertible for all $k \in \{0, \ldots, q\}$. If $m = p$, we can choose $K_1^{k1} = I_p$ for all $k \in \{0, \ldots, q\}$. The pair (K_1, G_1) is observable by construction.

PART II. The Galerkin Approximation and Stabilization.

Step 2: For a fixed and sufficiently large $N \in \mathbb{N}$, apply the Galerkin approximation described in Section II-A to the system (A, B, C) to arrive at the finite-dimensional system (A^N, B^N, C^N) on V^N.

Step 3: Choose the parameters $\alpha_1, \alpha_2 \ge 0$, $Q_1 \in \mathcal{L}(X, Y_0)$, and $Q_2 \in \mathcal{L}(U_0, X)$ with U_0, Y_0 Hilbert in such a way that the systems $(A + \alpha_1 I, B, Q_1)$ and $(A + \alpha_2 I, Q_2, C)$ are both exponentially stabilizable and detectable. Let Q_1^N and Q_2^N be the approximations of Q_1 and Q_2, respectively, according to the approximation V^N of V. Let $Q_0 \in \mathcal{L}(\mathbb{C}^{p_0}, Z_0)$ be such that (G_1, Q_0) is controllable, and $R_1 \in \mathcal{L}(U)$ and $R_2 \in \mathcal{L}(Y)$ be positive definite matrices. Denote $C_s^N = \begin{bmatrix} DK_1, & C^N \end{bmatrix}$ and

$$A_s^N = \begin{bmatrix} G_1 & 0 \\ B^N K_1 & A^N \end{bmatrix}, \quad Q_s^N = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_2^N \end{bmatrix},$$

Define $\mathcal{G}_2^N := \begin{bmatrix} G_2^N \\ L^N \end{bmatrix} = -\Pi_N C_s^N R_2^{-1} \in \mathcal{L}(Y, Z_0 \times V^N)$ and $K_2^N = -R_1^{-1}(B^N)^* \Sigma_N \in \mathcal{L}(V^N, U)$ where Σ_N and Π_N are the non-negative solutions of the finite-dimensional Riccati equations

$$\begin{aligned} &(A^N + \alpha_1 I)^* \Sigma_N + \Sigma_N (A^N + \alpha_1 I) \\ &\qquad - \Sigma_N B^N R_1^{-1} \left(B^N\right)^* \Sigma_N = -(Q_1^N)^* Q_1^N \\ &(A_s^N + \alpha_2 I)\Pi_N + \Pi_N (A_s^N + \alpha_2 I)^* \\ &\qquad - \Pi_N \left(C_s^N\right)^* R_2^{-1} C_s^N \Pi_N = -Q_s^N \left(Q_s^N\right)^*. \end{aligned}$$

The exponential detectability of the pair $(C_s^N, A_s^N + \alpha_2 I)$ for large N follows from [21, Sec. 5.2] and Lemma A.2. With these choices the matrices $A^N + B^N K_2^N$ and $A_s^N + \mathcal{G}_2^N C_s^N$ are Hurwitz if N is sufficiently large [1, Thm. 4.8].

PART III. The Model Reduction

Step 4: For a fixed and suitably large $r \in \mathbb{N}$, $r \le N$ apply the balanced truncation method in Section II-B to the stable finite-dimensional system

$$\left(A^N + B^N K_2^N, L^N, \begin{bmatrix} C^N + DK_2^N \\ K_2^N \end{bmatrix}\right)$$

to obtain a stable r-dimensional reduced order system

$$\left(A_K^r, L^r, \begin{bmatrix} C_K^r \\ K_2^r \end{bmatrix}\right).$$

Theorem III.2. *Let Assumption* II.1 *be satisfied. The finite-dimensional controller* (7) *solves the Robust Output Regulation Problem provided that the order N of the Galerkin approximation and the order r of the model reduction are sufficiently high.*

If $\alpha_1, \alpha_2 > 0$, then the controller achieves a uniform stability margin in the sense that for any fixed $0 < \alpha < \min\{\alpha_1, \alpha_2\}$ the operator $A_e + \alpha I$ will generate an exponentially stable semigroup if N and $r \le N$ are sufficiently large.

[1]This choice is possible by Assumption II.1 whenever $i\omega_k \in \rho(A)$. If $i\omega_k \notin \rho(A)$ for some k, then we instead choose K_1^{1k} in such a way that $P_L(i\omega_k)K_1^{k1} \in \mathcal{L}(Y)$ is boundedly invertible where $P_L(\lambda) = CR(\lambda, A + LC)(B + LD) + D$ with some $L \in \mathcal{L}(Y, X)$ such that $A + LC$ is exponentially stable. The invertibility of $P_L(i\omega_k)K_1^{k1} \in \mathcal{L}(Y)$ does not depend on the choice of L due to the identity $P_{\tilde L}(i\omega_k) = (I - CR(i\omega_k, A + LC)(\tilde L - L))^{-1} P_L(i\omega_k)$ where $\tilde L \in \mathcal{L}(Y, X)$ is another operator for which $A + \tilde L C$ is exponentially stable.

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IV. Proofs of the Main Results

The proofs of Theorems III.1 and III.2 are based on the internal model principle which states that a controller solves the robust output regulation problem provided that it contains an internal model of the frequencies of $y_{ref}(t)$ and $w_{dist}(t)$ and the closed-loop system is exponentially stable.

In showing the closed-loop stability we employ a combination of perturbation and approximation arguments. We first construct an infinite-dimensional controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ which stabilizes the closed-loop system and then compare the distance between two closed-loop systems — one with our controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ and one with $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ — in the graph topology for large N and r. To ensure the stabilizability and detectability of the closed-loop systems, we consider them with suitable modified input and output operators $\tilde{B}_e$ and $\tilde{C}_e$. We then prove that $(A_e, \tilde{B}_e, \tilde{C}_e)$ is input-output stable by showing that for sufficiently large N and r the distance of this system in the graph topology to the input-output stable closed-loop system $(A_e^\infty, \tilde{B}_e^\infty, \tilde{C}_e^\infty)$ can be made arbitrarily small. The input-output stability together with stabilizability and detectability of $(A_e, \tilde{B}_e, \tilde{C}_e)$ will finally imply that $T_e(t)$ is exponentially stable.

In summary, the proof consists of the following parts:

1. Verify that $(\mathcal{G}_1, \mathcal{G}_2, K)$ has an internal model.
2. Define an exponentially stabilizable and detectable closed-loop system $(A_e, \tilde{B}_e, \tilde{C}_e)$ with suitable $\tilde{B}_e$ and $\tilde{C}_e$. The input-output stability of this system will imply the exponential stability of $T_e(t)$ by [30, Cor. 1.8].
3. Construct a stabilizing infinite-dimensional controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ and the corresponding input-output stable closed-loop system $(A_e^\infty, \tilde{B}_e^\infty, \tilde{C}_e^\infty)$.
4. Show that for large N and r the distance in graph topology between $(A_e, \tilde{B}_e, \tilde{C}_e)$ and $(A_e^\infty, \tilde{B}_e^\infty, \tilde{C}_e^\infty)$ becomes arbitrarily small, and thus $(A_e, \tilde{B}_e, \tilde{C}_e)$ is input-output stable for sufficiently large N and r [17], [21].
5. Combine parts 1, 2, and 4 to conclude that $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the robust output regulation problem.

