The Infinite-Dimensional Sylvester Differential Equation and Periodic Output Regulation

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Abstract. In this paper the solvability of the infinite-dimensional Sylvester differential equation is considered. This is an operator differential equation on a Banach space. Conditions for the existence of a unique classical solution to the equation are presented. In addition, a periodic version of the equation is studied and conditions for the existence of a unique periodic solution are given. These results are applied to generalize a theorem which characterizes the controllers achieving output regulation of a distributed parameter system with a nonautonomous signal generator.

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1. Introduction

In this paper we consider the solvability of a Sylvester differential equation on a Banach space. This is an operator differential equation of form

$$\dot{\Sigma}(t) = A(t)\Sigma(t) - \Sigma(t)B(t) + C(t), \qquad \Sigma(0) = \Sigma_0, \tag{1.1}$$

where $(A(t), \mathcal{D}(A(t)))$ and $(B(t), \mathcal{D}(B(t)))$ are families of unbounded operators on Banach spaces X and Y, respectively, $C(\cdot)$ is an operator-valued function and Σ_0 is a bounded linear operator. The equations of this type have an application in the output regulation of linear distributed parameter systems when the reference signals are generated with a periodic exosystem of form

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in \mathbb{C}^q, \tag{1.2a}$$

$$y_{ref}(t) = F(t)v(t). \tag{1.2b}$$

By the periodicity of the exosystem we mean that $S(\cdot)$ and $F(\cdot)$ are periodic functions with the same period, i.e. there exists $\tau > 0$ such that $S(t+\tau) = S(t)$ and $F(t+\tau) = F(t)$ for all $t \in \mathbb{R}$. Paunonen and Pohjolainen [9] have shown that the solvability of the output regulation problem related to this type of exosystem can be characterized using the properties of the solution to a certain Sylvester differential equation.

The results in [9] generalize the theory of periodic output regulation of linear finite-dimensional systems presented by Zhang and Serrani [13]. In the finite-dimensional theory the Sylvester differential equations are ordinary matrix differential equations and $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are smooth matrix-valued functions. However, if we want to consider output regulation of infinite-dimensional linear systems, the matrix-valued function $A(\cdot)$ becomes a family $(A(t), \mathcal{D}(A(t)))$ of unbounded operators associated to the closed-loop system consisting of the distributed parameter system to be controlled and the controller.

The treatment of the Sylvester differential equation presented in this paper generalizes the results on the solvability of finite-dimensional equations of this type [13, 7]. Also the infinite-dimensional equation has been studied in the case where $A(t) \equiv A$ and $B(t) \equiv B$ are generators of strongly continuous semigroups [5, 4]. In the case of time-dependent families of operators some results are also known for time-dependent Riccati equations [2]. On the other hand, in the time-invariant case the equation becomes an infinite-dimensional Sylvester equation [1, 11, 12]. Our approach in solving the Sylvester differential equation (1.1) generalizes the methods used in [11].

We have in [9] studied a Sylvester differential equation of form (1.1), where $(A(t), \mathcal{D}(A(t)))$ is a family of unbounded operators and $B(t) \equiv B$ is a matrix. In this paper we consider the solvability of the equation in a more general case where also $B(t)$ are allowed to be unbounded operators. We restrict ourselves to a situation where the domains of the unbounded operators are independent of time. The main tools in our analysis are the strongly continuous evolution families associated to families of unbounded operators and nonautonomous abstract Cauchy problems [10, Ch. 5], [6, Sec. VI.9].

We apply the theoretic results on the solvability of (1.1) to the output regulation of infinite-dimensional systems. In particular we present a characterization of the controllers achieving output regulation of a linear distributed parameter system to the signals generated by a nonautonomous periodic signal generator.

The paper is organized as follows. In Section 2 we introduce notation, recall the definition of a strongly continuous evolution family and state the basic assumptions on the families of operators. The solvability of the Sylvester differential equation is considered in Section 3. The main results of the paper are Theorems 3.2 and 3.3. In Section 4 we apply these results to output regulation. Section 5 contains concluding remarks.

2. Notation and Definitions

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$ the domain and the range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The space of continuous functions $f : I \subset \mathbb{R} \to X$ is denoted by $C(I, X)$ and the space of continuously differentiable functions by $C^1(I, X)$. Finally, we denote by $C(I, \mathcal{L}_s(X, Y))$ the space of strongly continuous $\mathcal{L}(X,Y)$-valued functions.

In dealing with families of unbounded operators we use the theory of strongly continuous evolution families [10, Ch. 5], [6, Sec. VI.9].

Definition 2.1 (A Strongly Continuous Evolution Family). A family of bounded operators $(U(t,s))_{t\geq s} \subset \mathcal{L}(X)$ is called a *strongly continuous evolution family* if

(a) $U(s,s) = I$ for $s \in \mathbb{R}$.
(b) $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$.
(c) $\{ (t,s) \in \mathbb{R}^2 \mid t \geq s \} \ni (t,s) \mapsto U(t,s)$ is a strongly continuous mapping.

A strongly continuous evolution family is called *exponentially bounded* if there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq M e^{\omega(t-s)}$$

for all $t \geq s$. The evolution family is called *periodic* (with period $\tau > 0$) if

$$U(t+\tau, s+\tau) = U(t,s)$$

for all $t \geq s$.

Strongly continuous evolution families are related to nonautonomous abstract Cauchy problems. If we consider an equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t), \\ x(s) &= x_s \in X \end{aligned}$$

and if $U(t,s)$ is a strongly continuous evolution family associated to the family $(A(t), \mathcal{D}(A(t)))$ of operators, then if for every $s \in \mathbb{R}$ this equation has a classical solution $x(\cdot) \in C^1([s,\infty), X)$ such that $x(t) \in \mathcal{D}(A(t))$ for all $t \geq s$, this solution is given by

$$x(t) = U(t,s)x_s + \int_s^t U(t,r)f(r)ds \tag{2.1}$$

for all $t \geq s$. If the family $(A(t), \mathcal{D}(A(t)))$ of operators is periodic with period $\tau > 0$, then also the associated evolution family is periodic with the same period.

