On Polynomial Stability of Linear Systems

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***Abstract*— In this paper we discuss and compare different definitions for stability of an infinite-dimensional linear system. In particular, we concentrate on a situation where the semigroup generated by the system operator is polynomially stable. We derive conditions for strong input-output stability of the system. In addition, we introduce a weaker concept of polynomial input-output stability for linear systems, and show that it corresponds to the recently introduced notion of P-stability in the frequency domain.**

I. Introduction

There are many ways of defining "stability" of a linear infinite-dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

on a separable Hilbert space X. For example, in the time domain a common approach is to relate the stability of (1) to the stability properties of the strongly continuous semigroup $T(t)$ generated by A. On the other hand, in the frequency domain the characteristic (and sometimes the only available) property of a system is the way the output of the system depends on its input. Because of this, in the frequency domain stability is usually defined as the property of a system that "stable inputs" (in an appropriate sense) lead to "stable outputs". The main purpose of this paper is to discuss and compare selected definitions for stability of linear distributed parameter systems.

Systems of the form (1) where A generates an exponentially stablizable semigroup $T(t)$ and where B and C are bounded operators are well-understood [1], [2]. Also the more general situation where B and C are allowed to be unbounded operators has been studied extensively in the literature [3], [4]. In this paper we concentrate on a situation where $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$, but the semigroup $T(t)$ is not exponentially stabilizable. The stability properties of a system with a semigroup $T(t)$ that is only strongly stabilizable has been considered previously in, for example, [5], [6], [7]. In the previous references, the main type of stability was defined as follows.

Definition 1.1 (Input-Output Stability): The system (1) is called *input-output stable* if $u \in L^2(0,\infty;U)$ implies $y \in L^2(0,\infty;Y)$ and $\|y\|_{L^2} \leq M\|u\|_{L^2}$ for some $M \geq 0$ independent of u.

On a separable Hilbert space the Paley-Wiener Theorem implies that an equivalent condition for input-output stability is that the Laplace transforms $\hat{u}$ and $\hat{y}$ of u and y, respectively, have the property that if $\hat{u} \in H_2(\mathbb{C}^+, U)$, then also $\hat{y} \in H_2(\mathbb{C}^+, Y)$, and $\|\hat{y}\|_{H_2} \leq M\|\hat{u}\|_{H_2}$ for some constant $M \geq 0$. Since $\hat{y} = P\hat{u}$ where $P(\lambda) = CR(\lambda, A)B$ is the transfer function of (1), this is in turn equivalent to the property $P(\cdot) \in H_\infty(\mathbb{C}^+, \mathcal{L}(U, Y))$.

The main novelty in this paper is that we study a situation where the semigroup $T(t)$ is *polynomially stable* [8], [9], i.e., $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$ and there exist $\alpha > 0$ and $M \geq 1$ such that

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

As our first main results, we show that if A generates a polynomially stable semigroup, it is possible to derive natural and easily verifiable conditions on the operators B and C so that the system (1) is input-output stable in the sense of Definition 1.1.

In some situations the input-output stability in the sense of Definition 1.1 may be too strict a definition for stability of a linear system. In particular, this may be the case if the inputs to the plant are known to have special properties, such as a certain number of continuous derivatives. This type of situation is often encountered in studying output regulation with infinite-dimensional signal generators [10], [11]. In the output regulation problem we are first and foremost concerned of the stability properties of the closed-loop system, whose input is produced by another linear system called the exosystem. The exosystem, in turn, can produce signals with various levels of smoothness depending on the choice of its initial state. Motivated by situations like this, we introduce a new concept of polynomial input-output stability, which generalizes the input-output stability in Definition 1.1 in the following way. Here $W^{\alpha,2}$ is the Sobolev space of order α [12].

Definition 1.2 (Polynomial Input-Output Stability): Let $\alpha \in \mathbb{N}_0$. The system (1) is *polynomially input-output stable with* α if $u \in W^{\alpha,2}(0,\infty;U)$ implies $y \in L^2(0,\infty;Y)$ and $\|y\|_{L^2} \leq M\|u\|_{W^{\alpha,2}}$ for some $M \geq 0$ independent of u.

Recently in [13], [14] the class $H_\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$ of transfer functions was extended by allowing polynomial growth of $\|P(i\omega)\|$ on the imaginary axis $i\omega \in i\mathbb{R}$. These so-called P-stable transfer functions were introduced to meet the requirements of the theory of output regulation in the frequency domain. In this paper we show that the P-stability in [13], [14] (in a slightly stronger sense) corresponds to the polynomial input-output stability in Definition 1.2.

To summarize the main contributions, in this paper we

- present conditions for a linear system (1) with a poly-

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nomially stable semigroup to be input-output stable in the sense of Definition 1.1.
- introduce the concept of polynomial input-output stability.
- show that polynomial input-output stability (in the time domain) corresponds to P-stability (in the frequency domain).

