SIAM J. CONTROL OPTIM.
Vol. 57, No. 6, pp. 3827–3856

STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS*

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Abstract. We study the stability of coupled impedance passive regular linear systems under power-preserving interconnections. We present new conditions for strong, exponential, and nonuniform stability of the closed-loop system. We apply the stability results to the construction of passive error feedback controllers for robust output tracking and disturbance rejection for strongly stabilizable passive systems. In the case of nonsmooth reference and disturbance signals we present conditions for nonuniform rational and logarithmic rates of convergence of the output. The results are illustrated with examples on designing controllers for linear wave and heat equations, and on studying the stability of a system of coupled partial differential equations.

Key words. linear system, strongly continuous semigroup, coupled systems, strong stability, polynomial stability, impedance passive, feedback, robust output regulation, controller design

AMS subject classifications. Primary, 93C05, 47D06, 93D20, 93B52; Secondary, 47A10, 35B35, 93D15

DOI. 10.1137/17M1136407

1. Introduction.

In this paper we study the stability properties and control of regular linear systems [43] of the form[1]

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X, \tag{1.1a}$$

$$y(t) = C_\Lambda x(t) + Du(t) \tag{1.1b}$$

on a Hilbert space X, where $u(t)$ is the input of the system and $y(t)$ is the output. Our main interest is in systems that are *impedance passive* [10, 38, 40] (or *passive* for short) in the sense that their solutions satisfy

$$\frac{d}{dt}\|x(t)\|^2 \le 2\operatorname{Re}\langle u(t), y(t)\rangle, \qquad t > 0.$$

Passive systems are encountered especially in the study of mechanical or electrical systems modeled with partial differential equations. In particular, (1.1) is impedance passive if A generates a contraction semigroup, B and C are bounded operators, $C = B^*$, and $\operatorname{Re} D \ge 0$.

The paper consists of two main parts. In the first part we focus on the stability of the coupled system consisting of (1.1) and another passive regular linear system

$$\dot{z}(t) = A_c z(t) + B_c u_c(t), \qquad z(0) = z_0 \in Z, \tag{1.2a}$$

$$y_c(t) = C_{c\Lambda} z(t) + D_c u_c(t) \tag{1.2b}$$

with $D_c^* = D_c$ under a *power-preserving interconnection* where

$$u(t) = y_c(t), \qquad u_c(t) = -y(t).$$

*Received by the editors June 27, 2017; accepted for publication (in revised form) September 17, 2019; published electronically November 26, 2019.
https://doi.org/10.1137/17M1136407
Funding: This research was funded by the Academy of Finland grants 298182 and 310489.
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[1]Here C_Λ and $C_{c\Lambda}$ denote the Λ-extensions of C and C_c, respectively. See section 2 for details.

We study the stability of the resulting closed-loop system

$$\dot{x}_e(t) = A_e x_e(t), \qquad x_e(0) = x_{e0} \in X_e \tag{1.3}$$

on the Hilbert space $X_e = X \times Z$. The notation (A_c, B_c, C_c, D_c) and our results on the closed-loop stability are motivated by the second part of the paper where we study robust output tracking and disturbance rejection for the system (1.1). In this situation (1.2) is an unstable dynamic feedback controller. However, our results are also applicable when the roles of the systems are reversed, i.e., when (1.2) is a system to be controlled and (1.1) is the controller, and they can also be used to study the stability of systems of partial differential equations coupled on the boundary or inside the domain. Our main interest is in the situation where A_c has a countable number of spectral points on the imaginary axis.

We study (1.3) in terms of the stability properties of the strongly continuous semigroup $T_e(t)$ generated by $A_e : D(A_e) \subset X_e \to X_e$. As our main results we introduce conditions under which the semigroup $T_e(t)$ is exponentially stable, strongly stable, or *nonuniformly stable* [7, 36]. Among these, exponential stability is the strongest form of stability. However, in certain control applications exponential stability is unachievable, and many partial differential equations and coupled systems are known to lack exponential decay of energy. These situations arise especially in wave equations with partial damping and in coupled hyperbolic-parabolic systems [49, 6]. Recently many such coupled systems have been shown to be *polynomially stable* [25, 7, 8], which means that the classical solutions of the system decay at rational rates, i.e., for some constants $M_e, \alpha, t_0 > 0$

$$\|T_e(t)x_{e0}\| \le \frac{M_e}{t^{1/\alpha}} \|A_e x_{e0}\|, \qquad x_{e0} \in D(A_e),\ t \ge t_0.$$

In this paper we introduce new results for studying polynomial and the more general nonuniform stability for coupled passive abstract linear systems (1.1) and (1.2).

Strong and exponential closed-loop stabilities of infinite-dimensional systems have been studied in the literature for passive one-dimensional boundary control systems [41, 33], coupled systems with collocated inputs and outputs [16], and passive systems coupled with finite-dimensional systems [50]. Polynomial stability of coupled systems has been studied extensively in the context of coupled linear partial differential equations [3, 1, 6, 2] and for abstract hyperbolic-parabolic systems [22].

In the second part of the paper we study the *robust output regulation problem* where the aim is to design a controller in such a way that the output $y(t)$ of the system (1.1) converges to a given reference signal $y_{ref}(t)$ asymptotically in the sense that

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad t \to \infty,$$

despite possible external disturbance signals $w_{dist}(t)$. In addition, the controller is required to be *robust* in the sense that it should achieve output tracking even if the parameters (A, B, C_Λ, D) experience small changes or contain uncertainties. This control problem has been studied actively in the literature for various classes of infinite-dimensional linear systems [48, 26, 19, 35, 23, 20, 31, 42] including regular linear systems [45, 9, 32, 47, 29, 30] and passive systems [35].

The robust output regulation problem can be solved with a dynamical error feedback controller of the form

$$\dot{z}(t) = A_c z(t) + B_c(y_{ref}(t) - y(t)), \qquad z(0) = z_0 \in Z, \tag{1.4a}$$

$$u(t) = C_{c\Lambda} z(t) + D_c(y_{ref}(t) - y(t)). \tag{1.4b}$$

One of the fundamental results of the theory, the *internal model principle* [17, 15, 31, 32], implies that robust output tracking can be achieved by including a suitable number of copies of the frequencies $\{\omega_k\}_{k\in\mathcal{I}}$ of $y_{ref}(t)$ and $w_{dist}(t)$ into the dynamics of the controller and using the remaining parameters of (1.4) to stabilize the closed-loop system. While the inclusion of the *internal model* is both necessary and sufficient for robustness, the resulting closed-loop can be stabilized in various ways. Under fairly general assumptions the closed-loop stability can be achieved with observer-based design methods [20, 29] leading to infinite-dimensional controllers. If the system (1.1) can be stabilized exponentially with output feedback and if $y_{ref}(t)$ and $w_{dist}(t)$ contain a finite number of frequencies, then A_c can be chosen to be *minimal* in the sense that it contains only the internal model, and the closed-loop system can be stabilized with suitable choices of B_c and C_c [26, 19, 35]. It was shown in [35, Thm. 1.2] that if (1.1) is passive and exponentially stabilizable, then robust output regulation can be achieved in a natural way using a minimal passive controller (1.4).

