Polynomial Input-Output Stability for Linear Systems

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We introduce the concept of polynomial input-output stability for infinite-dimensional linear systems. We show that this stability type corresponds exactly to the recent notion of P-stability in the frequency domain. In addition, we show that on a Hilbert space a regular linear system whose system operator generates a polynomially stable semigroup is always polynomially input-output stable, and present additional conditions under which the system is input-output stable. The results are illustrated with an example of a polynomially input-output stable one-dimensional wave system.

***Index Terms*—Distributed parameter system, stability**

I. Introduction

The concept of "stability" for a regular linear system [16], [15]

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{1a}$$
$$y(t) = Cx(t) \tag{1b}$$

on a Banach space X can be defined in various ways. In this paper we study the stability of (1) in the *input-output sense*, where the system is defined to be stable if every "stable input" $u(\cdot) : [0,\infty) \to U$ results in a "stable output" $y(\cdot) : [0,\infty) \to Y$. Both U and Y are Hilbert spaces. One of the most well-known of such stability types is the *input-output stability*[1] [15].

Definition 1 (Input-Output Stability)**.** The system (1) is called *input-output stable* if for $x_0 = 0$ the property $u \in L^2(0,\infty;U)$ implies $y \in L^2(0,\infty;Y)$ and $\|y\|_{L^2} \leq M\|u\|_{L^2}$ for some $M \geq 0$ independent of u.

The purpose of this paper is to introduce a new stability type called *polynomial input-output stability*, which relaxes Definition 1 by restricting the class of stable inputs u. In addition, we study the properties of this stability type and, in particular, show that it corresponds precisely to a recently introduced frequency domain stability type. We also prove that on a Hilbert space any regular linear system whose system operator A generates a polynomially stable semigroup [5] is polynomially input-output stable.

The motivation for this paper derives from robust output regulation of distributed parameter systems with infinite-dimensional exosystems. Recently, the theory of robust output regulation in this setting has been developed concurrently in the time domain [12], [13], [6] and in the frequency domain [9]. However, even though results of similar nature appear in both domains, the exact relationships between the corresponding parts of the theory are not at all properly understood. This gap between the two domains is largely caused by the lack of correspondence between the stability types of the closed-loop system on the two sides of the fence.

In this paper we take the first steps towards remedying the situation. In particular, we investigate suitable forms of stability for the closed-loop systems in both the time and frequency domains, and study their relationships. In the time-domain the most common approach to stability is to require that the closed-loop semigroup is stable [12]. However, for the purpose of bridging the gap between the two domains the input-output stability in Definition 1 has a clear advantage over stability defined in terms of the semigroup: the input-output stability has an exact counterpart in the frequency domain. Indeed, due to the Paley–Wiener Theorem the system (1) is input-output stable if and only if its transfer function $P(\lambda)$ satisfies $P(\cdot) \in H^\infty(\mathbb{C}^+, \mathcal{L}(U,Y))$. Unfortunately, it has been observed that in robust output regulation with infinite-dimensional exosystems the input-output stability of the closed-loop system, or equivalently its H^∞-stability, is in general unachievable [9]. However, in [9] it was shown that the robust output regulation problem can instead be solved under the weaker assumption of *P-stability* of the closed-loop system [9, Sec. 3.2]. This stability type relaxes H^∞-stability by permitting polynomial growth of the transfer function on the imaginary axis (for details, see Definition 5). Similarly, in the time-domain case it has been observed that even though the exponential stabilization of the closed-loop semigroup is impossible, it can be stabilized *polynomially* under very reasonable assumptions, and that the robust output regulation problem can be solved using this as the main stability type [14].

Motivated by the above consideration, and especially by the solvability of the robust output regulation problem under the assumption of polynomial stability of the closed-loop semigroup, we introduce a new time-domain stability type called the *polynomial input-output stability*. Here $W^{\alpha,2}$ is the Sobolev space of order $\alpha \geq 0$ [1], see Section II-B for details.

Definition 2 (Polynomial Input-Output Stability)**.** The system (1) is *polynomially input-output stable with* $\alpha \geq 0$ if for $x_0 = 0$ the property $u \in W^{\alpha,2}(0,\infty;U)$ implies $y \in L^2(0,\infty;Y)$ and $\|y\|_{L^2} \leq M\|u\|_{W^{\alpha,2}}$ for some $M \geq 0$ independent of u.

Definition 2 relaxes the definition of input-output stability, and coincides with it in the case of $\alpha = 0$. As the main result of this paper we show in Section III that polynomial input-output stability is the exact time-domain correspondent

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[1]It should be noted that there is some variation in the terminology used in the literature. A more precise term for the concept in Definition 1 would be "L^2-input-output stability", and "input-output stability" is sometimes used for other forms of stability.

of P-stability in the frequency domain.

In the final part of this paper we study the situation where the semigroup $T(t)$ generated by A is polynomially stable, i.e., if $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$ and there exist $\tilde{\alpha} > 0$ and $M \geq 1$ such that [5]

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\tilde{\alpha}}}, \qquad \forall t > 0. \tag{2}$$

We will see that polynomial input-output stability appears naturally in situations where the semigroup generated by A is polynomially stable. In fact, we will show that if A generates a polynomially stable semigroup satisfying (2), then the system (1) is polynomially input-output stable with exponent $\alpha = \tilde{\alpha} + 2$. Moreover, if B and C are bounded operators, then (1) is polynomially input-output stable with $\alpha = \tilde{\alpha}$.

Polynomial input-output stability was first introduced in a preliminary version [11] of this article for integer exponents $\alpha \in \mathbb{N}$, and for systems with bounded operators B and C. In this paper we extend the study for real exponents $\alpha \geq 0$, and allow the system (1) to have unbounded input and output operators. Moreover, in [11] the definition of P-stability was stronger than the original definition introduced in [9]. In this paper we prove the correspondence of polynomial input-output stability and P-stability using the original definition of P-stability.