II. Strongly Stable Systems

In this section we review the strong stability of system as defined in [5], [6], [7], and study it in the case where A generates a polynomially stable semigroup. As our main result, we present sufficient conditions for the stability of (1). We begin by defining the input, output and input-output maps as in [6], [4], [3].

Definition 2.1: We define the *extended input map* Φ by

$$\Phi u = \lim_{t\to\infty} \int_0^t T(t-s)Bu(s)ds \qquad u \in L^2(0,\infty;U)$$

if the limit exists and belongs to X for every $u \in L^2(0,\infty;U)$.

The *extended output map* Ψ is defined by

$$\Psi x = CT(\cdot)x : [0,\infty) \to Y.$$

for every $x \in X$.

We define the *extended input-output map* $\mathbb{F}$ such that $(\mathbb{F}u)(\cdot) = y(\cdot) : [0,\infty) \to Y$, where

$$y(t) = \int_0^t CT(t-s)Bu(s)ds \quad u \in L^2(0,\infty;U),\ t \geq 0.$$

In [6], [7] a *strongly stable system* is defined in the following way.

Definition 2.2 (Strongly stable system): The system (1) is said to be

(i) *input stable* if the extended input map satisfies $\Phi \in \mathcal{L}(L^2(0,\infty;U),X)$

(ii) *output stable* if the extended output map satisfies $\Psi \in \mathcal{L}(X, L^2(0,\infty;Y))$

(ii) *input-output stable* if the extended input-output map satisfies $\mathbb{F} \in \mathcal{L}(L^2(0,\infty;U), L^2(0,\infty;Y))$

If $T(t)$ is a strongly stable semigroup and (1) has the above properties, then it is called a *strongly stable system*.

On a Hilbert space, the Paley-Wiener Theorem [1, Thm. A.6.21] can be used to give frequency domain characterizations for the stability concepts in Definition 2.2. The definition of the Hardy spaces H_2 and H_∞ can be found in [1, Def. A.6.14].

Lemma 2.3: The system (1) is

(i) input stable if and only if $B^*R(\cdot,A^*)x \in H_2(\mathbb{C}^+;U)$ for all $x \in X$.

(ii) output stable if and only if $CR(\cdot,A)x \in H_2(\mathbb{C}^+;Y)$ for all $x \in X$.

(iii) input-output stable if and only if $CR(\cdot,A)B \in H_\infty(\mathbb{C}^+;\mathcal{L}(U,Y))$.

A. Systems with Polynomially Stable Semigroups

Throughout this section we assume that $T(t)$ generated by A is polynomially stable. Since $T(t)$ is uniformly bounded, the operators $-A$ and $-A^*$ are invertible and sectorial in the sense of [15, Sec. 2.1] or [16, Sec. 3.8]. Because of this, the fractional powers $(-A)^\beta$ and $(-A^*)^\gamma$ are well-defined operators for all $\beta,\gamma \in \mathbb{R}$.

On a Hilbert space the polynomial stability of a semigroup has the following characterizations. For the proof of the theorem, see [17, Lem. 2.4], [9, Lem. 2.3, Thm. 2.4], and [18, Lem. 3.2].

Theorem 2.4: Assume A generates a uniformly bounded semigroup on a Hilbert space X, and $i\mathbb{R} \subset \rho(A)$. For fixed constants $\alpha,\beta > 0$ the following properties are equivalent.

(a) $\|T_A(t)A^{-1}\| \leq \dfrac{M}{t^{1/\alpha}}, \qquad \forall t > 0$

(a′) $\|T_A(t)(-A)^{-\beta}\| \leq \dfrac{M}{t^{\beta/\alpha}}, \qquad \forall t > 0$

(b) $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$

(c) $\sup_{\mathrm{Re}\,\lambda\geq 0} \|R(\lambda,A)(-A)^{-\alpha}\| < \infty.$

We consider input and output operators satisfying

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma). \tag{2}$$

for some $\beta,\gamma \geq 0$. Under these conditions, the Closed Graph Theorem [1, Thm. A.3.4.9] implies that $(-A)^\beta B \in \mathcal{L}(U,X)$ and $(-A^*)^\gamma C^* \in \mathcal{L}(Y,X)$.

The following theorem summarizes the main results of this section. In particular, the strong stability of the system (1) is guaranteed if the exponents $\beta,\gamma \geq 0$ in (2) are large enough.

Theorem 2.5: Consider the system (1) on a separable Hilbert space X and assume the semigroup $T(t)$ is polynomially stable with $\alpha > 0$. Then the following are true.

(i) If $\beta+\gamma \geq \alpha$, then (1) is input-output stable.

(ii) If $\beta > \alpha/2$, then (1) is input stable.

(iii) If $\gamma > \alpha/2$, then (1) is output stable.