In this paper we extend the passive controller design presented in [35]. We present a robust passive controller for systems (1.1) that are not exponentially stablizable, but only strongly stabilizable. Such systems are encountered, for example, in control of wave equations, as illustrated in section 6. Moreover, our design methods allow considering nonsmooth periodic reference and disturbance signals with infinite numbers of frequencies. In earlier references, the robust output regulation of nonsmooth signals has only been achieved using an observer in the controller [20, 30]. We solve this problem with two new robust controllers having the property that A_c contains only the internal model of the reference and disturbance signals. These controllers achieve either exponential, polynomial, or nonuniform closed-loop stability depending on the properties of the system (1.1) and the choices of the controller's parameters. In the case of nonuniform closed-loop stability we present nonuniform rates of convergence for the output $y(t)$ for sufficiently smooth $y_{ref}(\cdot)$ and $w_{dist}(\cdot)$.

One of the passive controllers presented in this paper is based on a transport equation with boundary control and observation, and under suitable assumptions on the system (1.1) (in general requiring $D \neq 0$) the controller achieves robust output regulation of all τ-periodic reference and disturbance signals with exponential convergence rate of the output. This structure is related to the controllers used in repetitive control [21, 45] and in [23].

The paper is organized as follows. In section 2 we state the main standing assumptions. The results on stability of the closed-loop system are presented in section 3. In section 4 we formulate the robust output regulation problem, and the results on construction of robust controllers are presented in section 5. In section 6 we illustrate the controller construction for concrete partial differential equations, including two one-dimensional wave equations and a two-dimensional heat equation. Appendix A collects helpful lemmata that are used throughout the paper.

2. Notation and definitions. If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $D(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel, and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{L}(X)$ on a Hilbert space X we define $\operatorname{Re} T = \frac{1}{2}(T + T^*)$. The Moore–Penrose pseudoinverse of $T \in \mathcal{L}(X, Y)$ is denoted by $T^\dagger$. For two functions $f : I \subset \mathbb{R} \to X$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ we write $\|f(t)\| = O(g(|t|))$ if there exist $M_g, T_g > 0$ such that $\|f(t)\| \leq M_g g(|t|)$ whenever

$|t| \geq T_g$. We denote $f(t) \lesssim g(t)$ and $f_k \lesssim g_k$ if there exist $M_1, M_2 > 0$ such that $f(t) \leq M_1 g(t)$ and $f_k \leq M_2 g_k$ for all values of the parameters t and k.

In sections 4 and 5 we also consider the system (1.1) on a Hilbert space X with an additional disturbance signal input $w_{dist}(t)$, i.e.,

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \in X, \tag{2.1a}$$

$$y(t) = C_\Lambda x(t) + Du(t). \tag{2.1b}$$

Throughout the paper the operators $B \in \mathcal{L}(U, X_{-1})$, $B_d \in \mathcal{L}(U_d, X_{-1})$, and $C \in \mathcal{L}(X_1, Y)$ are admissible [39, sec. 4] with respect to the semigroup $T(t)$ generated by $A : D(A) \subset X \to X$. Here U, U_d, and Y are Hilbert spaces, the space $X_1 = D(A)$ is equipped with the graph norm of A, and X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} = \|R(\lambda_0, A)x\|$, where $\lambda_0 \in \rho(A)$ is arbitrary and fixed. We assume that the system $(A, [B, B_d], C_\Lambda, D)$ in (2.1) with input $(u(t), w_{dist}(t)) \in U \times U_d$ and output $y(t) \in Y$ is a regular linear system [43, sec. 5]. We denote $X_B = D(A) + \mathcal{R}(R(\lambda_0, A)B)$ and $X_{B,B_d} = D(A) + \mathcal{R}(R(\lambda_0, A)[B, B_d])$. The Λ-extension of C is $C_\Lambda x = \lim_{\lambda \to \infty} \lambda C R(\lambda, A)x$, where $D(C_\Lambda)$ consists of those $x \in X$ for which the limit exists. The regularity of (2.1) implies that $\mathcal{R}(R(\lambda, A)B) \subset D(C_\Lambda)$ and $\mathcal{R}(R(\lambda, A)B_d) \subset D(C_\Lambda)$ for all $\lambda \in \rho(A)$ and that the transfer functions $P(\cdot) : \hat{u} \mapsto \hat{y}$ and $P_d(\cdot) : \hat{w}_{dist} \mapsto \hat{y}$ have the formulas

$$P(\lambda) = C_\Lambda R(\lambda, A)B + D, \qquad P_d(\lambda) = C_\Lambda R(\lambda, A)B_d.$$

Throughout the paper we assume that $Y = U$ and that (A, B, C_Λ, D) is impedance passive [10, 38, 40], which is equivalent to the property that $\operatorname{Re}\langle Ax + Bu, x\rangle \leq \operatorname{Re}\langle C_\Lambda x + Du, u\rangle$ for all $x \in X$ and $u \in U$ satisfying $Ax + Bu \in X$ [38, Thm. 4.2]. Under this assumption the semigroup $T(t)$ generated by A is contractive, $\operatorname{Re} D \geq 0$, and $\operatorname{Re} P(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$. (Such transfer functions are called *positive*.)

We frequently use the following operator identity (see, e.g., [46, Proof of Thm. 1.2]). For completeness, we give a proof of the lemma in Appendix A.

LEMMA 2.1. *Let (A, B, C_Λ, D) be a regular linear system and let $Q \in \mathcal{L}(Y, U)$ be invertible. If $\lambda \in \rho(A)$ and if $Q^{-1} + C_\Lambda R(\lambda, A)B$ is boundedly invertible, then $\lambda \in \rho(A - BQC_\Lambda)$ and*

$$R(\lambda, A - BQC_\Lambda) = R(\lambda, A) - R(\lambda, A)B(Q^{-1} + C_\Lambda R(\lambda, A)B)^{-1} C_\Lambda R(\lambda, A),$$

where $D(A - BQC_\Lambda) = \{\, x \in D(C_\Lambda) \mid (A - BQC_\Lambda)x \in X \,\}$.