We illustrate polynomial input-output stability by considering a one-dimensional damped wave equation with collocated control and observation. We prove that the system under consideration is polynomially input-output stable by showing that its transfer function grows polynomially on the imaginary axis.

II. Polynomial Input-Output Stability

A. Mathematical Preliminaries

Throughout the paper we consider a regular linear system (1) [16], [15] on a Banach space X. The operator $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$ on X. For a fixed $\lambda_0 > \omega_0(T(t))$ we define the scale spaces $X_1 = (\mathcal{D}(A), \|(\lambda_0 - A)\cdot\|)$ and $X_{-1} = \overline{(X, \|R(\lambda_0, A)\cdot\|)}$ (the completion of X with respect to the norm $\|R(\lambda_0, A)\cdot\|$). We denote by $A_{-1} : X \subset X_{-1} \to X_{-1}$ and $T_{-1}(t)$ the extensions of the operator A and the semigroup $T(t)$, respectively, to the space X_{-1}. The input space U and the output space Y are Hilbert spaces. The control and observability operators B and C are admissible with respect to A, and we define the Lebesgue extension C_L of C as

$$C_L x = \lim_{t \searrow 0} \frac{1}{t} C \int_0^t T(s)x ds$$

with $\mathcal{D}(C_L)$ consisting of those $x \in X$ for which the limit exists. If $C \in \mathcal{L}(X, Y)$, then $C_L = C$. Since (1) is a regular linear system, we have that $\mathcal{R}(R(\lambda, A)B) \subset \mathcal{D}(C_L)$ for all $\lambda \in \rho(A)$ and the transfer function of (1) is given by [16, Sec. 4]

$$P(\lambda) = C_L R(\lambda, A)B \qquad \forall \lambda \in \rho(A).$$

We can without loss of generality assume that the system has no feedthrough, i.e., $D = 0$, because this operator has no effect on the considered stability properties of (1).

The concept of polynomial input-output stability is given in terms of the *extended input-output map* of the system (1).

Definition 3. The *extended input-output map* $\mathbb{F}_\infty$ of (1) is defined so that $\mathbb{F}_\infty : L^2_{\text{loc}}(0, \infty; U) \to L^2_{\text{loc}}(0, \infty; Y)$ and $(\mathbb{F}_\infty u)(\cdot) = y(\cdot) : [0, \infty) \to Y$ for $x^0 = 0$ and for every $u \in L^2_{\text{loc}}(0, \infty; U)$.

Since (1) is a regular linear system, the extended input-output map has the representation [16, Thm. 4.4]

$$(\mathbb{F}_\infty u)(t) = C_L \int_0^t T(t - s)Bu(s)ds$$

for every $u \in L^2_{\text{loc}}(0, \infty; U)$ and for almost all $t \geq 0$.

B. Definition of Polynomial Input-Output Stability

We begin by defining the Sobolev spaces $W^{\alpha,2}$ for $\alpha \geq 0$ and renorming them in a suitable way. For $\alpha \in \mathbb{N}_0$ we define [1]

$$W^{\alpha,2}(0, \infty; U) = \left\{ u \in L^2 \;\middle|\; u^{(k)} \in L^2 \text{ for } 0 \leq k \leq \alpha \right\}$$
$$\|u\|^2_{W^{\alpha,2}} = \sum_{k=0}^{\alpha} \|u^{(k)}\|^2_{L^2}.$$

Denote by $C_0^\infty(0, \infty; U)$ the space of smooth functions with compact support, and denote by $\hat{u}$ the Laplace transform of a function $u \in L^2(0, \infty; U)$. Since $C_0^\infty(0, \infty; U)$ is dense in $W^{\alpha,2}(0, \infty; U)$, Lemma 12 shows that a function $u \in L^2(0, \infty; U)$ satisfies $u \in W^{\alpha,2}(0, \infty; U)$ if and only if $\lambda \mapsto (1 + \lambda)^\alpha \hat{u}(\lambda) \in H^2(\mathbb{C}^+; U)$, and that we can define an equivalent norm on $W^{\alpha,2}(0, \infty; U)$ by

$$|||u|||_{W^{\alpha,2}} = \|(1 + \lambda)^\alpha \hat{u}\|_{H^2}.$$

Furthermore, for real values $\alpha \geq 0$ the Sobolev spaces $W^{\alpha,2}(0, \infty; U)$ are given by

$$W^{\alpha,2}(0, \infty; U) = \left\{ u \in L^2 \;\middle|\; |||u|||_{W^{\alpha,2}} < \infty \right\},$$

and $|||u|||_{W^{\alpha,2}} = \|(1 + \lambda)^\alpha \hat{u}\|_{H^2}$ is again equivalent to the norm $\|\cdot\|_{W^{\alpha,2}}$ (For definition of $W^{\alpha,2}(0, \infty; U)$ and $\|\cdot\|_{W^{\alpha,2}}$ for $\alpha > 0$, see [4, Def. 6.2.2].)

The polynomial input-output stability in Definition 2 can now be reformulated in the following way.

Definition 4 (Polynomial Input-Output Stability)**.** The system (1) is called *polynomially input-output stable with* $\alpha \geq 0$ if the extended input-output map $\mathbb{F}_\infty$ satisfies $\mathbb{F}_\infty \in \mathcal{L}(W^{\alpha,2}(0, \infty; U), L^2(0, \infty; Y))$.

III. Properties of Polynomially Input-Output Stable Systems

In this section we study the properties of polynomial input-output stability. In particular, we show that it is equivalent to the concept of *P-stability* introduced recently in [9].

Definition 5 (P-Stability)**.** The system (1) with the transfer function $P(\cdot)$ is called *P-stable with* $\alpha \geq 0$ if the following conditions are satisfied.

(a) $P(\cdot)$ is analytic in $\overline{\mathbb{C}^+}$.
(b) $P(\cdot) \in H^\infty(\mathbb{C}^+_\xi; \mathcal{L}(U,Y))$ for every $\xi > 0$, where $\mathbb{C}^+_\xi = \{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \xi \}$
(c) There exists $M_P \geq 1$ such that $\|P(i\omega)\| \leq M_P(1+|\omega|^\alpha)$ for every $\omega \in \mathbb{R}$.