Throughout this paper we consider a case where the domains of the unbounded operators are independent of time, i.e.

$$A(t) : \mathcal{D}(A) \subset X \to X, \qquad B(t) : \mathcal{D}(B) \subset Y \to Y$$

for all t. We assume that there exist exponentially bounded strongly continuous evolution families $U_A(t,s)$ and $U_B(t,s)$ related to the families $(A(t), \mathcal{D}(A))$ and $(-B(t), \mathcal{D}(B))$ of operators, respectively, and that the evolution family $U_B(t,s)$ satisfies Definition 2.1 for all $t, s, r \in \mathbb{R}$. This means that the nonautonomous abstract Cauchy problem associated to this family of operators can be solved forward and backwards in time and the minus sign in the family of operators corresponds to the reversal of time in the equation. Because of this, we can also think of the situation in such a way that the evolution family related to the family $(B(t), \mathcal{D}(B))$ of operators satisfies Definition 2.1 for all $t \le r \le s$. Motivated by this, we denote this evolution family by $U_B(s,t)$ for $t \ge s$.

3. The Infinite-Dimensional Sylvester Differential Equation

In this section we consider the infinite-dimensional Sylvester differential equation

$$\dot{\Sigma}(t) = A(t)\Sigma(t) - \Sigma(t)B(t) + C(t), \qquad \Sigma(0) = \Sigma_0 \tag{3.1}$$

on an interval $[0,T]$. The equation is considered in the strong sense for $y \in \mathcal{D}(B)$.

The main result of this paper is Theorem 3.2 which states sufficient conditions for the existence of a classical solution to the Sylvester differential equation. As we are motivated by the periodic output regulation problem for distributed parameter systems [9], we will also show that if the families of operators $(A(t), \mathcal{D}(A))$ and $(B(t), \mathcal{D}(B))$ and the function $C(\cdot)$ are periodic with the same period, then under suitable additional assumptions on the growths of the evolution families $U_A(t,s)$ and $U_B(s,t)$ the Sylvester differential equation has a unique periodic solution. This result is presented in Theorem 3.3.

We begin by defining the classical solution of the Sylvester differential equation on the interval $[0,T]$.

Definition 3.1. A strongly continuous function $\Sigma(\cdot) \in C([0,T], \mathcal{L}_s(Y,X))$ satisfying $\Sigma(\cdot)y \in C^1([0,T],X)$ and $\Sigma(t)y \in \mathcal{D}(A)$ for all $y \in \mathcal{D}(B)$ and $t \in [0,T]$ is called the *classical solution* of the Sylvester differential equation (3.1) if it satisfies the equation on $[0,T]$.

The next theorem is the main result of the paper. It states sufficient conditions for the solvability of the Sylvester differential equation on the interval $[0,T]$. The *parabolic conditions* [10, Sec. 5.6] appearing in the theorem essentially require that the operators $A(t)$ for $t \in [0,T]$ are generators of analytic semigroups on X.

Theorem 3.2. *Assume the following are satisfied.*

1. *There exists $\mu \in \mathbb{R}$ such that $U_A(t,s)$ satisfies the* parabolic conditions:
   - (P_1) *The domain $\mathcal{D}(A)$ is dense in X.*
   - (P_2) *We have $\{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \ge \mu \} \subset \rho(A(t))$ for every $t \in [0,T]$ and there exists a constant $M \ge 1$ such that*

     $$\|R(\lambda, A(t))\| \le \frac{M}{|\lambda - \mu| + 1}, \quad \operatorname{Re}\lambda \ge \mu, \ t \in [0,T].$$
   - (P_3) *There exists a constant $L \ge 0$ such that for $t,s,r \in [0,T]$*

     $$\|(A(t) - A(s))R(\mu, A(r))\| \le L|t-s|.$$

2. *The domain* $\mathcal{D}(A(t)^*) =: \mathcal{D}(A^*)$ *is independent of* $t \in [0,T]$ *and dense in* X^*. *For all* $x \in X$ *and* $x^* \in \mathcal{D}(A^*)$ *the mapping*

$$t \mapsto \langle x, A(t)^* x^* \rangle$$

is continuous on $[0,T]$.

3. *The domain* $\mathcal{D}(B)$ *is dense in* Y. *For every* $y \in \mathcal{D}(B)$ *the function* $B(\cdot)y$ *is continuous, we have* $U_B(s,t)y \in \mathcal{D}(B)$ *and the evolution family* $U_B(s,t)$ *satisfies the differentiation rules*

$$\frac{\partial}{\partial t} U_B(s,t)y = -U_B(s,t)B(t)y, \qquad \frac{\partial}{\partial s} U_B(s,t)y = B(s)U_B(s,t)y.$$

for all $t, s \in [0,T]$.

4. *For every* $y \in Y$ *the function* $C(\cdot)y$ *is Hölder continuous on* $[0,T]$.
5. $\Sigma_0(\mathcal{D}(B)) \subset \mathcal{D}(A)$.

The infinite-dimensional Sylvester differential equation (3.1) *has a unique classical solution* $\Sigma(\cdot)$ *on* $[0,T]$ *given by the formula*

$$\Sigma(t)y = U_A(t,0)\Sigma_0 U_B(0,t)y + \int_0^t U_A(t,s)C(s)U_B(s,t)y\,ds \tag{3.2}$$

for all $y \in Y$.