The system (1.2) is assumed to be another impedance passive regular linear system on a Hilbert space Z with $D_c^* = D_c$. The scale spaces Z_1 and Z_{-1} are defined analogously as X_1 and X_{-1}. We define $Z_{B_c} = D(A_c) + \mathcal{R}(R(\lambda_0, A_c)B_c)$ for some $\lambda_0 \in \rho(A_c)$ and denote the Λ-extension of C_c by $C_{c\Lambda}$. The passivity implies that $\operatorname{Re}\langle A_c z + B_c y, z\rangle \leq \operatorname{Re}\langle C_c z + D_c y, y\rangle$ for all $z \in Z$ and $y \in Y$ satisfying $A_c z + B_c y \in Z$, and we have $D_c \geq 0$. We denote the transfer function of (A_c, B_c, C_c, D_c) with

$$G(\lambda) = C_{c\Lambda} R(\lambda, A_c)B_c + D_c, \qquad \lambda \in \rho(A_c).$$

Our assumption $D_c \geq 0$ simplies the analysis of the admissibility of output feedbacks of the two passive systems (1.1) and (1.2). However, many of the results also hold in the situation where $\operatorname{Re} D_c \geq 0$ as long as the appropriate feedback operators remain admissible, which is the case, e.g., if $\|D_c - D_c^*\|$ is sufficently small.

3. Stability of coupled passive systems. In this section we present our main results on the stability of the closed-loop system associated to the power-preserving interconnection of (1.1) and (1.2). Lemma 4.2 in section 4 shows that the system operator A_e of the closed-loop system

$$\dot{x}_e(t) = A_e x_e(t), \qquad x_e(0) = x_{e0} = (x_0, z_0)^T \in X_e$$

is given by

$$A_e = \begin{bmatrix} A - BD_cQ_1C_\Lambda & BQ_2C_{c\Lambda} \\ -B_cQ_1C_\Lambda & A_c - B_cQ_1DC_{c\Lambda} \end{bmatrix}, \tag{3.1a}$$

$$D(A_e) = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in X_B \times Z_{B_c} \;\middle|\; \begin{matrix} (A - BD_cQ_1C_\Lambda)x + BQ_2C_{c\Lambda}z \in X \\ -B_cQ_1C_\Lambda x + (A_c - B_cQ_1DC_{c\Lambda})z \in Z \end{matrix} \right\}, \tag{3.1b}$$

where $Q_1 = (I + DD_c)^{-1}$ and $Q_2 = (I + D_cD)^{-1}$, and that A_e generates a strongly continuous contraction semigroup $T_e(t)$ on X_e.

Remark 3.1. Our results assume that (1.1) is stable and its transfer function $P(\lambda)$ satisfies certain additional conditions. However, the results are also immediately applicable when (1.1) is unstable but can be stabilized with a suitable output feedback. Indeed, if $D_c > 0$, we can write $D_c = D_{c1} + D_{c2}$ with $D_{c1} \geq 0$ and $D_{c2} > 0$. Lemma A.1(d) implies that $u(t) = -D_{c2}y(t)$ with $D_{c2} > 0$ is an admissible feedback for (A, B, C_Λ, D) and the resulting system $(A^S, B^S, C_\Lambda^S, D^S) = (A - BD_{c2}Q_1^SC_\Lambda, BQ_2^S, Q_1^SC_\Lambda, Q_1^SD)$ with $Q_1^S = (I + DD_{c2})^{-1}$ and $Q_2^S = (I + D_{c2}D)^{-1}$ is regular [43]. A direct computation shows that

$$A_e = \begin{bmatrix} A^S - B^SD_{c1}Q_3C_\Lambda^S & B^SQ_4C_{c\Lambda} \\ -B_cQ_3C_\Lambda^S & A_c - B_cQ_3D^SC_{c\Lambda} \end{bmatrix}.$$

Since this operator has exactly the same form as the original A_e, in each of our results it is possible to replace (A, B, C_Λ, D) with the stabilized system $(A^S, B^S, C_\Lambda^S, D^S)$, the transfer function $P(\lambda)$ with $P_S(\lambda) = C_\Lambda^S R(\lambda, A^S)B^S + D^S$, and the feedthrough operator $D_c \geq 0$ with $D_{c1} \geq 0$. It is important to note that if $P(\lambda)$ is invertible and $\operatorname{Re} P(\lambda) \geq 0$ for some $\lambda \in \rho(A)$, then for any $D_{c2} > 0$ we have $\operatorname{Re} P_S(\lambda) > 0$.

3.1. Strong stability. The following theorem presents sufficient conditions for the strong stability of the closed-loop system.

THEOREM 3.2. *Assume (A, B, C_Λ, D) is passive and strongly stable in such a way that $i\mathbb{R} \subset \rho(A)$. Moreover, assume $(A_c, B_c, C_{c\Lambda}, D_c)$ is passive, $D_c \geq 0$, and the following hold for some $\mathcal{I} \subset \mathbb{Z}$:*

(1) $\sigma(A_c) \cap i\mathbb{R} = \{i\omega_k\}_{k\in\mathcal{I}}$ *and* $\operatorname{Re} P(i\omega_k) > 0$ *for all* $k \in \mathcal{I}$.

(2) $I + P(i\omega)G(i\omega)$ *has a bounded inverse for every* $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$ *for which* $\operatorname{Re} G(i\omega)$ *is not boundedly invertible.*

(3) $\{i\omega_k\}_{k\in\mathcal{I}} \subset \rho(A_c - B_cD_0(I + D_cD_0)^{-1}C_{c\Lambda})$ *whenever* $\operatorname{Re} D_0 > 0$.

Then $i\mathbb{R} \subset \rho(A_e)$ and the closed-loop system is strongly stable.

Assume in addition that $\mathcal{I} \subset \mathbb{Z}$ is finite, (A, B, C_Λ, D) is exponentially stable, and $\sup_{|\omega|\geq R}\|R(i\omega, A_c)\| < \infty$ *for some $R > 0$. If either* $\limsup_{|\omega|\to\infty}\|G(i\omega)P(i\omega)\| < 1$ *or* $\operatorname{Re} P(i\omega) \geq \eta(\omega) \geq 0$ *and* $\operatorname{Re} G(i\omega) \geq d_c(\omega) \geq 0$ *such that* $\eta(\omega) + d_c(\omega) \geq \eta_0 > 0$ *for some constant $\eta_0 > 0$ and for all sufficiently large $|\omega|$, then the closed-loop system is exponentially stable.*

Proof. We begin by showing that $i\mathbb{R} \subset \rho(A_e)$. Since the semigroup generated by A_e is uniformly bounded by Lemma 4.2, the strong stability of $T_e(t)$ then follows from the Arendt–Batty–Lyubich–Vũ Theorem [4, 27].