For example, the transfer function $P(\cdot)$ defined by

$$P(\lambda) = \sum_{k=1}^\infty \frac{1}{k^{4/3}(\lambda + 1/k^2 - ik)},$$

for $\lambda \neq -1/k^2 + ik$ is P-stable but $P(\cdot) \notin H^\infty(\mathbb{C}^+; \mathcal{L}(U,Y))$. The conditions of Definition 5 are satisfied for $\alpha = 2/3$.

Definition 5 differs from the original definition of P-stable transfer functions in [9] due to the fact that here we consider the exponent $\alpha \geq 0$ to be fixed. In the original version $P(\cdot)$ is required to satisfy (c) for some such exponent. With our modification the set of P-stable transfer functions with $\alpha \geq 0$ is not an algebra, but our version is better suited for studying the relationship between the exponents in P-stability and polynomial input-output stability. The algebra of P-stable transfer functions can be obtained by defining

$$\mathbf{F}_P = \left\{ P(\cdot) \mid P(\cdot) \text{ is P-stable with some } \alpha \geq 0 \right\}.$$

The following theorem contains the main result of this paper. It details a relationship between the polynomial input-output stability in the time-domain and the P-stability in the frequency domain with equal exponents $\alpha \geq 0$ in the definitions. We are only interested in the situation where the growth bound of $T(t)$ satisfies $\omega_0(T(t)) = 0$, since $\omega_0(T(t)) < 0$ would immediately imply $P(\cdot) \in H^\infty(\mathbb{C}^+; \mathcal{L}(U,Y))$ [15, Lem. 10.3.3].

Theorem 6. *Assume $\omega_0(T(t)) = 0$.*
(1) *If there exist $M_A \geq 1$, $m \in \mathbb{N}$ such that*

$$\|R(\lambda, A)\| \leq \frac{M_A}{(\operatorname{Re}\lambda)^m} \qquad 0 < \operatorname{Re}\lambda \leq 1. \qquad (3)$$

and if the system (1) *is P-stable with $\alpha \geq 0$, then* (1) *is polynomially input-output stable with $\alpha \geq 0$.*
(2) *If $i\mathbb{R} \subset \rho(A)$ and if the system* (1) *is polynomially input-output stable with $\alpha \geq 0$, then it is P-stable with $\alpha \geq 0$.*

The condition (3) is in particular satisfied with $m = 1$ and $M_A = \sup_{t\geq 0}\|T(t)\|$ if the semigroup $T(t)$ is uniformly bounded. The following property of P-stable transfer functions is needed in the proof of the theorem. The lemma also confirms that under the given assumptions P-stability with $\alpha = 0$ coincides with H^∞-stability.

Lemma 7. *Assume that $\omega_0(T(t)) = 0$ and that there exist $M_A \geq 1$, $m \in \mathbb{N}$ such that* (3) *holds. If the system is P-stable with $\alpha \geq 0$, then $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda) \in H^\infty(\mathbb{C}^+; \mathcal{L}(U,Y))$.*

Proof. Since both $\lambda \mapsto (1+\lambda)^{-\alpha}$ and $P(\cdot)$ are analytic in $\mathbb{C}^+$, so is the function $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda)$. By Definition 5(b) the mapping $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda)$ is uniformly bounded for $\lambda \in \mathbb{C}^+$ with $\operatorname{Re}\lambda \geq 1$, and it therefore remains to show that it is uniformly bounded for $0 < \operatorname{Re}\lambda \leq 1$.

We begin by showing that there exists $M \geq 1$ such that $\|P(\lambda)\| \leq M/(\operatorname{Re}\lambda)^m$ for all $\lambda \in \mathbb{C}^+$ with $0 < \operatorname{Re}\lambda \leq 1$. Since $\omega_0(T(t)) = 0$ and since the system (1) is regular and well-posed, there exist $M_1, M_B, M_C \geq 1$ such that $\|P(1+i\eta)\| \leq M_1$, $\|R(1+i\eta, A_{-1})B\| \leq M_B$ and $\|C_L R(1+i\eta, A)\| \leq M_C$ for all $\eta \in \mathbb{R}$ [15, Lem. 10.3.3]. Now for every $\lambda = \xi + i\eta \in \mathbb{C}^+$ with $0 < \xi \leq 1$ the resolvent identity $R(\lambda, A_{-1}) = R(1+i\eta, A_{-1}) + (1-\xi)R(1+i\eta, A)R(\lambda, A_{-1})$ implies

$$\begin{aligned}
\|R(\lambda, A_{-1})B\| &\leq \|R(1+i\eta, A_{-1})B\|(1 + |1-\xi|\|R(\lambda, A)\|) \\
&\leq M_B(1 + \frac{M_A}{(\operatorname{Re}\lambda)^m}) \\
\|P(\lambda)\| &\leq \|P(1+i\eta)\| + \|C_L R(1+i\eta, A)\|\|R(\lambda, A_{-1})B\| \\
&\leq M_1 + M_C M_B(1 + \frac{M_A}{(\operatorname{Re}\lambda)^m}) \leq \frac{M}{(\operatorname{Re}\lambda)^m}
\end{aligned}$$

with $M = M_1 + M_B M_C(1+M_A)$. The uniform boundedness of $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda)$ for $\lambda \in \mathbb{C}^+$ with $0 < \operatorname{Re}\lambda \leq 1$ now follows exactly as in [2, Lem. 5.3] by considering the analytic functions $\lambda \mapsto F_R(\lambda) = (1+\lambda)^{-\alpha}P(\lambda)(1 + \frac{\lambda^2}{R^2})^m$ on $\Omega_R = \left\{ \lambda \mid 0 \leq \operatorname{Re}\lambda \leq 1, \ |\lambda| \leq R \right\}$ for large $R \geq 1$ and by using the maximum modulus principle. ☐

