Nonautonomous Controllers and Output Regulation of Unknown Harmonic Signals for Regular Linear Systems

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We introduce general results on well-posedness and output regulation of regular linear systems with nonautonomous controllers. We present a generalization of the internal model principle for time-dependent controllers with asymptotically converging parameters. This general result is utilised in controller design for output tracking and disturbance rejection of harmonic signals with unknown frequencies. Our controller can be flexibly combined with different frequency estimation methods. The results are illustrated in rejection of unknown harmonic disturbances for a one-dimensional boundary controlled heat equation.

***Index Terms*—Distributed parameter systems, output regulation, time varying systems, uncertain systems.**

I. Introduction

Asymptotic output tracking and disturbance rejection, jointly called output regulation, is an important control objective in many engineering applications. For a given reference signal $y_{\mathit{ref}}(t)$ and a class of external disturbance signals $w_{\mathit{dist}}(t)$ the output $y(t)$ of the controlled system is required to satisfy

$$\|y(t) - y_{\mathit{ref}}(t)\| \to 0, \qquad \text{as} \quad t \to \infty.$$

This control problem has been studied extensively for infinite-dimensional systems [6], [16], [17], [21], [24], [27], [30] and controlled partial differential equations (PDEs) [2], [12], [18].

In the classical output regulation problem the reference and disturbance signals are assumed to have the forms

$$y_{\mathit{ref}}(t) = a_0 + \sum_{k=1}^{q} (a_k \cos(\omega_k t) + b_k \sin(\omega_k t)) \tag{1a}$$

$$w_{\mathit{dist}}(t) = c_0 + \sum_{k=1}^{q} (c_k \cos(\omega_k t) + d_k \sin(\omega_k t)), \tag{1b}$$

where the frequencies $0 = \omega_0 < \omega_1 < \ldots < \omega_q$ are assumed to be known and the amplitudes $(a_k)_{k=0}^q$, $(b_k)_{k=1}^q$, $(c_k)_{k=0}^q$, and $(d_k)_{k=1}^q$ may be unknown. In this paper we study a more challenging version of the control problem where also the frequencies $(\omega_k)_{k=1}^q$ are *unknown*. For finite-dimensional linear and nonlinear systems the case of unknown frequencies has been studied in [8], [20], [31], [42], [43] using adaptive internal models. Controllers have also been designed for selected PDE models, in particular, 2×2 hyperbolic systems [2], a Kirchoff plate [28], and a 1D boundary controlled heat equation [15]. Our focus is on output regulation for linear distributed parameter systems, and for such systems our problem has only been studied in [38], [39] under restrictive structural assumptions. We study a considerably larger class of systems, namely *regular linear systems* [32], [40]

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{\mathit{dist}}(t), \qquad x(0) = x_0 \tag{2a}$$

$$y(t) = C_\Lambda x(t) + Du(t) + D_d w_{\mathit{dist}}(t), \tag{2b}$$

on a Hilbert space X (see Section II for detailed assumptions). Regular linear systems can be used in controller design for a wide range of PDE models with boundary control and observation, e.g., one-dimensional convection-diffusion equations, wave equations, beam equations, as well as multi-dimensional heat equations [5].

As our ultimate contribution we introduce a controller design method for output regulation of signals (1) with unknown frequencies, amplitudes and phases. In particular, we introduce a dynamic error feedback controller with a *time-varying internal model* based on estimates $(\hat{\omega}_k(t))_{k=1}^q$ of the frequencies in (1). The controller design leads to a *nonautonomous* dynamic error feedback controller

$$\dot{z}(t) = \mathcal{G}_1(t)z(t) + \mathcal{G}_2(t)(y(t) - y_{\mathit{ref}}(t)), \quad z(0) = z_0 \tag{3a}$$

$$u(t) = K(t)z(t) \tag{3b}$$

on a Hilbert space Z. Here $\mathcal{G}_2(\cdot) \in L^\infty(0,\infty;\mathcal{L}(\mathbb{C}^p, Z))$ and $K(\cdot) \in L^\infty(0,\infty;\mathcal{L}(Z,\mathbb{C}^m))$ and $\mathcal{G}_1(t)$ may contain an unbounded time-varying part (see Assumption II.2). The analysis of well-posedness of the closed-loop system consisting of (2) and (3) is highly nontrivial for unbounded operators B and C. As our first main result we prove that the closed-loop system has a well-defined mild state and output determined by bounded input/output maps. We achieve this result by expressing the time-varying closed-loop system as a *nonautonomous output feedback* of an autonomous regular linear system and by employing the nonautonomous feedback theory developed by Schnaubelt in [29]. Besides output regulation, the well-posedness result in Section III is also applicable in the study of other control problems with nonautonomous controllers.

In Section IV we introduce general theory for output regulation in the situation where the controller parameters $(\mathcal{G}_1(t), \mathcal{G}_2(t), K(t))$ — especially the internal model — converge to a limit $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ as $t \to \infty$. As our main result we show that if the autonomous "limit controller" $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ contains an internal model [14], [26] of the true frequencies $(\omega_k)_k$ and the closed-loop system is exponentially stable, then the controller achieves output regulation.

In Section V we introduce our controller for output regulation of signals (1) with unknown frequencies. We begin

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The research was supported by the Academy of Finland Grants number 298182, 310489, and 349002 held by L. Paunonen.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3324523

by introducing a general controller structure with a time-varying internal model and an observer part for closed-loop stabilization. The controller also includes an auxiliary output $y_{aux}(t)$ for estimation of the frequencies $(\omega_k)_k$ in (1). One of the key features of our controller is that $y_{aux}(t)$ is by design independent of the time-varying parts of the controller, and therefore the convergence of the frequency estimates in the internal model can be completed *separately* of the analysis of the closed-loop dynamics. In particular, our controller is not restricted to a single estimation method, but can instead be combined with any method which can identify $(\omega_k)_k$ based on the output $y_{aux}(t)$. As the final part of the controller design we present an online tuning algorithm for the stabilization of the nonautonomous closed-loop system and for guaranteeing the output tracking. In Section V-C we analyse the robustness properties of the controller with respect to perturbations in the system (A,B,C,D). The detailed robustness properties depend on the chosen frequency estimation method due to the effect of the perturbations on $y_{aux}(t)$. Our main result shows that for sufficiently long update intervals our controller achieves approximate output regulation despite small perturbations provided that the frequency estimates approximate the true frequencies $(\omega_k)_k$ with sufficient accuracy. We illustrate the controller design in Section VI in adaptive output regulation for a boundary controlled heat equation with uncertainty.

The main difference compared to the references [2], [15], [28] which have studied control of individual PDE models is that our results are applicable for abstract regular linear systems and various PDEs within this class. Moreover, in [2] $(\omega_k)_{k=1}^q$ are estimated from the tracking error signal. Our use of the auxiliary output $y_{aux}(t)$ is inspired by the "residual generator" in [8, Sec. 4] (a similar signal is used in [15]). A preliminary version of Theorem IV.2 was presented in [1].

Notation. If X and Y are Hilbert spaces, then the space of bounded linear operators $A:X\to Y$ is denoted by $\mathcal{L}(X,Y)$. The domain, kernel, and range of $A:\mathcal{D}(A)\subset X\to Y$ are denoted by $\mathcal{D}(A)$, $\mathcal{N}(A)$, and $\mathcal{R}(A)$, respectively. The resolvent operator of $A:\mathcal{D}(A)\subset X\to X$ is defined as $R(\lambda,A)=(\lambda I-A)^{-1}$ for those $\lambda\in\mathbb{C}$ for which the inverse is bounded. The inner product on X is denoted by $\langle\cdot,\cdot\rangle_X$. By $L^p(0,\tau;X)$ and $L^\infty(0,\tau;X)$ we denote, respectively, the spaces of p-integrable and essentially bounded measurable functions $f:(0,\tau)\to X$. For $f\in L^\infty(0,\infty;X)$ we denote $\|f(t)\|\to 0$ as $t\to\infty$ if $\operatorname{ess\,sup}_{s\geq t}\|f(s)\|\to 0$ as $t\to\infty$. If $A:\mathcal{D}(A)\subset X\to X$ generates a strongly continuous semigroup $T(t)$ on X, we define $X_1=\mathcal{D}(A)$ equipped with the graph norm of A and define X_{-1} as the completion of X with respect to the norm $\|x\|_{-1}:=\|(\lambda_0-A)^{-1}x\|_X$ for a fixed $\lambda_0\in\rho(A)$. Then A extends to an operator $X\to X_{-1}$ (also denoted by A) and this extension generates a semigroup (also denoted by $T(t)$) on X_{-1} [34, Sec. 2.10].

II. Preliminaries and Standing Assumptions

A. Background on regular linear systems

Let X, U, and Y be Hilbert spaces and consider

$$\dot{x}(t)=Ax(t)+Bu(t),\qquad x(0)=x_0\in X \tag{4a}$$
$$y(t)=C_\Lambda x(t)+Du(t) \tag{4b}$$

on X, where the operator $A:D(A)\subset X\to X$ is assumed to generate a strongly continuous semigroup $T(t)$ on X and $B\in\mathcal{L}(U,X_{-1})$, $C\in\mathcal{L}(X_1,Y)$ and $D\in\mathcal{L}(U,Y)$. Here $C_\Lambda:\mathcal{D}(C_\Lambda)\subset X\to Y$ is the Λ-extension of C such that $\mathcal{D}(C_\Lambda):=\{\,x\in X\mid \lim_{\lambda\to\infty}\lambda CR(\lambda,A)x \text{ exists}\,\}$ and

$$C_\Lambda x=\lim_{\lambda\to\infty}\lambda CR(\lambda,A)x,\qquad \forall x\in\mathcal{D}(C_\Lambda).$$

The operator B is *an admissible input operator* for the semigroup $T(t)$ if $\int_0^\tau T(t-s)Bu(s)ds\in X$ for all $u\in L^2(0,\tau;U)$ and $\tau>0$ [35, Sec. 3]. Moreover, C is *an admissible output operator* for the semigroup $T(t)$ if there exist $\tau,\gamma>0$ such that $\|CT(\cdot)x\|_{L^2(0,\tau)}\leq\gamma\|x\|$ for all $x\in\mathcal{D}(A)$ [35, Sec. 3].

Assumption II.1. For some Hilbert spaces X, U and Y the operators (A,B,C) have the following properties.

(1) $A:\mathcal{D}(A)\subset X\to X$ generates a semigroup $T(t)$ on X.
(2) $B\in\mathcal{L}(U,X_{-1})$ and $C\in\mathcal{L}(X_1,Y)$ are admissible.
(3) $\mathcal{R}(R(\lambda_0,A)B)\subset\mathcal{D}(C_\Lambda)$ for some $\lambda_0\in\rho(A)$.
(4) $\sup_{\operatorname{Re}\lambda\geq\beta}\|C_\Lambda R(\lambda,A)B\|<\infty$ for some $\beta>0$.

If (A,B,C) satisfy Assumption II.1 and $D\in\mathcal{L}(U,Y)$, then (4) is a *regular linear system* by [35, Thm. 5.6] and its transfer function is given by $P(\lambda)=C_\Lambda R(\lambda,A)B+D$. In this situation we write "(A,B,C,D) is a regular linear system".

B. Assumptions on the system and the controller

We consider a regular linear system of the form (2) on a Hilbert space X, where $x(t)\in X$, $u(t)\in\mathbb{C}^m$, $y(t)\in\mathbb{C}^p$, and $w_{dist}(t)\in\mathbb{C}^{n_d}$ are the system's state, input, output, and external disturbance, respectively. In particular, the number of outputs of the system is $p\in\mathbb{N}$, and $B\in\mathcal{L}(\mathbb{C}^m,X_{-1})$, $B_d\in\mathcal{L}(\mathbb{C}^{n_d},X)$, $C\in\mathcal{L}(X_1,\mathbb{C}^p)$, $D_d\in\mathbb{C}^{p\times n_d}$. The operators $B_d\in\mathcal{L}(\mathbb{C}^{n_d},X)$ and $D_d\in\mathbb{C}^{p\times n_d}$ are allowed to be unknown. We assume that (A,B,C,D) satisfy Assumption II.1. Since $B_d\in\mathcal{L}(\mathbb{C}^{n_d},X)$, also $(A,[B,B_d],C,[D,D_d])$ is a regular linear system. The transfer function of (2) (from u to y) is denoted by $P(\lambda)=C_\Lambda R(\lambda,A)B+D$ for $\lambda\in\rho(A)$.

We make the following assumptions on the parameters of the dynamic error feedback controller (3).

Assumption II.2. For almost every $t\geq 0$ we have

$$\mathcal{G}_1(t)z=\mathcal{G}_1^\infty z+\mathcal{G}_{11}^\infty\Delta_{\mathcal{G}_1}(t)z$$
$$\mathcal{D}(\mathcal{G}_1(t))=\{\,z\in Z\mid \mathcal{G}_1^\infty z+\mathcal{G}_{11}^\infty\Delta_{\mathcal{G}_1}(t)z\in Z\,\},$$

where $\mathcal{G}_1^\infty:\mathcal{D}(\mathcal{G}_1^\infty)\subset Z\to Z$ generates a strongly continuous semigroup on Z, $\mathcal{G}_{11}^\infty\in\mathcal{L}(U_c,Z_{-1})$ for some Hilbert space U_c is an admissible input operator for this semigroup, and $\Delta_{\mathcal{G}_1}(\cdot)\in L^\infty(0,\infty;\mathcal{L}(Z,U_c))$. Moreover, $\mathcal{G}_2(\cdot)\in L^\infty(0,\infty;\mathcal{L}(\mathbb{C}^p,Z))$ and $K(\cdot)\in L^\infty(0,\infty;\mathcal{L}(Z,\mathbb{C}^m))$.

In Section V the controller will contain additional dynamics for estimation of $(\omega_k)_k$ in (1), but in our control scheme the convergence of the frequency estimates is analysed separately.

We can formally express the closed-loop system consisting of (2) and (3) with state $x_e(t)=[x(t),z(t)]^T\in X_e:=X\times Z$ and input $w_e(t)=[w_{dist}(t)^T,y_{ref}(t)^T]^T\in\mathbb{C}^{n_d+p}$ as

$$\dot{x}_e(t)=A_e(t)x_e(t)+B_e(t)w_e(t),\quad x_e(0)=x_{e0} \tag{5a}$$
$$e(t)=C_e(t)x_e(t)+D_ew_e(t) \tag{5b}$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3324523

where $e(t) = y(t) - y_{ref}(t)$, $D_e = [D_d,\ -I] \in \mathbb{C}^{p\times(n_d+p)}$,

$$A_e(t) = \begin{bmatrix} A & BK(t) \\ \mathcal{G}_2(t)C_\Lambda & \mathcal{G}_1(t) + \mathcal{G}_2(t)DK(t) \end{bmatrix}$$
$$B_e(t) = \begin{bmatrix} B_d & 0 \\ \mathcal{G}_2(t)D_d & -\mathcal{G}_2(t) \end{bmatrix},\ C_e(t) = \begin{bmatrix} C_\Lambda, DK(t) \end{bmatrix},$$

and $\mathcal{D}(A_e(t)) = \{\, [x,z]^T \in \mathcal{D}(C_\Lambda) \times \mathcal{D}(\mathcal{G}_1(t)) \mid Ax + BK(t)z \in X \,\}$. The existence of well-defined mild state $x_e(t)$ and output $e(t)$ of (5) are proved in Section III.

III. Well-Posedness of the Closed-Loop System

In this section we will prove that the closed-loop system is well-posed in the sense that for the initial state $x_{e0} = [x_0, z_0]^T \in X_e$ and for $w_e \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{n_d+p})$ the equations (5) have a well-defined mild state and output

$$x_e(t) = U_e(t,0)x_{e0} + \Phi_e^{t,0} w_e \tag{6a}$$
$$e(t) = (\Psi_e^0 x_{e0})(t) + (\mathbb{F}_e^0 w_e)(t), \tag{6b}$$

where $U_e(t,s)$ is a strongly continuous evolution family [29, Def. 2.1] and $(U_e, \Phi_e, \Psi_e, \mathbb{F}_e)$ is a *well-posed nonautonomous linear system* [29, Def. 3.6]. We will show that $U_e(t,s)$ is related to $(A_e(t))_{t\geq 0}$ through a natural perturbation formula and that the mappings

$$\Phi_e^{t,s} w_e = \int_s^t U_e(t,r)B_e(r)w_e(r)dr \tag{7a}$$
$$(\Psi_e^s x)(t) = C_e(t)U_e(t,s)x \tag{7b}$$
$$(\mathbb{F}_e^s w_e)(t) = C_e(t)\int_s^t U_e(t,r)B_e(r)w_e(r)dr + D_e w_e(t) \tag{7c}$$

are well-defined for all $s \geq 0$ and a.e. $t \geq s$ and have appropriate boundedness properties (see Theorem III.2). In particular, $x_e(\cdot) \in C([0,\infty); X_e)$ by [29, Def. 2.1 & Prop. 3.5(2)].

We prove the closed-loop well-posedness using the nonautonomous feedback theory in [29, Sec. 4]. More precisely, we will express the system (5) as a part of a system obtained from an autonomous regular linear system $(A_{eo}^\infty, B_{eo}^\infty, C_{eo}^\infty, D_{eo}^\infty)$ under a combination of (i) autonomous output feedback with feedback operator Δ_0, (ii) parallel interconnection with a feedthrough operator D_e^{add} and (iii) nonautonomous feedback with feedback operators $\Delta(t)$ (see Figure 1).

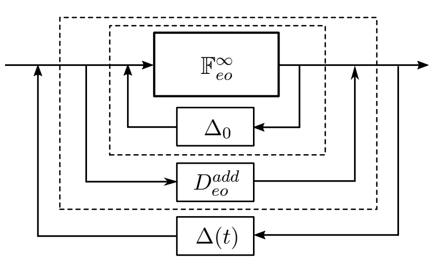

Fig. 1. The nonautonomous feedback structure.