Proof. Since $U_A(t,s)$ satisfies the parabolic conditions, we have from [10, Sec. 5.6] that for all $x \in X$, $x' \in \mathcal{D}(A)$, and $t > s$

$$\frac{\partial}{\partial t} U_A(t,s)x = A(t)U_A(t,s)x, \qquad \frac{\partial}{\partial s} U_A(t,s)x' = -U_A(t,s)A(s)x'.$$

Let $y \in \mathcal{D}(B)$, $x^* \in \mathcal{D}(A^*)$, and $s \in [0,T]$. Using the differentiation rules for $U_A(t,s)$ and $U_B(s,t)$ we see that for any $t \in (s,T]$

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle U_A(t,s)C(s)U_B(s,t)y, x^*\rangle \\
&= \langle A(t)U_A(t,s)C(s)U_B(s,t)y, x^*\rangle - \langle U_A(t,s)C(s)U_B(s,t)B(t)y, x^*\rangle \\
&= \langle U_A(t,s)C(s)U_B(s,t)y, A(t)^*x^*\rangle - \langle U_A(t,s)C(s)U_B(s,t)B(t)y, x^*\rangle \\
&\frac{\partial}{\partial t}\langle U_A(t,0)\Sigma_0 U_B(0,t)y, x^*\rangle \\
&= \langle A(t)U_A(t,0)\Sigma_0 U_B(0,t)y, x^*\rangle - \langle U_A(t,0)\Sigma_0 U_B(0,t)B(t)y, x^*\rangle \\
&= \langle U_A(t,0)\Sigma_0 U_B(0,t)y, A(t)^*x^*\rangle - \langle U_A(t,0)\Sigma_0 U_B(0,t)B(t)y, x^*\rangle.
\end{aligned}$$

To show that (3.2) is a solution of the Sylvester differential equation we will use the Leibniz integral rule [8, Lem. VIII.2.2]. This result states that if the function $f : \{ (t,s) \mid 0 \le s \le t \le T \} \to \mathbb{C}$ is continuous, and if $\frac{\partial}{\partial t} f(t,s)$ exists and is continuous and uniformly bounded on $\{ (t,s) \mid 0 \le s < t \le T \}$, then the mapping $t \mapsto \int_0^t f(t,s)ds$ is differentiable on $(0,T)$ and

$$\frac{d}{dt}\int_0^t f(t,s)ds = f(t,t) + \int_0^t \frac{\partial}{\partial t} f(t,s)ds.$$

Our assumptions imply that the function

$$(t,s) \to f(t,s) = \langle U_A(t,s)C(s)U_B(s,t)y, x^*\rangle$$

is continuous for $0 \le s \le t \le T$ and the computation above shows that its derivative with respect to t is continuous. It thus remains to show that this derivative is uniformly bounded. Since the mappings $(t,s) \to U_A(t,s)$ and $(t,s) \to U_B(s,t)$ are strongly continuous, there exist constants $M_A, M_B > 0$ such that

$$\max_{0\le s\le t\le T} \|U_A(t,s)\| \le M_A, \qquad \max_{0\le s\le t\le T} \|U_B(s,t)\| \le M_B.$$

Using these estimates we see that

$$\begin{aligned}
\left|\frac{\partial}{\partial t} f(t,s)\right| &\le \|U_A(t,s)C(s)U_B(s,t)y\| \cdot \|A(t)^*x^*\| \\
&\quad + \|U_A(t,s)C(s)U_B(s,t)B(t)y\| \cdot \|x^*\| \\
&\le \|U_A(t,s)\| \cdot \|C(s)\| \cdot \|U_B(s,t)\| \left(\|y\| \cdot \|A(t)^*x^*\| + \|B(t)y\| \cdot \|x^*\|\right) \\
&\le M_A M_B \max_{r\in[0,T]} \|C(r)\| \left(\|y\| \max_{r\in[0,T]} \|A(r)^*x^*\| + \|x^*\| \max_{r\in[0,T]} \|B(r)y\| \right) \\
&< \infty.
\end{aligned}$$

This concludes that we can use the Leibniz integral rule.

For the function $\Sigma(\cdot)$ defined in (3.2) we now have

$$\begin{aligned}
&\frac{d}{dt}\langle \Sigma(t)y, x^*\rangle = \frac{d}{dt}\langle U_A(t,0)\Sigma_0 U_B(0,t)y, x^*\rangle \\
&\quad + \frac{d}{dt}\int_0^t \langle U_A(t,s)C(s)U_B(s,t)y, x^*\rangle ds \\
&= \langle U_A(t,0)\Sigma_0 U_B(0,t)y, A(t)^*x^*\rangle - \langle U_A(t,0)\Sigma_0 U_B(0,t)B(t)y, x^*\rangle \\
&\quad + \int_0^t \left(\langle U_A(t,s)C(s)U_B(s,t)y, A(t)^*x^*\rangle - \langle U_A(t,s)C(s)U_B(s,t)B(t)y, x^*\rangle\right) ds \\
&\quad + \langle U_A(t,t)C(t)U_B(t,t)y, x^*\rangle \\
&= \langle \Sigma(t)y, A(t)^*x^*\rangle - \langle \Sigma(t)B(t)y, x^*\rangle + \langle C(t)y, x^*\rangle. \qquad (3.3)
\end{aligned}$$

We will next show that the mapping $t \mapsto \Sigma(t)y$ is continuously differentiable on $(0,T)$ and that $\Sigma(t)y \in \mathcal{D}(A)$ for all $t \in [0,T]$. We will do this by first considering the nonautonomous Cauchy problem

$$\dot{x}(t) = A(t)x(t) + C(t)U_B(t,0)v, \qquad x(0) = \Sigma_0 v,$$

where $v \in \mathcal{D}(B)$. Since $x(0) \in \mathcal{D}(A)$ and since $t \mapsto C(t)U_B(t,0)v$ is Hölder continuous on $[0,T]$ we have from [10, Thm. 5.7.1] that this equation has a unique classical solution given by

$$x(t) = U_A(t,0)\Sigma_0 v + \int_0^t U_A(t,s)C(s)U_B(s,0)vds$$

such that $x(\cdot)$ is continuously differentiable on $(0,T)$ and $x(t) \in \mathcal{D}(A)$ for all $t \in [0,T]$. If we denote by $H(\cdot) : [0,T] \to \mathcal{L}(Y,X)$ the strongly continuous mapping $x(t) = H(t)v$, then for all $v \in \mathcal{D}(B)$ the function $t \mapsto H(t)v$ is continuously differentiable on $(0,T)$ and $H(t)v \in \mathcal{D}(A)$. Since $t \mapsto U_B(0,t)y$ is strongly continuously differentiable, the choice $v = U_B(0,t)y \in \mathcal{D}(B)$ and a straight-forward computation finally show that the function