Proof of Theorem 6: We begin by showing that if the system is P-stable, then it is polynomially input-output stable. Let $u \in C_0^\infty(0,\infty; U)$ and denote $y = \mathbb{F}_\infty u : (0,\infty) \to Y$. Our aim is to show that $y \in L^2(0,\infty; Y)$ and that there exists $M \geq 1$ independent of u such that $\|y\|_{L^2} \leq M\|u\|_{W^{\alpha,2}}$. For every $\lambda \in \mathbb{C}^+$ we have

$$\hat{y}(\lambda) = P(\lambda)\hat{u}(\lambda) = (1+\lambda)^{-\alpha}P(\lambda)(1+\lambda)^\alpha \hat{u}(\lambda)$$

where $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda) \in H^\infty(\mathbb{C}^+; \mathcal{L}(U,Y))$ by Lemma 7. Since $\lambda \mapsto (1+\lambda)^\alpha \hat{u}(\lambda) \in H^2(\mathbb{C}^+; U)$ by Lemma 12, we have that $\hat{y} \in H^2(\mathbb{C}^+; Y)$, which further implies $y \in L^2(0,\infty; Y)$ by the Paley–Wiener Theorem [15, Thm. 10.3.4]. Moreover, the norms satisfy

$$\begin{aligned}
\|y\|_{L^2} &= \|\hat{y}\|_{H^2} = \|(1+\lambda)^{-\alpha}P(\lambda)(1+\lambda)^\alpha \hat{u}\|_{H^2} \\
&\leq \|(1+\lambda)^{-\alpha}P(\lambda)\|_{H^\infty}\|(1+\lambda)^\alpha \hat{u}\|_{H^2} \\
&= \|(1+\lambda)^{-\alpha}P(\lambda)\|_{H^\infty}\|u\|_{W^{\alpha,2}}.
\end{aligned}$$

Since $u \in C_0^\infty(\mathbb{C}^+; U)$ was arbitrary, and since the space C_0^∞ is dense in $W^{\alpha,2}$, this concludes that $\mathbb{F}_\infty \in \mathcal{L}(W^{\alpha,2}(0,\infty; U), L^2(0,\infty; Y))$.

Now assume that $\omega_0(T(t)) = 0$, $i\mathbb{R} \subset \rho(A)$, and that the system (1) is polynomially input-output stable. Our aim is to show that the system is P-stable. The first two properties in Definition 5 follow from $\overline{\mathbb{C}^+} \subset \rho(A)$ and from [16, Thm. 2.7(3)] since the system is regular and well-posed.

Let $M > 0$ be such that $\|\mathbb{F}_\infty u\| \leq M\|u\|_{W^{\alpha,2}}$ for all $u \in W^{\alpha,2}(0,\infty; U)$. Since $C_0^\infty(0,\infty; U)$ is dense in $L^2(0,\infty; U)$, the Paley–Wiener Theorem implies that the space $\mathcal{L}\{C_0^\infty\}$ (the image of C_0^∞ under the Laplace transform) is dense in $H^2(\mathbb{C}^+; U)$. Take $\hat{u} \in \mathcal{L}\{C_0^\infty(0,\infty; U)\}$ (i.e., $u \in C_0^\infty$), and define $\hat{u}_0 \in H^2(\mathbb{C}^+; U)$ such that $\hat{u}_0(\lambda) = (1+\lambda)^{-\alpha}\hat{u}(\lambda)$. Since $u \in W^{\alpha,2}$, and since $\lambda \mapsto (1+\lambda)^{-\alpha}$ is uniformly bounded in $\mathbb{C}^+$, we have that $u_0 = \mathcal{L}^{-1}\{\hat{u}_0\} \in W^{\alpha,2}(0,\infty; U)$. Because

$$\mathbb{F}_\infty \in \mathcal{L}(W^{\alpha,2}(0,\infty; U), L^2(0,\infty; Y))$$

by assumption, we can define $y = \mathbb{F}_\infty u_0 \in L^2(0,\infty;Y)$. Due to the Paley–Wiener Theorem we also have $\hat{y} \in H^2(\mathbb{C}^+;Y)$, and $\hat{y}(\lambda) = P(\lambda)\hat{u}_0(\lambda)$ for every $\lambda \in \mathbb{C}^+$. Using this we get

$$\begin{aligned}\|(1+\lambda)^{-\alpha}P(\lambda)\hat{u}\|_{H^2} &= \|P(\lambda)(1+\lambda)^{-\alpha}\hat{u}\|_{H^2}\\ &= \|P(\lambda)\hat{u}_0\|_{H^2} = \|\hat{y}\|_{H^2} = \|y\|_{L^2} = \|\mathbb{F}_\infty u_0\|_{L^2}\\ &\le M\|u_0\|_{W^{\alpha,2}} = M\|(1+\lambda)^{\alpha}\hat{u}_0\|_{H^2} = M\|\hat{u}\|_{H^2}.\end{aligned}$$

Since $\hat{u} \in \mathcal{L}\{C_0^\infty(0,\infty;U)\}$ was arbitrary and since $\mathcal{L}\{C_0^\infty\}$ is dense in $H^2(\mathbb{C}^+;U)$, this implies that the multiplication map $\lambda \mapsto (1+\lambda)^{-\alpha}P(\lambda)$ is in $\mathcal{L}(H^2(\mathbb{C}^+;U), H^2(\mathbb{C}^+,Y))$, which together with the continuity of $P(\cdot)$ on $i\mathbb{R}$ in particular shows that

$$\sup_{\omega\in\mathbb{R}} (1+|\omega|^{\alpha})^{-1}\|P(i\omega)\| < \infty.$$

This concludes that the system (1) is P-stable. ■

IV. Systems with Polynomially Stable Semigroups

In this section we study the properties of regular linear systems whose system operators generate polynomially stable semigroups. In particular we show that on Hilbert spaces all such systems are polynomially input-output stable. Moreover, we derive additional smoothness conditions for B and C under which the system is input-output stable in the sense of Definition 1.