We let $\mathcal{G}_2^\infty \in \mathcal{L}(\mathbb{C}^p, Z)$ and $K^\infty \in \mathcal{L}(Z, \mathbb{C}^m)$ and define $\Delta_{\mathcal{G}_2}(\cdot) = \mathcal{G}_2(\cdot) - \mathcal{G}_2^\infty \in L^\infty(0,\infty;\mathcal{L}(\mathbb{C}^p, Z))$ and $\Delta_K(\cdot) = K(\cdot) - K^\infty \in L^\infty(0,\infty;\mathcal{L}(Z,\mathbb{C}^m))$. We denote $U_{ee} = U \times U_d \times Y \times U_c \times Z$ and $Y_{ee} = Y \times U \times Z$ and define $\mathcal{D}(A_{eo}^\infty) = \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1^\infty)$, $\mathcal{D}(C_{eo}^\infty) = \mathcal{D}(C_\Lambda) \times Z$,

$$A_{eo}^\infty = \begin{bmatrix} A & 0 \\ 0 & \mathcal{G}_1^\infty \end{bmatrix}, \quad B_{eo}^\infty = \begin{bmatrix} B & B_d & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{G}_2^\infty & \mathcal{G}_{11}^\infty & I \end{bmatrix}$$
$$C_{eo}^\infty = \begin{bmatrix} C_\Lambda & 0 \\ 0 & K^\infty \\ 0 & I \end{bmatrix}, \qquad D_{eo}^\infty = \begin{bmatrix} D & D_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Our assumptions on $(A, [B, B_d], C, [D, D_d])$ and $\mathcal{G}_{11}^\infty$ imply that $(A_{eo}^\infty, B_{eo}^\infty, C_{eo}^\infty, D_{eo}^\infty)$ is a regular linear system with input space U_{ee} and output space Y_{ee}. The operator $C_{eo}^\infty : \mathcal{D}(C_{eo}^\infty) \subset X_e \to Y_{ee}$ coincides with the Λ-extension of $C_{eo}^\infty : \mathcal{D}(A_{eo}^\infty) \subset X_e \to X_e$. We define $\Delta_0 \in \mathcal{L}(Y_{ee}, U_{ee})$ and $D_{eo}^{add} \in \mathcal{L}(Y_{ee}, Y_{ee})$ by

$$\Delta_0 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{eo}^{add} = \begin{bmatrix} 0 & 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Since $I - D_{eo}^\infty \Delta_0 \in \mathcal{L}(Y_{ee})$ is boundedly invertible, Δ_0 is an admissible output feedback operator for $(A_{eo}^\infty, B_{eo}^\infty, C_{eo}^\infty, D_{eo}^\infty)$. The results in [40] and [32, Sec. 7.5] imply that first applying output feedback with operator Δ_0 and subsequently adding a parallel connection with the (constant) transfer function D_{eo}^{add} produces a regular linear system $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ with

$$A_e^\infty = A_{eo}^\infty + B_{eo}^\infty \Delta_0 (I - D_{eo}^\infty \Delta_0)^{-1} C_{eo}^\infty \tag{9a}$$
$$B_{ee}^\infty = B_{eo}^\infty (I - \Delta_0 D_{eo}^\infty)^{-1} \tag{9b}$$
$$C_{ee}^\infty = (I - D_{eo}^\infty \Delta_0)^{-1} C_{eo}^\infty \tag{9c}$$
$$D_{ee}^\infty = (I - D_{eo}^\infty \Delta_0)^{-1} D_{eo}^\infty + D_{eo}^{add}. \tag{9d}$$

where $\mathcal{D}(A_e^\infty) = \{\, x_e \in \mathcal{D}(C_\Lambda) \times Z \mid A_e^\infty x_e \in X_e \,\}$ and $\mathcal{D}(C_{ee}^\infty) = \mathcal{D}(C_{eo}^\infty) = \mathcal{D}(C_\Lambda) \times Z$. We denote by $T_e(t)$ the strongly continuous semigroup generated by A_e^∞ and by $\mathbb{F}_{ee}^\infty$ the extended input-output map of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$.

We define $\Delta(\cdot) \in L^\infty(0,\infty;\mathcal{L}(Y_{ee}, U_{ee}))$ and $P_{in} \in \mathcal{L}(\mathbb{C}^{n_d+p}, U_{ee})$ by

$$\Delta(t) = \begin{bmatrix} 0 & 0 & \Delta_K(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{\mathcal{G}_1}(t) \\ \Delta_{\mathcal{G}_2}(t) & 0 & 0 \end{bmatrix}, \quad P_{in} = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

and define $P_{out} = \begin{bmatrix} I, 0, 0 \end{bmatrix} \in \mathcal{L}(Y_{ee}, Y)$. A direct computation shows that for a.e. $t \geq 0$ the operator $I - D_{ee}^\infty \Delta(t) \in \mathcal{L}(Y_{ee})$ is boundedly invertible and

$$A_e^\infty + B_{ee}^\infty \Delta(t)(I - D_{ee}^\infty \Delta(t))^{-1} C_{ee}^\infty = A_e(t),$$

where $A_e(t)$ is as in (5). This identity confirms that $A_e(t)$ are (at this stage formally) associated to the system obtained from $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ with the nonautonomous feedback $\Delta(t)$. The following lemma shows that $\Delta(t)$ is an admissible feedback for $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ in the sense of [29, Def. 4.1].

Lemma III.1. *Let $\mathbb{F}_{ee}^\infty$ be the extended input-output map of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ and let $\Delta(\cdot)$ be as in* (10)*. Then for*

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every $t_0 > 0$ the operators $I - \mathbb{F}_{ee}^\infty \Delta(\cdot) \in \mathcal{L}(L^2(s, s+t_0; Y_{ee}))$ for $s \geq 0$ have uniformly bounded inverses.

Proof. The input-output map $\mathbb{F}_{eo}$ of $(A_{eo}^\infty, B_{eo}^\infty, C_{eo}^\infty, D_{eo}^\infty)$ can be partitioned as

$$\mathbb{F}_{eo} = \begin{bmatrix} \mathbb{F} & \mathbb{F}_d & 0 & 0 & 0 \\ 0 & 0 & \mathbb{F}_c & \mathbb{F}_{c2} & \mathbb{F}_{c3} \\ 0 & 0 & \mathbb{F}_{c4} & \mathbb{F}_{c5} & \mathbb{F}_{c6} \end{bmatrix}.$$

We have $\mathbb{F}_{ee}^\infty = (I - \mathbb{F}_{eo}\Delta_0)^{-1}\mathbb{F}_{eo} + D_{eo}^{add}$, which implies

$$\begin{aligned} I - \mathbb{F}_{ee}^\infty \Delta(\cdot) &= \begin{bmatrix} I & 0 & -S_1\mathbb{F}(\Delta_K(\cdot) + \mathbb{F}_{c2}\Delta_{\mathcal{G}_1}(\cdot)) \\ 0 & I & -S_2(\Delta_K(\cdot) + \mathbb{F}_{c2}\Delta_{\mathcal{G}_1}(\cdot)) \\ 0 & 0 & I \end{bmatrix} \\ &\quad - \begin{bmatrix} \mathbb{F}S_2\mathbb{F}_{c3}\Delta_{\mathcal{G}_2}(\cdot) & 0 & 0 \\ S_2\mathbb{F}_{c3}\Delta_{\mathcal{G}_2}(\cdot) & 0 & 0 \\ S_3\Delta_{\mathcal{G}_2}(\cdot) & 0 & \mathbb{F}_{c4}S_1\mathbb{F}\Delta_K(\cdot) + S_4\Delta_{\mathcal{G}_1}(\cdot) \end{bmatrix} \end{aligned}$$

with $S_1 = (I + \mathbb{F}\mathbb{F}_c)^{-1}$, $S_2 = (I + \mathbb{F}_c\mathbb{F})^{-1}$, $S_3 = \mathbb{F}_{c6} - \mathbb{F}_{c4}\mathbb{F}S_2\mathbb{F}_{c3}$, and $S_4 = \mathbb{F}_{c5} - \mathbb{F}_{c4}\mathbb{F}S_2\mathbb{F}_{c2}$. The first term of $I - \mathbb{F}_{ee}^\infty\Delta(\cdot)$ is invertible on $L^2(s, s+t_0; Y_{ee})$ and the inverse is uniformly bounded with respect to $s \geq 0$ and $0 < t_0 \leq 1$. Since K^∞ and I in C_{eo}^∞ are bounded operators, it is easy to verify that the restrictions of the input-output maps $\mathbb{F}_{c3}$, $\mathbb{F}_{c4}$, $\mathbb{F}_{c5}$, and $\mathbb{F}_{c6}$ to the time-interval $[0, t_0]$ satisfy $\|\mathbb{F}_{c3}|_{[0,t_0]}\| \to 0$, $\|\mathbb{F}_{c4}|_{[0,t_0]}\| \to 0$, $\|\mathbb{F}_{c5}|_{[0,t_0]}\| \to 0$, and $\|\mathbb{F}_{c6}|_{[0,t_0]}\| \to 0$ as $t_0 \to 0$. Since $\Delta_{\mathcal{G}_1}$, $\Delta_{\mathcal{G}_2}$ and Δ_K are essentially bounded the $\mathcal{L}(L^2(s, s+t_0; Y_{ee}))$-operator norm of the second term of $I - \mathbb{F}_{ee}^\infty\Delta(\cdot)$ converges to zero as $t_0 \to 0$ uniformly with respect to $s \geq 0$. Because of this, for a sufficiently small $t_0 > 0$ the operators $I - \mathbb{F}_{ee}^\infty\Delta(\cdot) \in \mathcal{L}(L^2(s, s+t_0; Y_{ee}))$ for $s \geq 0$ have uniformly bounded inverses $L^2(s, s+t_0; Y_{ee})$. By [29, Lem. 4.2] the same is then true for all $t_0 > 0$. □

We define $\mathcal{D}(C_{ee}(t)) = \mathcal{D}(C_{ee}^\infty) = \mathcal{D}(C_\Lambda) \times Z$, a.e. $t \geq 0$,

$$B_{ee}(\cdot) = B_{ee}^\infty Q_1(\cdot) \quad \text{and} \quad C_{ee}(\cdot) = Q_2(\cdot)C_{ee}^\infty \tag{11}$$

with $Q_1(\cdot) = (I - \Delta(\cdot)D_{ee}^\infty)^{-1}$ and $Q_2(\cdot) = (I - D_{ee}^\infty\Delta(\cdot))^{-1}$. The following theorem shows that the closed-loop system (5) has a well-defined strongly continuous evolution family $U_e(t,s)$ and an input map $\Phi_e^{t,s}$ defined in (7a). Moreover, since a direct computation shows that $B_{ee}(t)P_{in} = B_e(t)$ and $P_{out}C_{ee}(t) = C_e(t)$ for a.e. $t \geq 0$, the mappings defined in (7) and in the following theorem satisfy $\Psi_e^s = P_{out}\Psi_{ee}^s$ and $\mathbb{F}_e^s = P_{out}\mathbb{F}_{ee}^s$. Because of this, Theorem III.2 implies that for $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$ and $y_{ref}(t)$ the closed-loop system (5) has a well-defined mild state $x_e(t)$ and output $e(\cdot) \in L^2_{\text{loc}}(0, \infty; \mathbb{C}^p)$ determined by (6). The integral equation associates $(A_e(t))_{t \geq 0}$ to the evolution family $U_e(t,s)$.

Theorem III.2. *Let Assumption* II.2 *hold and let $\Delta(\cdot)$ be as in* (10)*. There exists a strongly continuous evolution family $U_e(t,s)$ such that for all $x \in X_e$ and $s \geq 0$ we have $U_e(r,s)x \in \mathcal{D}(C_{ee}^\infty) = \mathcal{D}(C_\Lambda) \times Z$ for a.e. $r \geq s$, $\|C_{ee}^\infty U_e(s + \cdot, s)x\|_{L^2(s,s+t_0)} \leq \gamma(t_0)\|x\|$ for every $t_0 > 0$ and some $\gamma(t_0) > 0$ (depending only on $t_0 > 0$), and*

$$\begin{aligned} U_e(t,s)x &= T_e(t-s)x \\ &\quad + \int_s^t T_e(t-r)B_{ee}^\infty\Delta(r)(I - D_{ee}^\infty\Delta(r))^{-1}C_{ee}^\infty U_e(r,s)x ds \end{aligned}$$

for all $t \geq s$. If $\Phi_e^{t,s}$ is defined as in (7a) *and if we define $(\Psi_{ee}^s x)(t) = C_{ee}(t)U_e(t,s)x$ and*

$$(\mathbb{F}_{ee}^s w_e)(t) = C_{ee}(t)\int_s^t U_e(t,r)B_e(r)w_e(r)dr + D_{ee}^\infty P_{in}w_e(t),$$

then $(U_e, \Phi_e^{t,s}, \Psi_{ee}^s, \mathbb{F}_{ee}^s)_{t \geq s \geq 0}$ is a well-posed nonautonomous system in the sense of [29, Def. 3.6]*. In particular, for all $s \geq 0$ and $t_0 > 0$ we have $\Phi_e^{t,s}w_e \in \mathcal{D}(C_{ee}(t)) = \mathcal{D}(C_\Lambda) \times Z$ a.e. $t \geq s$ and $w_e \in L^2_{loc}(0, \infty; \mathbb{C}^{n_d+p})$,*

$$\begin{aligned} &\Phi_e^{t,s} \in \mathcal{L}(L^2(s,t;\mathbb{C}^{n_d+p}), X_e), \qquad 0 \leq t - s \leq t_0 \\ &\Psi_{ee}^s \in \mathcal{L}(X_e, L^2(s, s+t_0; Y)), \\ &\mathbb{F}_{ee}^s \in \mathcal{L}(L^2(s, s+t_0; \mathbb{C}^{n_d+p}), L^2(s, s+t_0; Y)) \end{aligned}$$

with bounds independent of $s \geq 0$. Finally, we have

$$\mathbb{F}_{ee}^s = (I - \mathbb{F}_{ee}^\infty\Delta(\cdot))^{-1}\mathbb{F}_{ee}^\infty P_{in}$$

where $\mathbb{F}_{ee}^\infty$ is the input-output map of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$.

Proof. Due to the regularity of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$, $B_{ee}(\cdot)$ and $C_{ee}(\cdot)$ defined in (11) are "admissible input and output operators for the evolution family $(T_e(t-s))_{t \geq s \geq 0}$" in the sense of [29, Def. 3.3 and 2.4], where $T_e(t)$ is the semigroup generated by A_e^∞. For $t \geq s \geq 0$ and $u \in L^2(s,t;U_{ee})$ we define the mapping $\overline{\mathbb{K}}_s$ as in [29, Def. 3.3] by

$$(\overline{\mathbb{K}}_s B_{ee}(\cdot)u)(t) := \int_s^t T_e(t-r)B_{ee}(r)u(r)dr.$$

The regularity of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ implies that $(\overline{\mathbb{K}}_s B_{ee}(\cdot)u)(t) \in \mathcal{D}(C_{ee}^\infty) = \mathcal{D}(C_{ee}(t))$ for a.e. $t \geq s$ and $C_{ee}(\cdot)\overline{\mathbb{K}}_s B_{ee}(\cdot)u \in L^2_{\text{loc}}(s, \infty; Y_{ee})$. Moreover, if we define $\Phi_0^{t,s}$, Ψ_0^s and $\mathbb{F}_0^s$ by

$$\Phi_0^{t,s}u = (\overline{\mathbb{K}}_s B_{ee}(\cdot)u)(t), \quad (\Psi_0^s x)(t) = C_{ee}(t)T_e(t-s)x,$$

and $\mathbb{F}_0^s u = C_{ee}(\cdot)\overline{\mathbb{K}}_s B_{ee}(\cdot)u$ for $u \in L^2(s,t;U_{ee})$ and $x \in X_e$, then $(T_e, \Phi_0^{t,s}, \Psi_0^s, \mathbb{F}_0^s)_{t \geq s \geq 0}$ is a well-posed nonautonomous system by [29, Lem. 3.9].

We will now show that $(I - \Delta(\cdot)D_{ee}^\infty)\Delta(\cdot)$ is an admissible feedback [29, Def. 4.1] for $(T_e, \Phi_0^{t,s}, \Psi_0^s, \mathbb{F}_0^s)_{t \geq s \geq 0}$. Due to the definitions, we have $\mathbb{F}_0^s = Q_2(\cdot)[\mathbb{F}_{ee}^\infty - D_{ee}^\infty]Q_1(\cdot)$ for all $s \geq 0$ where $Q_1(\cdot) = (I - \Delta(\cdot)D_{ee}^\infty)^{-1}$ and $Q_2(\cdot) = (I - D_{ee}^\infty\Delta(\cdot))^{-1}$. Thus

$$\begin{aligned} &I - \mathbb{F}_0^s(I - \Delta(\cdot)D_{ee}^\infty)\Delta(\cdot) = I - Q_2(\cdot)(\mathbb{F}_{ee}^\infty - D_{ee}^\infty)\Delta(\cdot) \\ &\quad = I - Q_2(\cdot)\mathbb{F}_{ee}^\infty\Delta(\cdot) + (I - D_{ee}^\infty\Delta(\cdot))^{-1}D_{ee}^\infty\Delta(\cdot) \\ &\quad = Q_2(\cdot)(I - \mathbb{F}_{ee}^\infty\Delta(\cdot)). \end{aligned}$$

Since $\Delta(\cdot)$ is an admissible feedback for $\mathbb{F}_{ee}^\infty$ by Lemma III.1 and $Q_2(\cdot)$ have uniformly bounded inverses on $L^\infty(0, \infty; \mathcal{L}(Y_{ee}))$, we have that $(I - \Delta(\cdot)D_{ee}^\infty)\Delta(\cdot) \in L^\infty(0, \infty; \mathcal{L}(Y_{ee}, U_{ee}))$ is an admissible feedback for $(T_e, \Phi_0^{t,s}, \Psi_0^s, \mathbb{F}_0^s)_{t \geq s \geq 0}$.

By construction, the system $(T_e, \Phi_0^{t,s}, \Psi_0^s, \mathbb{F}_0^s)_{t \geq s \geq 0}$ has the properties in the first part of [29, Thm. 3.11], namely, that $\Phi_0^{t,s}u \in \mathcal{D}(C_{ee}(t))$ for a.e. $t \geq s$ and $t \mapsto C_{ee}(t)\Phi_0^{t,s}u \in L^2_{\text{loc}}(s, \infty; Y_{ee})$ for all $s \geq 0$ and $u \in L^2_{\text{loc}}(s, \infty; U_{ee})$. Because of this, the proof of [29, Thm. 4.4(a)] shows that there exists a strongly continuous evolution family $U_e(t,s)$ which satisfies the integral equation in the claim, and $C_{ee}^\infty U_e(\cdot, s)x \in$

$L^2_{\mathrm{loc}}(s,\infty;Y_{ee})$ for all $x \in X_e$ and $s \geq 0$. The proof of [29, Thm. 4.4(a)] also shows that $C_{ee}(t) = (I - D_{ee}^\infty \Delta(t))^{-1} C_{ee}^\infty$, a.e. $t \geq 0$, are admissible observation operators for $U_e(t,s)$. If Ψ_{ee}^s is defined as in the claim, then by [29, Lem. 2.5] the pair (U_e, Ψ_{ee}^s) is a "nonautonomous observation system" in the sense of [29, Def. 2.2]. Since $B_e(t) \in \mathcal{L}(\mathbb{C}^{n_d+p}, X_e)$ for a.e. $t \geq 0$, $\Phi_e^{t,s}$ can be defined as in (7a), and $(U_e, \Phi_e^{t,s})$ is a "nonautonomous control system" in the sense of [29, Def. 3.1]. Since $B_e(\cdot) \in L^\infty(0,\infty;\mathcal{L}(\mathbb{C}^{n_d+p}, X_e))$, the properties of (U_e, Ψ_e^s) and [29, Prop. 2.11] also imply that there exist $\kappa, t_1 > 0$ such that for all $s \geq 0$ and $w_e \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{n_d+p})$ we have $\Phi_e^{t,s} w_e \in \mathcal{D}(C_{ee}(t))$ for a.e. $t \geq s$ and

$$\|C_{ee}(\cdot)\Phi_e^{\cdot,s} w_e\|_{L^2(s,s+t_1)} \leq \kappa \|w_e\|_{L^2(s,s+t_1)}.$$

Thus [29, Lem. 3.9] implies that $(U_e, \Phi_e^{t,s}, \Psi_{ee}^s, \mathbb{F}_{ee,0}^s)_{t \geq s \geq 0}$ with $\mathbb{F}_{ee,0}^s := C_e(\cdot)\Phi_e^{\cdot,s}$ is a well-posed nonautonomous system, and since $\mathbb{F}_{ee}^s w_e = \mathbb{F}_{ee,0}^s w_e + D_{ee}^\infty P_{in} w_e$, the same is finally true also for $(U_e, \Phi_e^{t,s}, \Psi_{ee}^s, \mathbb{F}_{ee}^s)_{t \geq s \geq 0}$. In particular, $\Phi_e^{t,s}$, Ψ_{ee}^s, $\mathbb{F}_{ee}^s$ have the boundedness properties in the claim.