Proof of Theorem III.1. The matrices $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the error feedback controller (3) are given by

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ B_L^r K_1^N & A_L^r + B_L^r K_2^r \end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ -L^r \end{bmatrix},$$

$K = [K_1^N, K_2^r]$, and $Z = Z_0 \times \mathbb{C}^r$ or $Z = Z_0 \times \mathbb{R}^r$. If $\alpha_1 > 0$ and $\alpha_2 > 0$ we let $0 < \alpha < \min\{\alpha_1, \alpha_2\}$ be arbitrary. Otherwise we take $\alpha = 0$.

Part 1 – The Internal Model Property: The block structures of $\mathcal{G}_1$ and $\mathcal{G}_2$ are the same as in the controller constructed in [24, Sec. VI]. The matrices G_1 and G_2 are related to the corresponding matrices in [24, Sec. VI] through a similarity transform. Since the internal model property is invariant under such transformations, the argument at the end of the proof of [24, Thm. 15] shows that if the closed-loop is exponentially stable, then the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has an internal model in the sense that (see [24, Def. 5])

$$\begin{aligned} \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\}, \qquad 0 \le k \le q \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \\ \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k - 1} &\subset \mathcal{R}(i\omega_k - \mathcal{G}_1) \quad 0 \le k \le q. \end{aligned}$$

Part 2 – A Modified Closed-Loop System: Consider a composite system $(A_{e0}, \tilde{B}_e, \tilde{C}_e)$ with

$$A_{e0} = \begin{bmatrix} A & 0 \\ 0 & \mathcal{G}_1 \end{bmatrix}, \ \tilde{B}_e = \begin{bmatrix} B & 0 \\ 0 & \tilde{\mathcal{G}}_2 \end{bmatrix}, \ \tilde{C}_e = \begin{bmatrix} C & 0 \\ 0 & \tilde{K} \end{bmatrix},$$
$$\tilde{\mathcal{G}}_2 = \begin{bmatrix} G_2 & 0 \\ -L^r & B_L^r \end{bmatrix}, \ \tilde{K} = \begin{bmatrix} K_1^N & 0 \\ 0 & K_2^r \end{bmatrix}.$$

If N is large, then $A^N + \alpha I + L^N C^N$ is exponentially stable by [1, Thm. 4.8]. Since A_L^r is obtained from $A^N + L^N C^N$ using balanced truncation, also $A_L^r + \alpha I$ in $\mathcal{G}_1$ is Hurwitz for large N and r. The pair $(G_1 + \alpha I, G_2)$ is controllable by construction, and $(K_1^N, G_1 + \alpha I)$ is observable by Lemma A.2. Using these properties it is easy to see that $(\mathcal{G}_1 + \alpha I, \tilde{\mathcal{G}}_2, \tilde{K})$ is exponentially stabilizable and detectable for large N and r, and therefore the same holds for $(A_{e0} + \alpha I, \tilde{B}_e, \tilde{C}_e)$. A direct computation shows that $A_e = A_{e0} + \tilde{B}_e K_e \tilde{C}_e$ where

$$K_e = \begin{bmatrix} 0 & I & I \\ I & D & D \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

and thus under the output feedback with the operator K_e the system $(A_{e0} + \alpha I, \tilde{B}_e, \tilde{C}_e)$ becomes $(A_e + \alpha I, \tilde{B}_e, \tilde{C}_e)$. Since output feedback preserves stabilizability and detectability, for large N and $r \le N$ the input-output stability of $(A_e + \alpha I, \tilde{B}_e, \tilde{C}_e)$ will imply the exponential stability of the semigroup $e^{\alpha t} T_e(t)$ generated by $A_e + \alpha I$ [30, Cor. 1.8].

Part 3 – An Infinite-Dimensional Stabilizing Controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$: Choose $Z_\infty = Z_0 \times X$ and

$$\mathcal{G}_1^\infty = \begin{bmatrix} G_1 & 0 \\ (B + L^\infty D) K_1^\infty & A + L^\infty C + (B + L^\infty D) K_2^\infty \end{bmatrix},$$

and $\mathcal{G}_2^\infty = \left[\begin{smallmatrix} G_2 \\ -L^\infty \end{smallmatrix}\right]$ where $K^\infty := [K_1^\infty, K_2^\infty]$ and L^∞ are the limits of K^N and L^N in the sense that

$$\begin{aligned} &\|K^N \left[\begin{smallmatrix} I & 0 \\ 0 & P^N \end{smallmatrix}\right] - K^\infty\|_{\mathcal{L}(Z_0 \times X, U)} \to 0 \quad \text{and} \\ &\|P^N L^N - L^\infty\|_{\mathcal{L}(Y, X)} \to 0 \end{aligned}$$

as $N \to \infty$. Here $P^N : X \to V^N$ is again the Galerkin projection onto V^N. The limit L^∞ exists due to the approximation theory for solutions of Riccati operator equations [1, Thm. 4.8]. Moreover, if we define

$$A_s = \begin{bmatrix} G_1 & G_2 C \\ 0 & A \end{bmatrix}, \quad B_s = \begin{bmatrix} G_2 D \\ B \end{bmatrix}, \quad Q_s = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_2 \end{bmatrix}$$

then it is straightforward to show based on properties of A that the form defined by $a_s(\phi, \psi) = \langle -A_s \phi, \psi \rangle$, $\phi \in D(A_s)$, $\psi \in Z_0 \times X$ and the approximating subspaces $V_s^N = Z_0 \times V^N$ satisfy the assumptions of [1, Thm. 4.8]. Since $(A_s + \alpha_2 I, B_s, Q_s)$ is exponentially stabilizable and detectable by Lemma A.2, also the existence of K^∞ follows from [1, Thm.

4.8]. Moreover, the semigroups generated by $A+\alpha I+L^\infty C$ and $A_s+\alpha I+B_sK^\infty$ are exponentially stable.

We will now show that A_e^∞ — the closed-loop system operator with $(\mathcal{G}_1,\mathcal{G}_2,K)$ replaced by $(\mathcal{G}_1^\infty,\mathcal{G}_2^\infty,K^\infty)$ — is such that $A_e^\infty+\alpha I$ generates an exponentially stable semigroup. If we define a bounded similarity transform

$$\Lambda_e=\begin{bmatrix}0&I&0\\I&0&0\\-I&0&I\end{bmatrix},\qquad \Lambda_e^{-1}=\begin{bmatrix}0&I&0\\I&0&0\\0&I&I\end{bmatrix},$$

then a direct computation shows that

$$\begin{aligned}\Lambda_eA_e^\infty\Lambda_e^{-1}&=\Lambda_e\begin{bmatrix}A&BK^\infty\\\mathcal{G}_2^\infty C&\mathcal{G}_1^\infty+\mathcal{G}_2^\infty DK^\infty\end{bmatrix}\Lambda_e^{-1}\\&=\Lambda_e\begin{bmatrix}A&BK_1^\infty&BK_2^\infty\\G_2C&G_1+G_2DK_1^\infty&G_2DK_2^\infty\\-L^\infty C&BK_1^\infty&A+BK_2^\infty+L^\infty C\end{bmatrix}\Lambda_e^{-1}\\&=\begin{bmatrix}G_1+G_2DK_1^\infty&G_2(C+DK_2^\infty)&G_2DK_2^\infty\\BK_1^\infty&A+BK_2^\infty&BK_2^\infty\\0&0&A+L^\infty C\end{bmatrix}.\end{aligned}$$

The first 2×2 subsystem of $\Lambda_eA_e^\infty\Lambda_e^{-1}$ is given by

$$\begin{bmatrix}G_1&G_2C\\0&A\end{bmatrix}+\begin{bmatrix}G_2D\\B\end{bmatrix}\begin{bmatrix}K_1^\infty,&K_2^\infty\end{bmatrix}=A_s+B_sK^\infty.$$

Since $A+\alpha I+L^\infty C$ and $A_s+\alpha I+B_sK^\infty$ generate exponentially stable semigroups, the same is true for $\Lambda_e(A_e^\infty+\alpha I)\Lambda_e^{-1}$ and $A_e^\infty+\alpha I$.