$$\begin{aligned} t \mapsto H(t)U_B(0,t)y &= U_A(t,0)\Sigma_0 U_B(0,t)y + \int_0^t U_A(t,s)C(s)U_B(s,0)U_B(0,t)yds \\ &= U_A(t,0)\Sigma_0 U_B(0,t)y + \int_0^t U_A(t,s)C(s)U_B(s,t)yds \\ &= \Sigma(t)y \end{aligned}$$

is continuously differentiable on $(0,T)$ and $\Sigma(t)y \in \mathcal{D}(A)$ for all $[0,T]$. Now equation (3.3) becomes

$$\langle \frac{d}{dt}\Sigma(t)y, x^*\rangle = \langle A(t)\Sigma(t)y, x^*\rangle - \langle \Sigma(t)B(t)y, x^*\rangle + \langle C(t)y, x^*\rangle.$$

Since $x^* \in \mathcal{D}(A^*)$ was arbitrary and since $\mathcal{D}(A^*)$ is dense in X^*, this implies

$$\frac{d}{dt}\Sigma(t)y = A(t)\Sigma(t)y - \Sigma(t)B(t)y + C(t)y.$$

This concludes that $\Sigma(\cdot)$ is a classical solution of the Sylvester differential equation.

To prove the uniqueness of the solution, let $\Sigma_1(\cdot) \in C([0,T], \mathcal{L}_s(Y,X))$ be a classical solution of the Sylvester differential equation (3.2). Letting $y \in \mathcal{D}(B)$ and applying both sides of the equation to $U_B(s,t)y \in \mathcal{D}(B)$ for $t > s$ we obtain

$$\begin{aligned} &\dot{\Sigma}_1(s)U_B(s,t)y = A(s)\Sigma_1(s)U_B(s,t)y - \Sigma_1(s)B(s)U_B(s,t)y + C(s)U_B(s,t)y \\ \Rightarrow\ &U_A(t,s)\dot{\Sigma}_1(s)U_B(s,t)y = U_A(t,s)A(s)\Sigma_1(s)U_B(s,t)y \\ &- U_A(t,s)\Sigma_1(s)B(s)U_B(s,t)y + U_A(t,s)C(s)U_B(s,t)y \\ \Rightarrow\ &\frac{d}{ds}\left(U_A(t,s)\Sigma_1(s)U_B(s,t)y\right) = U_A(t,s)C(s)U_B(s,t)y \end{aligned}$$

Integrating both sides of the last equation from 0 to t and using $\Sigma_1(0) = \Sigma_0$ gives

$$\begin{aligned} \int_0^t U_A(t,s)C(s)U_B(s,t)yds &= U_A(t,t)\Sigma_1(t)U_B(t,t)y - U_A(t,0)\Sigma_1(0)U_B(0,t)y \\ &= \Sigma_1(t)y - U_A(t,0)\Sigma_0 U_B(0,t)y \end{aligned}$$

and thus $\Sigma_1(\cdot) = \Sigma(\cdot)$. □

As already mentioned, the conditions imposed on the evolution family $U_A(t,s)$ in Theorem 3.2 require that for $t \in [0,T]$ the operators $A(t)$ generate analytic semigroups on X. If these conditions are not satisfied, the solution (3.2) can under weaker conditions be seen as a *mild solution* of the Sylvester differential equation (3.1).

To illustrate the parabolic conditions we will present an example of a family of unbounded operators satisfying these conditions.

Example. Let $\alpha(\cdot), \gamma(\cdot) \in C([0,T],\mathbb{R})$ be Lipschitz continuous functions such that $\alpha(t) > 0$ for all $t \in [0,T]$. Consider a one-dimensional heat equation with time-varying coefficients

$$\begin{aligned} \frac{\partial x}{\partial t}(z,t) &= \alpha(t)\frac{\partial^2 x}{\partial t^2}(z,t) + \gamma(t)x(z,t), \\ x(z,0) &= x_0(z) \\ x(0,t) &= x(1,t) = 0 \end{aligned}$$

on the interval $[0,1]$. This can be written as a nonautonomous Cauchy problem

$$\dot{x} = A(t)x(t), \qquad x(t) = x_0 \in X$$

on the space $X = L^2(0,1)$ where the family of operators $(A(t), \mathcal{D}(A))$ is given by

$$\begin{aligned} &A(t)x = \alpha(t)x'' + \gamma(t)x, \\ &\mathcal{D}(A) = \left\{\, x \in X \;\middle|\; x, x' \text{ abs. cont. } x'' \in L^2(0,1),\ x(0) = x(1) = 0 \,\right\}. \end{aligned}$$

Furthermore, the operators $A(t)$ have spectral decompositions [3, Ex. A.4.26]

$$A(t)x = \sum_{n=1}^{\infty} \lambda_n(t)\langle x, \phi_n\rangle \phi_n, \qquad x \in \mathcal{D}(A) = \left\{\, x \in X \;\middle|\; \sum_{n=1}^{\infty} n^4 |\langle x, \phi_n\rangle|^2 < \infty \,\right\}$$

where the eigenvalues are given by $\lambda_n(t) = -\alpha(t)n^2\pi^2 + \gamma(t)$ and the corresponding eigenvectors $\phi_n = \sqrt{2}\sin(n\pi\cdot)$ form an orthonormal basis of X. These decompositions and the fact that $\alpha(\cdot)$ and $\gamma(\cdot)$ are Lipschitz continuous functions can be used to verify that the parabolic conditions are satisfied.