The semigroup $T(t)$ is called *polynomially stable* (*with* $\tilde{\alpha} > 0$) if it is uniformly bounded, if $i\mathbb{R} \subset \rho(A)$, and if it satisfies condition (a) in Theorem 8. On a Hilbert space polynomial stability can be characterized in the following way. The equivalences between (a), (b), and (c) follow from [2, Lem. 2.4], [5, Lem. 2.3, Thm. 2.4], and [10, Lem. 3.2]. The remaining part of the proof is presented in detail in the appendix.

Theorem 8. *Assume A generates a uniformly bounded semigroup $T(t)$ on a Hilbert space X, and $i\mathbb{R} \subset \rho(A)$. For a fixed constant $\tilde{\alpha} > 0$ the following are equivalent.*

(a) $\|T(t)A^{-1}\| \le \dfrac{M}{t^{1/\tilde{\alpha}}}, \qquad \forall t > 0$

(b) $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^{\tilde{\alpha}})$

(b') $\|(-A)^{-\beta}R(i\omega,A)\| = \mathcal{O}(|\omega|^{\tilde{\alpha}-\beta})$ *for* $-1 \le \beta \le \tilde{\alpha}$

(c) $\sup_{\operatorname{Re}\lambda\ge 0} \|R(\lambda,A)(-A)^{-\tilde{\alpha}}\| < \infty.$

Throughout this section we assume that X is a Hilbert space and that the semigroup $T(t)$ generated by A is polynomially stable. If $T(t)$ is polynomially stable, then it is in particular uniformly bounded, and the operators $-A$ and $-A^*$ are boundedly invertible and sectorial in the sense of [8, Sec. 2.1]. Because of this, the fractional powers $(-A)^\beta$ and $(-A^*)^\gamma$ are well-defined for all $\beta,\gamma \in \mathbb{R}$.

The following main result of this section shows that a regular linear system with a polynomially stable semigroup is always polynomially input-output stable. It should be noted that the conditions on the operators B and C in Theorem 9 are always satisfied for $\beta = 0$ and $\gamma = 0$, since in this situation we have $\mathcal{R}(B) \subset \mathcal{D}((-A_{-1})^\beta) = X_{-1}$ and $C(-A)^\gamma = C \in \mathcal{L}(X_1,Y)$ by assumption. The conditions on B and C in the theorem can in many examples be translated into requiring a sufficient level of smoothness for the inputs and outputs. If the input and output operators are bounded the conditions in Theorem 9 can be greatly simplified, as is shown in Corollary 10.

Theorem 9. *Assume the semigroup $T(t)$ on the Hilbert space X is polynomially stable with $\tilde{\alpha} > 0$. If $\beta,\gamma \ge 0$ are such that*

$$\mathcal{R}(B) \subset \mathcal{D}((-A_{-1})^\beta) \quad \textit{and}$$

$$C_L(-A)^\gamma : \mathcal{D}((-A)^{\gamma+1}) \to Y \quad \textit{extends to } C_\gamma \in \mathcal{L}(X_1,Y),$$

then the system (1) *is polynomially input-output stable with $\alpha = \max\{\tilde{\alpha}+2-\beta-\gamma, 0\}$. In particular, if $\beta+\gamma \ge \tilde{\alpha}+2$, then the system is input-output stable.*

Proof. The first two properties in Definition 5 follow from $\overline{\mathbb{C}^+} \subset \rho(A)$ and from [16, Thm. 2.7(3)] since the system is regular and well-posed. The Closed Graph Theorem implies that $B_\beta = (-A_{-1})^\beta B \in \mathcal{L}(U,X_{-1})$, and we have $C_\gamma \in \mathcal{L}(X_1,Y)$ by assumption. Further denote $\tilde{B}_\beta = (-A_{-1})^{-1}B_\beta \in \mathcal{L}(U,X)$ and $\tilde{C}_\gamma = C_\gamma(-A)^{-1} \in \mathcal{L}(X,Y)$. We have from [8, Sec. 6.3.3] that $(-A_{-1})^{\tilde{\beta}}|_X = (-A)^{\tilde{\beta}}$ for all $\tilde{\beta} \in \mathbb{R}$. If $\beta+\gamma \ge 1$, then $2-\beta-\gamma \le 1$ and for every $\lambda \in \rho(A) = \rho(A_{-1})$ we have

$$\begin{aligned}P(\lambda) &= C_L R(\lambda,A_{-1})B = C_L(-A)^{1-\beta}R(\lambda,A)\tilde{B}_\beta\\ &= \tilde{C}_\gamma(-A)^{2-\beta-\gamma}R(\lambda,A)\tilde{B}_\beta.\end{aligned}$$

Thus

$$\|P(i\omega)\| \le \|\tilde{C}_\gamma\|\|(-A)^{2-\beta-\gamma}R(i\omega,A)\|\|\tilde{B}_\beta\| = \mathcal{O}(|\omega|^\alpha)$$

with $\alpha = \max\{\tilde{\alpha}+2-\beta-\gamma, 0\}$ by Theorem 8. On the other hand, if $0 \le \beta+\gamma < 1$, then for every $\lambda \in \rho(A) = \rho(A_{-1})$ we have $(-A)R(\lambda,A) = I - \lambda R(\lambda,A)$ and

$$\begin{aligned}P(\lambda) &= C_L R(\lambda,A_{-1})B = C_L(-A)R(\lambda,A)(-A_{-1})^{-1}B\\ &= -C_L A_{-1}^{-1}B - \lambda C_L R(\lambda,A)(-A_{-1})^{-1}B\\ &= -C_L A_{-1}^{-1}B - \lambda\tilde{C}_\gamma(-A)^{1-\beta-\gamma}R(\lambda,A)\tilde{B}_\beta.\end{aligned}$$

This in particular implies

$$\begin{aligned}\|P(i\omega)\| &\le \|C_L A_{-1}^{-1}B\|\\ &\quad + |\omega|\|\tilde{C}_\gamma\|\|(-A)^{1-\beta-\gamma}R(i\omega,A)\|\|\tilde{B}_\beta\| = \mathcal{O}(|\omega|^\alpha)\end{aligned}$$

with $\alpha = \tilde{\alpha}+1-\beta-\gamma+1 = \tilde{\alpha}+2-\beta-\gamma > 1$ by Theorem 8. □

