On Robust Output Regulation for Continuous-Time Periodic Systems

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Abstract

We construct a controller to solve robust output tracking problem for a stable linear continuous-time periodic system on a finite-dimensional space. We begin by transforming the time-dependent plant to a time-invariant discrete-time system using the "lifting technique". The controller is then designed to achieve robust output tracking for the lifted system. We show that an exact solution to the control problem for a continuous-time periodic system necessarily requires an error feedback controller with an infinite-dimensional internal model. The results are illustrated with an example where robust output tracking is considered for a stable periodic scalar system.

1 Introduction

The purpose of this paper is to study the problem of robust output tracking for a stable periodically time-dependent linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \in X \tag{1a}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1b}$$

on a finite-dimensional space $X = \mathbb{C}^n$. It is well-known that for a time-invariant linear system the robust output output tracking problem can be solved using a controller incorporating an internal model of the exosystem's dynamics. This is a consequence of the fundamental *internal model principle* [4, 3].

Unfortunately, the internal model principle is not directly applicable in the case of time-dependent systems. However, for discrete-time periodically time-dependent systems the robust output regulation problem has been shown to have a solution involving two steps: In the first step, the discrete-time periodic system is rewritten as a time-invariant discrete-time systems using a technique called "lifting" [10, 2]. The second step involves constructing an internal model based robust controller for the lifted system. The resulting control law can then be used to control the original periodic plant. Indeed, this approach has been successfully used in solving the robust output tracking and disturbance rejection problem for periodic discrete-time systems in [6, 7, 5, 9, 8, 12].

The lifting technique is also available for periodic continuous-time systems [1, 11]. For both discrete-time and continuous-time systems the lifted system is a time-invariant discrete-time system with the same state space as the original system. However, for a discrete-time system the output space of the lifted system has dimension τp, where p is the dimension of the output space of (1) and $\tau \in \mathbb{N}$ is the period length. On the other hand, if we consider a continuous-time system with a p-dimensional output space, the output of the lifted system lies in an infinite-dimensional Hilbert space $L^2(0, \tau; \mathbb{C}^p)$, where $\tau > 0$ is again the period length.

The above difference has particularly severe consequences in connection to the robust output tracking problem, because the classical definition of an internal model is not applicable for systems with an infinite-dimensional output spaces. However, recent advances in the theory of robust output regulation for infinite-dimensional systems [14, 15, 16] provide tools for studying internal models for systems with infinite-dimensional output spaces without difficulties. In particular, the references [14, 15] introduce alternative ways of defining the internal model in such a way that the concept is useful even in the case of infinite-dimensional output spaces.

In this paper we apply the results in [15] to design a periodic discrete-time controller in such a way that

- The closed-loop system consisting of the plant and the controller is stable.
- The output $y(\cdot)$ converges asymptotically to the reference signal $y_{ref}(\cdot)$ in a suitable sense.
- The control law tolerates small perturbations in the parameters $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of the plant.

We approach the problem by first lifting the system (1) to a time-invariant linear discrete-time system, and by subsequently designing an internal-model based discrete-time error feedback controller to solve the transformed control problem. Since the lifted system has an infinite-dimensional output space, also the internal-model based controller is necessarily infinite-dimensional. Because of this drawback the proposed solution of the control problem is first and foremost

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theoretical. However, the approach and the controller structure presented in this paper may open new possibilities in constructing approximate controllers for robust output tracking. Moreover, the stabilization of the closed-loop system requires the assumption that the transfer function of the lifted system at the frequency $\mu = 1$ of the exosystem is boundedly invertible. For time-invariant systems this is a standard assumption, but for time-periodic systems this condition is quite restrictive.

2 Assumptions on the Plant and the Controller

The parameters of the plant (1) on the spate space $X = \mathbb{C}^n$ are such that $A(\cdot) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is τ-periodic and locally integrable, $B(\cdot) \in C_\tau(\mathbb{R}; \mathbb{C}^{n\times m})$, $C(\cdot) \in C_\tau(\mathbb{R}; \mathbb{C}^{p\times n})$, $D(\cdot) \in C_\tau(\mathbb{R}; \mathbb{C}^{p\times m})$, where $C_\tau(\mathbb{R}; H)$ denotes the space of continuous τ-periodic H-valued functions. The state of the plant (1) is then given by

$$x(t) = \Phi_A(t, 0)x_0 + \int_0^t \Phi_A(t, s)B(s)u(s)ds,$$

where $\Phi_A(t, s)$, $t \geq s$ is the fundamental matrix of (1) satisfying $\Phi_A(0, 0) = I$. Since $A(\cdot)$ is τ-periodic, we have that $\Phi_A(t + \tau, s + \tau) = \Phi_A(t, s)$ for all $t \geq s$. We assume the plant is uniformly exponentially stable. For a periodic system this means that its characteristic multipliers, i.e., the eigenvalues of $\Phi_A(\tau, 0)$, have absolute values strictly less than one.

The main control problem consist of tracking of a reference signal $y_{ref}(\cdot) \in C_\tau(\mathbb{R}; \mathbb{C}^p)$. This signal is obtained as an output of the periodic exosystem

$$\dot{v}(t) = 0 \cdot v(t), \qquad v(0) = v_0 \in \mathbb{C} \tag{2a}$$
$$y_{ref}(t) = -F(t)v(t) \tag{2b}$$

corresponding to the initial state $v_0 = 1$ when we choose $F(\cdot) = -y_{ref}(\cdot) \in C_\tau(\mathbb{R}; \mathbb{C}^p)$.