Finally, we will show that $\mathbb{F}_{ee}^s = (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1} \mathbb{F}_{ee}^\infty P_{in}$. Let $w_e \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{n_d+p})$. The evolution family $U_e(t,s)$ is associated to $(U_e, \Phi_e^{t,s}, \Psi_{ee}^s, \mathbb{F}_{ee,0}^s)_{t \geq s \geq 0}$ which is obtained from $(T_e, \Phi_0^{t,s}, \Psi_0^s, \mathbb{F}_0^s)_{t \geq s \geq 0}$ with output feedback $(I - \Delta(\cdot) D_{ee}^\infty)\Delta(\cdot)$. Applying the identity (4.13) in the proof of [29, Thm. 4.4(b)][1] to these two systems and $f = B_e(\cdot) w_e \in L^2_{\mathrm{loc}}(0,\infty;X_e)$ and using $B_e(\cdot) = B_{ee}(\cdot) P_{in}$ we get

$$\begin{aligned}
\mathbb{F}_{ee}^s w_e - D_{ee}^\infty P_{in} w_e &= \mathbb{F}_{ee,0}^s w_e = C_{ee}(\cdot) \int_s^\cdot U_e(\cdot, r) B_e(r) w_e(r) dr \\
&= (I - \mathbb{F}_0^s (I - \Delta(\cdot) D_{ee}^\infty)\Delta(\cdot))^{-1} C_{ee}(\cdot) \overline{\mathbb{K}}_s B_e(\cdot) w_e \\
&= (I - \mathbb{F}_0^s (I - \Delta(\cdot) D_{ee}^\infty)\Delta(\cdot))^{-1} \mathbb{F}_0^s P_{in} w_e.
\end{aligned}$$

A direct computation shows that $D_{ee}^\infty P_{in} = D_{ee}^\infty Q_1(\cdot) P_{in}$. Using $\mathbb{F}_0^s = Q_2(\cdot)[\mathbb{F}_{ee}^\infty - D_{ee}^\infty] Q_1(\cdot)$ with $Q_1(\cdot) = (I - \Delta(\cdot) D_{ee}^\infty)^{-1}$ and $Q_2(\cdot) = (I - D_{ee}^\infty \Delta(\cdot))^{-1}$ and denoting $u = P_{in} w_e$ for brevity we get

$$\begin{aligned}
\mathbb{F}_{ee}^s w_e &= (I - Q_2(\cdot)[\mathbb{F}_{ee}^\infty - D_{ee}^\infty]\Delta(\cdot))^{-1} \mathbb{F}_0^s u + D_{ee}^\infty u \\
&= (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1} [\mathbb{F}_{ee}^\infty - D_{ee}^\infty] Q_1(\cdot) u + D_{ee}^\infty Q_1(\cdot) u \\
&= (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1} \big[\mathbb{F}_{ee}^\infty - \mathbb{F}_{ee}^\infty \Delta(\cdot) D_{ee}^\infty] Q_1(\cdot) u \\
&= (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1} \mathbb{F}_{ee}^\infty P_{in} w_e.
\end{aligned}$$

Thus $\mathbb{F}_{ee}^s w_e = (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1} \mathbb{F}_{ee}^\infty P_{in} w_e$ on $[s, s+t_0]$ for any $s \geq 0$, $t_0 > 0$, and $w_e \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{n_d+p})$. □

Remark III.3. Let $x_0 \in X$ and $z_0 \in Z$. If Assumption II.2 is satisfied, then $(U_e, \Phi_e^{t,s}, \Psi_{ee}^s, \mathbb{F}_{ee}^s)_{t \geq s \geq 0}$ is a well-posed nonautonomous system by Theorem III.2. Thus for every $w_e \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{n_d+p})$ the closed-loop state $x_e(t)$ in (6) satisfies $x_e(\cdot) \in C([0,\infty);X_e)$ by [29, Def. 2.1 & Prop. 3.5(2)]. Since $K(\cdot) \in L^\infty(0,\infty;\mathcal{L}(Z,\mathbb{C}^m))$, for such $w_e(\cdot)$ we have $u(\cdot) \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^p)$. Theferore the properties of regular linear systems imply that if $w_{dist}(t)$ is as in (1), then (2) has a well-defined mild state $x(t)$ satisfying $x(t) \in \mathcal{D}(C_\Lambda)$ for a.e. $t \geq 0$, and the output $y(t)$ is determined by (2b) for a.e. $t \geq 0$.

[1]The identity (4.13) does not require "absolute regularity" and it extends to $L^2_{\mathrm{loc}}(0,\infty;X_e)$ since (U_e, Ψ_{ee}^s) is a nonautonomous observation system.

IV. Regulation with Converging Controllers

In this section we introduce general results on output regulation with a nonautonomous controller $(\mathcal{G}_1(t), \mathcal{G}_2(t), K(t))$ satisfying Assumption II.2. Our first main result in Theorem IV.2 is applicable in the situation where the controller parameters have well-defined asymptotic limits in the sense that

$$\begin{cases} \|\Delta_{\mathcal{G}_1}(t)\| \to 0, \\ \|\mathcal{G}_2(t) - \mathcal{G}_2^\infty\| \to 0 \qquad \text{as} \quad t \to \infty \\ \|K(t) - K^\infty\| \to 0 \end{cases}$$

for some $\mathcal{G}_2^\infty \in \mathcal{L}(\mathbb{C}^p, Z)$ and $K^\infty \in \mathcal{L}(Z, \mathbb{C}^m)$[2]. Our second main result in Theorem IV.4 considers a more general situation where the above norms become small as $t \to \infty$ but do not necessarily converge to zero. The main condition in our results is that the part $\mathcal{G}_1^\infty$ of $\mathcal{G}_1(\cdot)$ in Assumption II.2 has *an internal model* of the frequencies $(\omega_k)_{k=0}^q$ (with $\omega_0 = 0$) of $w_{dist}(t)$ and $y_{ref}(t)$ in (1) in the following sense.

Definition IV.1 ([26, Def. 6.1])**.** The operator $\mathcal{G}_1^\infty$ *has an internal model of* $(\omega_k)_{k=0}^q$ if $\dim \mathcal{N}(\pm i\omega_k - \mathcal{G}_1^\infty) \geq p$ for all $k \in \{0, \ldots, q\}$, where $p \in \mathbb{N}$ is the number of outputs of (2).

Our first result states that if the controller parameters converge, if $\mathcal{G}_1^\infty$ has an internal model of the frequencies of $y_{ref}(t)$ and $w_{dist}(t)$ and if the closed-loop system is exponentially stable, then the controller achieves output regulation. Exponential stability of $U_e(t,s)$ means that there exists $M, \alpha > 0$ such that $\|U_e(t,s)\| \leq M e^{-\alpha(t-s)}$ for $t \geq s \geq 0$.

Theorem IV.2. *Assume $y_{ref}(t)$ and $w_{dist}(t)$ in* (1) *and the initial states $x_0 \in X$ and $z_0 \in Z$ are such that there exist $\mathcal{G}_1(\cdot)$, $\mathcal{G}_2(\cdot)$ and $K(\cdot)$ satisfying Assumption* II.2*, and for some $\mathcal{G}_2^\infty \in \mathcal{L}(\mathbb{C}^p, Z)$ and $K^\infty \in \mathcal{L}(Z, \mathbb{C}^m)$*

$$\delta_{\mathcal{G}}(t) := \max\{\|\Delta_{\mathcal{G}_1}(t)\|, \|\mathcal{G}_2(t) - \mathcal{G}_2^\infty\|, \|K(t) - K^\infty\|\}$$

satisfies $\delta_{\mathcal{G}}(t) \to 0$ as $t \to \infty$. If $U_e(t,s)$ is exponentially stable and $\mathcal{G}_1^\infty$ has an internal model of $(\omega_k)_{k=0}^q$ in (1)*, then*

$$\int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds \to 0, \qquad as \quad t \to \infty \tag{12}$$

If $\operatorname{ess\,sup}_{t \geq 0} e^{\alpha t} \delta_{\mathcal{G}}(t) < \infty$ for some $\alpha > 0$, then there exists $\alpha_e > 0$ such that $t \mapsto e^{\alpha_e t}(y(t) - y_{ref}(t)) \in L^2(0,\infty;Y)$.

In Theorem IV.2 the controller parameters and their limits are allowed to depend on the initial states of the system and the controller and of $w_{dist}(t)$ and $y_{ref}(t)$. This possibility is motivated by our controller design for output regulation with unknown frequencies in Section V. If $(\mathcal{G}_1(t), \mathcal{G}_2(t), K(t))$ are independent of the initial states and $w_{dist}(t)$ and $y_{ref}(t)$, the claims of Theorem IV.2 (and Theorem IV.4) hold for all $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$ and $y_{ref}(t)$.

The proof of Theorem IV.2 utilises the feedback structure introduced in Section III. To this end we use the notation in Section III and in particular denote $\Delta_{\mathcal{G}_2}(\cdot) = \mathcal{G}_2(\cdot) - \mathcal{G}_2^\infty \in L^\infty(0,\infty;\mathcal{L}(\mathbb{C}^p, Z))$ and $\Delta_K(\cdot) = K(\cdot) - K^\infty \in L^\infty(0,\infty;\mathcal{L}(Z,\mathbb{C}^m))$. The "if"-part of the following lemma also follows from [29, Thm. 5.6] (see also [10, Sec. 4]).

[2]Recall that "$\|f(t)\| \to 0$" for $f \in L^\infty$ means $\operatorname{ess\,sup}_{s \geq t} \|f(s)\| \to 0$.

Lemma IV.3. *Let $(\mathcal{G}_1(t), \mathcal{G}_2(t), K(t))$ satisfy Assumption* II.2 *and assume $\mathcal{G}_2^\infty \in \mathcal{L}(\mathbb{C}^p, Z)$ and $K^\infty \in \mathcal{L}(Z, \mathbb{C}^m)$ are such that $\delta_\mathcal{G}(t)$ in Theorem* IV.2 *satisfies $\delta_\mathcal{G}(t) \to 0$ as $t \to \infty$. Then $U_e(t,s)$ is exponentially stable if and only if the semigroup $T_e(t)$ generated by A_e^∞ is exponentially stable.*

Proof. In the notation of Section III, a direct computation shows $M_D := \operatorname{ess\,sup}_{s\geq 0}\|(I - D_{ee}^\infty \Delta(\cdot))^{-1}\| < \infty$. Moreover, Theorem III.2 implies that for every $t_0 > 0$ there exists $\gamma(t_0) > 0$ such that $\sup_{s\geq 0}\|C_{ee}^\infty U_e(s+\cdot, s)x\|_{L^2(s,s+t_0)} \leq \gamma(t_0)\|x\|$ for all $x \in X_e$ and $s \geq 0$. Because of this, the integral equation in Theorem III.2 together with the admissibility of B_{ee}^∞ for the semigroup $T_e(t)$ imply that for any fixed $t_0 > 0$ there exists $M_{t_0} > 0$ such that

$$\begin{aligned}&\|U_e(s+t_0,s)x - T_e(t_0)x\| \\ &\quad \leq M_{t_0} M_D \|\Delta(\cdot)\|_{L^\infty(s,\infty)} \|C_{ee}^\infty U_e(\cdot,s)x\|_{L^2(s,s+t_0)} \\ &\quad \leq M_{t_0} M_D \|\Delta(\cdot)\|_{L^\infty(s,\infty)} \gamma(t_0)\|x\|\end{aligned}$$

for all $s \geq 0$ and $x \in X_e$. Since $\|\Delta(\cdot)\|_{L^\infty(s,\infty)} \to 0$ as $s \to \infty$ by assumption, we have $\|U_e(s+t_0,s) - T_e(t_0)\| \to 0$ as $s \to \infty$ for every fixed $t_0 > 0$. The "only if" part can now be verified by choosing sufficiently large values of $t_0 > 0$. In the "if" part the estimate $\|U_e(s+t_0,s)\| \leq \|U_e(s+t_0,s) - T_e(t_0)\| + \|T_e(t_0)\|$ implies that there exist $s_0, t_0 > 0$ such that $\|U_e(s+t_0,s)\| \leq c < 1$ for all $s \geq s_0$. The stability of $U_e(t,s)$ follows from these estimates and the fundamental properties of evolution families in [29, Def. 2.1]. □

Proof of Theorem IV.2. Let $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$ and $y_{ref}(t)$ satisfy the assumptions of the theorem. According to (6) the regulation error $e(t) = y(t) - y_{ref}(t)$ can be expressed using the output map Ψ_e^s and input-output maps $\mathbb{F}_e^s$ and $\mathbb{F}_e^\infty = P_{out}\mathbb{F}_{ee}^\infty P_{in}$ as

$$\begin{aligned}e(t) &= (\Psi_e^0 x_{e0})(t) + (\mathbb{F}_e^0 w_e)(t) \\ &= (\Psi_e^0 x_{e0})(t) + (\mathbb{F}_e^\infty w_e)(t) + \big[(\mathbb{F}_e^0 w_e)(t) - (\mathbb{F}_e^\infty w_e)(t)\big].\end{aligned}$$

If define $B_e^\infty = B_{ee}^\infty P_{in} \in \mathcal{L}(\mathbb{C}^{n_d+p}, X_e)$ and $C_e^\infty = P_{out}C_{ee}^\infty : \mathcal{D}(C_{ee}^\infty) \subset X_e \to \mathbb{C}^p$, then $\mathbb{F}_e^\infty$ is the input-output map of the regular linear system $(A_e^\infty, B_e^\infty, C_e^\infty, D_e)$. A direct computation using (9) shows that this system has the form of the closed-loop system in [24, Sec. II] corresponding to the regular linear system $(A, [B, B_d], C, [D, D_d])$ and the autonomous controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$. In particular, $(\mathbb{F}_e^\infty w_e)(t)$ is the regulation error corresponding to zero initial states of the system and the controller and the signals $w_{dist}(t)$ and $y_{ref}(t)$. Since $\mathcal{G}_1^\infty$ has an internal model of $(\omega_k)_{k=0}^q$ and $T_e(t)$ generated by A_e^∞ is stable by Lemma IV.3, we have from [24, Thm. 7] that $t \mapsto e^{\alpha_1 t}\|(\mathbb{F}_e^\infty w_e)(t)\| \in L^2(0,\infty)$ for some $\alpha_1 > 0$. Since $U_e(t,s)$ is exponentially stable, there exists $\alpha_2 > 0$ such that $t \mapsto e^{\alpha_2 t}\|(\Psi_e^0 x_{e0})(t)\| \in L^2(0,\infty)$.

It remains to analyse the term $\mathbb{F}_e^0 w_e - \mathbb{F}_e^\infty w_e$. Note that $\sup_{t\geq 0}\|w_e\|_{L^2(t,t+1;\mathbb{C}^{n_d+p})} < \infty$. For $\Delta(\cdot)$ in (10) we have $\operatorname{ess\,sup}_{t\leq s\leq t+1}\|\Delta(s)\| \to 0$ as $t \to \infty$ if $\delta_\mathcal{G}(t) \to 0$ as $t \to \infty$. Theorem III.2 implies $\mathbb{F}_{ee}^s = (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1}\mathbb{F}_{ee}^\infty P_{in}$ and

$$\begin{aligned}\mathbb{F}_e^0 - \mathbb{F}_e^\infty &= P_{out}\big[(I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1} - I\big]\mathbb{F}_{ee}^\infty P_{in} && \text{(13a)} \\ &= P_{out}\mathbb{F}_{ee}^\infty \Delta(\cdot)(I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1}\mathbb{F}_{ee}^\infty P_{in} && \text{(13b)} \\ &= P_{out}\mathbb{F}_{ee}^\infty \Delta(\cdot)\mathbb{F}_{ee}^0. && \text{(13c)}\end{aligned}$$

As shown in the proof of Theorem III.2, $\mathbb{F}_{ee}^s$ is an input-output map of a nonautonomous well-posed system with an exponentially stable evolution family $U_e(t,s)$. Thus Lemma A.1(a) implies $\sup_{t\geq 0}\|\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)} < \infty$ and

$$\begin{aligned}&\|\Delta(\cdot)\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)} \\ &\quad \leq \|\Delta(\cdot)\|_{L^\infty(t,t+1)}\|\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)} \to 0\end{aligned}$$

as $t \to \infty$. Moreover, Lemma IV.3 implies that the regular linear system $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ with extended input-output map $\mathbb{F}_{ee}^\infty$ is exponentially stable. Thus Lemma A.1(c) applied to this autonomous system and $u = \Delta(\cdot)\mathbb{F}_{ee}^0 w_e$ together with (13) show that $\|\mathbb{F}_e^0 w_e - \mathbb{F}_e^\infty w_e\|_{L^2(t,t+1)} \to 0$ as $t \to \infty$. This completes the proof of (12).

Finally, let $\alpha > 0$ be such that $\operatorname{ess\,sup}_{t\geq 0} e^{\alpha t}\delta_\mathcal{G}(t) < \infty$. Then $\sup_{t\geq 0} e^{\alpha t}\|\Delta(\cdot)\|_{L^\infty(t,t+1)} < \infty$ and since $\sup_{t\geq 0}\|\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)} < \infty$, we have

$$\begin{aligned}&\sup_{t\geq 0} e^{\alpha t}\|\Delta(\cdot)\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)} \\ &\quad \leq \sup_{t\geq 0} e^{\alpha t}\|\Delta(\cdot)\|_{L^\infty(t,t+1)}\|\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)} < \infty.\end{aligned}$$

Lemma A.1(d) for $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ and $u = \Delta(\cdot)\mathbb{F}_{ee}^0 w_e$ together with (13) imply $t \mapsto e^{\alpha_0 t}\|(\mathbb{F}_e^0 w_e)(t) - (\mathbb{F}_e^\infty w_e)(t)\| \in L^2(0,\infty)$ for some $\alpha_0 > 0$. This completes the proof. □

Exponential stability of $U_e(t,s)$ and Assumption II.2 imply $\sup_{t\geq 0}\|\Phi_e(t,0)w_e\| < \infty$ for all $w_e \in L^\infty(0,\infty;\mathbb{C}^{n_d+p})$ and thus $x_e(t)$ in Theorem IV.2 satisfies $\sup_{t\geq 0}\|x_e(t)\| < \infty$.

The following result generalises Theorem IV.2 to the situation where the parameters of the controller do not necessarily converge as $t \to \infty$, or where the limit of $\mathcal{G}_1(t)$ has an internal model of frequencies which are only close to $(\omega_k)_{k=0}^q$. In these cases the asymptotic tracking error will be small provided that the asymptotic error in the frequencies is sufficiently small.

Theorem IV.4. *Assume that $x_0 \in X$, $z_0 \in Z$, $y_{ref}(t)$, and $w_{dist}(t)$ in* (1) *are such that there exist $\mathcal{G}_1(\cdot)$, $\mathcal{G}_2(\cdot)$ and $K(\cdot)$ satisfying Assumption* II.2 *and $U_e(t,s)$ is exponentially stable. Moreover, assume $\mathcal{G}_1^\infty$ has an internal model of $(\omega_k)_{k=0}^q$ and $\mathcal{G}_2^\infty \in \mathcal{L}(\mathbb{C}^p, Z)$ and $K^\infty \in \mathcal{L}(Z, \mathbb{C}^m)$ are such that $T_e(t)$ is exponentially stable. Define*

$$\delta_\mathcal{G}(t) := \max\{\|\Delta_{\mathcal{G}_1}(t)\|, \|\mathcal{G}_2(t) - \mathcal{G}_2^\infty\|, \|K(t) - K^\infty\|\}$$

There exist $M_{err}, \delta_0 > 0$ depending only on (A, B, B_d, C, D, D_d) and $(\mathcal{G}_1^\infty, \mathcal{G}_{11}^\infty, \mathcal{G}_2^\infty, K^\infty)$ such that

$$\begin{aligned}&\limsup_{t\to\infty} \int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds \\ &\quad \leq M_{err}\|[w_{dist}(\cdot)^T, y_{ref}(\cdot)^T]^T\|_\infty^2 \limsup_{t\to\infty}\|\delta_\mathcal{G}(\cdot)\|_{L^\infty(t,\infty)}^2\end{aligned}$$

provided that $\limsup_{t\to\infty}\|\delta_\mathcal{G}(\cdot)\|_{L^\infty(t,\infty)} < \delta_0$.