Corollary 10. *Assume the semigroup $T(t)$ on the Hilbert space X is polynomially stable with $\tilde{\alpha} > 0$ and that $B \in \mathcal{L}(U,X)$ and $C \in \mathcal{L}(X,Y)$. If $\beta,\gamma \ge 0$ are such that*

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta) \quad \textit{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma),$$

then the system (1) *is polynomially input-output stable with $\alpha = \max\{\tilde{\alpha}-\beta-\gamma, 0\}$. In particular, if $\beta+\gamma \ge \tilde{\alpha}$, then the system is input-output stable.*

Proof. Since $\mathcal{D}((-A)^\beta) = \mathcal{D}((-A_{-1})^{\beta+1})$, we have $\mathcal{R}(B) \subset \mathcal{D}((-A_{-1})^{\beta+1})$. The assumption $\mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma)$ implies that $C(-A)^\gamma$ has a unique bounded extension $C_\gamma \in \mathcal{L}(X,Y)$. Therefore for all $x \in \mathcal{D}((-A)^{\gamma+1})$ we have

$$\|C(-A)^{\gamma+1}x\| = \|C_\gamma(-A)x\| \le \|C_\gamma\|\|Ax\|,$$

which implies that $C(-A)^{\gamma+1}$ has an extension to $\mathcal{L}(X_1,Y)$. This concludes that B and C satisfy the assumptions of Theorem 9 with $\tilde{\beta} = \beta+1$ and $\tilde{\gamma} = \gamma+1$, and thus the system is polynomially stable with $\alpha = \max\{\tilde{\alpha} - \tilde{\beta} - \tilde{\gamma} + 2, 0\} = \max\{\tilde{\alpha} - \beta - \gamma, 0\}$. □

The conditions in Theorem 9 and Corollary 10 are not optimal, and they can be improved in certain special cases, as the following example illustrates.

Example 11. Consider the diagonal operator $A = \operatorname{diag}(\lambda_k)_{k=1}^\infty$ on $X = \ell^2(\mathbb{C})$. Assume that the eigenvalues $\{\lambda_k\}_k \subset \mathbb{C}^-$ of A have a uniform gap and the semigroup generated by A is polynomially stable with $\tilde{\alpha} > 0$, but not exponentially stable. More precisely, we assume that there exist constants $r, R > 0$ such that $-R \le \operatorname{Re}\lambda_k \le -r|\operatorname{Im}\lambda_k|^{-\tilde{\alpha}}$ for all $k \in \mathbb{N}$, and $\operatorname{Re}\lambda_{k_l} \to 0$ as $l \to \infty$ for some subsequence $(k_l)_{l=1}^\infty \subset \mathbb{N}$. Let $B \in \mathcal{L}(\mathbb{C}, X)$ and $C \in \mathcal{L}(X, \mathbb{C})$ be rank one operators $B = b \in X$ and $C = \langle \cdot, c\rangle$ (finite-rank operators can be treated similarly). For $k \in \mathbb{N}$ denote $b_k = \langle b, e_k\rangle$ and $c_k = \langle e_k, c\rangle$, where $\{e_k\}_{k=1}^\infty$ are the natural basis vectors of X.

Let $\lambda \in \overline{\mathbb{C}^+}$ and denote by $k_0 = \arg\min_{k\in\mathbb{N}}|\operatorname{Im}\lambda - \operatorname{Im}\lambda_k|$. If $d = \min_{k\neq l}|\lambda_k - \lambda_l| > 0$, then

$$\begin{aligned}|P(\lambda)| &\le \sum_{k\in\mathbb{N}} \frac{|b_k||c_k|}{|\lambda-\lambda_k|} \le \frac{|b_{k_0}||c_{k_0}|}{|\lambda-\lambda_{k_0}|} + \sum_{k\neq k_0}\frac{|b_k||c_k|}{d/2} \\ &\le \frac{|b_{k_0}||c_{k_0}|}{|\operatorname{Re}\lambda_{k_0}|} + \frac{2}{d}\|B\|\|C\|\end{aligned}$$

since $|\lambda - \lambda_{k_0}|^2 = |\operatorname{Re}\lambda - \operatorname{Re}\lambda_{k_0}|^2 + |\operatorname{Im}\lambda - \operatorname{Im}\lambda_{k_0}|^2 \ge |\operatorname{Re}\lambda_{k_0}|^2$. This shows that $P(\cdot) \in H^\infty$ if and only if

$$\sup_{k\in\mathbb{N}} \frac{|b_k||c_k|}{|\operatorname{Re}\lambda_k|} < \infty.$$

Since we assumed that $|\operatorname{Re}\lambda_k| \le R$, the polynomial stability of $T(t)$ implies that there exists a constant $r > 0$ such that $|\operatorname{Re}\lambda_k| \ge r|\lambda_k|^{-\tilde{\alpha}}$ for all $k \in \mathbb{N}$. Let $\beta, \gamma \ge 0$ be such that $\beta + \gamma = \tilde{\alpha}$. Then

$$\begin{aligned}\sup_{k\in\mathbb{N}} \frac{|b_k||c_k|}{|\operatorname{Re}\lambda_k|} &\le \frac{1}{r}\sup_{k\in\mathbb{N}} |\lambda_k|^{\tilde{\alpha}}|b_k||c_k| \\ &\le \frac{1}{r}\Big(\sup_{k\in\mathbb{N}}|\lambda_k|^\beta|b_k|\Big)\Big(\sup_{k\in\mathbb{N}}|\lambda_k|^\gamma|c_k|\Big) < \infty\end{aligned}$$

whenever $(|\lambda_k|^\beta b_k)_{k=1}^\infty \in \ell^\infty(\mathbb{C})$ and $(|\lambda_k|^\gamma c_k)_{k=1}^\infty \in \ell^\infty(\mathbb{C})$. These conditions are always less strict than the conditions of Corollary 10, which are equivalent to $(|\lambda_k|^\beta b_k)_{k=1}^\infty \in \ell^2(\mathbb{C})$ and $(|\lambda_k|^\gamma c_k)_{k=1}^\infty \in \ell^2(\mathbb{C})$.