(iv) If $\beta > \alpha/2$ and $\gamma > \alpha/2$, then the system (1) is strongly stable (in the sense of Definition 2.2).

Proof: We begin by showing that (i) is satisfied. To this end, assume $\beta+\gamma \geq \alpha$ in (2). By Theorem 2.4 there exists $\tilde{M} \geq 1$ such that $\sup_{\mathrm{Re}\,\lambda\geq 0}\|R(\lambda,A)(-A)^{-\alpha}\| \leq \tilde{M}$. We have $\alpha-\beta-\gamma \leq 0$, and thus $(-A)^{\alpha-\beta-\gamma} \in \mathcal{L}(X)$.

Since $\sigma(A) \subset \mathbb{C}^-$, the function $CR(\cdot,A)B$ is analytic in $\mathbb{C}^+$. Denote $B_\beta = (-A)^\beta B \in \mathcal{L}(U,X)$ and $C_\gamma = \overline{C(-A)^\gamma} \in \mathcal{L}(X,Y)$ (the unique bounded extension of $C(-A)^\gamma : \mathcal{D}((-A)^\gamma) \to Y$ to X), which satisfies $\|C_\gamma\| = \|(-A^*)^\gamma C^*\|$. For all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$ we have

$$\begin{aligned}
\|CR(\lambda,A)B\| &= \|C(-A)^\gamma R(\lambda,A)(-A)^{-\beta-\gamma}(-A)^\beta B\| \\
&= \|C_\gamma R(\lambda,A)(-A)^{-\beta-\gamma}B_\beta\| \\
&= \|C_\gamma R(\lambda,A)(-A)^{-\alpha}(-A)^{\alpha-\beta-\gamma}B_\beta\| \\
&\leq \|C_\gamma\|\|R(\lambda,A)(-A)^{-\alpha}\|\|(-A)^{\alpha-\beta-\gamma}\|\|B_\beta\| \\
&\leq \tilde{M}\|(-A)^{\alpha-\beta-\gamma}\|\|(-A)^\beta B\|\|(-A^*)^\gamma C^*\|.
\end{aligned}$$

Since the bound is independent of $\lambda \in \overline{\mathbb{C}^+}$, this concludes $CR(\cdot, A)B \in H_\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$, and thus by Lemma 2.3 the system is input-output stable.

In order to prove (ii) and (iii), assume $\beta, \gamma > \alpha/2$. Theorem 2.4 shows that there exist $M_\beta, M_\gamma \geq 1$ such that

$$\|T(t)(-A)^{-\beta}\| \leq \frac{M_\beta}{t^{\beta/\alpha}} \quad \text{and} \quad \|T(t)(-A)^{-\gamma}\| \leq \frac{M_\gamma}{t^{\gamma/\alpha}},$$

where $\beta/\alpha > 1/2$ and $\gamma/\alpha > 1/2$. Denote $B_\beta = (-A)^\beta B \in \mathcal{L}(U, X)$ and $C_\gamma = \overline{C(-A)^\gamma} \in \mathcal{L}(X, Y)$ (the unique bounded extension of $C(-A)^\gamma : \mathcal{D}((-A)^\gamma) \to Y$ to X).

If $u \in L^2(0, \infty; U)$, then for every $t \geq 1$ we have

$$\begin{aligned}
&\int_0^{t-1} \|T(t-s)Bu(s)\| ds \\
&\quad = \int_0^{t-1} \|T(t-s)(-A)^{-\beta}(-A)^\beta Bu(s)\| ds \\
&\quad \leq \|B_\beta\| \int_0^{t-1} \|T(t-s)(-A)^{-\beta}\| \|u(s)\| ds \\
&\quad \leq \|B_\beta\| \left(\int_0^{t-1} \|T(t-s)(-A)^{-\beta}\|^2 \right)^{1/2} \\
&\qquad \times \left(\int_0^{t-1} \|u(s)\|^2 ds \right)^{1/2} \\
&\quad \leq M_\beta \|B_\beta\| \left(\int_0^{t-1} \frac{1}{(t-s)^{2\beta/\alpha}} \right)^{1/2} \|u\|_{L^2} \\
&\quad \leq M_\beta \|B_\beta\| \|u\|_{L^2} \left(\int_1^\infty \frac{1}{s^{2\beta/\alpha}} \right)^{1/2} < \infty,
\end{aligned}$$

since $2\beta/\alpha > 1$, and

$$\begin{aligned}
&\int_{t-1}^t \|T(t-s)Bu(s)\| ds \\
&\quad \leq \|B\| \left(\sup_{t>0} \|T(t)\| \right) \int_{t-1}^t \|u(s)\| ds \\
&\quad \leq \|B\| \left(\sup_{t>0} \|T(t)\| \right) \left(\int_{t-1}^t \|u(s)\|^2 ds \right)^{1/2} \cdot 1 \\
&\quad \leq \|B\| \left(\sup_{t>0} \|T(t)\| \right) \|u\|_{L^2} < \infty.
\end{aligned}$$

Together these estimates conclude that the extended output map Φ is well-defined and $\Phi \in \mathcal{L}(L^2(0, \infty; U), X)$. Thus the system (1) is input stable.