Finally, define $(A_{e0}^\infty,\tilde{B}_e^\infty,\tilde{C}_e^\infty)$

$$A_{e0}^\infty=\begin{bmatrix}A&0\\0&\mathcal{G}_1^\infty\end{bmatrix},\quad \tilde{B}_e=\begin{bmatrix}B&0\\0&\tilde{\mathcal{G}}_2^\infty\end{bmatrix},\quad \tilde{C}_e=\begin{bmatrix}C&0\\0&\tilde{K}^\infty\end{bmatrix}$$

where

$$\tilde{\mathcal{G}}_2^\infty=\begin{bmatrix}G_2&0\\-L^\infty&B+L^\infty D\end{bmatrix}\quad\text{and}\quad\tilde{K}^\infty=\begin{bmatrix}K_1^\infty&0\\0&K_2^\infty\end{bmatrix}.$$

Output feedback with the feedback operator in (8) transforms $(A_{e0}^\infty+\alpha I,\tilde{B}_e^\infty,\tilde{C}_e^\infty)$ to $(A_e^\infty+\alpha I,\tilde{B}_e^\infty,\tilde{C}_e^\infty)$. The system $(A_e^\infty+\alpha I,\tilde{B}_e^\infty,\tilde{C}_e^\infty)$ is input-output stable since $A_e^\infty+\alpha I$ generates an exponentially stable semigroup.

Part 4 – Input-Output Stability of $(A_e,\tilde{B}_e,\tilde{C}_e)$: Our aim is to show that for large N and r the distance in graph topology between $(A_e+\alpha I,\tilde{B}_e,\tilde{C}_e)$ and $(A_e^\infty+\alpha I,\tilde{B}_e^\infty,\tilde{C}_e^\infty)$ can be made arbitrarily small. By Lemma A.1 and Part 3 it is sufficient to show that the distance between $(A_{e0}^\infty+\alpha I,\tilde{B}_e^\infty,\tilde{C}_e^\infty)$ and $(A_{e0}+\alpha I,\tilde{B}_e,\tilde{C}_e)$ becomes small for large N and r. Due to the structure of these systems this is true if (and only if) the distance in graph topology between $(\mathcal{G}_1+\alpha I,\tilde{\mathcal{G}}_2,\tilde{K})$ and $(\mathcal{G}_1^\infty+\alpha I,\tilde{\mathcal{G}}_2^\infty,\tilde{K}^\infty)$ becomes small. If we define

$$\mathcal{G}_{10}=\begin{bmatrix}G_1&0\\0&A_L^r\end{bmatrix},\ \mathcal{G}_{10}^\infty=\begin{bmatrix}G_1&0\\0&A+L^\infty C\end{bmatrix},\ K_c=\begin{bmatrix}0&0\\I&I\end{bmatrix}$$

we see that $\mathcal{G}_1=\mathcal{G}_{10}+\tilde{\mathcal{G}}_2K_c\tilde{K}$ and $\mathcal{G}_1^\infty=\mathcal{G}_{10}^\infty+\tilde{\mathcal{G}}_2^\infty K_c\tilde{K}^\infty$. Therefore Lemma A.1 and the structure of the controllers imply that the distance between $(\mathcal{G}_1+\alpha I,\tilde{\mathcal{G}}_2,\tilde{K})$ and $(\mathcal{G}_1^\infty+\alpha I,\tilde{\mathcal{G}}_2^\infty,\tilde{K}^\infty)$ can be made small provided that the distance $d(\mathcal{P},\mathcal{P}_r)$ in the graph topology between

$$\begin{aligned}&\mathcal{P}:=(A+\alpha I+L^\infty C,[B+L^\infty D,\ L^\infty],K_2^\infty)\quad\text{and}\\&\mathcal{P}_r:=(A_L^r+\alpha I,[B_L^r,\ L^r],K_2^r)\end{aligned}$$

becomes arbitrarily small for large r and N. The triangle inequality implies $d(\mathcal{P},\mathcal{P}_r)\le d(\mathcal{P},\mathcal{P}_N)+d(\mathcal{P}_N,\mathcal{P}_r)$ where $\mathcal{P}_N:=(A^N+\alpha I+L^NC^N,[B^N+L^ND,\ L^N],K_2^N)$. Since $\mathcal{P}$ and $\mathcal{P}_N$ are parts of systems obtained with output feedback from $(A+\alpha I,[B+L^\infty D,\ \ L^\infty],\left[\begin{smallmatrix}C&0\\0&K_2^\infty\end{smallmatrix}\right])$ and $(A^N+\alpha I,[B^N+L^ND,\ L^N],\left[\begin{smallmatrix}C^N&0\\0&K_2^N\end{smallmatrix}\right])$, respectively, Lemmas A.1 and II.2 show that $d(\mathcal{P},\mathcal{P}_N)\to 0$ as $N\to\infty$. Finally, since $\mathcal{P}_r$ is the system obtained from $\mathcal{P}_N$ using model reduction, we have from Lemma II.3 that $d(\mathcal{P}_N,\mathcal{P}_r)$ can be made arbitrarily small by choosing a sufficiently large $r\le N$ (in the extreme case only the choice $r=N$ may be possible, in which case $d(\mathcal{P}_N,\mathcal{P}_r)=0$).

Part 5 – Conclusion: By Part 1 the controller contains an internal model and by Parts 2–4 the semigroup $e^{\alpha t}T_e(t)$ generated by $A_e+\alpha I$ is exponentially stable. We have from [24, Thm. 7] that the controller solves robust output regulation problem[2]. □

Proof of Theorem III.2. The matrices $(\mathcal{G}_1,\mathcal{G}_2,K)$ of the error feedback controller (3) are given by

$$\mathcal{G}_1=\begin{bmatrix}G_1&G_2^NC_K^r\\0&A_K^r+L^rC_K^r\end{bmatrix},\qquad\mathcal{G}_2=\begin{bmatrix}G_2^N\\L^r\end{bmatrix},$$

$K=\begin{bmatrix}K_1,&-K_2^r\end{bmatrix}$, and $Z=Z_0\times\mathbb{C}^r$ or $Z=Z_0\times\mathbb{R}^r$.

Part 1 – The Internal Model Property: Due to the properties of G_1 and the block structure of $\mathcal{G}_1$, the controller contains an internal model of the reference and disturbance signals in the sense that $\dim\mathcal{N}(i\omega_k-\mathcal{G}_1)\ge\dim Y=p$ for all $k\in\{0,\dots,q\}$ and $\mathcal{G}_1$ has at least p independent Jordan chains of length greater than or equal to n_k associated to each eigenvalue $i\omega_k$ (see [24, Def. 4]).