2.1 The Lifted Systems Using the lifting technique [1, 11], the plant (1) can be represented as a time-invariant discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \qquad \mathbf{x}_0 = x_0 \tag{3a}$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k, \tag{3b}$$

on the space $X = \mathbb{C}^n$ with input space $U = L^2(0, \tau; \mathbb{C}^m)$ and output space $Y = L^2(0, \tau; \mathbb{C}^p)$. The lifted state $\mathbf{x}_k$, input $\mathbf{u}_k$ and output $\mathbf{y}_k$ are defined by

$$\mathbf{x}_k = x(k\tau),$$
$$\mathbf{u}_k = u(k\tau + \cdot) : [0, \tau) \to U$$
$$\mathbf{y}_k = y(k\tau + \cdot) : [0, \tau) \to Y$$

The operators $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are bounded so that $\mathbf{A} \in \mathcal{L}(X)$, $\mathbf{B} \in \mathcal{L}(U, X)$, $\mathbf{C} \in \mathcal{L}(X, Y)$, and $\mathbf{D} \in \mathcal{L}(U, Y)$, and they are defined so that for all $\mathbf{x} \in X$ and $\mathbf{u} \in U$

$$\mathbf{A}\mathbf{x} = \Phi_A(\tau, 0)\mathbf{x}$$
$$\mathbf{B}\mathbf{u} = \int_0^\tau \Phi_A(\tau, s)B(s)\mathbf{u}(s)ds$$
$$(\mathbf{C}\mathbf{x})(\cdot) = C(\cdot)\Phi_A(\cdot, 0)\mathbf{x}$$
$$(\mathbf{D}\mathbf{u})(\cdot) = D(\cdot)\mathbf{u}(\cdot) + C(\cdot)\int_0^\cdot \Phi_A(\cdot, s)B(s)\mathbf{u}(s)ds.$$

The lifted system (3) is thus a linear time-invariant system on a finite-dimensional space, and its input and output spaces are infinite-dimensional Hilbert spaces. Because the periodic system (1) was assumed to be stable, also the lifted system is stable and $\sigma(\mathbf{A}) \subset \mathbb{D}$. The transfer function of the system (3) is given by

$$\mathbf{P}(\mu) = \mathbf{C}(\mu I - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \in \mathcal{L}(U, Y), \qquad \mu \in \rho(\mathbf{A}).$$

The exosystem (2) can be similarly transformed to a discrete-time system

$$\mathbf{v}_{k+1} = \mathbf{v}_k, \qquad \mathbf{v}_0 = v_0 \in W \tag{4a}$$
$$\mathbf{y}_k^{ref} = -\mathbf{F}\mathbf{v}_k \tag{4b}$$

where $\mathbf{v}_k = v_0$ for all $k \in \mathbb{N}_0$, $\mathbf{y}_k^{ref} = y_{ref}(k\tau + \cdot)$, $\mathbf{F} \in \mathcal{L}(\mathbb{C}, Y) = Y$, and

$$\mathbf{y}_k^{ref} = -\mathbf{F}\mathbf{v}_k = -\mathbf{F}v_0 = -F(\cdot)v_0.$$

In this paper we choose to study the asymptotic convergence of $y(\cdot)$ to $y_{ref}(\cdot)$ in the following L^2-sense, because it is the natural form of convergence for the lifted systems.

Definition 2.1. *We say that* $y(\cdot)$ converges asymptotically to $y_{ref}(\cdot)$ *if*

$$\|\mathbf{y}_k - \mathbf{y}_k^{ref}\|^2 = \int_{k\tau}^{(k+1)\tau} \|y(t) - y_{ref}(t)\|^2 dt \stackrel{k\to\infty}{\longrightarrow} 0.$$

It should be noted that this form of convervence does not imply $\|y(t) - y_{ref}(t)\|_{\mathbb{C}^p} \to 0$ as $t \to \infty$.

In order to solve the robust output regulation problem, we make the following standing assumption on the plant. This is a standard assumption in robust control of time-invariant systems, but in the case of lifted systems it becomes quite restrictive. This is due to the fact that the input and output spaces of the lifted system are infinite-dimensional.

Assumption 2.1. *The transfer function* $\mathbf{P}(1) \in \mathcal{L}(U, Y)$ *of* (3) *at* $\mu = 1$ *is boundedly invertible.*

Since $X = \mathbb{C}^n$ is finite-dimensional, this assumption can only be satisfied if $D(t)$ is invertible for all $t \in [0, \tau]$. Relaxing this requirement is an important topic for future research.

2.2 The Controller and the Closed-Loop System We consider a discrete-time controller of the form

$$\text{(5a)} \qquad z_{k+1} = \mathcal{G}_1 z_k + \mathcal{G}_2 \mathbf{e}_k, \qquad z_0 = z^0 \in Z$$
$$\text{(5b)} \qquad \mathbf{u}_k = K z_k$$

on a Banach space Z. Here $\mathcal{G}_1 \in \mathcal{L}(Z)$, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$, $K \in \mathcal{L}(Z,U)$ and $\mathbf{e}_k = \mathbf{y}_k - \mathbf{y}_k^{ref}$ is the *regulation error.* It is clear that the output $y(t)$ converges to the reference signal $y_{ref}(t)$ in the sense of Definition 2.1 if and only if $\|\mathbf{e}_k\|_Y \to 0$ as $k \to \infty$.