Lemma A.1(d) implies that $u(t) = -D_c y(t)$ is an admissible output feedback for (A, B, C_Λ, D), and by [43] the resulting system $(A^{cl}, B^{cl}, C_\Lambda^{cl}, D^{cl}) = (A - BD_cQ_1C_\Lambda, BQ_2, Q_1C_\Lambda, Q_1D)$ is regular. The assumption $i\mathbb{R} \subset \rho(A)$ and Lemma A.3 imply $i\mathbb{R} \subset \rho(A^{cl})$, and by Lemma A.1(d) the transfer function $P_{cl}(\lambda)$ is given by $P_{cl}(i\omega) = P(i\omega)(I + D_cP(i\omega))^{-1}$ for all $\omega \in \mathbb{R}$. If $\omega \in \mathbb{R}$ and if we denote $R_{i\omega} = R(i\omega, A^{cl})$, then $i\omega - A_e$ has a bounded inverse given by

$$R(i\omega, A_e) = \begin{bmatrix} R_{i\omega} - R_{i\omega}B^{cl}C_{c\Lambda}S_A(i\omega)^{-1}B_cC_\Lambda^{cl}R_{i\omega} & R_{i\omega}B^{cl}C_{c\Lambda}S_A(i\omega)^{-1} \\ -S_A(i\omega)^{-1}B_cC_\Lambda^{cl}R_{i\omega} & S_A(i\omega)^{-1} \end{bmatrix}$$

provided that the Schur complement

$$\begin{aligned} S_A(i\omega) &= i\omega - A_c + B_cD^{cl}C_{c\Lambda} + B_cC_\Lambda^{cl}R(i\omega, A^{cl})B^{cl}C_{c\Lambda} \\ &= i\omega - A_c + B_cP(i\omega)(I + D_cP(i\omega))^{-1}C_{c\Lambda} \end{aligned}$$

with domain $D(S_A(i\omega)) = \{\, z \in D(C_{c\Lambda}) \mid S_A(i\omega)z \in Z \,\}$ has a bounded inverse. If $\omega = \omega_n$ for some $n \in \mathcal{I}$, then $\operatorname{Re} P(i\omega_n) > 0$ and assumption (3) imply that $S_A(i\omega_n)$ is boundedly invertible. Thus $\{i\omega_k\}_{k\in\mathcal{I}} \subset \rho(A_e)$.

Now let $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$. If $\operatorname{Re} G(i\omega) \not> 0$, then $I + G(i\omega)P(i\omega)$ is invertible by condition (2) of the theorem. By Lemma A.1(a) the same is also true if $\operatorname{Re} G(i\omega) > 0$, since $I + G(i\omega)P(i\omega) = G(i\omega)(G(i\omega)^{-1} + P(i\omega))$. Because

$$I + D_cP(i\omega) + C_{c\Lambda}R(i\omega, A_c)B_cP(i\omega) = I + G(i\omega)P(i\omega),$$

Lemma 2.1 implies that $S_A(i\omega)$ has a bounded inverse

$$S_A(i\omega)^{-1} = R(i\omega, A_c)\big[I - B_cP(i\omega)(I + G(i\omega)P(i\omega))^{-1}C_{c\Lambda}R(i\omega, A_c)\big]. \tag{3.2}$$

Thus $i\omega \in \rho(A_e)$ also for all $\omega \in \mathbb{R}\setminus\{\omega_k\}_{k\in\mathcal{I}}$. Since the semigroup $T_e(t)$ is contractive, the closed-loop system is strongly stable.

Finally, assume that $\mathcal{I} \subset \mathbb{Z}$ is finite, (A, B, C_Λ, D) is exponentially stable, and $\sup_{|\omega|\geq R}\|R(i\omega, A_c)\| < \infty$ for some $R > 0$. The stability and regularity of (A, B, C_Λ, D) imply that the norms $\|R(\cdot, A)\|$, $\|R(\cdot, A)B\|$, $\|C_\Lambda R(\cdot, A)\|$, and $\|P(\cdot)\|$ are uniformly bounded on $i\mathbb{R}$. Similarly the regularity of the controller implies that $\|R(i\omega, A_c)\|$, $\|R(i\omega, A_c)B_c\|$, $\|C_{c\Lambda}R(i\omega, A_c)\|$, and $\|C_{c\Lambda}R(i\omega, A_c)B_c\|$ are uniformly bounded with respect to $\omega \in \mathbb{R}$ with $|\omega| \geq R$. If $\limsup_{|\omega|\to\infty}\|G(i\omega)P(i\omega)\| < 1$ the norms $\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\|$ are uniformly bounded for large $|\omega|$. On the other hand, if $\eta(\omega) + d_c(\omega) \geq \eta_0 > 0$, then Lemma A.1(b) implies $\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\| \lesssim \eta_0^{-1}$. Thus (3.2) implies that $\|R(i\omega, A_e)\|$ is uniformly bounded for large $|\omega|$. Since $i\mathbb{R} \subset \rho(A_e)$ and $T_e(t)$ is contractive, the closed-loop system is exponentially stable. ∎

Remark 3.3. Condition (2) is in particular satisfied if $\operatorname{Re} G(i\omega) > 0$ for all $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$. Moreover, if $\operatorname{Re} G(i\omega) \geq d_c > 0$ for some constant $d_c > 0$ and for all $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$, then $\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\| \leq d_c^{-1}$ for all $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$ by Lemma A.1(b).

The proof of Theorem 3.2 can also be adapted to show that if $\operatorname{Re} P(i\omega) > 0$ for all $\omega \in \mathbb{R}$, then $T_e(t)$ is strongly stable and $i\mathbb{R} \subset \rho(A_e)$ even without assumption (2). Indeed, if $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$ and $\operatorname{Re} P(i\omega) > 0$, then Lemma A.1(a) implies that $P(i\omega)$

and $I+G(i\omega)P(i\omega)=(P(i\omega)^{-1}+G(i\omega))P(i\omega)$ are boundedly invertible, and $S_A(i\omega)$ has the bounded inverse given by the formula (3.2). Thus we again have $i\omega\in\rho(A_e)$. Lemma A.1(b) also shows that if $\eta(\omega)>0$ is such that $\operatorname{Re}P(i\omega)\geq\eta(\omega)>0$, then $\|P(i\omega)(I+G(i\omega)P(i\omega))^{-1}\|\leq\eta(\omega)^{-1}\|P(i\omega)\|^2$.

The following lemma provides a sufficient condition for the assumption (3) in Theorem 3.2 for isolated spectral points under a suitable observability property.