Part 2 – Stability of the Closed-Loop System: If $\alpha_1>0$ and $\alpha_2>0$ we let $0<\alpha<\min\{\alpha_1,\alpha_2\}$ be arbitrary. Otherwise we take $\alpha=0$. We will prove exponential closed-loop stability by showing that the adjoint $A_e^*+\alpha I$ of $A_e+\alpha I$ generates an exponentially stable semigroup. The adjoint operator A_e^* is given by

$$A_e^*=\begin{bmatrix}A^*&C^*\mathcal{G}_2^*\\K^*B^*&\mathcal{G}_1^*+K^*D^*\mathcal{G}_2^*\end{bmatrix}$$

where $\mathcal{G}_2^*=\begin{bmatrix}(G_2^N)^*,&(L^r)^*\end{bmatrix}$, $K^*=\left[\begin{smallmatrix}K_1^*\\-(K_2^r)^*\end{smallmatrix}\right]$,

$$\mathcal{G}_1^*=\begin{bmatrix}G_1^*&0\\(C_K^r)^*(G_2^N)^*&(A_K^r)^*+(C_K^r)^*(L^r)^*\end{bmatrix}.$$

The dual $(\mathcal{G}_1^*,K^*,\mathcal{G}_2^*)$ of $(\mathcal{G}_1,\mathcal{G}_2,K)$ coincides with a controller constructed in Section III-A for the dual system (A^*,C^*,B^*,D^*) in all but two respects: G_1^* has a block lower-triangular structure (instead of block upper-triangular structure), and the choice of K_1^* is slightly different from the choice of G_2 in Section III-A. However, as seen in the proof of Theorem III.1, the properties of (G_1,G_2) only affect the closed-loop stability by guaranteeing the exponential stabilizability of the block-operator pair "$(A_s^N+\alpha_2I,B_s^N)$" in

[2]In the reference [24] the objective of the robust output regulation problem was to achieve $t\mapsto e^{\alpha t}\|e(t)\|\in L^2(0,\infty;Y)$ for some $\alpha>0$, but since in our case B, C, $\mathcal{G}_2$ and K are bounded operators, the expression for $e(t)$ in the proof of [24, Thm. 7] implies that also (4) is satisfied.

Step 3 of the construction algorithm in Section III-A. Because of duality, this property corresponds exactly to the exponential detectability of the block operator pair "$(C_s^N, A_s^N + \alpha_2 I)$" for the controller in the current theorem, and therefore the required stabilizability property is guaranteed by Lemma A.2. Moreover, the definitions of the Galerkin approximation in Section II-A imply that the approximation $((A^*)^N, (C^*)^N, (B^*)^N)$ of the dual system (A^*, C^*, B^*) is given by $(A^*)^N = (A^N)^*$, $(B^*)^N = (B^N)^*$, and $(C^*)^N = (C^N)^*$ with the same choices of the approximating subspaces V^N. In addition, it is straightforward to check that the reduced order model constructed using balanced truncation for a dual system coincides with the dual system of the reduced order model of the original system, and the reduced dual system convergences in the graph topology to the dual of the original system. Because of this, it follows from the proof of Theorem III.1 that $A_e^* + \alpha I$ generates an exponentially stable semigroup when N and r are sufficiently large. Since X_e is a Hilbert space, also $e^{\alpha t} T_e(t)$ generated by $A_e + \alpha I$ is exponentially stable. □

V. Robust Controller Design for Parabolic PDE Models

In this section we apply the control design algorithms in Section III for selected PDE models. In each case we use two distinct Galerkin approximations, one (of order N) for constructing the controller and a second one (of order $n \gg N$) for simulating the behaviour of the original system.

A. A 1D Reaction–Diffusion Equation

Consider a one-dimensional reaction–diffusion equation on the spatial domain $\Omega = (0, 1)$ with distributed control and observation and Neumann boundary disturbance,

$$\frac{\partial x}{\partial t}(\xi, t) = \frac{\partial}{\partial \xi}\left(\alpha(\xi)\frac{\partial x}{\partial \xi}(\xi, t)\right) \tag{9a}$$

$$+ \gamma(\xi)x(\xi, t) + b(\xi)u(t), \tag{9b}$$

$$\frac{\partial x}{\partial \xi}(0, t) = w_{dist}(t), \ \frac{\partial x}{\partial \xi}(1, t) = 0, \ x(\xi, 0) = x_0(\xi), \tag{9c}$$

$$y(t) = \int_0^1 x(\xi, t)c(\xi)d\xi. \tag{9d}$$

We assume $\alpha \in W^{1,\infty}(0, 1; \mathbb{R})$ with $\alpha(\xi) \geq \alpha_0 > 0$ for all $\xi \in (0, 1)$, $\gamma \in L^\infty(0, 1; \mathbb{R})$, and $b, c \in L^2(0, 1; \mathbb{R})$. The disturbance signal $w_{dist}(t)$ acts on the left boundary. The system (9) is a more general version of the 1D heat equation studied in [28].

Choose $X = L^2(0, 1)$. Due to the boundary disturbance at $\xi = 0$, the system (9) has the form of a *boundary control system* [7, Sec. 3.3],

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t)$$
$$y(t) = Cx(t)$$
$$w_{dist}(t) = \mathcal{B}_d x(t)$$

where $\mathcal{A}x = \frac{\partial}{\partial \xi}(\alpha(\cdot)\frac{\partial x}{\partial \xi}) + \gamma(\cdot)x$ for $x \in D(\mathcal{A}) = \{ x \in H^2(0, 1) \mid x'(1) = 0 \}$, $B = b(\cdot) \in \mathcal{L}(\mathbb{R}, X)$, $C = \langle \cdot, c(\cdot)\rangle \in \mathcal{L}(X, \mathbb{R})$, and $\mathcal{B}_d x = x'(1)$ for $x \in D(\mathcal{A})$. The disturbance signal $w_{dist}(\cdot)$ is assumed to be of the form (2b) and is therefore smooth. As in [7, Sec. 3.3, Ex. 3.3.5] we can make a change of variables $\tilde{x}(t) = x(t) - B_{d0}w_{dist}(t)$ where $B_{d0} \in \mathcal{L}(\mathbb{R}, X)$ is such that $D(\mathcal{A}) \subset \mathcal{R}(B_{d0})$ and $\mathcal{B}_d B_{d0} = I$. This allows us to write the PDE system (1) in the form

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + \begin{bmatrix}\mathcal{A}B_{d0}, & -B_{d0}\end{bmatrix}\begin{bmatrix}w_{dist}(t)\\ \dot{w}_{dist}(t)\end{bmatrix}$$

$$y(t) = C\tilde{x}(t) + \begin{bmatrix}CB_{d0}, & 0\end{bmatrix}\begin{bmatrix}w_{dist}(t)\\ \dot{w}_{dist}(t)\end{bmatrix}$$

where $Ax = \mathcal{A}x$ for $x \in D(A) := D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B}_d)$. Since $B_d := [\mathcal{A}B_{d0}, -B_{d0}] \in \mathcal{L}(\mathbb{R}^2, X)$ and $(w_{dist}(t), \dot{w}_{dist}(t))^T$ is of the form (2b), this system is indeed of the form (1) and the results in Section III are therefore applicable for (9). Note that it is not necessary to compute the expressions of the operators B_{d0}, $\mathcal{A}B_{d0}$ and CB_{d0} since the robustness of the controller implies that the disturbance signal is rejected for any disturbance input and feedthrough operators.

Now $D(A) = \{ x \in H^2(0, 1) \mid x'(0) = x'(1) = 0 \}$ and if we choose $V = H^1(0, 1)$ with inner product $\langle \phi, \psi \rangle_V = \int_0^1 \phi'(\xi)\overline{\psi'(\xi)}d\xi + \int_0^1 \phi(\xi)\overline{\psi(\xi)}d\xi$, then the operator A is defined by the bounded and coercive sesquilinear form $a : V \times V \to \mathbb{C}$

$$a(\phi, \psi) = \langle \alpha(\cdot)\phi', \psi' \rangle + \langle \gamma(\cdot)\phi, \psi \rangle.$$

We assume $b(\cdot)$ and $c(\cdot)$ are such that (A, B, C) is exponentially stabilizable and detectable, which in this case means that $\langle b, \phi \rangle \neq 0$ and $\langle \phi, c \rangle \neq 0$ for any eigenfunctions ϕ of A associated to unstable eigenvalues [7, Sec. 5.2].