Proof. As shown in the proof of Theorem IV.2, we have $e(\cdot) = e_0(\cdot) + (\mathbb{F}_e^0 w_e - \mathbb{F}_e^\infty w_e)$, where $w_e(t) = [w_{dist}(t)^T, y_{ref}(t)^T]^T$ and $t \mapsto e^{\alpha_0 t}\|e_0(t)\| \in L^2(0,\infty)$ for some $\alpha_0 > 0$. Moreover, the identity (13) and Lemma A.1(b) imply that there exists $M_1 > 0$ depending only on $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ such that

$$\begin{aligned}\limsup_{t\to\infty}\|e(\cdot)\|_{L^2(t,t+1)} &\leq \limsup_{t\to\infty} \|\mathbb{F}_e^0 w_e - \mathbb{F}_e^\infty w_e\|_{L^2(t,t+1)} \\ &\leq M_1 \limsup_{t\to\infty}\Big(\|\Delta(\cdot)\|_{L^\infty(t,t+1)}\|\mathbb{F}_{ee}^0 w_e\|_{L^2(t,t+1)}\Big)\end{aligned}$$

where $\mathbb{F}_{ee}^0 = (I - \mathbb{F}_{ee}^\infty \Delta(\cdot))^{-1}\mathbb{F}_{ee}^\infty P_{in}$. Since $\mathbb{F}_{ee}^\infty$ is the extended input-output map of the regular linear system $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$, by Lemma A.1(a) there exists $M_0 > 0$ such that $\sup_{\tau\geq 0}\|\mathbb{F}_{ee}^\infty u\|_{L^2(\tau,\tau+1)} \leq M_0 \sup_{\tau\geq 0}\|u\|_{L^2(\tau,\tau+1)}$ for all $u \in L^2_{\text{loc}}(0,\infty;U_{ee})$. We define

$$\delta_0 = 2^{-3/2}\min\{\|\mathbb{F}_{ee}^\infty\|^{-1}, M_0^{-1}\}$$

and assume $\operatorname{ess\,sup}_{t\geq t_0}\delta_{\mathcal{G}}(t) \leq \delta_0$ for some fixed $t_0 > 0$. Since the definition of $\Delta(t)$ in (10) implies that $\|\Delta(t)\| \leq \sqrt{2}\delta_{\mathcal{G}}(t)$ for a.e. $t \geq 0$, we have $\|\mathbb{F}_{ee}^\infty\|\|\Delta\|_{L^\infty(t_0,\infty)} \leq 1/2$ and $M_0\|\Delta\|_{L^\infty(t_0,\infty)} \leq 1/2$.

By Theorem III.2, $(U_e, \Phi_e^{t,s}, \Psi_{ee}^s, \mathbb{F}_{ee}^s)_{t\geq s\geq 0}$ is an exponentially stable nonautonomous well-posed system. We have from [29, Def. 3.6] that for a.e. $t \geq t_0$

$$(\mathbb{F}_{ee}^0 w_e)(t) = (\Psi_{ee}^{t_0}\Phi_e^{t_0,0}w_e)(t) + (\mathbb{F}_{ee}^0 w_{t_0})(t), \tag{14}$$

where $w_{t_0} : [0,\infty) \to \mathbb{C}^p$ is defined so that $w_{t_0}(t) = 0$ for $t \in [0, t_0)$ and $w_{t_0}(t) = w_e(t)$ for $t \geq t_0$. Define $\Delta_{t_0} \in L^\infty(0,\infty;\mathcal{L}(Y_{ee}, U_{ee}))$ so that $\Delta_{t_0}(t) = 0$ for $t \in [0, t_0)$ and $\Delta_{t_0}(t) = \Delta(t)$ for a.e. $t \geq t_0$. Then $\|\Delta_{t_0}\|_{L^\infty} = \|\Delta\|_{L^\infty(t_0,\infty)}$, and the properties $(\mathbb{F}_{ee}^\infty P_{in} w_{t_0})(t) = 0$ for $t \in [0, t_0]$ and $\|\mathbb{F}_{ee}^\infty\|\|\Delta_{t_0}(\cdot)\|_{L^\infty} \leq 1/2$ imply

$$\mathbb{F}_{ee}^0 w_{t_0} = (I - \mathbb{F}_{ee}^\infty \Delta_{t_0}(\cdot))^{-1}\mathbb{F}_{ee}^\infty P_{in} w_{t_0} \tag{15a}$$

$$= \sum_{n=0}^\infty (\mathbb{F}_{ee}^\infty \Delta_{t_0}(\cdot))^n \mathbb{F}_{ee}^\infty P_{in} w_{t_0} \tag{15b}$$

with convergence in $L^2_{\text{loc}}(0,\infty;Y_{ee})$. The choice of $M_0 > 0$ and $\|w_e\|_{L^2(t,t+1)} \leq \|w_e\|_\infty$ for $t \geq 0$ imply that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\sup_{t\geq 0}\|(\mathbb{F}_{ee}^\infty \Delta_{t_0}(\cdot))^n \mathbb{F}_{ee}^\infty P_{in} w_{t_0}\|_{L^2(t,t+1)} \\
&\quad\leq (M_0\|\Delta_{t_0}\|_{L^\infty})^n \sup_{t\geq 0}\|\mathbb{F}_{ee}^\infty P_{in} w_{t_0}\|_{L^2(t,t+1)} \\
&\quad\leq 2^{-n} M_0 \sup_{t\geq 0}\|P_{in} w_{t_0}\|_{L^2(t,t+1)} \leq 2^{-n} M_0 \|w_e\|_\infty.
\end{aligned}$$

Thus (15) implies $\sup_{t\geq 0}\|\mathbb{F}_{ee}^0 w_{t_0}\|_{L^2(t,t+1)} \leq 2M_0\|w_e\|_\infty$. Since $\|\Psi_{ee}^{t_0}\Phi_e^{t_0,0}w_e\|_{L^2(t,t+1)} \to 0$ as $t \to \infty$ and since $\|\Delta(t)\| \leq \sqrt{2}\delta_{\mathcal{G}}(t)$ for a.e. $t \geq 0$, equation (14) finally implies

$$\begin{aligned}
&\limsup_{t\to\infty}\|e(\cdot)\|_{L^2(t,t+1)} \\
&\leq M_1 \limsup_{t\to\infty}\Big(\|\Delta(\cdot)\|_{L^\infty(t,t+1)}\|\mathbb{F}_{ee}^0 w_{t_0}\|_{L^2(t,t+1)}\Big) \\
&\leq 2M_0M_1\|w_e\|_\infty \limsup_{t\to\infty}\|\Delta(\cdot)\|_{L^\infty(t,t+1)} \\
&\leq 2\sqrt{2}M_0M_1\|w_e\|_\infty \lim_{t\to\infty}\|\delta_{\mathcal{G}}(\cdot)\|_{L^\infty(t,\infty)}. \qquad \square
\end{aligned}$$

Remark IV.5. The proof of Theorem IV.4 shows that $M_{err}, \delta_0 > 0$ depend on the norm $\|\mathbb{F}_{ee}^\infty\|$ of the extended input-output map of the autonomous system $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ and on $M_0, M_1 > 0$ in Lemma A.1(a)–(b) corresponding to this system. Moreover, the proof of Lemma A.1, implies that $M_0, M_1 > 0$ are determined by constants $M_e, \alpha_e > 0$ such that $\|T_e(t)\| \leq M_e e^{-\alpha_e t}$ for $t \geq 0$ and by upper bounds for the norms of the input, output, and input–output map of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ on the time interval $[0, 1]$.

Remark IV.6. By [24, Thm. 7] the internal model of $\mathcal{G}_1^\infty$ in Theorems IV.2 and IV.4 can be replaced with the conditions

$$\mathcal{R}(\pm i\omega_k - \mathcal{G}_1^\infty) \cap \mathcal{R}(\mathcal{G}_2^\infty) = \{0\} \quad \forall k \in \{0, \ldots, q\} \tag{16a}$$

$$\mathcal{N}(\mathcal{G}_2^\infty) = \{0\}. \tag{16b}$$

V. Controller Design for Output Regulation with Unknown Frequencies

In this section we introduce a controller for output regulation of $y_{ref}(t)$ and $w_{dist}(t)$ with unknown frequencies. Our controller contains a time-varying internal model of *estimates* $(\hat{\omega}_k(t))_{k=1}^q$ of $(\omega_k)_{k=1}^q$ in (1) and an observer-based part for achieving closed-loop stability. The frequency estimates are formed based on an auxiliary output $y_{aux}(t)$ of the controller which contains the information on $(\omega_k)_k$ but is by design independent of the time-varying parts of the controller. Therefore our controller can be combined with any estimation method which can asymptotically estimate the frequencies $(\omega_k)_k$ from $y_{aux}(t)$. We solve the problem under Assumptions V.1 and V.3.

Assumption V.1. There exist $\tilde{K} \in \mathcal{L}(X, \mathbb{C}^m)$ and $L \in \mathcal{L}(\mathbb{C}^p, X)$ such that the semigroups generated by $A + B\tilde{K} : \mathcal{D}(A + B\tilde{K}) \subset X \to X$ with domain $\mathcal{D}(A + B\tilde{K}) = \{\, x \in X \mid Ax + B\tilde{K}x \in X \,\}$ and $A + LC : \mathcal{D}(A) \subset X \to X$ are exponentially stable.

Definition V.2 ([21, Def. V.1])**.** The point $i\lambda_0 \in i\mathbb{R}$ *is a transmission zero of* (A, B, C, D) if $P_{\tilde{K}}(i\lambda_0)$ is not surjective, where $\tilde{K} \in \mathcal{L}(X, \mathbb{C}^m)$ is such that $i\lambda_0 \in \rho(A + B\tilde{K})$ and $P_{\tilde{K}}(\lambda)$ is the transfer function of $(A + B\tilde{K}, B, C + D\tilde{K}, D)$.

Assumption V.3. Assume $y_{ref}(t)$ and $w_{dist}(t)$ are of the form (1) with $0 = \omega_0 < \omega_1 < \ldots < \omega_q$ and (A, B, C, D) does not have transmission zeros at $\{0\} \cup \{\pm i\omega_k\}_{k=1}^q$.

We begin by introducing a general controller structure in Section V-A and present general conditions for output regulation in the situation where $(\hat{\omega}_k(t))_{k=1}^q$ converge to $(\omega_k)_k$ in (1). We analyse the structure of $y_{aux}(t)$ in Lemma V.6 and in Remark V.7 we list selected methods which can be used to estimate the frequencies based on $y_{aux}(t)$. In Section V-B we present our Controller Tuning Algorithm for constructing the controller parameters in order to achieve closed-loop stability and output regulation under Assumptions V.1 and V.3. The tuning is based completely on design of *autonomous* feedback and output injections. Finally, in Section V-C we analyse the robustness properties of our controller.

A. Controller with a time-varying internal model

Our error feedback controller has the form

$$\dot{z}(t) = \mathcal{G}_1(t)z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0 \in Z \tag{17a}$$

$$u(t) = K(t)z(t) \tag{17b}$$

$$y_{aux}(t) = K_{aux}(t)z(t) + e(t) \tag{17c}$$

where $e(t) = y(t) - y_{ref}(t)$ is the regulation error. The controller structure in Definition V.4 generalises the autonomous robust controller in [17, Sec. 7] and the adaptive internal model based controller scheme in [8, Sec. 4], where a separate "residual generator" system was used to construct $y_{aux}(t)$.

Definition V.4. The controller $(\mathcal{G}_1(t), \mathcal{G}_2, K(t))$ on $Z = Z_0 \times X$ with $Z_0 = \mathbb{C}^{p(2q+1)}$ is defined by choosing $L \in \mathcal{L}(\mathbb{C}^p, X)$ so that $A + LC$ generates an exponentially stable semigroup $T_L(t)$, $K(\cdot) = [K_1(\cdot), K_2(\cdot)] \in L^\infty(0, \infty; \mathcal{L}(Z, \mathbb{C}^m))$, and

$$\mathcal{G}_1(t) = \begin{bmatrix} G_1(t) & 0 \\ (B+LD)K_1(t) & A + LC_\Lambda + (B+LD)K_2(t) \end{bmatrix}$$

$$\mathcal{D}(\mathcal{G}_1(t)) = \{ [\begin{smallmatrix} z_0 \\ x \end{smallmatrix}] \in Z_0 \times \mathcal{D}(C_\Lambda) \mid Ax + BK(t)[\begin{smallmatrix} z_0 \\ x \end{smallmatrix}] \in X \}$$

$$\mathcal{G}_2 = \begin{bmatrix} G_2 \\ -L \end{bmatrix}, \ K_{aux}(t) = [-DK_1(t), -C_\Lambda - DK_2(t)]$$

with $\mathcal{D}(K_{aux}(t)) = Z_0 \times \mathcal{D}(C_\Lambda)$. Finally, we define

$$G_1(t) = \operatorname{diag}(0_p, \hat{\omega}_1(t)\Omega_p, \ldots, \hat{\omega}_q(t)\Omega_p) \in \mathcal{L}(Z_0),$$
$$G_2 = \left[I_p, I_p, 0_p, I_p, 0_p, \ldots, I_p, 0_p\right]^T \in \mathbb{R}^{p(2q+1)\times p}$$

with $\Omega_p = \left[\begin{smallmatrix} 0_p & I_p \\ -I_p & 0_p \end{smallmatrix}\right]$, where $0_p, I_p \in \mathbb{R}^{p\times p}$ are the zero and identity matrices and $\hat{\omega}_k(\cdot) \in L^\infty(0, \infty; \mathbb{R}_+)$ for all k.

The function $G_1(\cdot)$ is the *time-varying internal model* which contains the estimates $(\hat{\omega}_k(t))_{k=1}^q$ of the frequencies in $w_{dist}(t)$ and $y_{ref}(t)$. By construction, for every $t \geq 0$ the pair $(G_1(t), G_2)$ is controllable if the values $(\hat{\omega}_k(t))_{k=1}^q$ are distinct and nonzero. If $|\hat{\omega}_k(t) - \omega_k| \to 0$ as $t \to \infty$ for all k, then $\|G_1(t) - G_1^\infty\| \to 0$ as $t \to \infty$ where $G_1^\infty \in \mathcal{L}(Z_0)$ is defined by replacing $(\hat{\omega}_k(t))_k$ in $G_1(t)$ with $(\omega_k)_k$.

For any $G_1^\infty \in \mathcal{L}(Z_0)$ and $K^\infty = [K_1^\infty, K_2^\infty] \in \mathcal{L}(Z, \mathbb{C}^m)$ we can define $\Delta_G(t) = G_1(t) - G_1^\infty$, $\Delta_K(t) = K(t) - K^\infty$,

$$\mathcal{G}_1^\infty = \begin{bmatrix} G_1^\infty & 0 \\ (B+LD)K_1^\infty & A + LC_\Lambda + (B+LD)K_2^\infty \end{bmatrix} \quad (18a)$$

$$\mathcal{G}_{11}^\infty = \begin{bmatrix} I & 0 \\ 0 & B + LD \end{bmatrix}, \qquad \Delta_{\mathcal{G}_1}(t) = \begin{bmatrix} [\Delta_G(t), 0] \\ \Delta_K(t) \end{bmatrix}. \quad (18b)$$

The feedback theory for regular linear systems in [40] implies that $\mathcal{G}_1^\infty$ with domain $\mathcal{D}(\mathcal{G}_1^\infty) = \{ [\begin{smallmatrix} z_0 \\ x \end{smallmatrix}] \in Z_0 \times \mathcal{D}(C_\Lambda) \mid Ax + BK^\infty[\begin{smallmatrix} z_0 \\ x \end{smallmatrix}] \in X \}$ generates a strongly continuous semigroup on Z and that $\mathcal{G}_{11}^\infty$ is an admissible input operator for this semigroup. Since $\mathcal{G}_1(t) = \mathcal{G}_1^\infty + \mathcal{G}_{11}^\infty \Delta_{\mathcal{G}_1}(t)$ and $\Delta_{\mathcal{G}_1}(\cdot) \in L^\infty(0, \infty; \mathcal{L}(Z, Z_0 \times U))$, the controller in Definition V.4 satisfies Assumption II.2. Therefore the well-defined mild closed-loop state $x_e(t)$ and regulation error $e(t)$ are guaranteed by Theorem III.2.

Our first result shows that if $(\hat{\omega}_k(t))_k$ converge to the true frequencies and if $K(\cdot)$ is such that the semigroup generated by the block operator (19) is stable, then the controller achieves output regulation. The construction of $K(\cdot)$ to stabilize (19) will be presented in Section V-B.

Theorem V.5. *Choose $(\mathcal{G}_1(t), \mathcal{G}_2, K(t))$ as in Definition* V.4. *Assume that $w_{dist}(t)$ and $y_{ref}(t)$ and the intial conditions $x_0 \in X$ and $z_0 \in Z$ are such that*

$$|\hat{\omega}_k(t) - \omega_k| \to 0 \quad \textit{and} \quad \|K(t) - K^\infty\| \to 0$$

as $t \to \infty$ for all k and for some $K^\infty \in \mathcal{L}(Z, \mathbb{C}^m)$. If the semigroup generated by

$$A_s^\infty + B_s K^\infty := \begin{bmatrix} G_1^\infty & G_2 C_\Lambda \\ 0 & A \end{bmatrix} + \begin{bmatrix} G_2 D \\ B \end{bmatrix} K^\infty \qquad (19)$$

with domain $\{ [z_0, x]^T \in Z_0 \times \mathcal{D}(C_\Lambda) \mid Ax + BK^\infty[\begin{smallmatrix} z_0 \\ x \end{smallmatrix}] \in X \}$ is exponentially stable, then

$$\int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds \to 0, \qquad \textit{as} \quad t \to \infty$$

and $U_e(t,s)$ is exponentially stable. If $\operatorname{ess\,sup}_{t\geq 0} e^{\alpha t}|\hat{\omega}_k(t) - \omega_k| < \infty$ and $\operatorname{ess\,sup}_{t\geq 0} e^{\alpha t}\|K(t) - K^\infty\| < \infty$ for some $\alpha > 0$ and for all k, then there exists $\alpha_e > 0$ such that $t \mapsto e^{\alpha_e t}(y(t) - y_{ref}(t)) \in L^2(0, \infty; Y)$.

Proof. Let $w_{dist}(t)$, $y_{ref}(t)$, $x_0 \in X$, and $z_0 \in Z$ be such that the assumptions hold. If we define $G_1^\infty \in \mathcal{L}(Z_0)$ by replacing $(\hat{\omega}_k(t))_k$ in $G_1(t)$ with $(\omega_k)_k$ and let $\Delta_G(t) = G_1(t) - G_1^\infty$ and $\Delta_K(t) = K(t) - K^\infty$, then $\Delta_{\mathcal{G}_1}(t)$ in (18) satisfies $\|\Delta_{\mathcal{G}_1}(t)\| \to 0$ as $t \to \infty$. As shown in the proof of [24, Thm. 15], the pair $(\mathcal{G}_1^\infty, \mathcal{G}_2)$ satisfies the "$\mathcal{G}$-conditions" (16). In view of Remark IV.6, the claims follow from Theorem IV.2 and Lemma IV.3 once we show that the semigroup $T_e(t)$ generated by A_e^∞ is exponentially stable.