LEMMA 3.4. *Assume $(A_c,B_c,C_{c\Lambda},D_c)$ is passive with $D_c\geq 0$. Assume further that $i\omega_k\in\sigma(A_c)$ is an isolated spectral point and A_c has a spectral decomposition $A_c=A_c^0+A_c^c$ according to $Z=\mathcal{N}(i\omega_k-A_c)\oplus\mathcal{N}(i\omega_k-A_c)^\perp$ so that $i\omega_k\in\rho(A_c^c)$, and there exists $\gamma>0$ such that $\|C_{c\Lambda}z\|\geq\gamma\|z\|$ for all $z\in\mathcal{N}(i\omega_k-A_c)$. Then $i\omega_k\in\rho(A_c-B_cD_0(I+D_cD_0)^{-1}C_{c\Lambda})$ for any $D_0\in\mathcal{L}(U)$ with* $\operatorname{Re}D_0>0$.

Proof. Let $D_0\in\mathcal{L}(U)$ be such that $\operatorname{Re}D_0\geq d_0>0$ and denote $D_1=D_0(I+D_cD_0)^{-1}$. Due to the passivity of $(A_c,B_c,C_{c\Lambda},D_c)$ and [5, Cor. 4.3.2] we have $i\omega_k\in\sigma(A_c-B_cD_1C_{c\Lambda})$ provided that $\|(i\omega_k-A_c+B_cD_1C_{c\Lambda})z\|\geq c\|z\|$ for some constant $c>0$ and for all $z\in D(A_c-B_cD_1C_{c\Lambda})\subset Z_{B_c}$. Let $z\in D(A_c-B_cD_1C_{c\Lambda})$ and denote $y=(i\omega_k-A_c+B_cD_1C_{c\Lambda})z$. The passivity of $(A_c,B_c,C_{c\Lambda},D_c)$ implies

$$\begin{aligned}\operatorname{Re}\langle y,z\rangle&=-\operatorname{Re}\langle A_cz+B_c(-D_1C_{c\Lambda}z),z\rangle\geq\operatorname{Re}\langle C_{c\Lambda}z-D_cD_1C_{c\Lambda}z,D_1C_{c\Lambda}z\rangle\\&=\operatorname{Re}\langle(I+D_cD_0)^{-1}C_{c\Lambda}z,D_0(I+D_cD_0)^{-1}C_{c\Lambda}z\rangle\geq d_0\|I+D_cD_0\|^{-2}\|C_{c\Lambda}z\|^2.\end{aligned}$$

Thus $\|C_{c\Lambda}z\|^2\lesssim\|z\|\|y\|$. Write $z=z^k+z^c$ according to the decomposition $Z=\mathcal{N}(i\omega_k-A_c)\oplus\mathcal{N}(i\omega_k-A_c)^\perp$. If we apply $R_1=R(i\omega_k+1,A_c)$ to both sides of $y=(i\omega_k-A_c+B_cD_1C_{c\Lambda})z$ and use $R_1z^k\in\mathcal{N}(i\omega_k-A_c)$ we obtain

$$(i\omega_k-A_c^c)R_1z^c=R_1y-R_1B_cD_1C_{c\Lambda}z. \tag{3.3}$$

Since $R_1B_c\in\mathcal{L}(U,Z)$ and $i\omega_k-A_c^c$ is boundedly invertible by assumption, we have $\|R_1z^c\|^2\lesssim\|(i\omega_k-A_c^c)R_1z^c\|^2\lesssim\|y\|^2+\|C_{c\Lambda}z\|^2\lesssim\|y\|^2+\|z\|\|y\|$. Moreover, $(i\omega_k-A_c)R_1z^c=z^c-R_1z^c$ and $\|z^c\|\leq\|z\|$ together with (3.3) further imply

$$\begin{aligned}\|z^c\|^2&=\|R_1z^c+R_1y-R_1B_cD_1C_{c\Lambda}z\|^2\\&\lesssim\|R_1z^c\|^2+\|y\|^2+\|C_{c\Lambda}z\|^2\lesssim\|y\|^2+\|z\|\|y\|\\\|C_{c\Lambda}z^c\|^2&=\|C_{c\Lambda}R_1(z^c+y)-C_{c\Lambda}R_1B_cD_1C_{c\Lambda}z\|^2\\&\lesssim\|z^c\|^2+\|y\|^2+\|C_{c\Lambda}z\|^2\lesssim\|y\|^2+\|z\|\|y\|.\end{aligned}$$

Finally, since $\|z^k\|^2\leq\gamma^{-2}\|C_{c\Lambda}z^k\|^2\lesssim\gamma^{-2}(\|C_{c\Lambda}z\|^2+\|C_{c\Lambda}z^c\|^2)\lesssim\|y\|^2+\|z\|\|y\|$, we have $\|z\|^2=\|z^k\|^2+\|z^c\|^2\lesssim\|y\|^2+\|z\|\|y\|$, and thus also $\|z\|\lesssim\|y\|$. ∎

3.2. Exponential stability. The following theorem presents sufficient conditions for exponential stability of the closed-loop system. The transfer function $P(i\omega)$ is allowed to be noninvertible for some values $\omega\in\mathbb{R}$ (i.e., the system (A,B,C_Λ,D) may have "transmission zeros" on $i\mathbb{R}$), but such points must be uniformly disjoint from the spectrum of A_c. It should be noted that the result also remains valid if the conditions are satisfied for $\Omega=\mathbb{R}$. Condition (2) is in particular satisfied if $\operatorname{Re}G(i\omega)\geq d_c>0$ for some constant $d_c>0$ and for all $\omega\in\mathbb{R}\setminus\Omega$. Here *exponential stabilizability* and *exponential detectability* of a regular linear system are defined as in [34, Defs. 1.4–1.5] and [44, sec. III].

THEOREM 3.5. *Assume (A,B,C_Λ,D) is passive and exponentially stable,* $\operatorname{Re}D>0$*, and there exist $\Omega\subset\mathbb{R}$ and $\eta_0>0$ such that* $\operatorname{Re}P(i\omega)\geq\eta_0>0$ *for all $\omega\in\Omega$. Moreover, assume $(A_c,B_c,C_{c\Lambda},D_c)$ is passive, $D_c\geq 0$, and the following hold:*

(1) $\sigma(A_c) \cap i\mathbb{R} \subset i\Omega$ *and* $\sup_{\omega\in\mathbb{R}\setminus\Omega}\|R(i\omega, A_c)\| < \infty$.

(2) *Let* $\eta(\cdot), d_c(\cdot) : \mathbb{R} \setminus \Omega \to [0, 1]$ *be such that* $\operatorname{Re} P(i\omega) \geq \eta(\omega) \geq 0$ *and* $\operatorname{Re} G(i\omega) \geq d_c(\omega) \geq 0$ *for all* $\omega \in \mathbb{R} \setminus \Omega$. *Assume there exist* $0 < \delta < 1$ *and* $\eta_1 > 0$ *such that for each* $\omega \in \mathbb{R} \setminus \Omega$ *either* $\|G(i\omega)P(i\omega)\| \leq \delta < 1$ *or* $\eta(\omega) + d_c(\omega) \geq \eta_1 > 0$.