The lifted plant and the controller can be written together as a closed-loop system

$$x_{e,k+1} = A_e x_{e,k} + B_e \mathbf{v}_k$$
$$\mathbf{e}_k = C_e x_{e,k} + D_e \mathbf{v}_k$$

on the space $X_e = X \times Z$ with bounded operators $C_e = (\mathbf{C} \quad \mathbf{D}K) \in \mathcal{L}(X_e, Y)$, $D_e = -\mathbf{F} \in \mathcal{L}(W,Y)$,

$$A_e = \begin{pmatrix} \mathbf{A} & \mathbf{B}K \\ \mathcal{G}_2\mathbf{C} & \mathcal{G}_1 + \mathcal{G}_2 \mathbf{D}K \end{pmatrix} \qquad B_e = \begin{pmatrix} 0 \\ \mathcal{G}_2 \mathbf{F} \end{pmatrix}.$$

3 The Robust Output Tracking Problem

In the control problem we study a situation where the parameters $(A(\cdot), B(\cdot), C(\cdot), D(\cdot), F(\cdot))$ are perturbed to $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot), \tilde{F}(\cdot))$. The class $\mathcal{O}$ of admissible perturbations is defined as follows.

Definition 3.1. *The class $\mathcal{O}$ of admissible perturbations has the following properties.*

(a) *The nominal parameters belong to the class, i.e., $(A(\cdot), B(\cdot), C(\cdot), D(\cdot), F(\cdot)) \in \mathcal{O}$.*

(b) *For all perturbations $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot), \tilde{F}(\cdot)) \in \mathcal{O}$, the operator $\tilde{\mathbf{P}}(1)$ is boundedly invertible.*

The robust output tracking problem is formulated in detail in the following.

The Robust Output Regulation Problem. *Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:*

(a) *The closed-loop is exponentially stable, i.e., $\sigma(A_e) \subset \mathbb{D}$.*

(b) *For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error decays to zero at an exponential rate, i.e., there exist $M, \omega > 0$ such that $\|\mathbf{e}_k\| \le M e^{-\omega k}$ for all $k \in \mathbb{N}_0$.*

(c) *If $(A(\cdot), B(\cdot), C(\cdot), D(\cdot), F(\cdot))$ are perturbed to $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot), \tilde{F}(\cdot)) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error $\mathbf{e}_k$ decays to zero at an exponential rate.*

Part (b) of Definition 3.1 and the stability of the closed-loop system are preserved under perturbations for which the norms $\|\mathbf{A} - \tilde{\mathbf{A}}\|$, $\|\mathbf{B} - \tilde{\mathbf{B}}\|$, $\|\mathbf{C} - \tilde{\mathbf{C}}\|$, and $\|\mathbf{D} - \tilde{\mathbf{D}}\|$ are small enough. Theorem 4.1 in Section 4 relates the sizes of the above norms to the sizes of the perturbations in the parameters $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of the original plant.

We will now show that the robust output regulation problem can be solved with an infinite-dimensional controller on the Hilbert space $Z = Y = L^2(0, \tau; \mathbb{C}^p)$ if we choose the parameters as $\mathcal{G}_1 = I$ and $\mathcal{G}_2 = \varepsilon I$ with $\varepsilon > 0$. With these choices the controller (5) becomes

$$z_{k+1} = z_k + \varepsilon \mathbf{e}_k \qquad z_0 = z^0 \in Z$$
$$\mathbf{u}_k = K z_k.$$

The sole frequency $\mu = 1$ of the lifted exosystem (4) is an eigenvalue of $\mathcal{G}_1$ with an infinite-dimensional multiplicity. This can be interpreted as the controller incorporating an infinite-dimensional internal model of the exosystem's dynamics. The following theorem shows that this property is indeed sufficient for the controller to solve the robust output regulation problem.

Theorem 3.1. *If we choose $Z = Y$, $\mathcal{G}_1 = I \in \mathcal{L}(Y)$, $\mathcal{G}_2 = \varepsilon I \in \mathcal{L}(Y)$, and*

$$K = -\mathbf{P}(1)^{-1} \in \mathcal{L}(Y,U),$$

then there exists $\varepsilon^* > 0$ such that for any $0 < \varepsilon \le \varepsilon^*$ the controller (5) solves the robust output regulation problem.

The choices of the controller parameters are based on the following characterization of controllers that solve the robust output regulation problem. The result is a discrete-time analogue of [16, Thm. 5.1].

Theorem 3.2. *A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ that stabilizes the closed-loop system solves the robust output regulation problem if and only if for all perturbations $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}, \tilde{\mathbf{F}}) \in \mathcal{O}$ for which the closed-loop system is stable the equations*

$$\text{(6a)} \qquad \tilde{\mathbf{P}}(1) K z = -\tilde{\mathbf{F}}$$
$$\text{(6b)} \qquad (I - \mathcal{G}_1) z = 0$$

have a solution $z \in Z$.

Proof. The proof is analogous to the proofs of [15, Thm. 4] and [16, Thm. 5.1]. □

Proof of Theorem 3.1. In our controller (5) we have $\mathcal{G}_1 = I \in \mathcal{L}(Y)$ and $\mathcal{G}_2 = \varepsilon I$. Thus $I - \mathcal{G}_1 = 0$, and the equations (6) have a solution if and only if there exists $z \in Z$ such that

$$\tilde{\mathbf{P}}(1)Kz = -\tilde{\mathbf{F}}.$$

If we choose $K = \mathbf{P}(1)^{-1}$ as suggested, then for any perturbations $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot), \tilde{F}(\cdot)) \in \mathcal{O}$

$$\tilde{\mathbf{P}}(1)Kz = \tilde{\mathbf{P}}(1)\mathbf{P}(1)^{-1}z = -\tilde{\mathbf{F}}$$

with the choice $z = -\mathbf{P}(1)\tilde{\mathbf{P}}(1)^{-1}\mathbf{F} \in Y = Z$.