V. Example

We consider a damped wave equation on $[0,1]$ with collocated input and output

$$\frac{d^2w}{dt^2} + \langle \frac{dw}{dt}, a\rangle_{L^2} a(\xi) = \frac{d^2w}{d\xi^2} + b(\xi)u(t) \tag{4a}$$

$$w(0,t) = w(1,t) = 0 \tag{4b}$$

$$y(t) = \int_0^1 b(\xi)\overline{\frac{dw}{dt}(\xi,t)}d\xi \tag{4c}$$

where the function in the damping term is $a(\xi) = \xi^2(1-\xi)/\sqrt{10}$ for $\xi \in [0,1]$. Moreover, $b(\xi) = 1$ for $0 \le \xi \le 1/2$ and $b(\xi) = 0$ if $1/2 < \xi \le 1$.

The system operator A of the wave system generates a strongly stable contraction semigroup with no spectrum on the imaginary axis [3]. Since B and C are bounded, we have that $P(\cdot)$ is analytic in $\overline{\mathbb{C}^+}$, and $P(\cdot) \in H^\infty(\mathbb{C}^+_\xi; \mathcal{L}(U,Y))$ for every $\xi > 0$. To compute the transfer function $P(\cdot)$ of the system, we observe that the damped wave equation can be seen as an undamped wave equation with output feedback. More precisely, if we consider a system

$$\begin{aligned}\frac{d^2w}{dt^2} &= \frac{d^2w}{d\xi^2} + a(\xi)u_0(t) + b(\xi)u(t) \\ w(0,t) &= w(1,t) = 0 \\ \begin{pmatrix} y_0(t) \\ y(t)\end{pmatrix} &= \int_0^1 \begin{pmatrix} a(\xi) \\ b(\xi)\end{pmatrix}\overline{\frac{dw}{dt}(\xi,t)}d\xi,\end{aligned}$$

then our original system is obtained by applying a negative output feedback $u_0(t) = -y_0(t)$ to the undamped system. The transfer function of the open-loop system is of the form

$$\begin{pmatrix}\hat{y}_0(\lambda) \\ \hat{y}(\lambda)\end{pmatrix} = \begin{pmatrix} p_{11}(\lambda) & p_{12}(\lambda) \\ p_{21}(\lambda) & p_{22}(\lambda)\end{pmatrix}\begin{pmatrix}\hat{u}_0(\lambda) \\ \hat{u}(\lambda)\end{pmatrix}.$$

If we apply the output feedback $\hat{u}_0 = -\hat{y}_0$, then the mapping $\hat{u} \mapsto \hat{y}$ is the transfer function of our original plant. Letting $\hat{u}_0 = -\hat{y}_0$, we can solve

$$\hat{y}(\lambda) = p_{22}(\lambda)\hat{u}(\lambda) - p_{21}(\lambda)(1 + p_{11}(\lambda))^{-1}p_{12}(\lambda)\hat{u}(\lambda),$$

and thus $P(\lambda) = p_{22}(\lambda) - p_{21}(\lambda)(1 + p_{11}(\lambda))^{-1}p_{12}(\lambda)$. The components of the open loop transfer function can be computed explicitly similarly as in [7, Sec. 2.1], and each $p_{kl}(\cdot)$ for $k, l \in \{1,2\}$ has an explicit formula given in terms of hyperbolic functions. This further allows us to derive an explicit expression for the transfer function $P(\lambda)$ of (4).

In order to show that the system (4) is polynomially input-output stable, it suffices to study the behaviour of its transfer function on $i\mathbb{R}$. The full explicit expression of $P(i\omega)$ for $\omega \in \mathbb{R}$ is too lengthy to be presented here in its entirety, but it can easily be used in analyzing the asymptotic behaviour of $|P(i\omega)|$. In particular, if $i\omega = i(2k+1)\pi$ for some $k \in \mathbb{Z}$, then the expression for $P(i\omega)$ simplifies to

$$\begin{aligned}P(i\omega) = \frac{\omega^4}{420} - i\frac{\omega^3}{840\cdot 525} - i\frac{\omega}{600\cdot 240} - i\frac{421}{840\omega} \\ + i\frac{351(-1)^k}{350\omega^2} + i\frac{3}{350\omega^3}.\end{aligned}$$

This in particular implies that $P(\cdot)$ is not uniformly bounded on the imaginary axis, and thus $P(\cdot) \notin H^\infty(\mathbb{C}^+; \mathcal{L}(U,Y))$.

Moreover, the general expression for $P(i\omega)$ can be used to show that for $\omega \in \mathbb{R}$ with sufficiently large $|\omega|$ the norm of the transfer function behaves as $|P(i\omega)| = \mathcal{O}(|\omega|^4)$. Together with Theorem 6 this concludes that the system (4) is polynomially input-output stable with $\alpha = 4$. In particular this means that if $u \in W^{4,2}(0,\infty;\mathbb{C})$, then $y \in L^2(0,\infty;\mathbb{C})$.