For the spatial discretization of (9) we use the Finite Element Method with piecewise linear basis functions. These approximations have the required property (5) by [5].

A Simulation Example

As a numerical example, we consider (9) with parameters

$$\alpha(\xi) = \frac{2 - \xi}{4}, \qquad \gamma(\xi) = 12\xi,$$
$$b(\xi) = 4\chi_{(.25,\,.5)}(\xi), \qquad c(\xi) = 4\chi_{(.5,\,.75)}(\xi)$$

where $\chi_{(a,b)}(\cdot)$ denotes the characteristic function on the interval (a, b). The control $u(t)$ and observation $y(t)$ act on the subintervals $(.25, .5)$ and $(.5, .75)$ of Ω, respectively. We consider the reference and disturbance signals

$$y_{ref}(t) = \cos(t) + \frac{1}{2}\sin(2t) - 2\cos(3t),$$
$$w_{dist}(t) = \frac{1}{4}\sin(4t).$$

The set of frequencies in (2) in $\{\omega_k\}_{k=0}^q$ is $\{1, 2, 3, 4\}$ with $q = 4$ and $n_k = 1$ for all $k \in \{1, \ldots, 4\}$. We modify the internal model in Section III in such a way that the parts associated to $\omega_0 = 0$ are omitted.

We construct the dual observer-based controller in Section III-B. In the absence of the frequency 0 the internal model has dimension $\dim Z_0 = p \times q \times 2 = 8$. In the controller construction, we use a Finite Element approximation of order $N = 300$. The parameters of the stabilization are chosen as

$$\alpha_1 = 0, \ \alpha_2 = .95, \ Q_1 = Q_2 = I_X, \ R_1 = R_2 = 1 \in \mathbb{R}.$$

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Finally, we use balanced truncation with order $r = 12$. The system (9) is unstable with a finite number of eigenvalues with positive real parts.

For the simulation of the original system (9) we use a Finite Element approximation of order $n = 1000$. Figure 2 depicts parts of the spectrum of the original system, the closed-loop system without model reduction in the controller (i.e., with $r = N$), and the closed-loop system with model reduction of order $r = 12$.

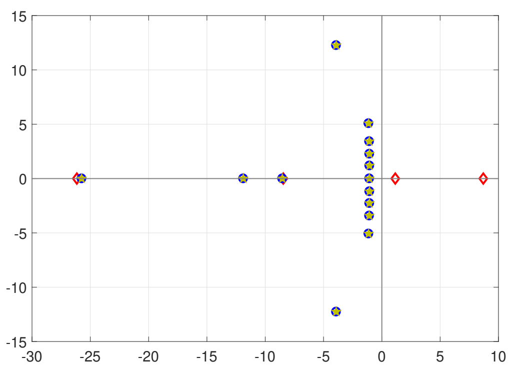

Fig. 2. Spectra of the uncontrolled system (red diamonds) and the closed-loop system with $r = N = 300$ (yellow stars) and $r = 12$ (blue circles).

The output of the controlled system for the initial states $x_0(\xi) = -\xi/10$ and $z_0 = 0 \in \mathbb{R}^{8+12}$ of the system and the controller is depicted in Figure 3.

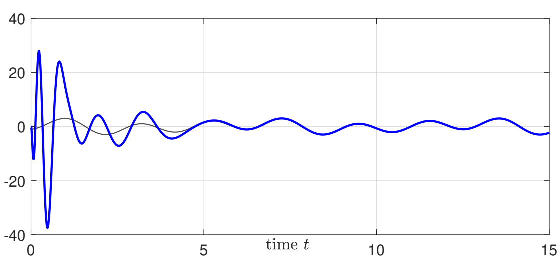

Fig. 3. Output of the 1D heat equation with the dual observer-based controller.

B. A 2D Reaction–Diffusion–Convection Equation

We consider a controlled reaction–diffusion–convection equation on a 2-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ with C^∞-smooth boundary $\partial\Omega$ and assume Ω is located locally on one side of $\partial\Omega$. The PDE is defined as (see [2, Sec. 3])

$$\frac{\partial x}{\partial t}(\xi, t) = \nabla(\alpha(\xi)\nabla x(\xi, t)) + \nabla \cdot (\beta(\xi)x(\xi, t)) \tag{10a}$$

$$+ \gamma(\xi)x(\xi, t) + f(\xi) + Bu(t), \tag{10b}$$

$$x(\xi, t) = 0, \quad \text{on } \xi \in \partial\Omega, \quad x(\xi, 0) = x_0(\xi) \tag{10c}$$

$$y(t) = Cx(\cdot, t) \tag{10d}$$

with state $x : (0, \infty) \times \Omega \to \mathbb{R}$. The possible source term $f(\xi)$ can be treated as a disturbance input with frequency $\omega_0 = 0$, and it will be handled by the internal model based controller. Here $\alpha \in W^{1,\infty}(\Omega, \mathbb{R})$ with $\alpha(\xi) \geq \alpha_0 > 0$ for all $\xi \in \Omega$, $\beta = (\beta_1(\cdot), \beta_2(\cdot))^T$ with $\beta_1, \beta_2 \in W^{1,\infty}(\Omega; \mathbb{R})$, and $\gamma,\ f \in L^\infty(\Omega, \mathbb{R})$. We assume (10) has m distributed inputs and therefore $u(t) = (u_k(t))_{k=1}^m \in U = \mathbb{R}^m$ and

$$Bu(t) = \sum_{k=1}^m u_k(t)b_k(\cdot)$$

where $b_k(\cdot) \in L^2(\Omega; \mathbb{R})$ are fixed functions. Similarly we assume the system has p measured outputs so that $y(t) = (y_k(t))_{k=1}^p \in Y = \mathbb{R}^p$ and

$$y_k(t) = \int_\Omega x(\xi, t)c_k(\xi)d\xi$$

for some fixed $c_k(\cdot) \in L^2(\Omega; \mathbb{R})$.

The system (10) can be written in the form (1) on $X = L^2(\Omega; \mathbb{R})$. If we choose $V = H_0^1(\Omega, \mathbb{R})$, then the system operator A is determined by the sesquilinear form $a : V \times V \to \mathbb{C}$ such that for all $\phi, \psi \in V$,

$$a(\phi, \psi) = \langle \alpha\nabla\phi, \nabla\psi\rangle_{L^2} + \langle \beta\phi, \nabla\psi\rangle_{L^2(\Omega;\mathbb{R}^2)} + \langle \gamma\phi, \psi\rangle_{L^2}.$$

Similarly as in [2, Sec. 3] we can deduce that $a(\cdot, \cdot)$ is bounded and coercive. The input and output operators $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$ are such that $Bu = \sum_{k=1}^m b_k(\cdot)u_k$ for all $u = (u_k)_k \in U$ and $Cx = \left(\int_\Omega x(\xi)c_k(\xi)d\xi\right)_{k=1}^p$ for all $x \in X$. We assume $\{b_k(\cdot)\}_{k=1}^m$ and $\{c_k(\cdot)\}_{k=1}^p$ are such that (A, B, C) is exponentially stabilizable and detectable. The autonomous source term $f(\xi)$ is considered as a disturbance input, i.e., we write $f(\cdot) = B_d w_{dist}(t)$ where $w_{dist}(t) \equiv 1$ and $B_d = f(\cdot) \in \mathcal{L}(\mathbb{R}, X)$.