We can also show that the second condition in Theorem 3.2 is satisfied for this family of operators. The operators $A(t)$ are self-adjoint and thus we can achieve this by showing that the mapping $t \mapsto A(t)x$ is continuous for all $x \in \mathcal{D}(A)$. If we define the operator $A_0 : \mathcal{D}(A) \to X$ by $A_0 x = x''$ we can write

$$A(t)x = \alpha(t)A_0 x + \gamma(t)x, \qquad x \in \mathcal{D}(A).$$

Since the functions $\alpha(\cdot)$ and $\gamma(\cdot)$ are continuous, we can conclude that the second condition in Theorem 3.2 is satisfied.

Families of operators satisfying the conditions concerning $(B(t), \mathcal{D}(B))$ include, for example, all functions $B(\cdot) \in C([0,T], \mathcal{L}_s(Y))$ and the case where $B(t) \equiv B$ is a generator of a strongly continuous group on Y.

We conclude this section by considering the periodic Sylvester differential equation. By this we mean the equation

$$\dot{\Sigma}(t) = A(t)\Sigma(t) - \Sigma(t)B(t) + C(t) \tag{3.4}$$

for $t \in \mathbb{R}$ when the families of unbounded operators and the function $C(\cdot)$ are periodic with the same period $\tau > 0$. The periodic solution of this equation is a periodic function $\Sigma(\cdot) \in C(\mathbb{R}, \mathcal{L}_s(Y,X))$ which is a classical solution of the Sylvester differential equation (3.1) with some initial condition $\Sigma(0) = \Sigma_0 \in \mathcal{L}(Y,X)$ on an interval $[0,T]$. The following theorem states that if the exponential growths of the evolution families $U_A(t,s)$ and $U_B(s,t)$ satisfy a certain condition, then under the assumptions of Theorem 3.2 the periodic Sylvester differential equation (3.4) has a unique periodic solution and that this solution has period τ.

Theorem 3.3. *Assume the conditions of Theorem* 3.2 *are satisfied and that the evolution families* $(A(t), \mathcal{D}(A))$ *and* $(B(t), \mathcal{D}(B))$ *and the function* $C(\cdot)$ *are periodic with period* $\tau > 0$. *If there exist constants* $M_A, M_B \geq 1$ *and* $\omega_A, \omega_B \in \mathbb{R}$ *such that* $\omega_A + \omega_B < 0$ *and such that for all* $t \geq s$

$$\|U_A(t,s)\| \leq M_A e^{\omega_A(t-s)}, \qquad \|U_B(s,t)\| \leq M_B e^{\omega_B(t-s)},$$

then the periodic Sylvester differential equation (3.4) *has a unique periodic solution* $\Sigma_\infty(\cdot) \in C(\mathbb{R}, \mathcal{L}_s(Y,X))$ *such that* $\Sigma_\infty(\cdot)y \in C^1(\mathbb{R}, X)$ *and* $\Sigma(t)y \in \mathcal{D}(A)$ *for all* $y \in \mathcal{D}(B)$ *and* $t \in \mathbb{R}$. *The function* $\Sigma_\infty(\cdot)$ *has period* τ *and is given by the formula*

$$\Sigma_\infty(t)y = \int_{-\infty}^{t} U_A(t,s)C(s)U_B(s,t)y\,ds, \qquad y \in Y.$$

Proof. We will first show that $\Sigma_\infty(\cdot)$ is a classical solution of the Sylvester differential equation (3.1) on the interval $[0, 2\tau]$. Since for every $y \in Y$ we have

$$\Sigma_\infty(t)y = U_A(t,0)\int_{-\infty}^{0} U_A(0,s)C(s)U_B(s,0)U_B(0,t)y\,ds + \int_{0}^{t} U_A(t,s)C(s)U_B(s,t)y\,ds,$$

it suffices to show that the linear operator $\Sigma_\infty(0) : Y \to X$ defined by

$$\Sigma_\infty(0)y = \int_{-\infty}^{0} U_A(0,s)C(s)U_B(s,0)y\,ds, \qquad y \in Y$$

is bounded and $\Sigma_\infty(0)(\mathcal{D}(B)) \subset \mathcal{D}(A)$. Our assumptions imply that for all $y \in Y$ we have

$$\int_{-\infty}^{0} \|U_A(0,s)C(s)U_B(s,0)y\|ds \leq M_A M_B \max_{r\in[0,\tau]} \|C(r)\| \int_{-\infty}^{0} e^{-(\omega_A+\omega_B)s}ds \cdot \|y\| =: M\|y\|,$$

where $M < \infty$. This concludes that $\Sigma_\infty(0) : Y \to X$ is a well-defined linear operator and since

$$\|\int_{-\infty}^{0} U_A(0,s)C(s)U_B(s,0)yds\| \leq \int_{-\infty}^{0} \|U_A(0,s)C(s)U_B(s,0)y\|ds \leq M\|y\|,$$

we have $\Sigma_\infty(0) \in \mathcal{L}(Y,X)$. To show that $\Sigma_\infty(0)(\mathcal{D}(B)) \subset \mathcal{D}(A)$, let $y \in \mathcal{D}(B)$ and write

$$\begin{aligned}\Sigma_\infty(0)y = &\int_{-\infty}^{-1} U_A(0,s)C(s)U_B(s,0)yds \\ &+ \int_{-1}^{0} U_A(0,s)C(s)U_B(s,0)yds =: v_0 + v_1.\end{aligned}$$

If we denote $f(s) = U_A(0,s)C(s)U_B(s,0)y$, then $f(s) \in \mathcal{D}(A)$ for all $s < 0$ and from the previous estimate we have $f \in L^1((-\infty,-1),X)$. We have from [10, Thm. 5.6.1] that $A(0)U_A(0,-1) \in \mathcal{L}(X)$ and thus

$$\begin{aligned}&\int_{-\infty}^{-1} \|A(0)U_A(0,s)C(s)U_B(s,0)y\|ds \\ &\leq \|A(0)U_A(0,-1)\| \int_{-\infty}^{-1} \|U_A(-1,s)C(s)U_B(s,0)y\|ds \\ &\leq M_A M_B \max_{r\in[0,\tau]} \|C(r)\| \cdot \|A(0)U_A(0,-1)\| \cdot \|y\| \cdot e^{-\omega_A} \int_{-\infty}^{-1} e^{-(\omega_A+\omega_B)s}ds < \infty.\end{aligned}$$