If $x \in X$, we have

$$\begin{aligned}
\int_0^t \|CT(t)x\|^2 dt &\leq \int_0^1 \|CT(t)x\|^2 dt \\
&\quad + \int_1^\infty \|C(-A)^\gamma T(t)(-A)^{-\gamma}x\|^2 dt \\
&\leq \left(\sup_{t>0} \|T(t)\| \right)^2 \|C\|^2 \|x\|^2 \\
&\quad + M_\gamma^2 \|C_\gamma\|^2 \|x\|^2 \int_1^\infty \frac{1}{t^{2\gamma/\alpha}} dt < \infty
\end{aligned}$$

since $2\gamma/\alpha > 1$. This shows that the extended output map Ψ is well-defined and $\Psi \in \mathcal{L}(X, L^2(0, \infty; Y))$. Because of this, the system (1) is output stable.

Finally, if $\beta, \gamma > \alpha/2$, we have $\beta + \gamma > \alpha/2 + \alpha/2 = \alpha$, and the strong stability of (1) in the sense of Definition 2.2 follows from (i), (ii), and (iii), and the fact that $T(t)$ is a strongly stable semigroup. ■

Remark 2.6: The results (ii) and (iii) in Theorem 2.5 also follow from each others by duality, since output stability of (1) is equivalent to input stability of the dual system, and vice versa.

Theorem 2.5 and Lemma 2.3 immediately imply the following corollary.

Corollary 2.7: If $\gamma > \alpha/2$, then $CR(\cdot, A)x \in H_2(\mathbb{C}^+; Y)$ for every $x \in X$. Similarly, if $\beta > \alpha/2$, then $B^* R(\cdot, A^*)x \in H_2(\mathbb{C}^+; U)$ for every $x \in X$.

III. Stability in the Frequency Domain: P-Stability

In this section we review the definition of P-stable transfer functions. This extension of the class $H_\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$ of transfer functions was introduced recently in [13, Sec. 4.2],[14]. We use a definition where the condition (c) is slightly stronger than in the original version, where the polynomial estimate for $\|P(\lambda)\|$ was only required on the imaginary axis $\lambda \in i\mathbb{R}$.

Definition 3.1 (P-Stability): A system with a transfer function $P(\lambda)$ is called *P-stable (with $\alpha > 0$)* if the following conditions are satisfied.

(a) $P(\cdot)$ is analytic in a domain containing $\overline{\mathbb{C}^+}$
(b) $P(\cdot) \in H_\infty(\mathbb{C}^+_\xi; \mathcal{L}(U, Y))$ for every $\xi > 0$, where $\mathbb{C}^+_\xi = \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \xi \}$
(c) There exist $\varepsilon > 0$ and $M_P \geq 1$ such that

$$\|P(\lambda)\| \leq M_P(1 + |\lambda|^\alpha)$$

for every $\lambda \in \overline{\mathbb{C}^+}$ with $0 \leq \operatorname{Re} \lambda < \varepsilon$.

Example 3.2: As an example of a transfer function that is P-stable but not in $H_\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$ we can consider

$$P(\lambda) = \sum_{k=1}^\infty \frac{1}{k^{4/3}(\lambda + 1/k^2 - ik)},$$

which satisfies the conditions of Definition 3.1 for $\alpha = 2/3$.

Besides condition (c), Definition 3.1 also differs from the definition of P-stable transfer functions in [13], [14] in another aspect: In the above definition, we consider the exponent $\alpha > 0$ to be *fixed*, whereas in the original definition it was only required that (c) is satisfied for *some* such exponent. The set of transfer functions that are "P-stable with $\alpha > 0$" (where $\alpha > 0$ is fixed) is not an algebra. Indeed, the product of two transfer functions not need be P-stable with the same $\alpha > 0$. Regardless of this drawback, we choose to pay close attention to the exponent α, because it can be connected to the properties of the semigroup $T(t)$ and the extended input-output map $\mathbb{F}$, as the results presented later in this paper demonstrate. Moreover, the algebra of P-stable transfer functions used in [13], [14] can be obtained from Definition 3.1 by defining

$$\mathbf{F}_P = \{ P(\cdot) \mid P(\cdot) \text{ is P-stable with some } \alpha > 0 \}.$$

The following theorem shows that the concept of P-stability is indeed closely related to systems where A generates a polynomially stable semigroup.

Theorem 3.3: If the semigroup $T(t)$ generated by A is polynomially stable with $\alpha > 0$, then the system (1) is P-stable with $\alpha > 0$.