To discretize the equation using Finite Element method, the domain Ω is approximated with a polygonal domain Ω_D and we consider a partition of Ω_D into non-overlapping triangles. The approximating subspaces V^N are chosen as the span of N piecewise linear hat functions ϕ_k. The subspaces V^N then have the required property (5) by [5].

Remark V.1. Also in the case of the 2D reaction–diffusion–convection equation it would be in addition possible to consider boundary disturbances using the same approach as in Section V-A.

A Simulation Example

As a particular numerical example, we consider a reaction–diffusion–convection equation on the unit disk $\Omega = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 < 1\}$ with parameters

$$\alpha(\xi) = \frac{1}{2},\ \beta(\xi) = \begin{bmatrix}\cos(\xi_1) - \sin(2\xi_2)\\ \sin(3\xi_1) + \cos(4\xi_2)\end{bmatrix},\ \gamma(\xi) = 10,$$

and $f = 0$. We consider (10) with two inputs and two measurements acting on rectangular subdomains of Ω. More precisely,

$$b_1(\cdot) = \chi_{\Omega_1},\ b_2(\cdot) = \chi_{\Omega_2},\ c_1(\cdot) = \chi_{\Omega_3},\ c_2(\cdot) = \chi_{\Omega_4}$$

where $\Omega_1 = \left(\frac{3}{20}, \frac{7}{20}\right) \times \left(\frac{1}{15}, \frac{4}{15}\right)$ and $\Omega_2 = \left(\frac{3}{5}, \frac{4}{5}\right) \times \left(-\frac{2}{25}, \frac{2}{25}\right)$, $\Omega_3 = \left(-\frac{7}{10}, -\frac{1}{2}\right) \times \left(-\frac{29}{60}, -\frac{11}{60}\right)$, and $\Omega_4 = \left(-\frac{1}{2}, -\frac{3}{10}\right) \times \left(\frac{7}{25}, \frac{13}{25}\right)$. The configuration of the control inputs and measurements is illustrated in Figure 4.

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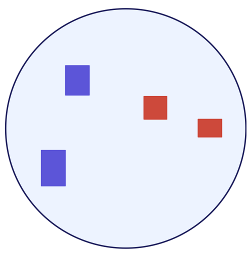

Fig. 4. Regions of control (red) and observation (blue).

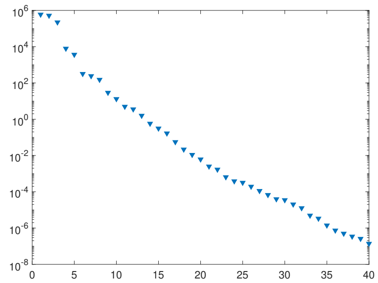

Fig. 5. Hankel singular values of the Galerkin approximation.

Our aim is to track a reference signal

$$y_{ref}(t) = \begin{bmatrix} 20\cos(t) + 5\sin(2t) - 2\cos(3t) \\ 45\sin(10t) - 2\cos(t) \end{bmatrix}.$$

The corresponding set of frequencies in (2) is $\{1, 2, 3, 10\}$ with $q = 4$ and $n_k = 1$ for all $k \in \{1, \dots, 4\}$. We modify the internal model in Section III in such a way that the parts associated to $\omega_0 = 0$ are omitted. We construct the dual observer-based controller in Section III-B using a Galerkin approximation with order $N = 1258$ and subsequent balanced truncation with order $r = 40$. In the absence of the frequency 0, the internal model has dimension $\dim Z_0 = p \times q \times 2 = 16$. The FEM discretization is implemented using the Matlab PDE Toolbox functions. The parameters of the stabilization are chosen as

$$\alpha_1 = 2,\ \alpha_2 = 2.5,\ Q_1 = Q_2 = I_X,\ R_1 = R_2 = 1 \in \mathbb{R}.$$

The first Hankel singular values of the Galerkin approximation are plotted for illustration in Figure 5.

In the simulation the original PDE is represented by another Finite Element approximation of (10) with order $n = 2072$. Figure 6 depicts parts of the spectrum of the uncontrolled system and the closed-loop system. In the plotted region the locations of the closed-loop eigenvalues for the controller without model reduction (i.e., with $r = N$) are very close to those with the final controller.

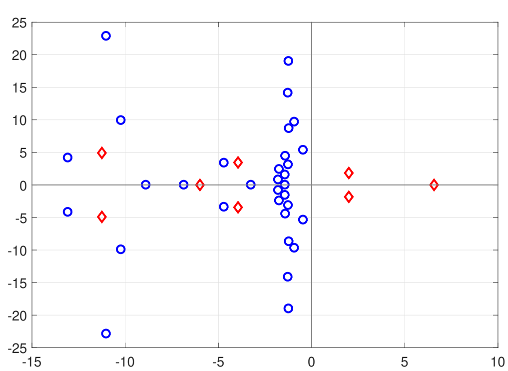

Fig. 6. Spectra of the uncontrolled system (red diamonds) and the closed-loop system with $N = 1258$ and $r = 12$ (blue circles).

The output of the controlled system for the initial states $x_0(\xi) = \cos(5\xi_1)$ and $z_0 = 0 \in \mathbb{R}^{16+40}$ of the system and the controller is depicted in Figure 7.

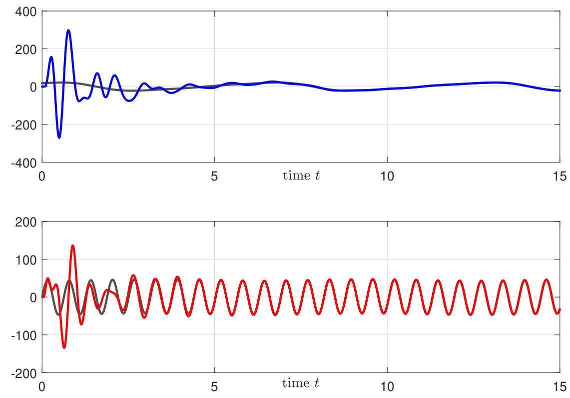

Fig. 7. Output $y(t) = (y_1(t), y_2(t))^T$ of the system (10) with the dual observer-based controller (top: $y_1(t)$, bottom: $y_2(t)$).

C. A Beam Equation with Kelvin–Voigt Damping

Consider a one-dimensional Euler-Bernoulli beam model on $\Omega = (0, \ell)$ [15, Sec. 3]

$$\frac{\partial^2 v}{\partial t^2}(\xi,t) + \frac{\partial^2}{\partial \xi^2}\left(\alpha \frac{\partial^2 v}{\partial \xi^2}(\xi,t) + \beta \frac{\partial^3 v}{\partial \xi^2 \partial t}(\xi,t)\right) \tag{11a}$$

$$+ \gamma \frac{\partial v}{\partial t}(\xi,t) = B_0 u(t) + B_{d0}(\xi) w_{dist}(t), \tag{11b}$$

$$v(\xi,0) = v_0(\xi), \qquad \frac{\partial v}{\partial t}(\xi,0) = v_1(\xi), \tag{11c}$$

$$y(t) = C_1 v(\cdot,t) + C_2 \dot{v}(\cdot,t) \tag{11d}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are constants so that $\alpha, \beta > 0$ and $\gamma \geq 0$. The input operator is defined by $B_0 u = \sum_{k=1}^m b_k(\cdot) u_k$ for $u = (u_k)_{k=1}^m \in U = \mathbb{R}^m$ for some fixed $b_k(\cdot) \in L^2(0,\ell)$ and the disturbance input operator B_{d0} is defined analogously. The assumptions on the measurement operators for the deflection $v(\cdot,t)$ and velocity $\dot{v}(\cdot,t)$ are given later.