It remains to show that the closed-loop system is exponentially stable. If we define a similarity transformation $Q = \begin{pmatrix} I & 0 \\ H & -I \end{pmatrix}$ with $H = \varepsilon \mathbf{C}(\mathbf{A} - I)^{-1}$, then

$$\begin{aligned} QA_eQ^{-1} &= \begin{pmatrix} I & 0 \\ H & -I \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B}K \\ \varepsilon\mathbf{C} & I + \varepsilon\mathbf{D}K \end{pmatrix} \begin{pmatrix} I & 0 \\ H & -I \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} + \varepsilon\mathbf{B}K\mathbf{C}(\mathbf{A} - I)^{-1} & -\mathbf{B}K \\ -\varepsilon^2\mathbf{P}(1)K\mathbf{C}(\mathbf{A} - I)^{-1} & I + \varepsilon\mathbf{P}(1)K \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} - \varepsilon\mathbf{B}\mathbf{P}(1)^{-1}\mathbf{C}(\mathbf{A} - I)^{-1} & \mathbf{B}\mathbf{P}(1)^{-1} \\ 0 & (1-\varepsilon)I \end{pmatrix} \\ &\quad + \varepsilon^2 \begin{pmatrix} 0 & 0 \\ \mathbf{C}(\mathbf{A} - I)^{-1} & 0 \end{pmatrix}. \end{aligned}$$

From this form we see that for a small enough $\varepsilon > 0$ the closed-loop system is exponentially stable. In particular, there exists $\varepsilon^* > 0$ such that $\sigma(A_e) \subset \mathbb{D}$ whenever $0 < \varepsilon \leq \varepsilon^*$. □

4 Analysis of the Perturbations

In this section we compare the sizes of the perturbations in the parameters $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ to the sizes of the perturbations in the lifted system's operators $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. In particular we show that if the perturbations in the original system are small, then the same is true for the perturbations in the lifted system. It should be noted that the results in this section only represent one particular way in which such estimates can be carried out. The norms in which the changes in $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ are measured can easily be changed to best suit the requirements of the application at hand.

We assume the following.

- $\tilde{A}(\cdot) = A(\cdot) + \Delta_A(\cdot)$ where $\Delta_A(\cdot) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^{n\times n})$ is τ-periodic. The perturbation is measured in the norm $\|\Delta_A\|_{L^1} = \int_0^\tau \|\Delta_A(t)\|_{\mathbb{C}^{n\times n}} dt$.
- We let

$$\begin{aligned} \tilde{B}(\cdot) &= B(\cdot) + \Delta_B(\cdot), \\ \tilde{C}(\cdot) &= C(\cdot) + \Delta_C(\cdot), \\ \tilde{D}(\cdot) &= D(\cdot) + \Delta_D(\cdot), \end{aligned}$$

where $\Delta_B(\cdot) \in C_\tau(\mathbb{R}, \mathbb{C}^{n\times m})$, $\Delta_C(\cdot) \in C_\tau(\mathbb{R}, \mathbb{C}^{p\times n})$, $\Delta_D(\cdot) \in C_\tau(\mathbb{R}, \mathbb{C}^{p\times m})$. The sizes of the perturbations are measured in the norms $\|\Delta_B\|_\infty = \max_{0\leq t\leq \tau}\|\Delta_B(t)\|$, and $\|\Delta_C\|_\infty$ and $\|\Delta_D\|_\infty$ (defined analogously).

THEOREM 4.1. *Assume that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the robust output regulation problem. There exists $\delta > 0$ such that if $\|\Delta_A\|_{L^1} < \delta$, $\|\Delta_B\|_\infty < \delta$, $\|\Delta_C\|_\infty < \delta$, and $\|\Delta_D\|_\infty < \delta$, then*

(a) $\tilde{\mathbf{P}}(1)$ *is boundedly invertible*

(b) *The perturbed closed-loop system is exponentially stable.*

Since parts (a) and (b) of Theorem 4.1 are satisfied provided that $\|\mathbf{A} - \tilde{\mathbf{A}}\|$, $\|\mathbf{B} - \tilde{\mathbf{B}}\|$, $\|\mathbf{C} - \tilde{\mathbf{C}}\|$, and $\|\mathbf{D} - \tilde{\mathbf{D}}\|$ are small enough, it is sufficient to show that these four norms can be made arbitrarily small by making $\|\Delta_A\|_{L^1}$, $\|\Delta_B\|_\infty$, $\|\Delta_C\|_\infty$, and $\|\Delta_D\|_\infty$ sufficiently small.