The operator A_e^∞ is exactly the operator $A_e(t)$ with $\mathcal{G}_1(t)$ and $K(t)$ replaced with $\mathcal{G}_1^\infty$ and K^∞, respectively, i.e.,

$$A_e^\infty = \begin{bmatrix} A & BK_1^\infty & BK_2^\infty \\ G_2 C_\Lambda & G_1^\infty + G_2 D K_1^\infty & G_2 D K_2^\infty \\ -LC_\Lambda & BK_1^\infty & A + LC_\Lambda + BK_2^\infty \end{bmatrix}.$$

If we define $Q_e \in \mathcal{L}(X \times Z_0 \times X, Z_0 \times X \times X)$ by

$$Q_e = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ -I & 0 & I \end{bmatrix}, \quad Q_e^{-1} = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & I & I \end{bmatrix}, \qquad (20)$$

a direct computation shows that

$$Q_e A_e^\infty Q_e^{-1} = \begin{bmatrix} A_s^\infty + B_s K^\infty & B_s K_2^\infty \\ 0 & A + LC \end{bmatrix} \qquad (21)$$

with $\mathcal{D}(Q_e A_e^\infty Q_e^{-1}) = \{ [x_s, \tilde{x}]^T \in (Z_0 \times \mathcal{D}(C_\Lambda)) \times \mathcal{D}(A) \mid A_s x_s + B_s(K^\infty x_s + K_2^\infty \tilde{x}) \in Z_0 \times X \}$. By assumption the semigroups generated by $A + LC : \mathcal{D}(A) \subset X \to X$ and $A_s + B_s K^\infty$ are exponentially stable. Moreover, B_s is an admissible input operator for the semigroup generated by $A_s + B_s K^\infty$ by the results in [40, Sec. 7]. Thus the semigroup generated by $Q_e A_e^\infty Q_e^{-1}$ is exponentially stable and similarity implies the same for $T_e(t)$. The claims now follow from Theorem IV.2 with Lemma IV.3 and Remark IV.6. ☐

We conclude this section by analysing $y_{aux}(t)$. Lemma V.6 in particular shows that $y_{aux}(t)$ is independent of the time-varying parameters $(\hat{\omega}_k(t))_{k=1}^q$ and $K(t)$. The form of $y_{aux}(t)$ involves $B_{dL} = [B_d + LD_d, -L]$ and the transfer function

$$P_{tot,L}(\lambda) = C_\Lambda R(\lambda, A + LC_\Lambda) B_{dL} + [D_d, -I]$$

of the regular linear system $(A + LC, B_{dL}, C, [D_d, -I])$.

Lemma V.6. *Let $x_0 \in X$ and $z_0 = (z_{10}, z_{20}) \in Z$ and let $y_{ref}(t)$ and $w_{dist}(t)$ be as in* (1). *Consider the controller $(\mathcal{G}_1(t), \mathcal{G}_2, K(t))$ in Definition* V.4, *let $T_L(t)$ be the semigroup generated by $A + LC$ and denote $\omega_{-k} := -\omega_k$ for $k \in \{1, \ldots, q\}$. Then for a.e. $t \geq 0$,*

$$y_{aux}(t) = y_0(t) + \sum_{k=-q}^{q} e^{i\omega_k t} P_{tot,L}(i\omega_k) c_e^k \qquad (22)$$

with $c_e^0 = \begin{bmatrix} c_0 \\ a_0 \end{bmatrix}$, $c_e^{\pm k} = \frac{1}{2}\begin{bmatrix} c_k \mp id_k \\ a_k \mp ib_k \end{bmatrix}$ *for* $k \in \{1,\dots,q\}$*, and*

$$y_0(t) = C_\Lambda T_L(t)\big(x_0 - z_{20} - \sum_{k=-q}^{q} R(i\omega_k, A+LC)B_{dL}c_e^k\big).$$

We have $t \mapsto e^{\alpha t}y_0(t) \in L^2(0,\infty;\mathbb{C}^p)$ *for some* $\alpha > 0$*. Moreover, if* $C \in \mathcal{L}(X,\mathbb{C}^p)$*, if* $x_0 - z_{20} \in \mathcal{D}(A)$*, or if* A *generates an analytic semigroup, then* $y_0(\cdot)$ *is continuous and* $e^{\alpha t}\|y_0(t)\| \to 0$ *as* $t \to \infty$ *for some* $\alpha > 0$*.*

Proof. Denote $A_L = A + LC$, $B_{dL} = [B_d + LD_d, -L]$, and

$$A_s(t) = \begin{bmatrix} G_1(t) & G_2C_\Lambda \\ 0 & A \end{bmatrix}.$$

Since Assumption II.2 holds the closed-loop state $x_e(t) = [x(t), z_1(t), z_2(t)]^T$ and the regulation error $e(t)$ in (6) are well-defined by Theorem III.2. With $B_s \in \mathcal{L}(U, Z_0 \times X_{-1})$ in (19) and Q_e in (20) we have

$$Q_eB_e(t) \equiv Q_e \begin{bmatrix} B_d & 0 \\ G_2D_d & -G_2 \\ -LD_d & L \end{bmatrix} = \begin{bmatrix} G_2D_d & -G_2 \\ B_d & 0 \\ -B_d - LD_d & L \end{bmatrix}$$
$$Q_eA_e(t)Q_e^{-1} = \begin{bmatrix} A_s(t) + B_sK(t) & B_sK_2(t) \\ 0 & A_L \end{bmatrix}$$

with $\mathcal{D}(Q_eA_e(t)Q_e^{-1}) = \{\, [x_s, \tilde{x}]^T \in (Z_0 \times \mathcal{D}(C_\Lambda)) \times \mathcal{D}(A) \mid A_s(t)x_s + B_s(K(t)x_s + K_2(t)\tilde{x}) \in Z_0 \times X \,\}$ for a.e. $t \geq 0$. Consequently $Q_eU_e(t,s)Q_e^{-1}$ has a block triangular form for all $t \geq s \geq 0$. Applying the similarity transformation Q_e in (20) to (6a) and (7a) therefore shows that $\tilde{x}(t) = x(t) - z_2(t)$ is the mild solution of

$$\dot{\tilde{x}}(t) = A_L\tilde{x}(t) + B_{dL}w_e(t), \qquad \tilde{x}(0) = x_0 - z_{20} \tag{23}$$

with $w_e(t) = [w_{dist}(t)^T, y_{ref}(t)^T]^T$. Since C is admissible with respect to $T_L(t)$ generated by A_L and $B_{dL} \in \mathcal{L}(\mathbb{C}^{n_d+p}, X)$, we have $\tilde{x}(t) \in \mathcal{D}(C_\Lambda)$ for a.e. $t \geq 0$. Moreover, $x(t) \in \mathcal{D}(C_\Lambda)$ for a.e. $t \geq 0$ by Theorem III.2. Thus $z_2(t) = x(t) - \tilde{x}(t) \in \mathcal{D}(C_\Lambda)$ for a.e. $t \geq 0$ and the formula (17c) for $y_{aux}(t)$ is well-defined for a.e. $t \geq 0$. Since $K_{aux}(t)z(t) = -C_\Lambda z_2(t) - DK(t)z(t)$, Remark III.3 and (2b) imply that

$$\begin{aligned} y_{aux}(t) &= -C_\Lambda z_2(t) - Du(t) + y(t) - y_{ref}(t) \\ &= C_\Lambda\tilde{x}(t) + [D_d, \, -I]w_e(t) \end{aligned}$$

for a.e. $t \geq 0$. Thus $y_{aux}(t)$ is the output of the regular linear system $(A_L, B_{dL}, C, [D_d, -I])$ with initial state $\tilde{x}(0) = x_0 - z_{20} \in X$ and input $w_e(t)$. When $\tilde{x}(0) = 0$ and $w_e(t) = e^{i\omega_k t}w_0$ for some $w_0 \in \mathbb{C}^{n_d+p}$, [32, Cor. 4.6.13] implies

$$y_{aux}(t) = e^{i\omega_k t}P_{tot,L}(i\omega_k)w_0 - C_\Lambda T_L(t)R(i\omega_k, A_L)B_{dL}w_0.$$

Finally, linearity implies that for $\tilde{x}(0) = x_0 - z_{20}$ and $w_e(t) = [w_{dist}(t)^T, y_{ref}(t)^T]^T$ the output $y_{aux}(t)$ has the form in (22) with the given $\{c_e^k\}_{k=-q}^q$ and $y_0(t)$. Since C is admissible with respect to the exponentially stable semigroup $T_L(t)$, we have $t \mapsto e^{\alpha t}y_0(t) \in L^2(0,\infty;\mathbb{C}^p)$ for some $\alpha > 0$.

In the last claim, if $C \in \mathcal{L}(X,\mathbb{C}^p)$, then pointwise convergence of $y_0(t)$ follows directly from stability of $T_L(t)$. In the other cases $x_1 := \sum_{k=-q}^q R(i\omega_k, A_L)B_{dL}c_e^k \in \mathcal{D}(A_L)$ and $CA_L^{-1} \in \mathcal{L}(X,\mathbb{C}^p)$ imply that $y_0(t) = CA_L^{-1}A_LT_L(t)(x_0 - z_{20} - x_1) \to 0$ at an exponential rate as $t \to \infty$. □

Remark V.7 (Methods for Frequency Estimation)**.** Multi-frequency estimators based on dynamical adaptive observers have been developed by several authors in, e.g., [3], [7], [9], [19], [23], [36], [37], [41][3]. Our controller requires frequency estimation in the presence of the nonsmooth decaying part $y_0(t)$ of $y_{aux}(t)$ (i.e., the estimator is required to be input-to-state stable). One such adaptive estimator was introduced in [7] (see [7, Rem. 3]). The estimator in [7] is compatible with our control scheme and it only requires knowing the number q of frequencies and nonzero amplitudes. Several estimators are also capable of online estimation of q and have desirable transient performance and robustness properties [3], [8], [9].

Estimation of $(\omega_k)_{k=1}^q$ from $y_{aux}(t)$ requires that all frequencies appear in the non-decaying part of $y_{aux}(t)$. This is generically true since the amplitudes corresponding to $\pm\omega_{k_0}$ are zero only if a_{k_0}, b_{k_0}, c_{k_0}, and d_{k_0} in (1) are related in a very specific way through the identity $P_{tot,L}(\pm i\omega_{k_0})c_e^{\pm k_0} = 0$.

B. The Controller Tuning Algorithm

In this section we introduce an algorithm for constructing $(\hat{\omega}_k(\cdot))_{k=1}^q$ and $K(\cdot)$ in the controller. Even though several estimators provide continuous-time estimates of $(\omega_k)_{k=1}^q$, we choose the estimates $(\hat{\omega}_k(\cdot))_{k=1}^q$ in the internal model to be piecewise constant functions which are updated at predefined time instances $0 = t_0 < t_1 < t_2 < \cdots$ (via sample-and-hold). This way we can guarantee stable closed-loop behaviour during the update intervals $[t_j, t_{j+1}]$ despite possible rapid changes in the frequency estimates. The algorithm utilises the *Estimate Admissibility Condition* defined below.

Definition V.8. Let $\varepsilon_f > 0$ and $M_f > 0$. We say that $(\hat{\omega}_k)_{k=1}^q$ satisfy the *Estimate Admissibility Condition* EAC(M_f, ε_f) for the system (A, B, C, D) if the following hold:

- $\varepsilon_f \leq |\hat{\omega}_k| \leq M_f$ for all $k \in \{1,\dots,q\}$.
- $|\hat{\omega}_k - \hat{\omega}_j| \geq \varepsilon_f$ for all $k \neq j$.
- $|i\hat{\omega}_k \pm i\lambda| \geq \varepsilon_f$ for every transmission zero $i\lambda \in i\mathbb{R}$ of (A, B, C, D) and for all k.

The algorithm uses Theorem V.9 below to stabilize the pairs $(A_s(t_j), B_s)$, where

$$A_s(t) = \begin{bmatrix} G_1(t) & G_2C \\ 0 & A \end{bmatrix}, \quad B_s = \begin{bmatrix} G_2D \\ B \end{bmatrix} \tag{24}$$

with $G_1(t) = \operatorname{diag}(0_p, \hat{\omega}_1(t)\Omega_p, \dots, \hat{\omega}_q(t)\Omega_p)$. The result guarantees that the stabilizing feedback gains K^j are *a priori* bounded and the stabilized semigroups satisfy uniform decay estimates with $M_s, \alpha_s > 0$ independent of t_j. Moreover, in this method the stabilizing gain of the infinite-dimensional pair (A, B) does not need to be recomputed when the frequencies in $G_1(t)$ are updated. The assumptions of the theorem will be guaranteed by our tuning algorithm. A similar method has

[3]While many estimators are introduced only for scalar-valued signals, they can also be used if $p > 1$ by replacing $y_{aux}(t)$ with $r^T y_{aux}(t)$ where $r \in \mathbb{R}^p$ is a fixed random vector. The randomness of r guarantees the presence of all frequency components in $r^T y_{aux}(t)$ with probability 1.

been previously used in [17], [24] (also [22, Thm. 3.7]) for internal models with fixed frequencies. The result uses notation

$$H_0(i\omega) = \frac{1}{2}\begin{bmatrix} C_K R_K(i\omega) + C_K R_K(-i\omega) \\ iC_K R_K(i\omega) - iC_K R_K(-i\omega) \end{bmatrix}$$
$$B_0(i\omega) = \frac{1}{2}\begin{bmatrix} P_K(i\omega) + P_K(-i\omega) \\ iP_K(i\omega) - iP_K(-i\omega) \end{bmatrix},$$

where $R_K(\lambda) = R(\lambda, A + BK_{21})$, $C_K = C_\Lambda + DK_{21}$, and $P_K(\lambda) = (C_\Lambda + DK_{21})R(\lambda, A + BK_{21})B + D$.

Theorem V.9. *Let $\varepsilon_f, M_f, r > 0$ and let $Q_1 \in \mathcal{L}(\mathbb{C}^{p(2q+1)})$ and $R_1 \in \mathbb{C}^{m\times m}$ be positive definite. Assume $K_{21} \in \mathcal{L}(X, \mathbb{C}^m)$ is such that $A + BK_{21}$ generates an exponentially stable semigroup $T_K(t)$ with growth bound $\omega_0(T_K(t)) < 0$. Assume $t_j \geq 0$ is such that $(\hat{\omega}_k(t_j))_{k=1}^q$ satisfy* EAC(M_f, ε_f) *in Definition* V.8 *for the system (A, B, C, D). Define $H_j \in \mathcal{L}(X, \mathbb{C}^{p(2q+1)})$ and $B_{1j} \in \mathbb{C}^{p(2q+1)\times m}$ by*

$$H_j x = \begin{bmatrix} C_K R_K(0)x \\ H_0(i\hat{\omega}_1(t_j))x \\ \vdots \\ H_0(i\hat{\omega}_q(t_j))x \end{bmatrix} \quad \text{and} \quad B_{1j} = \begin{bmatrix} P_K(0) \\ B_0(i\hat{\omega}_1(t_j)) \\ \vdots \\ B_0(i\hat{\omega}_q(t_j)) \end{bmatrix}$$

Choose $K_1^j = -R_1^{-1}B_{1j}^*\Pi_{1j} \in \mathbb{C}^{m\times p(2q+1)}$ where $\Pi_{1j} \in \mathcal{L}(\mathbb{C}^{p(2q+1)})$ is the unique non-negative solution of

$$(rI + G_1(t_j))^*\Pi_{1j} + \Pi_{1j}(rI + G_1(t_j)) - \Pi_{1j}B_{1j}R_1^{-1}B_{1j}^*\Pi_{1j} = -Q_1.$$

If we choose $K^j = [K_1^j, K_{21} + K_1^j H_j] \in \mathcal{L}(Z, \mathbb{C}^m)$, then $\|K^j\| \leq M_K$ for some $M_K > 0$ independent of t_j. Moreover, the semigroup $T_s^j(t)$ generated by $A_s(t_j) + B_s K^j$ is exponentially stable so that for any $0 < \alpha_s < \min\{r, -\omega_0(T_K(t))\}$ there exists $M_s, M_B > 0$ (independent of t_j) such that

$$\|T_s^j(t)\| \leq M_s e^{-\alpha_s t}, \qquad t \geq 0 \tag{25}$$

and $\|R(\lambda, A_s(t_j) + B_s K^j)B_s\| \leq M_B$ for $\lambda \in \mathbb{C}_+$. If $(\hat{\omega}_k(t_j))_{k=1}^q$ satisfy EAC(M_f, ε_f) *for all j large and if $\max_k|\hat{\omega}_k(t_j) - \omega_k| \to 0$ as $j \to \infty$, then $\lim_{j\to\infty}\|K^j - K^\infty\| = 0$, where $K^\infty = [K_1^\infty, K_{21} + K_1^\infty H_\infty]$ is obtained by replacing $(\hat{\omega}_k(t_j))_k$ with $(\omega_k)_k$ in $G_1(t_j)$, B_{1j}, and H_j.*

The proof of Theorem V.9 is presented in the Appendix. In the tuning algorithm we denote by $0 < \mu_1(t) < \cdots < \mu_q(t)$ the estimated frequencies computed based the signal $y_{aux}(t)$ by the separate frequency estimator. We make the following assumptions on the parameters of the algorithm.

Assumption V.10 (Tuning Parameters)**.**

- The sequence $0 = t_0 < t_1 < \ldots$ of update times satisfies $\tau_1 \leq t_j - t_{j-1} \leq \tau_2$ for some $\tau_1, \tau_2 > 0$ and all $j \in \mathbb{N}$.
- The matrices $R_1 \in \mathbb{C}^{m\times m}$ and $Q_1 \in \mathcal{L}(\mathbb{C}^{p(2q+1)})$ are positive definite and $r > 0$.
- The frequency overlap parameter $\varepsilon_f > 0$ is suitably small and the upper bound $M_f > 0$ for the frequencies is suitably large.
- The operator $K_{21} \in \mathcal{L}(X, \mathbb{C}^m)$ is such that the semigroup generated by $A + BK_{21}$ is exponentially stable.
- The initial frequency estimates $(\mu_k(0))_{k=1}^q$ satisfy EAC(M_f, ε_f) for (A, B, C, D).

The Controller Tuning Algorithm below constructs $G_1(\cdot)$ (based on $(\hat{\omega}_k(\cdot))_{k=1}^q$) and $K(\cdot)$ in the observer-based controller in Definition V.4. The condition EAC(M_f, ε_f) in Definition V.8 is used in **Step 1** to detect if the frequency estimates nearly overlap or are close to the transmission zeros of (A, B, C, D). In both cases the closed-loop stabilization becomes difficult, and therefore the the algorithm does not update the frequencies of the internal model if EAC(M_f, ε_f) is violated (**Step 2** vs. **Step 3**). This update strategy guarantees that the assumptions of Theorem V.9 are satisfied in **Step 2**.

The Controller Tuning Algorithm: *Choose $(t_j)_{j=0}^\infty$, R_1, Q_1, $\varepsilon_f, M_f, r > 0$, and K_{21} as in Assumption* V.10*. Set $j = 0$.*

Step 1. *Obtain $0 < \mu_1(t_j) < \cdots < \mu_q(t_j)$ from the frequency estimator. If $(\mu_k(t_j))_{k=1}^q$ satisfy* EAC(M_f, ε_f) *for (A, B, C, D), then go to* **Step 2***. Otherwise go to* **Step 3***.*

Step 2 (Frequency update). *Set $\hat{\omega}_k(t) \equiv \mu_k(t_j)$ for $t \in [t_j, t_{j+1})$ and all k. Choose $K^j = [K_1^j, K_{21} + K_1^j H_j] \in \mathcal{L}(Z, \mathbb{C}^m)$ as in* Theorem V.9 *and set $K(t) \equiv K^j$ for $t \in [t_j, t_{j+1})$. Increment j to $j + 1$ and go to* **Step 1***.*

Step 3 (No frequency update). *Set $\hat{\omega}_k(t) \equiv \hat{\omega}_k(t_{j-1})$ and $K(t) \equiv K(t_{j-1})$ for $t \in [t_j, t_{j+1})$ and all k. Increment j to $j + 1$ and go to* **Step 1***.*

Since the initial frequency estimates $(\mu_k(0))_{k=1}^q$ are assumed to satisfy EAC(M_f, ε_f), the tuning algorithm will proceed to **Step 2** when $j = 0$, and therefore $G_1(\cdot)$ and $K(\cdot)$ are well-defined on $[0, \infty)$. Our main result below shows that if the estimates $(\mu_k(t))_k$ converge to the true frequencies $(\omega_k)_k$ in (1), then the controller constructed with the above algorithm achieves output regulation of $y_{ref}(t)$ and $w_{dist}(t)$.