We consider a situation where the beam is clamped at $\xi = 0$ and free at $\xi = \ell$. The boundary conditions are

$$v(0,t) = 0, \qquad \frac{\partial v}{\partial \xi}(0,t) = 0,$$

$$\left[\alpha \frac{\partial^2 v}{\partial \xi^2} + \beta \frac{\partial^3 v}{\partial \xi^2 \partial t}\right]_{\xi=\ell} = 0, \qquad \left[\alpha \frac{\partial^3 v}{\partial \xi^3} + \beta \frac{\partial^4 v}{\partial \xi^3 \partial t}\right]_{\xi=\ell} = 0.$$

Let $V_0 = \{ v \in H^2(0,\ell) \mid v(0) = v'(0) = 0 \}$. We define an inner product on V_0 by

$$\langle \phi_1, \phi_2 \rangle_{V_0} = \int_0^\ell \phi_1''(\xi) \phi_2''(\xi) d\xi, \qquad \forall \phi_1, \phi_2 \in V_0.$$

Defining the state as $x(t) = (v(\cdot,t), \dot v(\cdot,t))^T$ the beam model (11) can be written in the form (1) on $X = V_0 \times L^2(0,\ell)$ with

$$A = \begin{bmatrix} 0 & I \\ -\alpha \frac{d^4}{d\xi^4} & -\beta \frac{d^4}{d\xi^4} - \gamma \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \; B_d = \begin{bmatrix} 0 \\ B_{d0} \end{bmatrix}$$
$$D(A) = \{ (v,w) \in V_0 \times V_0 \mid \alpha v'' + \beta w'' \in H_r^2(0,\ell) \}$$

where $H_r^2(0,\ell) = \{ f \in H^2(0,\ell) \mid f(\ell) = f'(\ell) = 0 \}$. We assume the measurement operators $C_1 \in \mathcal{L}(V_0, Y)$ and $C_2 \in \mathcal{L}(L^2(0,\ell), Y)$ for $Y = \mathbb{R}^p$, and thus $C_1 v = (C_1^k v)_{k=1}^p$ where $C_1^k \in \mathcal{L}(V_0, \mathbb{R})$ and $C_2 w = (\langle w, c_k^1 \rangle_{L^2})_{k=1}^p$ for some fixed functions $c_k^1(\cdot) \in L^2(0,\ell)$. Since for any $0 < \xi_0 \le \ell$ the point evaluation $C_{\xi_0} v = v(\xi_0)$ is a linear functional on V_0, it is in particular possible to consider pointwise tracking of the deflection with $y(t) = v(\xi_0, t)$ in (11).

Choose $V = V_0 \times V_0$. As shown in [15, Sec. 3] the operator A is defined by a bounded and coercive sesquilinear form $a : V \times V \to \mathbb{C}$ defined so that for all $\phi = (\phi_1, \phi_2) \in V$ and $\psi = (\psi_1, \psi_2) \in V$ we have

$$a(\phi, \psi) = -\langle \phi_2, \psi_1 \rangle_{V_0} + \langle \alpha\phi_1 + \beta\phi_2, \psi_2 \rangle_{V_0} + \gamma \langle \phi_2, \psi_2 \rangle_{L^2}.$$

As the Galerkin approximation of (11) we use the Finite Element Method with cubic Hermite shape functions to approximate functions of V_0 and $L^2(0,\ell)$ in the spaces V_0^N. As shown in [15, Sec. 3] the approximating subspaces $V^N = V_0^N \times V_0^N$ have the required property (5). For additional details on the approximations, see [36, Sec. 4].

A Simulation Example

For a numerical example we consider a beam model with $\ell = 7$, $\alpha = 0.5$, $\beta = 1$, and $\gamma = 2$. Similarly as in [15, Sec. 3] we choose $U = \mathbb{R}$ and $B_0 = b_1(\cdot)$ with $b_1(\xi) = \xi$, and choose a measurement

$$y(t) = \int_5^6 v(\xi, t) d\xi, \qquad \text{i.e.,} \quad C_1 = \chi_{(5,6)}(\cdot), \; C_2 = 0.$$

The disturbance $w_{dist}(t)$ acts on the interval $(3,6)$ so that $B_{d0} = \chi_{(3,6)}(\cdot) \in \mathcal{L}(\mathbb{R}, L^2(0,\ell))$.

With our choices of parameters the damping in the beam model (11) is strong enough to stabilize the system exponentially. However, the stability margin of the system is very small. In such a situation the finite-dimensional low-gain robust controllers [13], [31] typically only achieve very limited closed-loop stability margins and slow convergence of the output. In this example we use our controller design to improve the degree of stability of the original model and achieve an improved closed-loop stability margin.

We take the reference signal and disturbance signal

$$y_{ref}(t) = 3\cos(t) - 2\cos(3t) + 15\sin(5t) - 6\sin(10t),$$
$$w_{dist}(t) = 3\sin(4t) + 5\sin(7t).$$

The corresponding set of frequencies in (2) is $\{1, 3, 4, 5, 7, 10\}$ with $q = 6$ and $n_k = 1$ for all $k \in \{1, \ldots, 6\}$. We modify the internal model in Section III in such a way that the parts associated to $\omega_0 = 0$ are omitted. We construct the observer-based controller in Section III-B using a Galerkin approximation with order $N = 58$ and subsequent balanced truncation with order $r = 10$. In the absence of the frequency 0, the internal model has dimension $\dim Z_0 = p \times q \times 2 = 12$.

The stability margins in the stabilizability of (A, B) and the detectability of (C, A) are limited because the beam model (11) is known to have an accumulation point of eigenvalues at $\lambda_{\text{acc}} \in \mathbb{R}_-$ [38]. In particular, the assumptions of the detectability of $(C, A + \alpha_1 I)$ and the stabilizability of $(A + \alpha_2 I, B)$ can only be satisfied if $0 \le \alpha_1, \alpha_2 < |\lambda_{\text{acc}}|$. To find the upper bound $|\lambda_{\text{acc}}|$, the spectrum of A can be computed similarly as in [19, Sec. 4.3]. In particular, the eigenvalues $\lambda_n \ne -\alpha/\beta$ of A are solutions of the quadratic equation $\lambda_n^2 - (\beta\eta_n + \gamma)\lambda_n - \alpha\eta_n = 0$, where $\eta_n \in \mathbb{R}_+$ are such that $\phi_n'''' = \eta_n \phi_n$ for some $\phi_n(\cdot) \in H^4(0,\ell)$ satisfying $\phi_n(0) = \phi_n'(0) = \phi_n''(\ell) = \phi_n'''(\ell) = 0$. Since $\eta_n \to \infty$ as $n \to \infty$, a direct computation shows that the eigenvalues λ_n have a limit $\lambda_n \to -\alpha/\beta =: \lambda_{\text{acc}}$ as $n \to \infty$. Thus the condition on α_1 and α_2 becomes $0 \le \alpha_1, \alpha_2 < |\lambda_{\text{acc}}| = \alpha/\beta = 0.5$. Motivated by this, the parameters of the stabilization are chosen as

$$\alpha_1 = \alpha_2 = 0.4, \; Q_1 = Q_2 = I_X, \; R_1 = 10^{-3}, \; R_2 = 10^3.$$

For the simulation of the original system (10) we use another Finite Element approximation of order $n = 140$. Figure 8 depicts parts of the spectrum of the uncontrolled system and the closed-loop system. In the plotted region the locations of the closed-loop eigenvalues for the controller without model reduction (i.e., with $r = N$) are very close to those with the final controller.