Estimating $\|\tilde{\mathbf{A}} - \mathbf{A}\|$: By definition

$$\|\tilde{\mathbf{A}} - \mathbf{A}\| = \|\Phi_{\tilde{A}}(\tau, 0) - \Phi_A(\tau, 0)\| = \|\Delta_\Phi(\tau, 0)\|_{\mathbb{C}^{n\times n}}$$

where we have denoted $\Delta_\Phi(t,s) = \Phi_{\tilde{A}}(t,s) - \Phi_A(t,s)$ for $0 \leq s \leq t \leq \tau$. In the following we will show that $\|\Delta_\Phi(\cdot,\cdot)\|_\infty = \sup_{0\leq s\leq t\leq\tau}\|\Delta_\Phi(t,s)\|$ can be made arbitrarily small by making $\|\Delta_A\|_{L^1}$ small. Let $t \geq s$ and $x_s \in X$ be such that $\|x_s\| = 1$. Then

$$\begin{aligned} \Delta_\Phi(t,s)x_s &= \Phi_{\tilde{A}}(t,s)x_s - \Phi_A(t,s)x_s \\ &= \int_s^t \Phi_A(t,r)\Delta_A(r)\Phi_{\tilde{A}}(r,s)x_s dr \\ &= \int_s^t \Phi_A(t,r)\Delta_A(r)\Phi_A(r,s)x_s dr \\ &\quad + \int_s^t \Phi_A(t,r)\Delta_A(r)\Delta_\Phi(r,s)x_s dr. \end{aligned}$$

Thus

$$\begin{aligned} \|\Delta_\Phi(t,s)\| &\leq \int_s^t \|\Phi_A(t,r)\|\|\Delta_A(r)\|\|\Phi_A(r,s)\|dr \\ &\quad + \int_s^t \|\Phi_A(t,r)\|\|\Delta_A(r)\|\|\Delta_\Phi(r,s)\|dr \\ &\leq \|\Phi_A\|_\infty^2\|\Delta_A\|_{L^1} + \|\Phi_A\|_\infty \int_s^t \|\Delta_A(r)\|\|\Delta_\Phi(r,s)\|dr \end{aligned}$$

where $\|\Phi_A\|_\infty = \sup_{0\leq s\leq t\leq\tau}\|\Phi_A(t,s)\|$. Grönwall's lemma implies that

$$\|\Delta_\Phi\|_\infty \leq \|\Phi_A\|_\infty^2\|\Delta_A\|_{L^1}e^{\|\Phi_A\|_\infty\|\Delta_A\|_{L^1}},$$

which in turn concludes that $\|\Delta_\Phi(\cdot,\cdot)\|_\infty$ can be made arbitrarily small by making $\|\Delta_A\|_{L^1}$ small.

Estimating $\|\tilde{\mathbf{B}} - \mathbf{B}\|$: For every $\mathbf{u} \in L^2(0,\tau;U)$

$$
\begin{aligned}
\|\tilde{\mathbf{B}}\mathbf{u} - \mathbf{B}\mathbf{u}\|_X &= \|\int_0^\tau \Phi_A(\tau,s)\Delta_B(s)\mathbf{u}(s)ds\|_X \\
&\le \int_0^\tau \|\Phi_A(\tau,s)\|\|\Delta_B(s)\|\|\mathbf{u}(s)\|ds \\
&\le \|\Delta_B\|_\infty \|\Phi_A(\tau,\cdot)\|_{L^2}\|\mathbf{u}\|_{L^2},
\end{aligned}
$$

which implies $\|\tilde{\mathbf{B}} - \mathbf{B}\|_X \le \|\Delta_B\|_\infty \|\Phi_A(\tau,\cdot)\|_{L^2}$.

Estimating $\|\tilde{\mathbf{C}} - \mathbf{C}\|$: For every $\mathbf{x} \in X$

$$
\begin{aligned}
\|\tilde{\mathbf{C}}\mathbf{x} - \mathbf{C}\mathbf{x}\|_{L^2}^2 &= \int_0^\tau \|\Delta_C(t)\Phi_A(t,0)\mathbf{x}\|_X^2 dt \\
&\le \|\Delta_C\|_\infty^2 \|\Phi_A(\cdot,0)\|_{L^2}^2 \|\mathbf{x}\|_X^2
\end{aligned}
$$

and thus $\|\tilde{\mathbf{C}} - \mathbf{C}\|_{L^2} \le \|\Delta_C\|_\infty \|\Phi_A(\cdot,0)\|_{L^2}$.

Estimating $\|\tilde{\mathbf{D}} - \mathbf{D}\|$: For all continuous functions f_Φ, f_B, f_C, where $(t,s) \mapsto f_\Phi(t,s)$ is defined on $\{ (t,s) \mid 0 \le s \le t \le \tau \} \subset [0,\tau] \times [0,\tau]$,

$$
\begin{aligned}
&\|f_C(\cdot) \int_0^\cdot f_\Phi(\cdot,s) f_B(s)\mathbf{u}(s)ds\|_{L^2}^2 \\
&= \int_0^\tau \|f_C(t) \int_0^t f_\Phi(t,s) f_B(s)\mathbf{u}(s)ds\|^2 dt \\
&\le \|f_C(\cdot)\|_\infty^2 \int_0^\tau \left(\int_0^t \|f_\Phi(t,s)\|\|f_B(s)\|\|\mathbf{u}(s)ds\| \right)^2 dt \\
&\le \|f_C(\cdot)\|_\infty^2 \|f_B(\cdot)\|_\infty^2 \|\mathbf{u}\|_{L^2}^2 \int_0^\tau \int_0^t \|f_\Phi(t,s)\|^2 ds dt \\
&= \|f_C(\cdot)\|_\infty^2 \|f_B(\cdot)\|_\infty^2 \|\mathbf{u}\|_{L^2}^2 \|f_\Phi(\cdot,\cdot)\|_{L^2}^2.
\end{aligned}
$$