Appendix

Lemma 12. *Let $\alpha \in \mathbb{N}$. If $u \in \mathbb{C}_0^\infty(0,\infty;U)$, then $\lambda \mapsto (1+\lambda)^\alpha \hat{u}(\lambda) \in H^2(\mathbb{C}^+;U)$ and*

$$\frac{1}{\sqrt{\alpha+1}}\|u\|_{W^{\alpha,2}} \le \|(1+\lambda)^\alpha \hat{u}\|_{H^2} \le 2^\alpha \|u\|_{W^{\alpha,2}}. \qquad (5)$$

Proof. Let $u \in C_0^\infty$. We have $u \in L^2(0,\infty;U)$ and $\hat{u} \in H^2(\mathbb{C}^+;U)$. For every $k \in \{0,\dots,\alpha\}$ we have

$$\begin{aligned} u \in W^{\alpha,2}(0,\infty;U) \quad &\Rightarrow \quad u^{(k)} \in L^2(0,\infty;U) \\ &\Leftrightarrow \quad \mathcal{L}\{u^{(k)}\} \in H^2(\mathbb{C}^+;U). \end{aligned}$$

Using the properties of the Laplace transform and the fact that $u^{(l)}(0) = 0$ for $0 \le l \le \alpha - 1$, for every $\lambda \in \mathbb{C}^+$ we have

$$\mathcal{L}\{u^{(k)}\}(\lambda) = \lambda^k \hat{u}(\lambda) + \sum_{l=0}^{k-1} \lambda^{k-1-l} u^{(l)}(0) = \lambda^k \hat{u}(\lambda).$$

For every $\lambda \in \mathbb{C}^+$

$$\begin{aligned} \|(1+\lambda)^\alpha \hat{u}(\lambda)\|_U^2 &= |1+\lambda|^{2\alpha}\|\hat{u}(\lambda)\|_U^2 \le (1+|\lambda|)^{2\alpha}\|\hat{u}(\lambda)\|_U^2 \\ &\le 2^{2\alpha}(1+|\lambda|^{2\alpha})\|\hat{u}(\lambda)\|_U^2 = 2^{2\alpha}(\|\hat{u}(\lambda)\|_U^2 + \|\lambda^\alpha \hat{u}(\lambda)\|_U^2). \end{aligned}$$

Since $\lambda \mapsto (1+\lambda)^\alpha$ and $\hat{u}$ are analytic in $\mathbb{C}^+$, also their product $\lambda \mapsto (1+\lambda)^\alpha \hat{u}(\lambda)$ is analytic. The above pointwise inequality implies $\|(1+\lambda)^\alpha \hat{u}\|_{H^2}^2 \le 2^{2\alpha}(\|\hat{u}\|_{H^2}^2 + \|\lambda^\alpha \hat{u}\|_{H^2}^2)$, and using the Paley–Wiener Theorem [15, Thm. 10.3.4] we get

$$\begin{aligned} \|(1+\lambda)^\alpha \hat{u}\|_{H^2}^2 &\le 2^{2\alpha}(\|\hat{u}\|_{H^2}^2 + \|\lambda^\alpha \hat{u}\|_{H^2}^2) \\ &= 2^{2\alpha}(\|u\|_{L^2}^2 + \|u^{(\alpha)}\|_{L^2}^2) \le 2^{2\alpha}\sum_{k=0}^{\alpha}\|u^{(k)}\|_{L^2}^2 \\ &= 2^{2\alpha}\|u\|_{W^{\alpha,2}}^2. \end{aligned}$$

This proves the second inequality in (5).

For every $\lambda \in \mathbb{C}^+$ and $k \in \{0,\dots,\alpha\}$ we have $|\lambda|^k \le |1+\lambda|^k \le |1+\lambda|^\alpha$. Thus

$$\begin{aligned} \|\lambda^k \hat{u}(\lambda)\|_U &= |\lambda|^k \|\hat{u}(\lambda)\|_U \le |1+\lambda|^\alpha \|\hat{u}(\lambda)\|_U \\ &= \|(1+\lambda)^\alpha \hat{u}(\lambda)\|_U, \end{aligned}$$

which in turn implies $\|\lambda^k \hat{u}\|_{H^2} \le \|(1+\lambda)^\alpha \hat{u}\|_{H_2}$ and

$$\begin{aligned} \|u\|_{W^{\alpha,2}}^2 &= \sum_{k=0}^{\alpha}\|u^{(k)}\|_{L^2}^2 = \sum_{k=0}^{\alpha}\|\lambda^k \hat{u}\|_{H^2}^2 \\ &\le \sum_{k=0}^{\alpha}\|(1+\lambda)^\alpha \hat{u}\|_{H_2}^2 = (\alpha+1)\|(1+\lambda)^\alpha \hat{u}\|_{H_2}^2. \end{aligned}$$

This proves the first inequality in (5). □

Proof of Theorem 8. We have from [5, Lem. 2.3, Thm. 2.4] that (a), (b), and (c) are equivalent. On the other hand, (b) is a special case of (b') with $\beta = 0$, and (b') for $\beta = \tilde{\alpha}$ follows immediately from (c). For $0 < \beta < \tilde{\alpha}$ the property (b') follows from the cases $\beta = 0$ and $\beta = \tilde{\alpha}$, and from the Moment Inequality [8, Prop. 6.6.4]. For $\beta = -1$ we have

$$\|(-A)R(i\omega,A)\| \le 1 + |\omega|\|R(i\omega,A)\| = \mathcal{O}(|\omega|^{\tilde{\alpha}+1}),$$

where we have used $(-A)R(i\omega,A) = I - i\omega R(i\omega,A)$ and (b). Finally, if $-1 < \beta < 0$, then $0 < -\beta < 1$ and the Moment Inequality [8, Prop. 6.6.4] implies that there exists a constant $M_\beta \ge 1$ such that

$$\begin{aligned} \|(-A)^{-\beta}R(i\omega,A)\| &\le M_\beta \|R(i\omega,A)\|^{1+\beta}\|(-A)R(i\omega,A)\|^{-\beta} \\ &= \mathcal{O}(|\omega|^{\tilde{\alpha}-\beta}) \end{aligned}$$

since we have $\|R(i\omega,A)\|^{1+\beta} = \mathcal{O}(|\omega|^{\tilde{\alpha}(1+\beta)})$ by (b), $\|(-A)R(i\omega,A)\|^{-\beta} = \mathcal{O}(|\omega|^{(\tilde{\alpha}+1)(-\beta)})$ as above, and since $\tilde{\alpha}(1+\beta) + (\tilde{\alpha}+1)(-\beta) = \tilde{\alpha} - \beta$. □

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