(3) *The system* $(A_c, B_c, C_{c\Lambda}, D_c)$ *is exponentially stabilizable and detectable.*

Then the closed-loop system is exponentially stable.

Proof. Our aim is to show $i\mathbb{R} \subset \rho(A_e)$ and $\sup_{\omega\in\mathbb{R}}\|R(i\omega, A_e)\| < \infty$. First let $\omega \in \mathbb{R} \setminus \Omega$. The proof of Theorem 3.2 shows that $S_A(i\omega)$ has an inverse

$$S_A(i\omega)^{-1} = R(i\omega, A_c)\big[I - B_cP(i\omega)(I + G(i\omega)P(i\omega))^{-1}C_{c\Lambda}R(i\omega, A_c)\big].$$

If $\|G(i\omega)P(i\omega)\| \leq \delta < 1$, then $\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\| \leq \|P(i\omega)\|/(1 - \delta)$, and if $\eta(\omega) + d_c(\omega) \geq \eta_1 > 0$, Lemma A.1(b) implies $\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\| \leq \eta_1^{-1} \max\{1, \|P(i\omega)\|\}$. Assumption (1) and the admissiblity of B_c and C_c imply $i\mathbb{R} \setminus i\Omega \subset \rho(A_e)$ and $\sup_{\omega\in\mathbb{R}\setminus\Omega}\|R(i\omega, A_e)\| < \infty$.

It remains to consider $\omega \in \Omega$. We decompose D into two parts $D = \mu D + \nu D$ with $\mu \in (0, 1)$ and $\nu = 1 - \mu$ in such a way that the first part stabilizes $(A_c, B_c, C_{c\Lambda}, D_c)$ exponentially and the second part can be used to show closed-loop stability. Indeed, for any $\mu \in (0, 1)$ the transfer function of the system $(A^\mu_c, B^\mu_c, C^\mu_{c\Lambda}, D^\mu_c)$ obtained from $(A_c, B_c, C_{c\Lambda}, D_c)$ with the admissible output feedback $u_c(t) = -\mu D y_c(t)$ is given by $G(\lambda)(I + \mu DG(\lambda))^{-1}$. Since $\operatorname{Re} D > 0$, this transfer function is uniformly bounded on $\mathbb{C}_+$ by Lemma A.1(b), and since $(A^\mu_c, B^\mu_c, C^\mu_{c\Lambda}, D^\mu_c)$ is exponentially stabilizable and detectable due to assumption (3), the semigroup generated by A^μ_c is exponentially stable [34, Cor. 1.8].

For all sufficiently small $\mu \in (0, 1)$ the transfer function $P_\nu(\lambda)$ of $(A, B, C_\Lambda, \nu D)$ satisfies $\operatorname{Re} P_\nu(i\omega) \geq \tilde\eta_0 > 0$ for some constant $\tilde\eta_0 > 0$ and for all $\omega \in \Omega$. Since $D^\mu_c = D_c(I + \mu DD_c)^{-1}$, Lemmas A.1 and A.2 imply that we can choose $\mu \in (0, 1)$ so that $I + \nu DD^\mu_c$ and $I + P_\nu(i\omega)D^\mu_c$ for all $\omega \in \Omega$ are invertible, and $\sup_{\omega\in\Omega}\|(I + P_\nu(i\omega)D^\mu_c)^{-1}\| < \infty$. Thus $u(t) = -D^\mu_c y(t)$ is an admissible output feedback for $(A, B, C_\Lambda, \nu D)$. Denoting the resulting regular linear system with $(A^\mu, B^\mu, C^\mu_\Lambda, D^\mu) = (A - BD^\mu_c Q^\mu_5 C_\Lambda, BQ^\mu_6, Q^\mu_5 C_\Lambda, \nu Q^\mu_5 D)$, where $Q^\mu_5 = (I + \nu DD^\mu_c)^{-1}$ and $Q^\mu_6 = (I + \nu D^\mu_c D)^{-1}$, we can write

$$A_e = \begin{bmatrix} A - BD^\mu_c Q^\mu_5 C_\Lambda & BQ^\mu_6 C^\mu_{c\Lambda} \\ -B^\mu_c Q^\mu_5 C_\Lambda & A^\mu_c - \nu B^\mu_c Q^\mu_5 DC^\mu_{c\Lambda} \end{bmatrix} = \begin{bmatrix} A^\mu & B^\mu C^\mu_{c\Lambda} \\ -B^\mu_c C^\mu_\Lambda & A^\mu_c - B^\mu_c D^\mu C^\mu_{c\Lambda} \end{bmatrix}.$$

Similarly as in Lemma A.3 we can show that $\sup_{\omega\in\Omega}\|R(i\omega, A^\mu)\| < \infty$ and the transfer function of $(A^\mu, B^\mu, C^\mu_\Lambda, D^\mu)$ satisfies $P_\mu(i\omega) = P_\nu(i\omega)(I + D^\mu_c P_\nu(i\omega))^{-1}$ for all $\omega \in \Omega$. The transfer function of $(A^\mu_c, B^\mu_c, C^\mu_{c\Lambda}, D^\mu_c)$ is denoted by $G^\mu(\lambda)$.

Let $\omega \in \Omega$. If we denote $R^\mu_{i\omega} = R(i\omega, A^\mu)$, then $i\omega - A_e$ has a bounded inverse

$$R(i\omega, A_e) = \begin{bmatrix} R^\mu_{i\omega} - R^\mu_{i\omega} B^\mu C^\mu_{c\Lambda} S^\mu_A(i\omega)^{-1} B^\mu_c C^\mu_\Lambda R^\mu_{i\omega} & R^\mu_{i\omega} B^\mu C^\mu_{c\Lambda} S^\mu_A(i\omega)^{-1} \\ -S^\mu_A(i\omega)^{-1} B^\mu_c C^\mu_\Lambda R^\mu_{i\omega} & S^\mu_A(i\omega)^{-1} \end{bmatrix}$$

provided that the Schur complement

$$\begin{aligned} S^\mu_A(i\omega) &= i\omega - A^\mu_c + B^\mu_c D^\mu C^\mu_{c\Lambda} + B^\mu_c C^\mu_\Lambda R(i\omega, A^\mu) B^\mu C^\mu_{c\Lambda} \\ &= i\omega - A^\mu_c + B^\mu_c P_\nu(i\omega)(I + D^\mu_c P_\nu(i\omega))^{-1} C^\mu_{c\Lambda} \end{aligned}$$

has a bounded inverse. If $S^\mu_A(i\omega)$ is boundedly invertible for all $\omega \in \Omega$, then the regularity of $(A^\mu, B^\mu, C^\mu_\Lambda, D^\mu)$ and $\sup_{\omega\in\Omega}\|R(i\omega, A^\mu)\| < \infty$ imply $\sup_{\omega\in\Omega}\|R(i\omega, A_e)\| <$

∞ provided that $\|S_A^\mu(i\omega)^{-1}\|$, $\|S_A^\mu(i\omega)^{-1}B_c^\mu\|$, $\|C_{c\Lambda}^\mu S_A^\mu(i\omega)^{-1}\|$, and $\|C_{c\Lambda}^\mu S_A^\mu(i\omega)^{-1}B_c^\mu\|$ are uniformly bounded with respect to $\omega \in \Omega$.