Proof: (a) Since $T(t)$ is uniformly bounded and $i\mathbb{R} \subset \rho(A)$, we have that $R(\cdot, A)$ is analytic in $\rho_\infty(A) \supset \overline{\mathbb{C}^+}$ (the connected component of $\rho(A)$ containing $(0, \infty)$). The same is true for the transfer function $\lambda \mapsto P(\lambda) = CR(\lambda, A)B$.

(b) Since $T(t)$ is uniformly bounded, the Hille-Yosida Theorem implies that $R(\cdot, A)$ is uniformly bounded in every half-plane $\mathbb{C}_\xi^+$ where $\xi > 0$. The same is true for the transfer function $\lambda \mapsto P(\lambda) = CR(\lambda, A)B$.

(c) By Theorem 2.4 there exists $M_A \geq 1$ such that $\|R(i\omega, A)\| \leq M_A(1 + |\omega|^\alpha)$ for every $\omega \in \mathbb{R}$. If $M = \sup_{t\geq 0}\|T(t)\|$, then the Hille-Yosida Theorem implies $\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda}$ for all $\lambda \in \mathbb{C}^+$. Let $\lambda = \xi + i\omega \in \mathbb{C}$ be such that $0 < \xi < 1$. We can use the resolvent identity

$$\begin{aligned} R(\xi + i\omega, A) &= R(i\omega, A) - \xi R(\xi + i\omega, A)R(i\omega, A) \\ &= (I - \xi R(\xi + i\omega, A))R(i\omega, A) \end{aligned}$$

to estimate

$$\begin{aligned} &\|R(\xi + i\omega, A)\| \leq (1 + |\xi|\|R(\xi + i\omega, A)\|)\|R(i\omega, A)\| \\ &\quad \leq (1 + M)M_A(1 + |\omega|^\alpha) \leq (1 + M)M_A(1 + |\lambda|^\alpha). \end{aligned}$$

This immediately implies that

$$\begin{aligned} \|P(\lambda)\| &\leq \|C\|\|R(\lambda, A)\|\|B\| \\ &\leq \|B\|\|C\|(1 + M)M_A(1 + |\lambda|^\alpha) \end{aligned}$$

whenever $0 < \operatorname{Re}\lambda < 1$. ■

The following property of P-stable transfer functions will be needed in the next section.

Lemma 3.4: If $P(\cdot) = CR(\cdot, A)B$ is P-stable with $\alpha > 0$, then $\lambda \mapsto (1 + \lambda)^{-\alpha}P(\lambda)$ is in $H_\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$, i.e., $\lambda \mapsto (1 + \lambda)^{-\alpha}P(\lambda)$ is analytic in $\mathbb{C}^+$ and

$$\sup_{\lambda\in\mathbb{C}^+} \|(1 + \lambda)^{-\alpha}P(\lambda)\| < \infty.$$

Proof: Since both $\lambda \mapsto (1+\lambda)^{-\alpha}$ and $P(\cdot)$ are analytic in $\mathbb{C}^+$, so is the function $\lambda \mapsto (1 + \lambda)^{-\alpha}P(\lambda)$.

Since $P(\cdot)$ is P-stable, there exist $M_P \geq 1$ and $\varepsilon > 0$ such that $\|P(\lambda)\| \leq M_P(1 + |\lambda|^\alpha)$ if $0 < \operatorname{Re}\lambda < \varepsilon$. This implies that if $0 < \operatorname{Re}\lambda < \varepsilon$, then

$$\begin{aligned} \|(1 + \lambda)^{-\alpha}P(\lambda)\|_{\mathcal{L}(U,Y)} &\leq \frac{M_P(1 + |\lambda|^\alpha)}{|1 + \lambda|^\alpha} \\ &\leq M_P \sup_{\lambda\in\mathbb{C}^+} \frac{1 + |\lambda|^\alpha}{|1 + \lambda|^\alpha} < \infty. \end{aligned}$$

On the other hand, since $|1 + \lambda| \geq 1$ for every $\lambda \in \mathbb{C}^+$, we have

$$\begin{aligned} \sup_{\operatorname{Re}\lambda\geq\varepsilon} \|(1 + \lambda)^{-\alpha}P(\lambda)\| &= \sup_{\operatorname{Re}\lambda\geq\varepsilon} |1 + \lambda|^{-\alpha}\|P(\lambda)\| \\ &\leq \sup_{\operatorname{Re}\lambda\geq\varepsilon} \|P(\lambda)\| < \infty \end{aligned}$$

due to the condition (b) in Definition 3.1. ■

IV. Polynomially Stable Systems

In this section introduce the *polynomial input-output stability* of a linear system. This concept is motivated by systems where A generates a polynomially stable semigroup and B and C are bounded operators. We relax the conditions in Definition 2.2 by restricting our attention to inputs with suitable smoothness properties.