Now for every $\mathbf{u} \in L^2(0,\tau;\mathbb{C}^m)$ we have

$$
\begin{aligned}
&\|\tilde{\mathbf{D}}\mathbf{u} - \mathbf{D}\mathbf{u}\|_{L^2} = \|\tilde{C}(\cdot) \int_0^\cdot \Phi_{\tilde{A}}(\cdot,s)\tilde{B}(s)\mathbf{u}(s)ds \\
&\qquad - C(\cdot) \int_0^\cdot \Phi_A(\cdot,s)B(s)\mathbf{u}(s)ds + \Delta_D(\cdot)\mathbf{u}(\cdot)\|_{L^2} \\
&= \|\Delta_D(\cdot)\mathbf{u}(\cdot) + \tilde{C}(\cdot) \int_0^\cdot \Phi_{\tilde{A}}(\cdot,s)\Delta_B(s)\mathbf{u}(s)ds \\
&\qquad + \tilde{C}(\cdot) \int_0^\cdot \Delta_\Phi(\cdot,s)B(s)\mathbf{u}(s)ds \\
&\qquad + \Delta_C(\cdot) \int_0^\cdot \Phi_A(\cdot,s)B(s)\mathbf{u}(s)ds\|_{L^2} \\
&\le \|\Delta_D\|_\infty \|\mathbf{u}\|_{L^2} + \|\tilde{C}(\cdot)\int_0^\cdot \Phi_{\tilde{A}}(\cdot,s)\Delta_B(s)\mathbf{u}(s)ds\|_{L^2} \\
&\quad + \|\tilde{C}(\cdot) \int_0^\cdot \Delta_\Phi(\cdot,s)B(s)\mathbf{u}(s)ds\|_{L^2} \\
&\quad + \|\Delta_C(\cdot) \int_0^\cdot \Phi_A(\cdot,s)B(s)\mathbf{u}(s)ds\|_{L^2}
\end{aligned}
$$

This together with the earlier estimate implies

$$
\begin{aligned}
\|\tilde{\mathbf{D}} - \mathbf{D}\| &\le \|\Delta_D\|_\infty + \|\tilde{C}(\cdot)\|_\infty \|\Delta_B(\cdot)\|_\infty \|\Phi_{\tilde{A}}(\cdot,\cdot)\|_{L^2} \\
&\quad + \|\tilde{C}(\cdot)\|_\infty \|B(\cdot)\|_\infty \|\Delta_\Phi(\cdot,\cdot)\|_{L^2} \\
&\quad + \|\Delta_C(\cdot)\|_\infty \|B(\cdot)\|_\infty \|\Phi_A(\cdot,\cdot)\|_{L^2}.
\end{aligned}
$$

The norm $\|\tilde{\mathbf{D}} - \mathbf{D}\|$ can be further estimated by using $\|\tilde{C}(\cdot)\|_\infty \le \|C(\cdot)\|_\infty + \|\Delta_C\|_\infty$, $\|\Phi_{\tilde{A}}(\cdot,\cdot)\|_{L^2} \le \|\Phi_A(\cdot,\cdot)\|_{L^2} + \|\Delta_\Phi\|_{L^2}$, and by using the fact that $\|\Delta_\Phi\|_{L^2} \le \frac{\tau^2}{2}\|\Delta_\Phi\|_\infty$ can be made small by making $\|\Delta_A\|_{L^1}$ sufficiently small.

5 Example

As an example, we consider a periodic scalar plant

$$
\begin{aligned}
\dot{x}(t) &= a(t)x(t) + u(t), \qquad x(0) = x_0 \in \mathbb{C} \\
y(t) &= x(t) + du(t),
\end{aligned}
$$

where $d > 0$ and $a(\cdot) \in C_\tau(\mathbb{R},\mathbb{C})$ with $\tau = 2\pi$ is such that

$$
a(t) = \begin{cases} -1 & 0 \le t < \pi \\ -2 & \pi \le t < 2\pi. \end{cases}
$$

The system is periodic with period $\tau = 2\pi$, and exponentially stable, since the fundamental matrix $\Phi_A(t,s) = e^{\int_s^t a(r)dr}$ satisfies $|\Phi_A(2\pi,0)| = |e^{-3\pi}| < 1$.

Our aim is to design a controller to achieve robust output tracking of a reference signal $y_{ref} \in C_\tau(\mathbb{R},\mathbb{C})$,

$$
y_{ref}(t) = \begin{cases} t - \pi/2 & 0 \le t < \pi \\ 3\pi/2 - t & \pi \le t < 2\pi. \end{cases}
$$

The function y_{ref} is depicted in Figure 1.

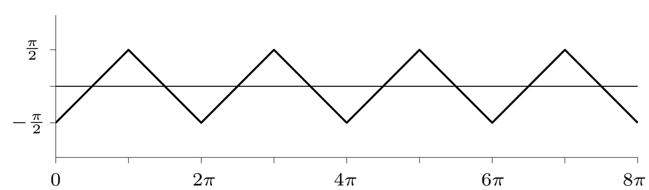

Figure 1: The reference signal y_{ref}.