This shows that $A(0)f \in L^1((-\infty,-1),X)$ and since $A(0)$ is a closed linear operator we have that $v_0 \in \mathcal{D}(A(0)) = \mathcal{D}(A)$. As in the proof of Theorem 3.2 we have that since the mapping $t \mapsto C(t)U_B(t,0)y$ is Hölder continuous on $[-1,0]$, the nonautonomous abstract Cauchy problem

$$\dot{x}(t) = A(t)x(t) + C(t)U_B(t,0)y, \qquad x(-1) = 0$$

has a unique classical solution

$$x(t) = \int_{-1}^{t} U_A(t,s)C(s)U_B(s,0)yds$$

on $[-1,0]$. Thus we also have $v_1 = x(0) \in \mathcal{D}(A)$. Combining these results shows that we have $\Sigma_\infty(0)y = v_0 + v_1 \in \mathcal{D}(A)$ and thus $\Sigma_\infty(0)$ is the unique classical solution of the Sylvester differential equation on $[0,2\tau]$ associated to the initial condition $\Sigma_\infty(0)$.

To prove the periodicity of $\Sigma_\infty(\cdot)$, let $t \in \mathbb{R}$. For all $y \in Y$ we then have

$$\begin{aligned}\Sigma_\infty(t+\tau)y &= \int_{-\infty}^{t+\tau} U_A(t+\tau,s)C(s)U_B(s,t+\tau)yds \\ &= \int_{-\infty}^{t} U_A(t+\tau,s+\tau)C(s+\tau)U_B(s+\tau,t+\tau)yds \\ &= \int_{-\infty}^{t} U_A(t,s)C(s)U_B(s,t)yds = \Sigma_\infty(t)y.\end{aligned}$$

This shows that $\Sigma_\infty(\cdot)$ is periodic with period τ. This and the fact that $\Sigma_\infty(\cdot)$ is the classical solution of the Sylvester differential equation (3.1) on the interval $[0, 2\tau]$ imply that $\Sigma_\infty(\cdot)y \in C^1(\mathbb{R}, X)$ and $\Sigma_\infty(t)y \in \mathcal{D}(A)$ for all $t \in \mathbb{R}$. This concludes that $\Sigma_\infty(\cdot)$ is a periodic solution of the periodic Sylvester differential equation.

It remains to prove that the periodic Sylvester differential equation (3.4) has no other periodic solutions. To this end, let $\Sigma(\cdot)$ be any periodic solution of the equation corresponding to an arbitrary initial condition $\Sigma(0) = \Sigma_0 \in \mathcal{L}(W, X)$. Let $y \in Y$. We have

$$\Sigma(t)y = U_A(t,0)\Sigma_0 U_B(0,t)y + \int_0^t U_A(t,s)C(s)U_B(s,t)yds$$

and the difference $\Delta(t)y = \Sigma_\infty(t)y - \Sigma(t)y$ satisfies

$$\begin{aligned}\Delta(t)y = &\int_{-\infty}^{t} U_A(t,s)C(s)U_B(s,t)yds - U_A(t,0)\Sigma_0 U_B(0,t)y \\ &- \int_0^t U_A(t,s)C(s)U_B(s,t)yds \\ = &\int_{-\infty}^{0} U_A(t,s)C(s)U_B(s,t)yds - U_A(t,0)\Sigma_0 U_B(0,t)y \\ = &\, U_A(t,0)\Sigma_\infty(0)U_B(0,t) - U_A(t,0)\Sigma_0 U_B(0,t)y = U_A(t,0)\Delta(0)U_B(0,t)y.\end{aligned}$$

Thus

$$\|\Delta(t)\| \leq M_A M_B e^{(\omega_A+\omega_B)t}\|\Delta(0)\|$$

and the assumption $\omega_A + \omega_B < 0$ implies $\lim_{t\to\infty} \Delta(t) = 0$. Since $\Sigma(\cdot)$ and $\Sigma_\infty(\cdot)$ are periodic and since $\lim_{t\to\infty}\|\Sigma(t) - \Sigma_\infty(t)\| = 0$, we must have $\Sigma(t) \equiv \Sigma_\infty(t)$. This concludes that no other periodic solutions than $\Sigma_\infty(\cdot)$ may exist. □

4. Periodic Output Regulation

In this section we finally apply the results on the solvability of the Sylvester differential equation to obtain a characterization for the controllers solving the output regulation problem related to a distributed parameter system and a nonautonomous periodic signal generator. We will use notation typical to mathematical systems theory and because of this the choices of symbols differ from the ones used in the earlier sections.

We consider the output regulation of an infinite-dimensional linear system in a situation where the reference and disturbance signals are generated by an exosystem

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in W \tag{4.1}$$

on a finite-dimensional space $W = \mathbb{C}^q$. We assume the input and output spaces U and Y, respectively, are Hilbert spaces and that the plant can be written in a standard form as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E(t)v(t), \qquad x(0) = x_0 \in X \\ e(t) &= Cx(t) + Du(t) + F(t)v(t) \end{aligned}$$

on a Banach space X. Here $e(t) \in Y$ is the regulation error, $u(t) \in U$ the input, $E(t)v(t)$ is the disturbance signal to the state and $F(t)v(t)$ contains the disturbance signal to the output and the reference signal. The operator $A : \mathcal{D}(A) \subset X \to X$ is assumed to generate an analytic semigroup on X and the rest of the operators are bounded. We consider a dynamic error feedback controller of form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1(t)z(t) + \mathcal{G}_2(t)e(t), \qquad z(0) = z_0 \in Z \\ u(t) &= K(t)z(t) \end{aligned}$$

on a Banach space Z. Here $(\mathcal{G}_1(t), \mathcal{D}(\mathcal{G}_1))$ is a family of unbounded operators, and $\mathcal{G}_2(t) \in \mathcal{L}(Y, Z)$ and $K(t) \in \mathcal{L}(Z, U)$ for all $t \geq 0$. The plant and the controller can be written as a closed-loop system