Theorem V.11. *Let Assumptions* V.1*,* V.3*, and* V.10 *hold. Consider the controller $(\mathcal{G}_1(t), \mathcal{G}_2, K(t))$ in Definition* V.4*, where $(\hat{\omega}_k(\cdot))_{k=1}^q$ and $K(\cdot)$ are based on the Controller Tuning Algorithm. Assume that the true frequencies $(\omega_k)_{k=1}^q$ of $y_{ref}(t)$ and $w_{dist}(t)$ satisfy* EAC$(\tilde{M}_f, \tilde{\varepsilon}_f)$ *for (A, B, C, D) with some $\tilde{\varepsilon}_f > \varepsilon_f$ and $0 < \tilde{M}_f < M_f$ and $P_{tot,L}(\pm i\omega_k)c_e^{\pm k} \neq 0$ for all k in Lemma* V.6*.*

The controller satisfies $G_1(\cdot) \in L^\infty(0, \infty; \mathcal{L}(Z_0))$ and $K(\cdot) \in L^\infty(0, \infty; \mathcal{L}(Z, \mathbb{C}^m))$. If $w_{dist}(t)$ and $y_{ref}(t)$ and the intial conditions $x_0 \in X$ and $z_0 \in Z$ are such that $|\mu_k(t) - \omega_k| \to 0$ as $t \to \infty$ for all k, then $U_e(t, s)$ is exponentially stable and

$$\int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds \to 0 \qquad \text{as} \quad t \to \infty.$$

Proof. We have $|\hat{\omega}_k(t)| \leq M_f$ for all k and $t \geq 0$ by construction and $\|K^j\| \leq M_K$ for all $j \in \mathbb{N}_0$ by Theorem V.9. Thus $G_1(\cdot) \in L^\infty(0, \infty; \mathcal{L}(Z_0))$ and $K(\cdot) \in L^\infty(0, \infty; \mathcal{L}(Z, \mathbb{C}^m))$ for any frequency estimates $(\mu_k(\cdot))_k$. Thus by Lemma V.6, $y_{aux}(t)$ has the form (22) and is independent of $G_1(\cdot)$ and $K(\cdot)$. Assume now that $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$, and $y_{ref}(t)$ are such that $|\mu_k(t) - \omega_k| \to 0$ as $t \to \infty$ for all k. Since $\tilde{\varepsilon}_f > \varepsilon_f$ and $0 < \tilde{M}_f < M_f$, there exists $N \in \mathbb{N}$ such that $(\mu_k(t_j))_{k=1}^q$ satisfy EAC(M_f, ε_f) for all $j \geq N$. Thus the algorithm will go to **Step 2** for all $j \geq N$, and the frequency estimates of the internal model satisfy

$\hat{\omega}_k(t) \to \omega_k$ as $t \to \infty$ for all k. By Theorem V.9 we have $K(t) \to K^\infty = [K_1^\infty, K_{21}+K_1^\infty H_\infty] \in \mathcal{L}(Z, \mathbb{C}^m)$ as $t \to \infty$, where K_1^∞ and H_∞ are obtained by replacing $(\hat{\omega}_k(t_j))_k$ with $(\omega_k)_k$ in $G_1(t_j)$, B_{1j}, and H_j. The claims will follow from Theorem V.5 provided that the semigroup generated by $A_s^\infty + B_s K^\infty$ is exponentially stable. However, since $(\omega_k)_{k=1}^q$ satisfy EAC(M_f, ε_f) by assumption, the stability of this semigroup follows directly from Theorem V.9 when we replace $(\hat{\omega}_k(t_j))_k$ with $(\omega_k)_k$ in $G_1(t_j)$, B_{1j}, and H_j. □

The following table summarises the parameters of the controller (17).

Controller parameters	**Origin**
$\mathcal{G}_1(t)$, $\mathcal{G}_2$, $K_{aux}(t)$	Definition V.4
$(t_j)_{j=0}^\infty$, $R_1, Q_1, \varepsilon_f, M_f, r, K_{21}$	Assumption V.10
$(\mu_k(t))_{k=1}^q$	Frequency estimator
$(\hat{\omega}_k(t))_{k=1}^q$ and $K(t)$	Controller Tuning Algorithm

The following lemma shows that for sufficiently large sampling intervals the evolution family $U_e(t,s)$ is always exponentially stable (independently of the behaviour of the frequency estimates $(\mu_k(t))_k$). Note that the required size of $\tau_1 > 0$ depends on the other tuning parameters in Assumption V.10.

Lemma V.12. *Let Assumptions* V.1 *and* V.10 *hold. There exists $\tau_1 > 0$ such that if $t_j - t_{j-1} \geq \tau_1$ for all $j \in \mathbb{N}$ in the Controller Tuning Algorithm, then there exist $M_e, \alpha_e > 0$ such that $\|U_e(t,s)\| \leq M_e e^{-\alpha_e(t-s)}$ for all $t \geq s \geq 0$ and for any $y_{ref}(t)$ and $w_{dist}(t)$ in* (1) *and for any $x_0 \in X$ and $z_0 \in Z$.*

Proof. Since $G_1(\cdot)$ and $K(\cdot)$ are piecewise constant we have $A_e(t) \equiv A_e(t_j)$ for all $t \in [t_j, t_{j+1})$ and $j \in \mathbb{N}_0$. Thus if we denote by $T_e^j(t)$ the semigroup generated by $A_e(t_j)$, then for all $t \geq s \geq 0$ we have $U_e(t,s) = T_e^j(t-s)$ if $t, s \in [t_j, t_{j+1})$ for some $j \in \mathbb{N}_0$, and otherwise

$$U_e(t,s) = T_e^j(t-t_j)T_e^{j-1}(t_j - t_{j-1}) \cdots T_e^\ell(t_{\ell+1} - s)$$

where $j, \ell \in \mathbb{N}_0$ are such that $s \in [t_\ell, t_{\ell+1})$ and $t \in [t_j, t_{j+1})$. Since $0 < \tau_1 \leq t_{j+1} - t_j \leq \tau_2$ for all $j \geq 0$ by assumption, the evolution family $U_e(t,s)$ is exponentially stable provided that there exists $M_{e0} \geq 0$ such that $\|T_e^j(t)\| \leq M_{e0}$ for all $t \geq 0$ and $j \in \mathbb{N}_0$ and $\sup_{j\geq 0}\|T_e^j(t_{j+1} - t_j)\| < 1$.

Theorem V.9 and the Hille–Yosida theorem imply the existence of $M_s, \alpha_s, M_B, M_K > 0$ such that for K^j in the Controller Tuning Algorithm we have $\|R(\lambda, A_s(t_j) + B_s K^j)\| \leq M_s/(\operatorname{Re}\lambda + \alpha_s) \leq M_s/\alpha_s$, $\|R(\lambda, A_s(t_j) + B_s K^j)B_s\| \leq M_B$, and $\|K^j\| \leq M_K$ for all $\lambda \in \mathbb{C}_+$ and $j \in \mathbb{N}_0$ Using the similarity transform Q_e in (20) we have (similarly as in (21))

$$Q_e A_e(t_j) Q_e^{-1} = \begin{bmatrix} A_s(t_j) + B_s K^j & B_s K_2^j \\ 0 & A + LC \end{bmatrix}.$$

The similarity, the triangular structure of $Q_e A_e(t_j) Q_e^{-1}$ and the norm estimates above imply that there exists $M_R > 0$ such that $\sup_{\lambda \in \mathbb{C}_+} \|R(\lambda, A_e(t_j))\| \leq M_R$ for all $j \geq 0$. By the Gearhart–Prüss–Greiner theorem [13, Thm. V.1.11] there exist $M_{e0}, \alpha_{e0} > 0$ such that $T_e^j(t)$ generated by $A_e(t_j)$ satisfy $\|T_e^j(t)\| \leq M_{e0} e^{-\alpha_{e0} t}$ for all $t \geq 0$ and $j \in \mathbb{N}_0$ (this uniform bound can be deduced, e.g., by applying [13, Thm. V.1.11] to the semigroup $\operatorname{diag}(T_e^0(t), T_e^1(t), \ldots)$ on the Hilbert space $\ell^2(X_e)$). This further implies that if we choose $\tau_1 > 0$ such that $M_{e0} e^{-\alpha_{e0}\tau_1} < 1$, then also $\|T_e^j(t_{j+1} - t_j)\| \leq M_{e0} e^{-\alpha_{e0}\tau_1} < 1$ and $U_e(t,s)$ is exponentially stable. Since M_{e0} and α_{e0} do not depend on x_0, z_0, $w_{dist}(t)$, and $y_{ref}(t)$, we can choose $M_e, \alpha_e > 0$ as in the claim. □

C. Robustness Analysis

We conclude this section by analysing the robustness properties of the controller constructed in the Controller Tuning Algorithm. The robustness properties depend on the chosen frequency estimation method — especially on its capability of handling small persistent errors in $y_{aux}(t)$ — but we can nevertheless present a general result for robustness analysis. Throughout the section we assume that Assumptions V.1, V.3 and V.10 are satisfied. We consider a perturbed regular linear system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with parameters

$$\begin{cases} \tilde{A} = A + \delta_A, \ \tilde{B} = B + \delta_B, \\ \tilde{C} = C + \delta_C, \ \tilde{D} = D + \delta_D \end{cases} \tag{26}$$

with $\delta_A \in \mathcal{L}(X)$, $\delta_B \in \mathcal{L}(\mathbb{C}^m, X)$, $\delta_C \in \mathcal{L}(X, \mathbb{C}^p)$ and $\delta_D \in \mathbb{C}^{p \times m}$. We do not need to consider perturbations in B_d and D_d since these parameters were allowed to be unknown. We begin by describing the effects of the perturbations on $y_{aux}(t)$.

Lemma V.13. *Consider the perturbed system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in* (26) *and the controller in Definition* V.4*. Assume $K(\cdot) \in L^\infty(0,\infty; \mathcal{L}(Z, \mathbb{C}^m))$ and $\hat{\omega}_k(\cdot) \in L^\infty(0,\infty)$ for all k are piecewise constant. Then the auxiliary output $\tilde{y}_{aux}(t)$ corresponding to the perturbed system satisfies $\tilde{y}_{aux}(t) = y_{aux}(t) + y_{pert}(t)$ for a.e. $t \geq 0$, where $y_{aux}(t)$ is as in Lemma* V.6 *and*

$$y_{pert}(t) = C_\Lambda \int_0^t T_L(t-s)[\delta_{AC}, \delta_{BD} K(s)] x_e(s) ds \tag{27}$$

where $\delta_{AC} = \delta_A + L\delta_C$, $\delta_{BD} = \delta_B + L\delta_D$ and $x_e(t)$ is the state of the (perturbed) closed-loop system (5)*.*

Proof. Since $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a regular linear system, Theorem III.2 implies that the closed-loop system consisting of the perturbed system and the controller in Definition V.4 has a well-defined mild state $x_e(t)$. Denote by $A_e(t)$ and $\tilde{A}_e(t)$ the closed-loop system operators corresponding to the nominal system (A, B, C, D) and the perturbed system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, respectively. Then $\mathcal{D}(\tilde{A}_e(t)) = \mathcal{D}(A_e(t))$ and $\tilde{A}_e(t) = A_e(t) + \delta_e(t)$ for a.e. $t \geq 0$ where

$$\delta_e(t) = \begin{bmatrix} \delta_A & \delta_B K_1(t) & \delta_B K_2(t) \\ G_2 \delta_C & G_2 \delta_D K_1(t) & G_2 \delta_D K_2(t) \\ -L\delta_C & -L\delta_D K_1(t) & -L\delta_D K_2(t) \end{bmatrix} \in \mathcal{L}(X_e).$$

If we apply the similarity transform Q_e in (20) to the perturbed closed-loop system (5), we obtain

$$\frac{d}{dt}(Q_e x_e(t)) = Q_e A_e(t) Q_e^{-1}(Q_e x_e(t)) + Q_e B_e w_e(t) + Q_e \delta_e(t) x_e(t)$$

where $Q_e \delta_e(\cdot) x_e(\cdot) \in L^2_{\mathrm{loc}}(0,\infty; X_e)$ by Remark III.3. The triangular structure of $Q_e A_e(t) Q_e^{-1}$ therefore implies that $\tilde{x}(t) := x(t) - z_2(t)$ is *formally* a solution of

$$\dot{\tilde{x}}(t) = A_L \tilde{x}(t) + B_{dL} w_e(t) + [\delta_{AC}, \delta_{BD} K(t)] x_e(t) \tag{28}$$

with initial condition $\tilde{x}(0) = x_0 - z_{20}$ and with $A_L = A + LC$. We will now prove that $\tilde{x}(t)$ is indeed a mild solution of (28). Denote by $U_e(t,s)$ and $\tilde{U}_e(t,s)$ the evolution families in Theorem III.2 corresponding to the nominal and perturbed systems, respectively. Let $0 = t_0 < t_1 < t_2 < \cdots$ be such that $A_e(t) \equiv A_e(t_j)$ and $\delta_e(t) \equiv \delta_e(t_j)$ for $t \in [t_j, t_{j+1})$. If we denote by $T_e^j(t)$ and $\tilde{T}_e^j(t)$ the semigroups generated by $A_e(t_j)$ and $\tilde{A}_e(t_j)$, respectively, then $U_e(t,s)$ and $\tilde{U}_e(t,s)$ are of the form given in the proof of Lemma V.12. Moreover, since $\tilde{A}_e(t_j) = A_e(t_j) + \delta_e(t_j)$ for all $j \in \mathbb{N}_0$, the perturbation formula in [13, Cor. III.1.7] and a direct computation show that

$$\tilde{U}_e(t,s)x = U_e(t,s)x + \int_s^t U_e(t,r)\delta_e(r)\tilde{U}_e(r,s)x dr, \quad (29)$$

for all $x \in X_e$ and $t \geq s \geq 0$. Applying the similarity transformation Q_e to (6) and (7a) and using the relationship (29) between $\tilde{U}_e(t,s)$ and $U_e(t,s)$ it is straightforward to confirm that $\tilde{x}(t)$ is the mild solution of (28). Since $\tilde{x}(t) \in \mathcal{D}(C_\Lambda)$ for a.e. $t \geq 0$, analogous arguments as in the proof of Lemma V.6 show $\tilde{y}_{aux}(t) = C_\Lambda \tilde{x}(t) + [D_d, -I]w_e(t)$. Comparing (23) and (28) shows that $\tilde{y}_{aux}(t) = y_{aux}(t) + y_{pert}(t)$ for a.e. $t \geq 0$. ☐

Our main result below shows that for sufficiently long sampling intervals in the Controller Tuning Algorithm the effect of small perturbations on $y_{aux}(t)$ will be small. Moreover, if the frequencies can be estimated with a sufficiently small asymptotic error, then the controller achieves output tracking *in an approximate sense*, i.e., with a small asymptotic error.

Theorem V.14. *Consider the perturbed system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in* (26) *and let Assumptions* V.1*,* V.3 *and* V.10 *hold. Let $\tau_1 > 0$ be as in Lemma* V.12 *and consider the controller in Definition* V.4*, where $(\hat{\omega}_k(\cdot))_{k=1}^q$ and $K(\cdot)$ are based on the Controller Tuning Algorithm.*

There exist $\varepsilon_{stab}, M_{pert} > 0$ such that if

$$c_\delta := \|\delta_A\| + \|\delta_B\| + \|\delta_C\| + \|\delta_D\| \leq \varepsilon_{stab}, \quad (30)$$

then for all $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$, and $y_{ref}(t)$ we have $\tilde{y}_{aux}(t) = y_{aux}(t) + y_{pert}(t)$, where $y_{aux}(t)$ is as in Lemma V.6 *and*

$$\sup_{\tau \geq 0} \int_\tau^{\tau+1} \|y_{pert}(t)\|^2 dt \leq M_{pert} c_\delta^2 (\|x_{e0}\|^2 + \|w_e(\cdot)\|_\infty^2)$$

with $x_{e0} = [x_0, z_0]^T$ and $w_e(t) = [w_{dist}(t)^T, y_{ref}(t)^T]^T$. If $C \in \mathcal{L}(X, \mathbb{C}^p)$, then $\|y_{pert}\|_\infty^2 \leq M_{pert} c_\delta^2 (\|x_{e0}\|^2 + \|w_e(\cdot)\|_\infty^2)$.

Assume $(\omega_k)_k$ in (1) *satisfy* EAC$(\tilde{M}_f, \tilde{\varepsilon}_f)$ *with some $\tilde{\varepsilon}_f > \varepsilon_f$ and $0 < \tilde{M}_f < M_f$. For any $\varepsilon_{err} > 0$ there exists $\delta_{err} > 0$ such that if the perturbations satisfy* (30) *and $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$, and $y_{ref}(t)$ are such that $(\mu_k(t))_{k=1}^q$ satisfy*

$$\max_k |\mu_k(t) - \omega_k| \leq \delta_{err}, \qquad \forall t \geq \tau_0 \quad (31)$$

for some $\tau_0 > 0$, then

$$\limsup_{t\to\infty} \int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds \leq \varepsilon_{err} \|w_e(\cdot)\|_\infty^2.$$

Proof. Fix $x_0 \in X$, $z_0 \in Z$, $w_{dist}(t)$, and $y_{ref}(t)$ and let $M_K > 0$ be as in Theorem V.9. Then $G_1(\cdot)$ and $K(\cdot)$ constructed in the Controller Tuning Algorithm satisfy $\|K(\cdot)\|_{L^\infty} \leq M_K$ and $\|G_1(\cdot)\|_{L^\infty} \leq M_f$ and thus by Theorem III.2 the closed-loop system corresponding to the perturbed system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ has a well-defined evolution family $\tilde{U}_e(t,s)$ and state $x_e(t)$. We begin by introducing some notation. We denote by $A_e(t)$ and $\tilde{A}_e(t)$ the closed-loop operators for (A,B,C,D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, respectively. Moreover, we denote by $T_e^j(t)$ and $\tilde{T}_e^j(t)$ the semigroups generated by $A_e(t_j)$ and $\tilde{A}_e(t_j)$, respectively. For the proof of the last claim we additionally assume that $(\omega_k)_k$ satisfy EAC$(\tilde{M}_f, \tilde{\varepsilon}_f)$. We then define $\mathcal{G}_1^\infty$, $\mathcal{G}_{11}^\infty$ and $\Delta_{\mathcal{G}_1}(t)$ as in (18) with $G_1^\infty = \operatorname{diag}(0_p, \omega_1\Omega_p, \ldots, \omega_q\Omega_p)$ (i.e., $(\hat{\omega}_k(t))_k$ in $G_1(t)$ replaced with $(\omega_k)_k$) and define $K^\infty = [K_1^\infty, K_{21} + K_1^\infty H_\infty]$ as in Theorem V.9. Finally, we denote by A_e^∞ and $\tilde{A}_e^\infty$ the operators in (9a) for (A,B,C,D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, respectively, and denote the semigroups generated by these two operators with $T_e(t)$ and $\tilde{T}_e(t)$, respectively. Note that $\mathcal{G}_1^\infty$, $\mathcal{G}_{11}^\infty$, $\mathcal{G}_2$, and K^∞ are independent of $x_0 \in X$, $z_0 \in Z$, $y_{ref}(t)$, and $w_{dist}(t)$.

If $M_{e0}, \alpha_{e0} > 0$ are as in the proof of Lemma V.12, we have $\|T_e^j(t)\| \leq M_{e0}e^{-\alpha_{e0}t}$ for all $t \geq 0$ and $j \in \mathbb{N}_0$ and the choice of $\tau_1 > 0$ implies $M_{e0}e^{-\alpha_{e0}\tau_1} < 1$. By construction A_e^∞ has the same structure as $A_e(t_j)$, $j \in \mathbb{N}_0$, with $(\hat{\omega}_k(t_j))_k$ and K^j replaced with $(\omega_k)_k$ and K^∞, respectively, and K^∞ in Theorem V.9 is chosen similarly as K^j. Under the additional assumption that $(\omega_k)_k$ satisfy EAC$(\tilde{M}_f, \tilde{\varepsilon}_f)$, the frequencies $(\omega_k)_k$ satisfy the assumptions of Theorem V.9 with the same parameters as $(\hat{\omega}_k(t_j))_k$, $j \in \mathbb{N}_0$ (both satisfy EAC(M_f, ε_f) for (A,B,C,D)). Therefore we can deduce as in the proof of Lemma V.12 that also $\|T_e(t)\| \leq M_{e0}e^{-\alpha_{e0}t}$ for all $t \geq 0$.