Assumption 4.1: Throughout the rest of this section we assume $\alpha \in \mathbb{N}_0$.

The definition of polynomial input-output stability uses Sobolev spaces [12]

$$\begin{aligned} W^{\alpha,2}(0, \infty; U) &= \left\{ u \in L^2 \;\middle|\; u^{(k)} \in L^2 \text{ for } 1 \leq k \leq \alpha \right\} \\ \|u\|^2_{W^{\alpha,2}} &= \sum_{k=0}^{\alpha} \|u^{(k)}\|^2_{L^2}. \end{aligned}$$

Later in this section we also consider functions in the space

$$\begin{aligned} C_0^\infty(0, \infty; U) = \big\{u \in C^\infty \;\big|\; &u^{(k)}(0) = 0 \;\; \forall k, \\ &\operatorname{supp} u \text{ is compact}\big\}. \end{aligned}$$

The space C_0^∞ is a dense subspace of $W^{\alpha,2}$ for every $\alpha \in \mathbb{N}$.

Definition 4.2 (Polynomially input-output stable system): Assume $\alpha \in \mathbb{N}_0$. The system (1) is called *polynomially input-output stable (with α)* if the extended input-output map $\mathbb{F}$ satisfies $\mathbb{F} \in \mathcal{L}(W^{\alpha,2}(0, \infty; U), L^2(0, \infty; Y))$

The last main result of this paper presented in the following theorem connects the P-stability in the frequency domain to the polynomial input-output stability in the time domain.

Theorem 4.3: Assume $\alpha \in \mathbb{N}$. If the system (1) is P-stable with α, then it is polynomially input-output stable with α. On the other hand, if $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$, and the system (1) is polynomially input-output stable with α, then it is P-stable with α.

In the proof of the theorem, we use the following lemma.

Lemma 4.4: Let $\alpha \in \mathbb{N}$. If $u \in \mathbb{C}_0^\infty(0, \infty; U)$, then $\lambda \mapsto (1 + \lambda)^\alpha \hat{u}(\lambda) \in H_2(\mathbb{C}^+; U)$ and

$$\frac{1}{\sqrt{\alpha + 1}}\|u\|_{W^{\alpha,2}} \leq \|(1 + \lambda)^\alpha \hat{u}\|_{H_2} \leq 2^\alpha\sqrt{2}\|u\|_{W^{\alpha,2}}. \quad (3)$$

Proof: Let $u \in C_0^\infty$. We have $u \in L^2(0, \infty; U)$ and $\hat{u} \in H_2(\mathbb{C}^+; U)$. For every $\lambda \in \mathbb{C}^+$

$$\begin{aligned} \|(1 + \lambda)^\alpha \hat{u}(\lambda)\|_U &= |1 + \lambda|^\alpha\|\hat{u}(\lambda)\| \leq (1 + |\lambda|)^\alpha\|\hat{u}(\lambda)\| \\ &\leq 2^\alpha(1 + |\lambda|^\alpha)\|\hat{u}(\lambda)\| = 2^\alpha(\|\hat{u}(\lambda)\| + |\lambda|^\alpha\|\hat{u}(\lambda)\|) \\ &= 2^\alpha(\|\hat{u}(\lambda)\| + \|\lambda^\alpha\hat{u}(\lambda)\|). \end{aligned}$$

This and the scalar inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for $a, b \geq 0$ further imply

$$\begin{aligned} \|(1 + \lambda)^\alpha\hat{u}\|^2_{H_2} &\leq \sup_{\sigma>0} \int_{-\infty}^{\infty} \|(1 + \sigma + i\eta)^\alpha\hat{u}(\sigma + i\eta)\|^2 d\eta \\ &\leq 2^{2\alpha} \sup_{\sigma>0} \int_{-\infty}^{\infty} 2\big(\|\hat{u}(\sigma + i\eta)\|^2 \\ &\qquad + \|(\sigma + i\eta)^\alpha\hat{u}(\sigma + i\eta)\|^2\big) d\eta \\ &\leq 2^{2\alpha+1}(\|\hat{u}\|^2_{H_2} + \|\lambda^\alpha\hat{u}\|^2_{H_2}). \end{aligned}$$

For every $k \in \{0,\dots,\alpha\}$ we have

$$u \in W^{\alpha,2}(0,\infty;U) \quad \Rightarrow \quad u^{(k)} \in L^2(0,\infty;U) \quad \Leftrightarrow \quad \mathcal{L}\{u^{(k)}\} \in H_2(\mathbb{C}^+;U).$$

Using the properties of the Laplace transform and the fact that $u^{(l)}(0)=0$ for $0 \le l \le \alpha-1$, for every $\lambda \in \mathbb{C}^+$ we have

$$\mathcal{L}\{u^{(k)}\}(\lambda) = \lambda^k \hat{u}(\lambda) + \sum_{l=0}^{k-1} \lambda^{k-1-l} u^{(l)}(0) = \lambda^k \hat{u}(\lambda).$$