$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)v(t) \qquad x_e(0) = x_{e0} \in X_e \tag{4.2a}$$

$$e(t) = C_e(t)x_e(t) + D_e(t)v(t) \tag{4.2b}$$

on the Banach space $X_e = X \times Z$ by choosing

$$A_e(t) = \begin{pmatrix} A & BK(t) \\ \mathcal{G}_2(t)C & \mathcal{G}_1(t) + \mathcal{G}_2(t)DK(t) \end{pmatrix}, \qquad B_e(t) = \begin{pmatrix} E(t) \\ \mathcal{G}_2(t)F(t) \end{pmatrix}$$

$C_e(t) = \big(C,\, DK(t)\big)$ and $D_e(t) = F(t)$. We assume the family $(A_e(t), \mathcal{D}(A_e(t)))$ of unbounded operators and the operator-valued functions $S(\cdot)$, $B_e(\cdot)$, $C_e(\cdot)$ and $D_e(\cdot)$ are periodic with the same period $\tau > 0$. The Periodic Output Regulation Problem is defined as follows.

Definition 4.1 (Periodic Output Regulation Problem). Choose the parameters $(\mathcal{G}_1(\cdot), \mathcal{G}_2(\cdot), K(\cdot))$ of the dynamic error feedback controller in such a way that

1. The evolution family $U_e(t, s)$ associated to the family $(A_e(t), \mathcal{D}(A_e(t)))$ is exponentially stable, i.e. there exist $M_e, \omega_e > 0$ such that for all $t \geq s$
$$\|U_e(t, s)\| \leq M_e e^{-\omega_e (t-s)}.$$
2. For all initial values $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system and the exosystem, respectively, the regulation error $e(t)$ goes to zero asymptotically, i.e. $e(t) \to 0$ as $t \to \infty$.

It has been shown in [9] that under suitable assumptions the solvability of the Periodic Output Regulation Problem can be characterized using the periodic Sylvester differential equation

$$\dot{\Sigma}(t) = A_e(t)\Sigma(t) - \Sigma(t)S(t) + B_e(t). \tag{4.3}$$

Using Theorem 3.3 we can weaken the assumptions required for this characterization and thus extend the results presented in [9] for more general classes of systems and exosystems. We will first state the required assumptions. Since the space $W = \mathbb{C}^q$ is finite-dimensional, the strong continuity of the operator-valued functions $S(\cdot)$ and $B_e(\cdot)$ coincide with the continuity with respect to the uniform operator topology.

1. The family $(A_e(t), \mathcal{D}(A_e(t)))$ satisfies the parabolic conditions.
2. The domain $\mathcal{D}(A_e(t)^*) =: \mathcal{D}(A_e^*)$ is independent of $t \in \mathbb{R}$ and dense in X_e^*. For all $x \in X_e$ and $x^* \in X_e^*$ the mapping $t \mapsto \langle x, A_e(t)^* x^* \rangle$ is continuous.
3. The matrix-valued function $S(\cdot)$ is continuous, we have $|\lambda| = 1$ for all eigenvalues λ of $U_S(\tau, 0)$ and there exists $M_S \geq 1$ such that $\|U_S(t,s)\| \leq M_S$ for all $t, s \in \mathbb{R}$.
4. The function $B_e(\cdot)$ is Hölder continuous.
5. The functions $C_e(\cdot)$ and $D_e(\cdot)$ are strongly continuous.

The following theorem characterizes the controllers solving the Periodic Output Regulation Problem using the properties of the Sylvester differential equation (4.3).

Theorem 4.2. *Assume that the above conditions are satisfied. If the controller stabilizes the closed-loop system exponentially, then the periodic Sylvester differential equation* (4.3) *has a unique periodic classical solution* $\Sigma_\infty(\cdot)$ *and the controller solves the Periodic Output Regulation Problem if and only if this solution satisfies*

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0 \tag{4.4}$$

for all $t \in [0, \tau]$.

Proof. Since the conditions of Theorem 3.3 are satisfied, the Sylvester differential equation (4.3) has a unique periodic classical solution $\Sigma_\infty(\cdot)$ with period τ. Since the space W is finite-dimensional we have $\Sigma_\infty(\cdot) \in C^1(\mathbb{R}, \mathcal{L}(W, X_e))$ and $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ for all $t \in \mathbb{R}$.

We will first study the asymptotic behaviour of the regulation error. For any initial conditions $x_{e0} \in X_e$ and $v_0 \in W$ and for any $t \geq 0$ the state of the closed-loop system is given by

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0 ds.$$

Using the Sylvester differential equation we see that

$$\begin{aligned}U_e(t,s)B_e(s)U_S(t,0)v_0 &= U_e(t,s)(\dot{\Sigma}_\infty(s)+\Sigma_\infty(s)S(s)-A_e(s)\Sigma_\infty(s))U_S(s,0)v_0\\ &= U_e(t,s)\dot{\Sigma}_\infty(s)U_S(s,0)v_0+U_e(t,s)\Sigma_\infty(s)S(s)U_S(s,0)v_0\\ &\quad -U_e(t,s)A_e(s)\Sigma_\infty(s)U_S(s,0)v_0\\ &= \frac{d}{ds}U_e(t,s)\Sigma_\infty(s)U_S(s,0)v_0.\end{aligned}$$