Our first task is to find the inverse $\mathbf{P}(1)^{-1}$ of the transfer function of the plant at $\mu = 1$. The operators of the lifted plant are then given by

$$
\begin{aligned}
\mathbf{A}\mathbf{x} &= \Phi_A(2\pi,0)\mathbf{x} = e^{-3\pi}\mathbf{x} \\
\mathbf{B}\mathbf{u} &= \int_0^{2\pi} \Phi_A(2\pi,s)B(s)\mathbf{u}(s)ds = \int_0^{2\pi} \Phi_A(2\pi,s)\mathbf{u}(s)ds \\
(\mathbf{C}\mathbf{x})(t) &= C(t)\Phi_A(t,0)\mathbf{x} = \Phi_A(t,0)\mathbf{x} \\
(\mathbf{D}\mathbf{u})(t) &= D(t)\mathbf{u}(t) + C(t) \int_0^t \Phi_A(t,s)B(s)\mathbf{u}(s)ds \\
&= d\mathbf{u}(t) + \int_0^t \Phi_A(t,s)\mathbf{u}(s)ds.
\end{aligned}
$$

For $\mu \neq e^{-3\pi}$ the transfer function $\mathbf{P}(\mu)$ is therefore determined by $u \mapsto \mathbf{P}(1)\mathbf{u} = \mathbf{y}$ in such a way that for $t \in [0, 2\pi]$

$$\begin{aligned}\mathbf{y}(t) &= (\mathbf{P}(\mu)\mathbf{u})(t) = (\mathbf{C}(\mu I - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{u})(t) \\ &= \frac{\Phi_A(t,0)}{\mu - e^{-3\pi}} \int_0^{2\pi} \Phi_A(2\pi, s)\mathbf{u}(s)ds + d\mathbf{u}(t) \\ &\quad + \int_0^t \Phi_A(t,s)\mathbf{u}(s)ds \\ &= d\mathbf{u}(t) + \frac{1}{\mu - e^{-3\pi}} \int_0^{2\pi} K_F(t,s)\mathbf{u}(s)ds \\ &\quad + \int_0^t K_V(t,s)\mathbf{u}(s)ds\end{aligned}$$

where

$$\begin{aligned}K_F(t,s) &= \Phi_A(t,0)\Phi_A(2\pi,s) \\ K_V(t,s) &= \Phi_A(t,s)\end{aligned}$$

Computing the inverse $\mathbf{P}(1)^{-1}$ is equivalent to finding the formula for $\mathbf{u} \in L^2(0, 2\pi)$ such that $\mathbf{P}(1)\mathbf{u} = \mathbf{y}$ for a given $\mathbf{y} \in L^2(0, 2\pi)$. Since $d \neq 0$, this is equivalent to solving the equation

$$\text{(7a)} \quad \mathbf{u}(t) = \frac{\mathbf{y}(t)}{d} - \frac{1}{d(1 - e^{-3\pi})} \int_0^{2\pi} K_F(t,s)\mathbf{u}(s)ds$$

$$\text{(7b)} \quad - \frac{1}{d} \int_0^t K_V(t,s)\mathbf{u}(s)ds$$

on $[0, 2\pi]$. This is a Volterra–Fredholm integral equation of the second kind. The equation can be solved numerically, for example, using the Adomian decomposition method [17, Sec. 8.2.2] where we define

$$\begin{aligned}\mathbf{u}(t) &= \sum_{l=0}^{\infty} u_l(t), \qquad u_0(t) = \frac{\mathbf{y}(t)}{d} \\ u_{l+1}(t) &= -\frac{1}{d(1 - e^{-3\pi})} \int_0^{2\pi} K_F(t,s)u_l(s)ds \\ &\quad - \frac{1}{d} \int_0^t K_V(t,s)u_l(s)ds.\end{aligned}$$

In particular, the solution $\mathbf{u} = \mathbf{P}(1)^{-1}\mathbf{y}$ can be approximated using a truncated series $\mathbf{u}(t) \approx \sum_{l=0}^{N_A} u_l(t)$ for some $N_A \in \mathbb{N}$. The choice of K was made to achieve exponential closed-loop stability. The standard perturbation theory therefore implies that if $\mathbf{P}(1)^{-1}$ is replaced with sufficiently accurate approximation, the closed-loop stability is preserved.

In this example we replace the infinite-dimensional controller in Theorem 3.1 with a finite-dimensional approximation on $\mathbb{C}^{N_Z}$ with $N_Z = 21$. The elements of the infinite-dimensional spaces $U = Y = L^2(0, 2\pi)$ are approximated with truncated Fourier series expansions

$$\begin{aligned}y(\cdot) &= \sum_{|l| \le n_Y} \langle y(\cdot), \phi_l \rangle_{L^2} \phi_l(\cdot) \\ u(\cdot) &= \sum_{|l| \le n_U} \langle u(\cdot), \phi_l \rangle_{L^2} \phi_l(\cdot),\end{aligned}$$

where $\phi_l(\cdot) = \frac{1}{\sqrt{2\pi}} e^{il\cdot}$. For the simulation we chose $n_U = n_Y = 20$. The operator $K = \mathbf{P}(1)^{-1}$ is approximated by solving (7) with the Adomian decomposition method with truncation parameter $N_A = 2$ (due to the approximations of U and Y it suffices to compute $\mathbf{P}(1)^{-1}\phi_l$ for $|l| \le 20$). Finally, we chose $\varepsilon = 0.2$.

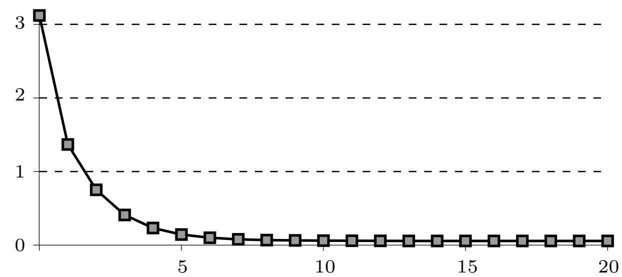

Figure 2: The regulation error $\mathbf{e}_k$.