Since $\lambda \mapsto (1+\lambda)^\alpha$ and $\hat{u}$ are analytic in $\mathbb{C}^+$, also their product $\lambda \mapsto (1+\lambda)^\alpha \hat{u}(\lambda)$ is analytic. By the Paley-Wiener Theorem [1, Thm. A.6.21] the norms satisfy

$$\begin{aligned}
\|(1+\lambda)^\alpha \hat{u}\|_{H_2}^2 &\le 2^{2\alpha+1}(\|\hat{u}\|_{H_2}^2 + \|\lambda^\alpha \hat{u}\|_{H_2}^2) \\
&\le 2^{2\alpha+1}(\|\mathcal{L}\{u\}\|_{H_2}^2 + \|\mathcal{L}\{u^{(\alpha)}\}\|_{H_2}^2) \\
&= 2^{2\alpha+1}(\|u\|_{L^2}^2 + \|u^{(\alpha)}\|_{L^2}^2) \le 2^{2\alpha+1} \sum_{k=0}^{\alpha} \|u^{(k)}\|_{L^2}^2 \\
&= 2^{2\alpha+1} \|u\|_{W^{\alpha,2}}^2.
\end{aligned}$$

This proves the second inequality in (3).

For every $\lambda \in \mathbb{C}^+$ and $k \in \{1,\dots,\alpha\}$ we have

$$|\lambda|^k \le |1+\lambda|^k \le |1+\lambda|^\alpha$$

since $|\lambda| \le |1+\lambda|$, and $|1+\lambda| \ge 1$. Thus

$$\begin{aligned}
\|\lambda^k \hat{u}(\lambda)\|_U &= |\lambda|^k \|\hat{u}(\lambda)\|_U \le |1+\lambda|^\alpha \|\hat{u}(\lambda)\|_U \\
&= \|(1+\lambda)^\alpha \hat{u}(\lambda)\|_U,
\end{aligned}$$

which in turn implies

$$\|\lambda^k \hat{u}\|_{H^2} \le \|(1+\lambda)^\alpha \hat{u}\|_{H_2}.$$

Using this, we can estimate

$$\begin{aligned}
\|u\|_{W^{\alpha,2}}^2 &= \sum_{k=0}^{\alpha} \|u^{(k)}\|_{L^2}^2 = \sum_{k=0}^{\alpha} \|\lambda^k \hat{u}\|_{H_2}^2 \\
&\le \sum_{k=0}^{\alpha} \|(1+\lambda)^\alpha \hat{u}\|_{H_2} = (\alpha+1)\|(1+\lambda)^\alpha \hat{u}\|_{H_2}^2.
\end{aligned}$$

This proves the first inequality in (3). ∎

Proof of Theorem 4.3: We begin by showing that if the system is P-stable, then it is polynomially input-output stable. Let $u \in C_0^\infty(0,\infty;U)$ and denote $y = \mathbb{F}u : (0,\infty) \to Y$. Our aim is to show that $y \in L^2(0,\infty;Y)$ and that there exists $M \ge 1$ independent of u such that $\|y\|_{L^2} \le M\|u\|_{W^{\alpha,2}}$. We have from Lemma 4.4 that $\lambda \mapsto (1+\lambda)^\alpha \hat{u}(\lambda) \in H_2(\mathbb{C}^+;U)$, and

$$\|(1+\lambda)^\alpha \hat{u}\|_{H_2} \le 2^\alpha \sqrt{2} \|u\|_{W^{\alpha,2}}.$$

Now for every $\lambda \in \mathbb{C}^+$ we have

$$\hat{y}(\lambda) = P(\lambda)\hat{u}(\lambda) = (1+\lambda)^{-\alpha} P(\lambda)(1+\lambda)^\alpha \hat{u}(\lambda)$$

where $\lambda \mapsto (1+\lambda)^{-\alpha} P(\lambda) \in H_\infty(\mathbb{C}^+;\mathcal{L}(U,Y))$ by Lemma 3.4. Since $\lambda \mapsto (1+\lambda)^\alpha \hat{u}(\lambda) \in H_2(\mathbb{C}^+;U)$, we have that $\hat{y} \in H_2(\mathbb{C}^+;Y)$ [1, Thm. A.6.26], which further implies $y \in L^2(0,\infty;Y)$ by the Paley-Wiener Theorem. Moreover, the norms satisfy

$$\begin{aligned}
\|y\|_{L^2} = \|\hat{y}\|_{H_2} &= \|(1+\lambda)^{-\alpha} P(\lambda)(1+\lambda)^\alpha \hat{u}\|_{H_2} \\
&\le \|(1+\lambda)^{-\alpha} P(\lambda)\|_{H_\infty} \|(1+\lambda)^\alpha \hat{u}\|_{H_2} \\
&\le 2^\alpha \sqrt{2} \|(1+\lambda)^{-\alpha} P(\lambda)\|_{H_\infty} \|u\|_{W^{\alpha,2}}.
\end{aligned}$$

Since $u \in C_0^\infty(\mathbb{C}^+;U)$ was arbitrary, and since the space C_0^∞ is dense in $W^{\alpha,2}$, this concludes that $\mathbb{F} \in \mathcal{L}(W^{\alpha,2}(0,\infty;U), L^2(0,\infty;Y))$.