Let $\omega \in \Omega$ be arbitrary. Since $\operatorname{Re} P_\nu(i\omega) \geq \tilde{\eta}_0 > 0$ and $\operatorname{Re} G^\mu(i\omega) \geq 0$, Lemma A.1 implies that $P_\nu(i\omega)$ and $I+G^\mu(i\omega)P_\nu(i\omega) = (P_\nu(i\omega)^{-1}+G^\mu(i\omega))P_\nu(i\omega)$ are boundedly invertible. Therefore the same is true for

$$I + D_c^\mu P_\nu(i\omega) + C_{c\Lambda}^\mu R(i\omega, A_c^\mu)B_c^\mu P_\nu(i\omega) = I + G^\mu(i\omega)P_\nu(i\omega).$$

Lemma 2.1 implies that $S_A^\mu(i\omega)$ has a bounded inverse

$$S_A^\mu(i\omega)^{-1} = R(i\omega, A_c^\mu)\big[I - B_c^\mu P_\nu(i\omega)(I + G^\mu(i\omega)P_\nu(i\omega))^{-1}C_{c\Lambda}^\mu R(i\omega, A_c^\mu)\big],$$

where $\|P_\nu(i\omega)(I + G^\mu(i\omega)P_\nu(i\omega))^{-1}\| \leq \|P_\nu(i\omega)\|^2/\tilde{\eta}_0$. Thus $i\omega \in \rho(A_e)$. Since $\sup_{\omega\in\mathbb{R}}\|P_\nu(i\omega)\| < \infty$ and $(A_c^\mu, B_c^\mu, C_{c\Lambda}^\mu, D_c^\mu)$ is regular and exponentially stable, the norms $\|S_A^\mu(i\omega)^{-1}\|$, $\|S_A^\mu(i\omega)^{-1}B_c^\mu\|$, $\|C_{c\Lambda}^\mu S_A^\mu(i\omega)^{-1}\|$, and $\|C_{c\Lambda}^\mu S_A^\mu(i\omega)^{-1}B_c^\mu\|$ are uniformly bounded with respect to $\omega \in \Omega$. This further implies that $\sup_{\omega\in\Omega}\|R(i\omega, A_e)\| < \infty$, and the closed-loop system is exponentially stable. ∎

Since both (A, B, C_Λ, D) and $(A_c, B_c, C_{c\Lambda}, D_c)$ are exponentially stabilizable in Theorem 3.5, the exponential closed-loop stability could alternatively be studied using [44, Prop. 4.6].

3.3. Nonuniform closed-loop stability. In this section we introduce conditions for polynomial and nonuniform stability of the closed-loop system in the case where A_c is diagonal. In addition, our main result can be used as an alternative to Theorem 3.5 in showing exponential closed-loop stability. The closed-loop system is said to be nonuniformly stable when $T_e(t)$ is uniformly bounded and $i\mathbb{R} \subset \rho(A_e)$ but the norms $\|R(i\omega, A_e)\|$ are not bounded with respect to $\omega \in \mathbb{R}$. If $M_R(\cdot)$ is a continuous nondecreasing function such that $\|R(i\omega, A_e)\| \leq M_R(|\omega|)$, then there exist $M_e, c, t_0 > 0$ such that

$$\|T_e(t)x_{e0}\| \leq \frac{M_e}{M_T(t)}\|A_e x_{e0}\| \qquad \forall x_{e0} \in D(A_e),\ t \geq t_0, \tag{3.4}$$

where the continuous nondecreasing function $M_T(\cdot) : [0, \infty) \to (0, \infty)$ is determined by the results in [7, 8, 36]. In particular, if $M_R(\omega) \lesssim 1 + \omega^\alpha$ for some $\alpha > 0$, we can choose $M_T(t) = t^{1/\alpha}$ [8], and if $M_R(\omega) \lesssim 1 + e^{\alpha\omega}$ for some $\alpha > 0$, then we can choose $M_T(t) = \log(t)/\alpha$ [7, Ex. 1.6].

In this section we assume $(A_c, B_c, C_{c\Lambda}, D_c)$ is regular and passive with $D_c \geq 0$ on a Hilbert space $Z = \bigotimes_{k\in\mathcal{I}} Z_k$ with norm $\|(z_k)_k\|_Z^2 = \sum_{k\in\mathcal{I}}\|z_k\|_{Z_k}^2$, where Z_k are Hilbert and $\mathcal{I} \subset \mathbb{Z}$ is infinite. We assume A_c has the structure

$$A_c = \operatorname{diag}(i\omega_k I_{Z_k})_{k\in\mathcal{I}}, \qquad D(A_c) = \left\{ (z_k)_k \in Z \,\middle|\, \sum_{k\in\mathcal{I}} |\omega_k|^2 \|z_k\|_{Z_k}^2 < \infty \right\}, \tag{3.5}$$

where $\omega_k \neq \omega_l$ for $k \neq l$ and $\{\omega_k\}_k$ has no finite accumulation points. Since A_c is skew-adjoint, the operators $B_c \in \mathcal{L}(Y, Z_{-1})$ and $C_c \in \mathcal{L}(Z_1, Y)$ are formally adjoint, i.e., $\langle B_c u, z\rangle_{-1,1} = \langle u, C_c z\rangle$ for all $z \in D(A_c)$ and $u \in Y$, and thus

$$B_c u = (B_{ck}u)_{k\in\mathcal{I}}, \qquad \text{and} \qquad C_c z = \sum_{k\in\mathcal{I}} B_{ck}^* z_k, \quad z = (z_k)_{k\in\mathcal{I}} \in D(A_c)$$

for some $B_{ck} \in \mathcal{L}(Y, Z_k)$. Our main result uses *wavepackets* of A_c [39, sec. 6.9].