We will now choose $\varepsilon_{stab} > 0$ so that (30) implies the existence of $M_e, \alpha_e > 0$ (independent of x_0, z_0, $w_{dist}(t)$ and $y_{ref}(t)$) such that $\|\tilde{U}_e(t,s)\| \leq M_e e^{-\alpha_e(t-s)}$ for $t \geq s \geq 0$ (our choice will also later guarantee the stability of $\tilde{T}_e(t)$). We have $\tilde{A}_e(t_j) = A_e(t_j) + \delta_e(t_j)$, $j \in \mathbb{N}_0$, where $\delta_e(\cdot) \in L^\infty(0,\infty; \mathcal{L}(X_e))$ is as in the proof of Lemma V.13, and

$$\|\delta_e(\cdot)\|_{L^\infty} \leq M_1\big(\|\delta_A\| + \|\delta_B\| + \|\delta_C\| + \|\delta_D\|\big) = M_1 c_\delta$$

for some $M_1 > 0$ depending only on L and M_K. For a fixed $r_0 \in (0,1)$ we choose $\varepsilon_{stab} > 0$ to be small enough so that

$$\varepsilon_{stab} \leq \frac{\alpha_{e0}}{2M_{e0}M_1} \quad \text{and} \quad M_{e0}e^{(-\alpha_{e0} + M_{e0}M_1\varepsilon_{stab})\tau_1} \leq r_0.$$

With this choice the condition (30) together with [13, Thm. III.1.3] implies that $\|\tilde{T}_e^j(t)\| \leq M_{e0}e^{-(\alpha_{e0}/2)t}$ for all $t \geq 0$ and $j \in \mathbb{N}_0$ and $\sup_{j\geq 0}\|\tilde{T}_e^j(t_{j+1} - t_j)\| \leq \sup_{j\geq 0} M_{e0}e^{(-\alpha_{e0} + M_{e0}\|\delta_e(t_j)\|)(t_{j+1}-t_j)} \leq r_0 < 1$. Since $\tilde{A}_e(t) \equiv \tilde{A}_e(t_j)$ for $t \in [t_j, t_{j+1})$ and $j \in \mathbb{N}_0$, the structure of $\tilde{U}_e(t,s)$ is analogous to that in the proof of Lemma V.12. Because the above estimates for $\|\tilde{T}_e^j(t)\|$ and $\|\tilde{T}_e(t_{j+1} - t_j)\|$ are uniform with respect to $(\delta_A, \delta_B, \delta_C, \delta_D)$ satisfying (30), we have (similarly as in the proof of Lemma V.12) that there exist $M_e, \alpha_e > 0$ such that $\|\tilde{U}_e(t,s)\| \leq M_e e^{-\alpha_e(t-s)}$ for all $t \geq s \geq 0$ and for any $(\delta_A, \delta_B, \delta_C, \delta_D)$ for which (30) holds.

We will now prove the claims concerning $y_{pert}(t)$. If $M_e, \alpha_e > 0$ are as above, (6a) and (7a) and the structure of (constant) $B_e(t) \equiv B_e \in \mathcal{L}(\mathbb{C}^{n_d+p}, X_e)$ imply that the state $\|x_e(t)\| \leq M_2(\|x_{e0}\| + \|w_e(\cdot)\|_\infty)$ for a constant $M_2 \geq 0$ independent of x_0, z_0, $w_{dist}(t)$ and $y_{ref}(t)$. Lemma V.13 shows that $y_{pert}(t) = (\mathbb{F}_L u)(t)$ for a.e. $t \geq 0$, where $\mathbb{F}_L$ is the input-output map of the exponentially stable regular linear system $(A+LC, I, C, 0)$ and where $u(\cdot) = [\delta_{AC}, \delta_{BD}K(\cdot)]x_e(\cdot) \in L^2_{\text{loc}}(0,\infty;X)$ by Remark III.3. Since $\|u\|_{L^2(\tau,\tau+1)} \leq (M_K+1)\max\{\|\delta_{AC}\|,\|\delta_{BD}\|\}\|x_e(\cdot)\|_\infty \leq (M_K+1)M_2M_3c_\delta(\|x_{e0}\|+\|w_e(\cdot)\|_\infty)$ for all $\tau \geq 0$ and for a constant $M_3 \geq 0$ depending only on $\|L\|$, the first estimate for $\|y_{pert}(\cdot)\|$ for some $M_{pert} \geq 0$ follows from Lemma A.1(a). If $C \in \mathcal{L}(X, \mathbb{C}^p)$ and if $M_L, \alpha_L > 0$ are such that $\|T_L(t)\| \leq M_L e^{-\alpha_L t}$ for all $t \geq 0$, then the second claim follows from (27) and a direct estimate

$$\|y_{pert}(t)\| \leq \|C\|\max\{\|\delta_{AC}\|,\|\delta_{BD}\|\}\frac{M_L(M_K+1)}{\alpha_L}\|x_e(\cdot)\|_\infty.$$

To prove the last claim we will apply Theorem IV.4 to the perturbed system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. Assume $(\omega_k)_k$ satisfy EAC$(\tilde{M}_f,\tilde{\varepsilon}_f)$ and let $\varepsilon_{stab} > 0$ be as above. It is easy to see that the perturbations (26) lead to bounded perturbations of $(A_e^\infty, B_{ee}^\infty, C_{ee}^\infty, D_{ee}^\infty)$ in (9) with norm bounds depending on c_δ, M_K and $\|L\|$. Moreover, by the choice of ε_{stab} the perturbation in A_e^∞ has norm at most $\alpha_{e0}/(2M_{e0})$ when (30) holds. Thus Remark IV.5 and Lemma A.2 imply that M_{err} and δ_0 in Theorem IV.4 can be chosen to hold for all perturbations $(\delta_A,\delta_B,\delta_C,\delta_D)$ satisfying (30). The controller $(\mathcal{G}_1(t),\mathcal{G}_2(t),K(t))$ satisfies the assumptions of Theorem IV.4 since Assumption II.2 is satisfied by construction, $(\mathcal{G}_1^\infty,\mathcal{G}_2)$ satisfy (16) in Remark IV.6 (similarly as in the proof of Theorem V.5), and as shown above, both $\tilde{U}_e(t,s)$ and $\tilde{T}_e(t)$ are exponentially stable whenever (30) holds. The definition of $\Delta_{\mathcal{G}}(t)$ in (18) and $\|G_1(t)-G_1^\infty\| = \max_k|\hat{\omega}_k(t)-\omega_k|$ imply that $\delta_{\mathcal{G}}(t)$ in Theorem IV.4 satisfies

$$\begin{aligned}\delta_{\mathcal{G}}(t) &= \max\{\|\Delta_{\mathcal{G}_1}(t)\|,\|K(t)-K^\infty\|\} = \|\Delta_{\mathcal{G}_1}(t)\|\\ &\leq \|K^j-K^\infty\| + \max_k|\hat{\omega}_k(t)-\omega_k|,\end{aligned}$$

where $j \in \mathbb{N}_0$ is such that $t \in [t_j,t_{j+1})$. Since $(\omega_k)_k$ satisfy EAC$(\tilde{M}_f,\tilde{\varepsilon}_f)$ with $\tilde{M}_f < M_f$ and $\tilde{\varepsilon}_f > \varepsilon_f$, by choosing a sufficiently small $\delta_{err} > 0$ we can guarantee that if (31) holds, then $(\mu_k(t_j))_k$ satisfy EAC(M_f,ε_f) for all $j \in \mathbb{N}$ such that $t_j \geq \tau_0$. Such choice guarantees that $(\hat{\omega}_k(t_j))_k$ and $K(t_j)$ are updated whenever $t_j \geq \tau_0$, and thus we also have $\max_k|\hat{\omega}_k(t)-\omega_k| \leq \delta_{err}$ for all $t \geq \tau_0+\tau_2$. Theorem V.9 shows that $\lim_{j\to\infty}K^j = K^\infty$ if $\lim_{j\to\infty}\max_k|\hat{\omega}_k(t)-\omega_k| = 0$ and thus $\delta_{\mathcal{G}}(t)$ for $t \in [t_j,t_{j+1})$ can be made arbitrarily small by requiring that $\max_k|\hat{\omega}_k(t_j)-\omega_k|$ is small. For any $\varepsilon_{err} > 0$ we can now combine the above properties to choose $\delta_{err} > 0$ (independent of x_0, z_0, $w_{dist}(t)$, and $y_{ref}(t)$) such that if (31) holds for some $\tau_0 > 0$, then $\limsup_{t\to\infty}\|\delta_{\mathcal{G}}(\cdot)\|^2_{L^\infty(t,\infty)} < \max\{\delta_0^2, \varepsilon_{err}/M_{err}\}$. We then have from Theorem IV.4 that (30) and (31) for some $\tau_0 > 0$ imply $\limsup_{t\to\infty}\|e(\cdot)\|^2_{L^2(t,t+1)} \leq M_{err}\|w_e(\cdot)\|^2_\infty \limsup_{t\to\infty}\|\delta_{\mathcal{G}}(\cdot)\|^2_{L^\infty(t,\infty)} \leq \varepsilon_{err}\|w_e(\cdot)\|^2_\infty$. □

VI. Adaptive Regulation for a Heat Equation

In this example we study a one-dimensional boundary controlled reaction-diffusion equation on $\xi \in (0,1)$,

$$\begin{aligned}v_t(\xi,t) &= v_{\xi\xi}(\xi,t) + r(\xi)v(\xi,t) + b_d(\xi)d_1(t)\\ -v_\xi(0,t) &= u(t)+d_2(t), \quad v_\xi(1,t) = d_3(t)\\ y(t) &= v(1,t), \qquad v(\xi,0) = v_0(\xi),\end{aligned}$$

where $v(\xi,t)$ describes temperature at the point $\xi \in (0,1)$ and at time $t > 0$. The boundary input $u(t)$ acts at $\xi = 0$, the output $y(t)$ is the temperature measurement at $\xi = 1$, and $d_2(t)$ and $d_3(t)$ are boundary disturbances. The reaction term with profile $r(\cdot)$ is unmodelled and we consider it as perturbation in the system. The disturbance input profile $b_d(\xi)$ is unknown. The system defines a regular linear system on $X = L^2(0,1)$ with state $x(t) = v(\cdot,t)$. The full disturbance input is defined as $w_{dist}(t) = [d_1(t),d_2(t),d_3(t)]^T \in \mathbb{R}^3$. Since the boundary disturbances $d_2(t)$ and $d_3(t)$ are smooth functions, we can apply a change of variables as in [11, Sec. 10.1, Ex. 10.1.7] to express the heat equation as a regular linear system with bounded B_d, $D_d \in \mathbb{R}^{1\times 3}$ and a modified initial state.

We construct a controller for output regulation of $y_{ref}(t) = 0.2\sin(0.5t+0.5)+0.4\sin(6t+0.5)$ and $w_{dist}(t) = [\cos(1.5t+0.5), \sin(0.5t+0.2), \cos(1.5t-0.4)]^T$, both assumed to be unknown. In the simulation we consider $b_d(\xi) = \cos(3\xi)$ (this is not used in controller design). The stabilizing parameters are chosen as $K_{21}x = -2\int_0^1 x(\xi)d\xi$ for $x \in L^2(0,1)$, $L \equiv -4 \in L^2(0,1)$. The system does not have transmission zeros on $i\mathbb{R}$. We use $\varepsilon_f = 0.2$ and $M_f = 30$ in the Estimate Admissibility Condition, and $r = 0.2$, $R = 1 \in \mathbb{R}$, and $Q = I \in \mathbb{R}^{3\times 3}$. The frequencies will also not be updated if $(\mu_k(t_j))_{k=1}^3$ are complex or negative. Since $G_1(t)$ and $K(t)$ are piecewise constant, the results in [25, Sec. III & V] imply that for $t \in [t_j,t_{j+1})$ the controller state $z(t) = (z_0(t),\hat{v}(\cdot,t)) \in Z_0 \times L^2(0,1)$ is the weak solution of

$$\begin{aligned}\dot{z}_0(t) &= G_1(t_j)z_0(t) + G_2(y(t)-y_{ref}(t)), \;\; z_0(0) \in Z_0\\ \hat{v}_t(\xi,t) &= \hat{v}_{\xi\xi}(\xi,t) - 4(\hat{v}(1,t)-y(t)+y_{ref}(t))\\ -\hat{v}_\xi(0,t) &= u(t), \quad \hat{v}_\xi(1,t) = 0, \quad \hat{v}(\cdot,0) \in L^2(0,1)\\ u(t) &= K_1^j z_0(t) - 2\int_0^1 \hat{v}(\xi,t)d\xi + K_1^j H_j\hat{v}(\cdot,t),\end{aligned}$$

and $y_{aux}(t) = v(1,t)-\hat{v}(1,t)-y_{ref}(t)$, where K_1^j, H_j and the estimates $(\hat{\omega}_k(t_j))_{k=1}^q$ in $G_1(t_j)$ are obtained from the Tuning Algorithm. Since the true frequencies satisfy EAC$(0.3,10)$, by Theorem V.12 this controller stabilizes the unperturbed closed-loop system (without the unmodelled reaction term) for all sufficiently long update intervals.

In this example we assume that the number of nonzero frequencies $q = 3$ is known and use the adaptive estimator from [7] with parameters "$\gamma_1 = 0.005$", "$\gamma_2 = 10$", and "$\{k_i\}_i$" being the coefficients of the Hurwitz polynomial $(\lambda+2)^{2\cdot 3-1}$. The input-to-state stability of the estimator and Theorem V.14 show that for sufficiently long update intervals the controller achieves closed-loop stability and approximate output tracking for any reaction term with a small $\|r(\cdot)\|_{L^2}$. In the case $r(\xi) \equiv 0$, closed-loop stability and perfect output tracking follow from Theorem V.11. Figure 2 shows the behaviour of the frequency estimates and the regulation error for the reaction profile $r(\xi) = 1.5\sin(0.5\pi\xi)$, the update sequence $t_j = 6j$, $j \in \mathbb{N}_0$, and initial states $x_0(\cdot,0) \equiv 0$ and $z_0 = 0 \in Z$. The simulations are implemented using Finite Difference with 100 points on $[0,1]$. Initial frequency estimates are chosen as $\mu_k(0) = k \in \mathbb{R}$ for $k \in \{1,2,3\}$. Our controller can be compared to the ODE-PDE controller in [15], which similarly includes an adaptive estimator, but has continuously time-varying parameters and different general structure.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3324523

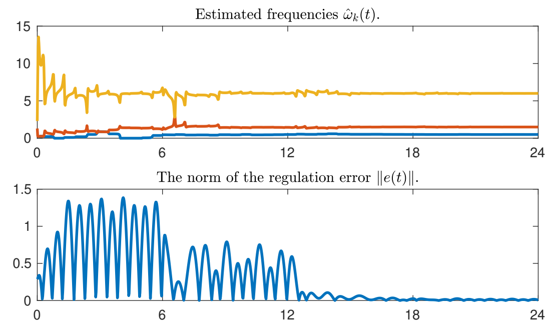

Fig. 2. Controlled heat equation with the estimator in [7].

Acknowledgement

The authors would like to thank Dr. Petteri Laakkonen for advice on transmission zeros and Prof. Dr. Andre Ran for helpful comments related to Lemma A.3.

APPENDIX

Lemma A.1. *Assume that $(U,\Phi,\Psi,\mathbb{F})$ is a well-posed nonautonomous system in the sense of* [29, Def. 3.6]. *If there exist $M,\omega>0$ such that $\|U(t,s)\|\le Me^{-\omega(t-s)}$ for all $t\ge s\ge 0$, then for $u\in L^2_{loc}(0,\infty;U)$ the following hold.*

(a) *There exists $M_0>0$ (independent of u) such that $\sup_{\tau\ge0}\|\mathbb{F}u\|_{L^2(\tau,\tau+1)}\le M_0\sup_{\tau\ge0}\|u\|_{L^2(\tau,\tau+1)}$.*

(b) *There exists $M_1>0$ (independent of u) such that*

$$\limsup_{\tau\to\infty}\|\mathbb{F}u\|_{L^2(\tau,\tau+1)}\le M_1\limsup_{\tau\to\infty}\|u\|_{L^2(\tau,\tau+1)}.$$

(c) *If $\lim_{\tau\to\infty}\|u\|_{L^2(\tau,\tau+1)}=0$, then $\lim_{\tau\to\infty}\|\mathbb{F}u\|_{L^2(\tau,\tau+1)}=0$.*

(d) *If $\sup_{\tau\ge0}e^{\alpha\tau}\|u\|_{L^2(\tau,\tau+1)}<\infty$ for some $0<\alpha<\omega$, then $t\mapsto e^{\beta t}\|(\mathbb{F}u)(t)\|\in L^2(0,\infty)$ for any $0<\beta<\alpha$.*

Proof. Let $u\in L^2_{\rm loc}(0,\infty;U)$. It is clearly sufficient to prove the claims when τ is replaced by $n\in\mathbb{N}_0$. The estimates in [29, Lem. 3.7] show that there exists a constant $M_2>0$ (depending only on $(U,\Phi,\Psi,\mathbb{F})$) such that

$$\|\mathbb{F}u\|_{L^2(n,n+1)}\le M_2(a*b)_n,\qquad \forall n\in\mathbb{N}_0,$$

where $(a*b)_n$ is the nth element of the convolution of $a=(a_k)_{k=0}^\infty$ with $a_k=e^{-\omega k}$ and $b=(b_k)_{k=0}^\infty$ with $b_k=\|u\|_{L^2(k,k+1)}$. The constant $M_2>0$ is determined by M and ω and the uniform (w.r.t $s\ge0$) bounds for $\|\Phi_{\cdot,s}\|_{\mathcal{L}(L^2(s,s+1),X)}$, $\|\Psi_s\|_{\mathcal{L}(X,L^2(s,s+1))}$, and $\|\mathbb{F}_s\|_{\mathcal{L}(L^2(s,s+1),L^2(s,s+1))}$.

Since $a\in\ell^1(\mathbb{R})$ and $\|b\|_{\ell^\infty}=\sup_{k\ge0}\|u\|_{L^2(k,k+1)}$, the claim in part (a) holds since $\|a*b\|_{\ell^\infty}\le\|a\|_{\ell^1}\|b\|_{\ell^\infty}$ by the Young's inequality for convolutions. Part (c) follows from (b). To prove (b) we assume $\limsup_{\tau\to\infty}\|u\|_{L^2(\tau,\tau+1)}<\infty$ (otherwise the claim is trivial). Since $u\in L^2_{\rm loc}(0,\infty;U)$ we have $\|b\|_{\ell^\infty}=\sup_{k\ge0}\|u\|_{L^2(k,k+1)}<\infty$. If $n,n_0\in\mathbb{N}$ satisfy $n_0<n$, then $a_{n-k}=e^{-\omega(n-n_0+1)}a_{n_0-1-k}$ and

$$(a*b)_n=e^{-\omega(n-n_0+1)}(a*b)_{n_0-1}+(a*\tilde{b})_{n-n_0},$$

where $\tilde{b}=(b_{k+n_0})_{k=0}^\infty$. Young's inequality thus implies

$$\begin{aligned}|(a*b)_n|&\le e^{-\omega(n-n_0+1)}\|a*b\|_{\ell^\infty}+\|a*\tilde{b}\|_{\ell^\infty}\\&\le e^{-\omega(n-n_0+1)}\|a\|_{\ell^1}\|b\|_{\ell^\infty}+\|a\|_{\ell^1}\|\tilde{b}\|_{\ell^\infty}.\end{aligned}$$

If we choose $n_0=\lfloor n/2\rfloor$, then the properties of the limit supremum and $\|\tilde{b}\|_{\ell^\infty}=\sup_{k\ge\lfloor n/2\rfloor}\|u\|_{L^2(k,k+1)}$ imply (b).