Now assume that $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$, and the system (1) is polynomially input-output stable. Our aim is to show that the system is P-stable. Similarly as in the proof of Theorem 3.3, the first two properties in Definition 3.1 follow directly from the assumptions on A and $T(t)$.

Since $C_0^\infty(0,\infty;U)$ is dense in $L^2(0,\infty;U)$, the Paley-Wiener Theorem implies that the space $\mathcal{L}\{C_0^\infty\}$ (the image of C_0^∞ under the Laplace transform) is dense in $H_2(\mathbb{C}^+;U)$. Take $\hat{u} \in \mathcal{L}\{C_0^\infty(0,\infty;U)\}$ (i.e., $u \in C_0^\infty$), and define $\hat{u}_0 \in H_2(\mathbb{C}^+;U)$ such that $\hat{u}_0(\lambda) = (1+\lambda)^{-\alpha}\hat{u}(\lambda)$. Since $u \in W^{\alpha,2}$, and since $\lambda \mapsto (1+\lambda)^{-\alpha}$ is uniformly bounded in $\mathbb{C}^+$, we have that $u_0 = \mathcal{L}^{-1}\{\hat{u}_0\} \in W^{\alpha,2}(0,\infty;U)$. Because

$$\mathbb{F} \in \mathcal{L}(W^{\alpha,2}(0,\infty;U), L^2(0,\infty;Y))$$

by assumption, we can define $y = \mathbb{F}u_0 \in L^2(0,\infty;Y)$. Due to the Paley-Wiener Theorem we also have $\hat{y} \in H_2(\mathbb{C}^+;Y)$, and $\hat{y}(\lambda) = P(\lambda)\hat{u}_0(\lambda)$ for every $\lambda \in \mathbb{C}^+$. We can use Lemma 4.4 to estimate

$$\begin{aligned}
\|(1+\lambda)^{-\alpha} P(\lambda)\hat{u}\|_{H_2} &= \|P(\lambda)(1+\lambda)^{-\alpha}\hat{u}\|_{H_2} \\
&= \|P(\lambda)\hat{u}_0\|_{H_2} = \|\hat{y}\|_{H_2} = \|y\|_{L^2} = \|\mathbb{F}u_0\|_{L^2} \\
&\le \|\mathbb{F}\|\|u_0\|_{W^{\alpha,2}} \le \frac{1}{\sqrt{\alpha+1}} \|\mathbb{F}\|\|(1+\lambda)^\alpha \hat{u}_0\|_{H_2} \\
&= \frac{1}{\sqrt{\alpha+1}} \|\mathbb{F}\|\|\hat{u}\|_{H_2}.
\end{aligned}$$

Since $\hat{u} \in \mathcal{L}\{C_0^\infty(0,\infty;U)\}$ was arbitrary and since $\mathcal{L}\{C_0^\infty\}$ is dense in $H_2(\mathbb{C}^+;U)$, this implies that the multiplication map $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda)$ is in $\mathcal{L}(H_2(\mathbb{C}^+;U), H_2(\mathbb{C}^+,Y))$, which in particular implies that

$$\sup_{0<\operatorname{Re}\lambda<1} (1+|\lambda|^\alpha)^{-1}\|P(\lambda)\| < \infty.$$

This concludes that the system (1) is P-stable. ∎

V. CONCLUSIONS

In this paper we have studied different definitions for stability of a linear system in the case where the semigroup $T(t)$ generated by A is not exponentially stabilizable. Our main interest was in the case where the semigroup $T(t)$ is polynomially stable. We demonstrated that for such systems it is possible to derive concrete conditions for the strong stability in the sense of [5], [7].

In addition, we introduced a new form of stability, the so-called polynomial input-output stability. The main motivation for this definition arises from a situation where the outputs of the systems are not required to be well-behaving for arbitrary square integrable inputs, but only for a smaller class of functions with some smoothness properties. In this paper we only considered one part of strong stability in Definition 2.2, the input-output stability. Main topics for future research include corresponding redefinitions of input and output stabilities, and further study of the systems with these properties.

The topics for future research also include generalizing the results on polynomial input-output stability for non-integer exponents $\alpha > 0$, as well as studying systems with unbounded input and output operators B and C.

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