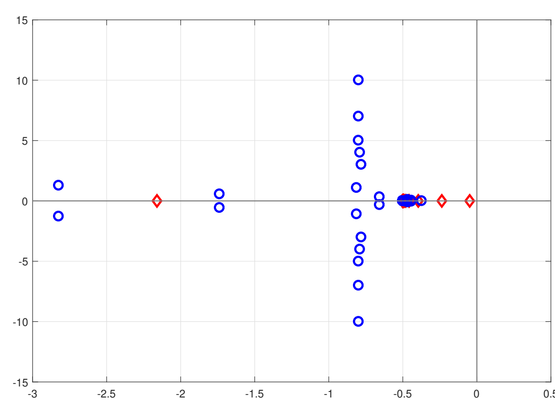

Fig. 8. Spectra of the uncontrolled system (red diamonds) and the closed-loop system with $N = 58$ and $r = 10$ (blue circles).

The output of the controlled system for the initial states $x_0(\xi) = \cos(5\xi_1) - 2$ and $z_0 = -3 \cdot \mathbf{1} \in \mathbb{R}^{12+10}$ of the system and the controller is depicted in Figure 7.

VI. Conclusions

We have studied the construction of finite-dimensional low-order controllers for robust output regulation of parabolic PDEs and other infinite-dimensional systems with analytic semigroups. We have presented two controller structures constructed using a Galerkin approximation of the control system and balanced truncation. Theorems III.1 and III.2 guarantee

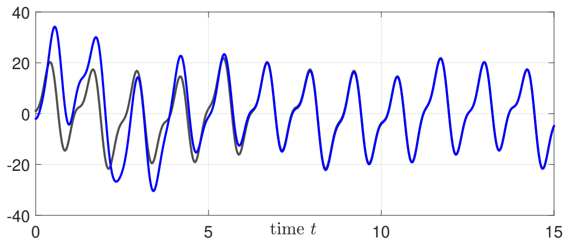

Fig. 9. Output (blue) of the beam model with the observer-based controller and the reference signal (gray).

that the controllers achieve robust output tracking and disturbance rejection provided that the orders N and $r \leq N$ of the Galerkin approximation and the model reduction, respectively, are sufficiently high, but the methods used in the proofs do not provide any concrete bounds for the sizes of N and r. The rate of decay of the Hankel singular values can be used together with Lemma II.3 as a rough indicator of how much reduction is possible in the last step of the controller construction algorithm. Deriving precise and reliable lower bounds N and r to guarantee closed-loop stability is an important topic for future research. Another open question is to develop a way to reliably estimate the stability margin of the closed-loop system for particular orders N and r.

Appendix

Lemma A.1. *The system (A^n, B^n, C^n) converges to (A, B, C) in the graph topology if and only if for some $Q \in \mathcal{L}(Y, U)$ the system $(A^n + B^n QC^n, B^n, C^n)$ converges to $(A + BQC, B, C)$ in the graph topology.*

Proof. The result follows from the property that in the graph topology the convergence of open loop systems is equivalent to the convergence of closed-loop systems [35, Prop. 7.3.40], [39, Thm. 3.3]. □

Lemma A.2. *Suppose Assumption II.1 and the standing assumptions are satisfied and G_1 is as in Sections* III-A *and* III-B. *Let $\alpha \geq 0$ be such that $(A + \alpha I, B, C)$ is exponentially stabilizable and detectable. Then the following hold.*

(a) *In the case of the observer-based controller, the pair*
$$\left(\begin{bmatrix} G_1 & G_2C \\ 0 & A \end{bmatrix} + \alpha I, \begin{bmatrix} G_2D \\ B \end{bmatrix}\right) \tag{12}$$
is exponentially stabilizable.

(b) *In the case of the dual observer-based controller, the pair*
$$\left(\begin{bmatrix} DK_1, & C \end{bmatrix}, \begin{bmatrix} G_1 & 0 \\ BK_1 & A \end{bmatrix} + \alpha I\right) \tag{13}$$
is exponentially detectable.

(c) *If $K = [K_1, K_2]$ stabilizes the pair* (12)*, then (K_1, G_1) is observable. If $\mathcal{G}_2 = \left[\begin{smallmatrix} G_2 \\ L \end{smallmatrix}\right]$ stabilizes the pair* (13)*, then (G_1, G_2) is controllable.*

Proof. We can assume $\alpha = 0$, since otherwise we may consider $\tilde{A} := A + \alpha I$ and $\tilde{G}_1 := G_1 + \alpha I$. We begin by proving part (b). Due to our assumptions we can choose L_1 so that $A + L_1C$ is exponentially stable and $P_L(i\omega_k) = CR(i\omega_k, A_L)B_L + D$ with $A_L = A + L_1C$ and $B_L = B + L_1D$ is surjective for every $k \in \{0, \dots, q\}$. Choose $L = L_1 + HG_2$ where H is the unique solution of the Sylvester equation $HG_1 = A_LH + B_LK_1$ and $G_2 \in \mathcal{L}(Y, Z_0)$ is such that the matrix $G_1 + G_2(CH + DK_1)$ is Hurwitz. The choice of G_2 is possible provided that the pair $(CH + DK_1, G_1)$ is observable. To see that this is true, let $k \in \{0, \dots, q\}$ and $0 \neq \phi_k \in \mathcal{N}(\pm i\omega_k - G_1)$. Since H is the solution of the Sylvester equation and G_1 and K_1 have special structure, we have $\phi_k = (\phi_k^0, \pm i\phi_k^0, 0, \dots, 0)^T$, $H\phi_k = R(\pm i\omega_k, A_L)B_LK_1^{k1}\phi_k^0$ and
$$(CH + DK_1)\phi_k = (CR(\pm i\omega_k, A_L)B_L + D)K_1^{k1}\phi_k^0 \neq 0$$
by the choices of $K_1^{k1} \in \mathcal{L}(Y, U)$. Thus the pair $(CH + DK_1, G_1)$ is observable. A direct computation then shows that
$$\begin{bmatrix} I & 0 \\ H & -I \end{bmatrix}\left(\begin{bmatrix} G_1 & 0 \\ BK_1 & A \end{bmatrix} + \begin{bmatrix} G_2 \\ L \end{bmatrix}\begin{bmatrix} DK_1, & C \end{bmatrix}\right)\begin{bmatrix} I & 0 \\ H & -I \end{bmatrix} = \begin{bmatrix} G_1 + G_2(CH + DK_1) & -G_2C \\ 0 & A + L_1C \end{bmatrix},$$
which generates an exponentially stable semigroup.

Part (a) can be proved analogously by considering adjoint operators. To prove (c), assume $K = [K_1, K_2]$ stabilizes the pair (12). If (K_1, G_1) is not controllable, there exist $k \in \{0, \dots, q\}$ and $0 \neq \phi_k \in \mathcal{N}(\pm i\omega_k - G_1)$ such that $K_1\phi_k = 0$. Then we also have
$$\left(\begin{bmatrix} G_1 & G_2C \\ 0 & A \end{bmatrix} + \begin{bmatrix} G_2D \\ B \end{bmatrix}[K_1, K_2]\right)\begin{bmatrix} \phi_k \\ 0 \end{bmatrix} = \pm i\omega_k \begin{bmatrix} \phi_k \\ 0 \end{bmatrix},$$
which contradicts the assumption that $[K_1, K_2]$ stabilizes (12). The second claim follows similarly by considering adjoint operators. □

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