Finally, to prove (d) we note that $e^{\alpha n}\|\mathbb{F}u\|_{L^2(n,n+1)}\le M_2(a_\alpha*b_\alpha)_n$ for all $n\in\mathbb{N}_0$, where $a_\alpha=(e^{-(\omega-\alpha)k})_{k=0}^\infty\subset\mathbb{R}$ and $b_\alpha=(e^{\alpha k}\|u\|_{L^2(k,k+1)})_{k=0}^\infty\subset\mathbb{R}$. Since $0<\alpha<\omega$, we have $a_\alpha\in\ell^1(\mathbb{R})$ and our assumptions imply $b_\alpha\in\ell^\infty(\mathbb{R})$. Thus Young's inequality implies $\|a_\alpha*b_\alpha\|_{\ell^\infty}\le\|a_\alpha\|_{\ell^1}\|b_\alpha\|_{\ell^\infty}$ and we have $\sup_{n\ge0}e^{\alpha n}\|\mathbb{F}u\|_{L^2(n,n+1)}<\infty$. This implies the claim for any $0<\beta<\alpha$. □

Lemma A.2. *Let (A,B,C,D) be a regular linear system. Assume that there exist $M,\alpha>0$ such that the semigroup $T(t)$ generated by A satisfies $\|T(t)\|\le Me^{-\alpha t}$ for $t\ge0$. If we denote the extended input, output and input–output maps of a perturbed system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ by $\tilde{\Phi}$, $\tilde{\Psi}$, and $\tilde{\mathbb{F}}$, respectively, then for any $\varepsilon\in(0,\alpha/M)$ and $\kappa>0$ we have*

$$\sup_{(\tilde{A},\tilde{B},\tilde{C},\tilde{D})\in\Omega(\varepsilon,\kappa)}\left(\|\tilde{\Phi}\|+\|\tilde{\Psi}\|+\|\tilde{\mathbb{F}}\|\right)<\infty,$$

where $\Omega(\varepsilon,\kappa)=\{\,(A+\delta_A,B+\delta_B,C+\delta_C,D+\delta_D)\mid\|\delta_A\|\le\varepsilon$ and $\|\delta_B\|+\|\delta_C\|+\|\delta_D\|\le\kappa\,\}$.

Proof. Denote the extended input, output and input–output maps of (A,B,C,D) by Φ, Ψ, and $\mathbb{F}$, respectively. Let $\varepsilon\in(0,\alpha/M)$ be fixed. We begin by considering perturbations in operator A only, i.e., $(\tilde{A},B,C,D)=(A+\delta_A,B,C,D)$, where $\|\delta_A\|\le\varepsilon$. Denote the input, output, and input–output maps of the extended system $(A,[B,I],[\begin{smallmatrix}C\\I\end{smallmatrix}],[\begin{smallmatrix}D&0\\0&0\end{smallmatrix}])$ by

$$\Phi_e=[\Phi,\Phi_I],\qquad \Psi_e=\begin{bmatrix}\Psi\\\Psi_I\end{bmatrix},\qquad \mathbb{F}_e=\begin{bmatrix}\mathbb{F}&\mathbb{F}_{CI}\\\mathbb{F}_{IB}&\mathbb{F}_{II}\end{bmatrix}.$$

By the results in [40, Sec. 7], applying an admissible output feedback $u_e(t)=\Delta y_e(t)+\tilde{u}_e(t)$ with $\Delta=[\begin{smallmatrix}0&0\\0&\delta_A\end{smallmatrix}]$ leads to the regular linear system $(A+\delta_A,[B,I],[\begin{smallmatrix}C\\I\end{smallmatrix}],[\begin{smallmatrix}D&0\\0&0\end{smallmatrix}])$ with input map $\bar{\Phi}_e$, output map $\bar{\Psi}_e$, and input-output $\bar{\mathbb{F}}_e$. We have from [40, Thm. 6.1] that $\bar{\mathbb{F}}_e=(I-\mathbb{F}_e\Delta)^{-1}\mathbb{F}_e$. Since $(\mathbb{F}_{II}u)(t)=\int_0^tT(t-s)u(s)ds$, [4, Prop. 1.3.5(a)] implies that $\|\mathbb{F}_{II}\|_{\mathcal{L}(L^2(0,\infty))}\le\|T(\cdot)\|_{L^1(0,\infty)}\le M/\alpha$. Thus $\|\mathbb{F}_{II}\delta_A\|\le M\|\delta_A\|/\alpha\le M\varepsilon/\alpha<1$ and

$$(I-\mathbb{F}_e\Delta)^{-1}=\begin{bmatrix}I&-\mathbb{F}_{CI}\delta_A\\0&I-\mathbb{F}_{II}\delta_A\end{bmatrix}^{-1}=\begin{bmatrix}I&\mathbb{F}_{CI}\delta_AQ\\0&Q\end{bmatrix}$$

where $Q=(I-\mathbb{F}_{II}\delta_A)^{-1}$. This implies $\|(I-\mathbb{F}_e\Delta)^{-1}\|\le(1+(1+\|\mathbb{F}_{CI}\delta_A\|^2)\|Q\|^2)^{1/2}$. Finally, the estimate $\|Q\|\le\alpha/(\alpha-M\varepsilon)$ and the formulas $\bar{\Phi}_e=\Phi_e(I+\Delta\bar{\mathbb{F}}_e)$ and $\bar{\Psi}_e=(I+\bar{\mathbb{F}}_e\Delta)\Psi_e$ in [40, Rem. 6.5] imply that

$$\sup_{\|\delta_A\|\le\varepsilon}\left(\|\bar{\Phi}_e\|+\|\bar{\Psi}_e\|+\|\bar{\mathbb{F}}_e\|\right)<\infty. \tag{32}$$

We will now consider perturbed systems satisfying $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})=(A+\delta_A,B+\delta_B,C+\delta_C,D+\delta_D)\in\Omega(\varepsilon,\kappa)$ with $\kappa>0$. It is easy to verify that $\tilde{\Psi}x=[I,\delta_C]\bar{\Psi}_ex$,

$$\tilde{\Phi}u=\bar{\Phi}_e\begin{bmatrix}u\\\delta_Bu\end{bmatrix}\quad\text{and}\quad\tilde{\mathbb{F}}u=\begin{bmatrix}I&\delta_C\end{bmatrix}\bar{\mathbb{F}}_e\begin{bmatrix}u\\\delta_Bu\end{bmatrix}+\delta_Du.$$

Since $\|\delta_A\|\le\varepsilon$ and $\|\delta_B\|+\|\delta_C\|+\|\delta_D\|\le\kappa$, the claim follows directly from (32). □

The following corollary of the continuity of the solutions of Riccati equations is essential for the proof of Theorem V.9. To the best of the authors' knowledge, this result is new.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3324523

Lemma A.3. *Let $r>0$, $Q\in\mathbb{C}^{n\times n}$ and $R\in\mathbb{C}^{m\times m}$ satisfy $Q>0$ and $R>0$. Let $\Omega\subset\mathbb{R}^q$ be a compact set and let $\delta\mapsto A_\delta:\Omega\to\mathbb{C}^{n\times n}$ and $\delta\mapsto B_\delta:\Omega\to\mathbb{C}^{n\times m}$ be continuous functions such that the pair (A_δ,B_δ) is controllable for all $\delta\in\Omega$. If we define $K_\delta=-R^{-1}B_\delta^*\Pi_\delta$, $\delta\in\Omega$, where $\Pi_\delta\in\mathbb{C}^{n\times n}$ are the unique non-negative solutions of*

$$(rI+A_\delta)^*\Pi_\delta+\Pi_\delta(rI+A_\delta)-\Pi_\delta B_\delta R^{-1}B_\delta^*\Pi_\delta=-Q,$$

then there exist $M,M_K>0$ such that $\|K_\delta\|\le M_K$ and $\|e^{(A_\delta+B_\delta K_\delta)t}\|\le Me^{-rt}$ for all $t\ge0$ and $\delta\in\Omega$.

Proof. The claim is trivially true if Ω is empty. Let $\delta\in\Omega$. The assumptions imply that Π_δ exists and is unique, and for all $x\in\mathbb{C}^n$ we have $2\operatorname{Re}\langle(rI+A_\delta+B_\delta K_\delta)x,\Pi_\delta x\rangle=\langle-\Pi_\delta B_\delta R^{-1}B_\delta^*\Pi_\delta x-Qx,x\rangle\le0$. Moreover, under our assumptions Π_δ is positive definite. Therefore $rI+A_\delta+B_\delta K_\delta$ is dissipative with respect to the inner product $\langle\cdot,\cdot\rangle_\delta:=\langle\cdot,\Pi_\delta\cdot\rangle_{\mathbb{C}^n}$ on $\mathbb{C}^n$. Thus if we define $\|x\|_\delta:=\|\Pi_\delta^{1/2}x\|_{\mathbb{C}^n}$ for $x\in\mathbb{C}^n$, then $\|e^{(rI+A_\delta+B_\delta K_\delta)t}x\|_\delta\le\|x\|_\delta$ for all $t\ge0$. The definition of $\|\cdot\|_\delta$ now implies that $\|e^{(A_\delta+B_\delta K_\delta)t}\|\le\|\Pi_\delta^{1/2}\|\|\Pi_\delta^{-1/2}\|e^{-rt}$ for all $t\ge0$.

Our aim is to show that $\sup_{\delta\in\Omega}\|\Pi_\delta^{1/2}\|\|\Pi_\delta^{-1/2}\|<\infty$. By [33, Thm. 3.1] the non-negative matrix Π_δ is a continuous function of the matrices A_δ and B_δ when δ is restricted to Ω. Therefore the function $\delta\mapsto\Pi_\delta$ is continuous on Ω, and since Π_δ and $\Pi_\delta^{1/2}$ are nonsingular for all $\delta\in\Omega$, also $\delta\mapsto\Pi_\delta^{1/2}$ and $\delta\mapsto\Pi_\delta^{-1/2}$ are continuous on Ω. Since Ω is compact, these functions are uniformly continuous and $\|\Pi_\delta^{1/2}\|$ and $\|\Pi_\delta^{-1/2}\|$ are uniformly bounded with respect to $\delta\in\Omega$. Thus the claims hold with $M_K:=\|R^{-1}\|\max_{\delta\in\Omega}\|B_\delta\|\|\Pi_\delta\|<\infty$ and $M:=\max_{\delta\in\Omega}\|\Pi_\delta^{1/2}\|\|\Pi_\delta^{-1/2}\|<\infty$. ☐

Proof of Theorem V.9. Since $K_{21}\in\mathcal{L}(X,\mathbb{C}^m)$, $(A+BK_{21},B,C+DK_{21},D)$ is an exponentially stable regular linear system and $\lambda\mapsto(C_\Lambda+DK_{21})R(\lambda,A+BK_{21})$ and $\lambda\mapsto P_K(\lambda)$ are continuous functions on $[-iM_f,iM_f]\subset i\mathbb{R}$. Thus $\|H_j\|\le M_H$ and $\|B_{1j}\|\le M_{B1}$ for some $M_H,M_{B1}>0$ independent of t_j. Definition V.2 implies that the transmission zeros of (A,B,C,D) are zeros of $\lambda\mapsto\det(P_K(\lambda)P_K(\lambda)^*)$, which is analytic on $\{\,\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda>\omega_0(T_K(t))\,\}$. Thus the transmission zeros of (A,B,C,D) on $i\mathbb{R}$ are a (possibly empty) discrete set with no finite accumulation points.

By assumption, $(\hat\omega_k(t_j))_{k=1}^q$ satisfy $\text{EAC}(M_f,\varepsilon_f)$. Therefore the set $\{\pm i\hat\omega_k(t_j)\}_{k=1}^q\cup\{0\}$ does not contain transmission zeros of (A,B,C,D) and $P_K(\pm i\hat\omega_k(t_j))^*$ and $P_K(0)^*$ are injective. Since the eigenvalues $\sigma_p(G_1(t_j))=\{\pm i\hat\omega_k(t_j)\}_{k=1}^q\cup\{0\}$ are distinct, it is easy to use the structures of $G_1(t_j)$ and the definition of B_{1j} to show that $B_{1j}^*\phi\ne0$ whenever $0\ne\phi\in\mathcal{N}(G_1(t_j)^*)$ or $0\ne\phi\in\mathcal{N}(\pm i\hat\omega_k(t_j)-G_1(t_j)^*)$ for some $k\in\{1,\dots,q\}$. Thus the pair $(G_1(t_j),B_{1j})$ is controllable. If we define $\delta=[\delta_1,\dots,\delta_q]^T\in\mathbb{R}^q$ by

$$\delta_1=\hat\omega_1(t_j),\quad \delta_k=\hat\omega_k(t_j)-\hat\omega_{k-1}(t_j)\quad\forall k\in\{2,\dots,q\},$$

we can define continuous functions $\tilde G_1:\mathbb{R}^q\to\mathcal{L}(\mathbb{C}^{p(2q+1)})$ and $\tilde B_1:\mathbb{R}^q\to\mathbb{C}^{p(2q+1)\times p}$ so that $\tilde G_1(\delta)=G_1(t_j)$ and $\tilde B_1(\delta)=B_{1j}$. The assumption that $(\hat\omega_k(t_j))_{k=1}^q$ satisfy $\text{EAC}(M_f,\varepsilon_f)$ implies that δ is contained in a compact set $\Omega\subset\mathbb{R}^q$ (determined by ε_f, M_f, and (A,B,C,D)), and similarly as above, $(\tilde G_1(\delta),\tilde B_1(\delta))$ is controllable whenever $\delta\in\Omega$. Because of this, Lemma A.3 implies that there exist $M_G,M_{K1}>0$ (independent of t_j) such that $\|K_1^j\|\le M_{K1}$ and $\|e^{(G_1(t_j)+B_{1j}K_1^j)t}\|\le M_Ge^{-rt}$ for all $t\ge0$. Morever, the definition $K^j=[K_1^j,K_{21}+K_1^jH_j]$ and $\|H_j\|\le M_H$ imply that $\|K^j\|\le M_K$ for some $M_K>0$ independent of t_j.

If $\max_k|\hat\omega_k(t_j)-\omega_k|\to0$ as $j\to\infty$, then $(\omega_k)_{k=1}^\infty$ also satisfy $\text{EAC}(M_f,\varepsilon_f)$ and thus K_1^∞ and K^∞ are well-defined. Clearly $G_1(t_j)\to G_1^\infty\in\mathcal{L}(Z_0)$ as $j\to\infty$ where G_1^∞ is obtained by replacing $(\hat\omega_k(t_j))_k$ by $(\omega_k)_k$ in $G_1(t_j)$. Since $\lambda\mapsto P_K(\lambda)$ and $\lambda\mapsto(C_\Lambda+DK_{21})R(\lambda,A+BK_{21})$ are continuous on $i\mathbb{R}$, also $B_{1j}\to B_1^\infty\in\mathbb{C}^{p(2q+1)\times m}$ and $H^j\to H^\infty\in\mathcal{L}(X,Z_0)$ as $j\to\infty$. We have from [33, Thm. 3.1] that Π_{1j} depends continuously on $G_1(t_j)$ and B_{1j}, and therefore $\Pi_{1j}\to\Pi_1^\infty\in\mathcal{L}(Z_0)$ as $j\to\infty$. Thus the definitions of K_1^j and K^j imply that $\|K^j-K^\infty\|\to0$ as $j\to\infty$.

Since (A,B,C,D) is regular, the operator $H_j\in\mathcal{L}(X,Z_0)$ extends to $\mathcal{R}(B)\subset X_{-1}$. It is straightforward to check that $G_1(t_j)H_j=H_jA_K+G_2(C_\Lambda+DK_{21})$ and $B_{1j}=H_jB+G_2D$. The definition $K^j=[K_1^j,K_{21}+K_1^jH_j]$ and similar computations as in [26, Thm. 13] and [24, Thm. 15] show

$$\begin{bmatrix}I&H_j\\0&-I\end{bmatrix}(A_s(t_j)+B_sK^j)\begin{bmatrix}I&H_j\\0&-I\end{bmatrix}=\begin{bmatrix}G_1(t_j)+B_{1j}K_1^j&0\\-BK_1^j&A+BK_{21}\end{bmatrix}=:\tilde A_{sK}^j$$

with domain $\mathcal{D}(\tilde A_{sK}^j)=\{\,[z_1,z_2]^T\in Z_0\times X_B\mid Az_2+B(K_{21}z_2-K_1^jz_1)\in X\,\}$ (the domains of the block operators can be analysed as in the proof of [24, Thm. 15]). Fix $0<\alpha_s<\min\{r,-\omega_0(T_K(t))\}$. Denoting $R_G^j(\lambda)=R(\lambda,G_1(t_j)+B_{1j}K_1^j)$ and $R_K(\lambda)=R(\lambda,A+BK_{21})$, for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>-\alpha_s$ we have

$$R(\lambda,\tilde A_{sK}^j)=\begin{bmatrix}R_G^j(\lambda)&0\\-R_K(\lambda)BK_1^jR_G^j(\lambda)&R_K(\lambda)\end{bmatrix}.$$

Since $\|R_G^j(\lambda)\|\le M_G/(\operatorname{Re}\lambda+r)\le M_G/(r-\alpha_s)$ and since B is admissible with respect to the stable semigroup generated by $A+BK_{21}$, also $\sup_{\operatorname{Re}\lambda>-\alpha_s}\|R_K(\lambda)B\|<\infty$ [34, Prop. 4.4.6]. Because of this, the Gearhart–Prüss–Greiner theorem [13, Thm. V.1.11] imply that there exists $\tilde M_s>0$ independent of t_j such that $\|\tilde T_s^j(t)\|\le\tilde M_se^{-\alpha_st}$ for all $t\ge0$ (this uniform bound can be deduced, e.g., by applying [13, Thm. V.1.11] to the semigroup $\operatorname{diag}(\tilde T_s^0(t),\tilde T_s^1(t),\dots)$ on $\ell^2(Z_0\times X)$). Due to $\|H_j\|\le M_H$ and the similarity between $\tilde T_s^j(t)$ and $T_s^j(t)$ we finally have that there exists $M_s>0$ such that that $\|T_s^j(t)\|\le M_se^{-\alpha_st}$ for all $t\ge0$.

It remains to prove the existence of $M_B>0$. We have

$$\begin{aligned}R(\lambda,A_s(t_j)+B_sK^j)B_s&=\begin{bmatrix}I&H_j\\0&-I\end{bmatrix}R(\lambda,\tilde A_{sK}^j)\begin{bmatrix}I&H_j\\0&-I\end{bmatrix}\begin{bmatrix}G_2D\\B\end{bmatrix}\\&=\begin{bmatrix}I&H_j\\0&-I\end{bmatrix}\begin{bmatrix}R_G^j(\lambda)B_{1j}\\-R_K(\lambda)BK_1^jR_G^j(\lambda)B_{1j}-R_K(\lambda)B\end{bmatrix}\end{aligned}$$

for $\lambda\in\mathbb{C}_+$. Since $\|R_G^j(\lambda)\|\le M_G/r$, $\sup_{\lambda\in\mathbb{C}_+}\|R_K(\lambda)B\|<\infty$, $\|H_j\|\le M_H$, $\|K_1^j\|\le M_{K1}$ and $\|B_{1j}\|\le M_{B1}$, we indeed have $\|R(\lambda,A_s(t_j)+B_sK^j)B_s\|\le M_B$ for all $\lambda\in\mathbb{C}_+$ and for some $M_B>0$ independent of t_j. ☐

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