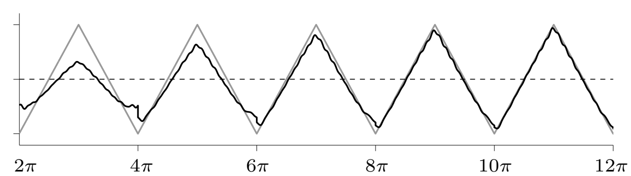

Figure 3: The output $y(t)$.

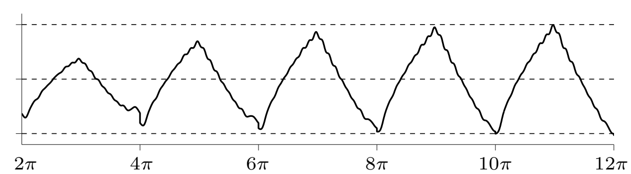

Figure 4: The control input $u(t)$.

Figure 2 depicts the simulated regulation error $\mathbf{e}_k$ for iterations $k = 0, \ldots, 20$ with initial state $x_0 = 2$ of the plant, and with a random initial state of the controller (uniform distribution over the interval $[-1/2, 1/2]$). The real parts of the output $y(t)$ and the control law $u(t)$ are plotted in Figures 3 and 4, respectively, for $2\pi \le t \le 18\pi$. Due to the fact that the controller is an approximation of the infinite-dimensional controller, the solution of the output regulation problem is not exact. However, the controller does achieve approximate tracking of the reference signal.

6 Conclusions

In this paper we have considered robust output tracking of periodic continuous-time systems. The main tools in the analysis are the lifting of the time-dependent systems to autonomous discrete-time systems, and the generalization of the internal model principle for systems with infinite-dimensional input and output spaces.

By definition, the statement of the robust output regulation problem for the lifted system allows a very wide class of perturbations in the parameters $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of the plant. In practical applications it may be that requiring robustness with respect to all perturbations preserving the closed-loop system is unnecessary, and instead the relevant perturbations have very specific forms. This motivates further study of the necessity of an infinite-dimensional internal model for robustness in situations where the class of admissible uncertainties is significantly smaller than the class $\mathcal{O}$ in Definition 3.1. Theorem 3.2 which states a necessary condition for robustness of a controller with respect to any given class of perturbations is a natural starting point for gaining deeper understanding of the properties of robust controllers. In particular, possibilities for reducing the size of the internal model can be approached similarly as in the references [15, 13].

Other important topics for further research include developing suitable ways of approximating the infinite-dimensional controller. In addition, relaxing the standing assumption on the invertibility of $\mathbf{P}(1)$ would make the results available for a larger class of periodic systems.

References

[1] B. A. Bamieh and J. B. Pearson, Jr. A general framework for linear periodic systems with applications to H^∞ sampled-data control. *IEEE Trans. Automat. Control*, 37(4):418–435, 1992.

[2] S. Bittanti and P. Colaneri. Invariant representations of discrete-time periodic systems. *Automatica J. IFAC*, 36(12):1777–1793, 2000.

[3] E. J. Davison. The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Trans. Automat. Control*, 21(1):25–34, 1976.

[4] B. A. Francis and W. M. Wonham. The internal model principle for linear multivariable regulators. *Appl. Math. Optim.*, 2(2):170–194, 1975.

[5] O.M. Grasselli, S. Longhi, A. Tornambé, and P. Valigi. Robust output regulation and tracking for linear periodic systems under structured uncertainties. *Automatica J. IFAC*, 32(7):1015–1019, 1996.

[6] O.M. Grasselli and S. Longhi. The geometric approach for linear periodic discrete-time systems. *Linear Algebra Appl.*, 158(0):27–60, 1991.

[7] O.M. Grasselli and S. Longhi. Robust tracking and regulation of linear periodic discrete-time systems. *Internat. J. Control*, 54(3):613–633, 1991.

[8] L. B. Jemaa and E. J. Davison. Performance limitations in the robust servomechanism problem for discrete time periodic systems. *Automatica J. IFAC*, 39(6):1053–1059, 2003.

[9] A. Langari. *Sampled-Data Repetitive Control Systems.* PhD thesis, University of Toronto, 1997.

[10] R. Meyer and C. S. Burrus. A unified analysis of multirate and periodically time-varying digital filters. *IEEE Trans. Circuits Syst.*, 22(3):162–168, 1975.

[11] L. Mirkin and Z.J. Palmor. A new representation of the parameters of lifted systems. *IEEE Trans. Automat. Control*, 44(4):833–840, 1999.

[12] M. Nagahara and Y. Yamamoto. Robust repetitive control by sampled-data H^∞-filters. In *Proceedings of the 48th IEEE Conference on Decision and Control held jointly with the 28th Chinese Control Conference*, pages 8136–8141, Dec. 16–18, 2009.

[13] L. Paunonen. Designing controllers with reduced order internal models. *IEEE Trans. Automat. Control*, 60(3):775–780, 2015.

[14] L. Paunonen and S. Pohjolainen. Internal model theory for distributed parameter systems. *SIAM J. Control Optim.*, 48(7):4753–4775, 2010.

[15] L. Paunonen and S. Pohjolainen. Reduced order internal models in robust output regulation. *IEEE Trans. Automat. Control*, 58(9):2307–2318, 2013.

[16] L. Paunonen and S. Pohjolainen. The internal model principle for systems with unbounded control and observation. *SIAM J. Control Optim.*, 52(6):3967–4000, 2014.

[17] A.M. Wazwaz. *Linear and Nonlinear Integral Equations, Methods and Applications.* Springer Berlin Heidelberg, 2011.