The state of the closed-loop system can thus be expressed using a formula

$$\begin{aligned}x_e(t) &= U_e(t,0)x_{e0}+\int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0 ds\\ &= U_e(t,0)x_{e0}+\Sigma_\infty(t)U_S(t,0)v_0-U_e(t,0)\Sigma_\infty(0)v_0\\ &= U_e(t,0)(x_{e0}-\Sigma_\infty(0)v_0)+\Sigma_\infty(t)v(t)\end{aligned}$$

and the regulation error corresponding to these initial states is given by

$$\begin{aligned}e(t) &= C_e(t)x_e(t)+D_e v(t)\\ &= C_e(t)U_e(t,0)(x_{e0}-\Sigma_\infty(0)v_0)+\left(C_e(t)\Sigma_\infty(t)+D_e(t)\right)v(t).\end{aligned}$$

Since the closed-loop system is stable there exist constants $M_e \geq 1$ and $\omega_e > 0$ such that for all $t \geq s$ we have $\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$. Using the formula for the regulation error we have

$$\begin{aligned}&\|e(t)-(C_e(t)\Sigma_\infty(t)+D_e(t))v(t)\| = \|C_e(t)U_e(t,0)(x_{e0}-\Sigma_\infty(0)v_0)\|\\ &\leq M_e e^{-\omega_e t}\max_{s\in[0,T]}\|C_e(s)\|\cdot\|x_{e0}-\Sigma_\infty(0)v_0\| \longrightarrow 0\end{aligned}$$

as $t\to\infty$ since $\omega_e > 0$. This property describing the asymptotic behaviour of the regulation error allows us to prove the theorem.

Assume first that (4.4) is satisfied for all $t \in [0,\tau]$. The periodicity of the functions implies that it is satisfied for all $t \in \mathbb{R}$ and thus for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies

$$\|e(t)\| = \|e(t)-(C_e(t)\Sigma_\infty(t)+D_e(t))v(t)\| \longrightarrow 0$$

as $t\to\infty$. This concludes that the controller solves the Periodic Output Regulation Problem.

To prove the converse implication assume that the controller solves the Periodic Output Regulation Problem. Let $t_0 \in [0,\tau)$ and $n \in \mathbb{N}_0$ and denote $t = n\tau+t_0$. Using the periodicity of the functions and the above property of the regulation error we have that for any initial state $v_0 \in W$ of the exosystem and any $x_{e0} \in X_e$

$$\begin{aligned}&\|(C_e(t_0)\Sigma_\infty(t_0)+D_e(t_0))v(t)\| = \|(C_e(t)\Sigma_\infty(t)+D_e(t))v(t)\|\\ &= \|e(t)-(C_e(t)\Sigma_\infty(t)+D_e(t))v(t)\|+\|e(t)\| \longrightarrow 0\end{aligned}$$

as $n \to \infty$. Let $\lambda \in \sigma(U_S(\tau, 0))$ and let $\{\phi_k\}_{k=1}^m$ be a Jordan chain associated to this eigenvalue. We will use the above limit to show that for all $k \in \{1, \ldots, m\}$ we have $(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\phi_k = 0$. By assumption we have $|\lambda| = 1$ and

$$U_S(\tau, 0)\phi_1 = \lambda\phi_1, \qquad U_S(\tau, 0)\phi_k = \lambda\phi_k + \phi_{k-1}, \quad k \in \{2, \ldots, m\}. \tag{4.5}$$

The periodicity of the evolution family $U_S(t, s)$ implies

$$\begin{aligned} U_S(t, 0) = U_S(n\tau + t_0, 0) &= U_S(n\tau + t_0, n\tau)U_S(n\tau, (n-1)\tau) \cdots U_S(\tau, 0) \\ &= U_S(t_0, 0)U_S(\tau, 0)^n \end{aligned}$$

and thus

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t, 0)\phi_1\| \\ &= \|(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t_0, 0)\phi_1\| \cdot \left(\lim_{n\to\infty} |\lambda|^n\right). \end{aligned}$$

This implies $(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t_0, 0)\phi_1 = 0$ since $|\lambda| = 1$. Using this and (4.5) we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t, 0)\phi_2\| \\ &= \|(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t_0, 0)\phi_2\| \cdot \left(\lim_{n\to\infty} |\lambda|^n\right) \end{aligned}$$

and thus also $(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t_0, 0)\phi_2 = 0$. Continuing this we finally obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t, 0)\phi_m\| \\ &= \|(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t_0, 0)\phi_m\| \cdot \left(\lim_{n\to\infty} |\lambda|^n\right) \end{aligned}$$

which implies $(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))\, U_S(t_0, 0)\phi_m = 0$. Since $\lambda \in \sigma(U_S(\tau, 0))$ and the associated Jordan chain were arbitrary, we must have

$$(C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0))U_S(t_0, 0) = 0.$$

The invertibility of $U_S(t_0, 0)$ further concludes that $C_e(t_0)\Sigma_\infty(t_0) + D_e(t_0) = 0$. Since $t_0 \in [0, \tau)$ was arbitrary, this finally shows that $C_e(t)\Sigma_\infty(t) + D_e(t) = 0$ for every $t \in [0, \tau]$. □

It should also be noted that Theorem 4.2 is independent of the form of the controller in the sense that if the closed-loop system can be written in the form (4.2), then this result implies that the output $e(t)$ of the closed-loop system driven by the nonautonomous exosystem (4.1) decays to zero asymptotically if and only if the solution of the Sylvester differential equation satisfies the constraint (4.4). This makes it possible to study the Periodic Output Regulation Problems with different types of controllers simultaneously. The general results obtained this way can subsequently be used to derive separate conditions for the solvability of the problem using different controller types.

5. Conclusions

In this paper we have considered the solvability of the infinite-dimensional Sylvester differential equation. We have introduced conditions under which the equation has a unique classical solution. We have also considered the periodic version of the equation and shown that if a certain condition on the growth of the evolution families associated to the equation is satisfied, then the periodic Sylvester differential equation has a unique periodic solution.

We applied the results on the solvability of the equation to the output regulation of a distributed parameter system with a time-dependent exosystem. In particular we showed that the controllers solving the output regulation problem can be characterized using the properties of the solution of the Sylvester differential equation. Developing the results for the solvability of these types of equations is crucial to the generalization of the theory of output regulation for more general classes of infinite-dimensional systems and exogeneous signals.

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