Definition 3.6. *Let $\omega \in \mathbb{R}$ and $\delta > 0$. An element $z = (z_k)_{k\in\mathcal{I}} \in Z$ is a (ω,δ)-wavepacket of A_c if $z_k = 0$ for those $k \in \mathcal{I}$ for which $|\omega - \omega_k| \geq \delta$.*

The following theorem is the main result of this section. The role of $\Omega_\varepsilon \subset \mathbb{R}$ is to show that only the behavior of $\operatorname{Re} P(i\omega)$ near $\sigma(A_c) = \{i\omega_k\}_{k\in\mathcal{I}}$ affects the asymptotic growth of $\|R(i\omega, A_e)\|$. By [28, Cor. 2.17] $\delta(\cdot)$ and $\gamma(\cdot)$ can be chosen as constant functions if and only if (A_c, B_c) is exactly controllable. The assumption that $M_R(\cdot) : [0,\infty) \to (0,\infty)$ has "positive increase" means that there exists $\alpha, c, \omega_0 > 0$ such that $M_R(\lambda\omega) \geq c\lambda^\alpha M_R(\omega)$ for all $\lambda > 0$ and $\omega \geq \omega_0$ [36, sec. 2], and this condition is in particular satisfied if $M_R(\cdot)$ grows polynomially or exponentially. The estimation of $\|S_A(i\omega)^{-1}\|$ in the proof extends techniques developed in [12].

Theorem 3.7. *Assume (A, B, C_Λ, D) is passive and exponentially stable and the system $(A_c, B_c, C_{c\Lambda}, D_c)$ is passive with A_c of form (3.5) and $D_c \geq 0$. Assume further that condition (2) of Theorem 3.5 is satisfied for $\Omega = \Omega_\varepsilon := \{\,\omega \in \mathbb{R} \mid \exists k \in \mathcal{I} : |\omega - \omega_k| < \varepsilon\,\}$ with some $\varepsilon > 0$ and that there exist continuous nonincreasing functions $\eta(\cdot), \delta(\cdot), \gamma(\cdot) : \mathbb{R}_+ \to (0,1]$ with the following properties:*

- *$\operatorname{Re} P(i\omega) \geq \eta(|\omega|)$ for all $\omega \in \Omega_\varepsilon$.*
- *$\|C_c z\| \geq \gamma(|\omega|)\|z\|$ for every $\omega \in \mathbb{R}$ and every $(\omega, \delta(|\omega|))$-wavepacket z of A_c.*

Then $T_e(t)$ is strongly stable, $i\mathbb{R} \subset \rho(A_e)$, and

$$\|R(i\omega, A_e)\| \leq M_R(|\omega|), \qquad \text{where} \quad M_R(\cdot) = M_0\eta(\cdot)^{-1}\gamma(\cdot)^{-2}\delta(\cdot)^{-2}$$

for some $M_0 > 0$. Moreover, the following hold:

(a) *If $\sup_{\omega>0} M_R(\omega) < \infty$, then $T_e(t)$ is exponentially stable.*

(b) *If $M_R(\cdot)$ is strictly increasing and has positive increase, then (3.4) holds with $M_T(t) = M_R^{-1}(ct)$ for some constants $M_e, c, t_0 > 0$.*

(c) *For all other $M_R(\cdot)$, (3.4) holds with $M_T(t) = M_{\log}^{-1}(ct)$ for some $M_e, c, t_0 > 0$, where $M_{\log}(\omega) = M_R(\omega)\,(\log(1 + M_R(\omega)) + \log(1 + \omega))$ for $\omega > 0$.*

Proof. By Theorem 3.2 and Lemma 3.4 the closed-loop system is strongly stable and $i\mathbb{R} \subset \rho(A_e)$. Once we show $\|R(i\omega, A_e)\| \leq M_R(|\omega|)$ the stability properties of the closed-loop system follow from the characterization of exponential stability (part (a)), from [36, Thm. 1.1] (part (b)), and from [7, Thm. 1.5] (part (c)).

Since $(A^{cl}, B^{cl}, C^{cl}_\Lambda, D^{cl})$ is regular and exponentially stable by Lemma A.3, we have from the proof of Theorem 3.2 that

$$\|R(i\omega, A_e)\| \lesssim \max\{\|S_A(i\omega)^{-1}\|, \|S_A(i\omega)^{-1}B_c\|, \|C_{c\Lambda}S_A(i\omega)^{-1}\|, \|C_{c\Lambda}S_A(i\omega)^{-1}B_c\|\}$$

for $\omega \in \mathbb{R}$, where $S_A(i\omega) = i\omega - A_c + B_c P_{cl}(i\omega)C_{c\Lambda}$ and $P_{cl}(i\omega) = P(i\omega)(I + D_c P(i\omega))^{-1}$. Moreover, (3.2) and our assumptions imply $\sup_{\omega\in\mathbb{R}\setminus\Omega_\varepsilon}\|R(i\omega, A_e)\| < \infty$ similarly as in the proof of Theorem 3.5. Thus it is sufficient to show that for each $\omega \in \Omega_\varepsilon$ the norms $\|S_A(i\omega)^{-1}\|$, $\|S_A(i\omega)^{-1}B_c\|$, $\|C_{c\Lambda}S_A(i\omega)^{-1}\|$, $\|C_{c\Lambda}S_A(i\omega)^{-1}B_c\|$ are bounded by $M_R(|\omega|)$ for some constant $M_0 > 0$.

We begin by showing $\|C_{c\Lambda}S_A(i\omega)^{-1}B_c\| \leq M_R(|\omega|)$. Formula (3.2) implies that for all $\omega \in \Omega_\varepsilon \setminus \{\omega_k\}_k$

$$\begin{aligned} C_{c\Lambda}S_A(i\omega)^{-1}B_c &= C_{c\Lambda}R(i\omega, A_c)B_c\bigl[I - (I + P(i\omega)G(i\omega))^{-1}P(i\omega)C_{c\Lambda}R(i\omega, A_c)B_c\bigr] \\ &= (G(i\omega) - D_c)(I + P(i\omega)G(i\omega))^{-1}(I + P(i\omega)D_c). \end{aligned}$$

Since $\operatorname{Re} P(i\omega) > 0$ and $\operatorname{Re} G(i\omega) \geq 0$, $I + P(i\omega)G(i\omega) = P(i\omega)(P(i\omega)^{-1} + G(i\omega))$ is boundedly invertible by Lemma A.1(a). If we denote $Q(i\omega) = (I + P(i\omega)G(i\omega))^{-1}$, the above formula and stability of (A, B, C_Λ, D) implies

$$\|C_{c\Lambda} S_A(i\omega)^{-1} B_c\| = \|(G(i\omega) - D_c)Q(i\omega)(I + P(i\omega)D_c)\| \lesssim \|G(i\omega)Q(i\omega)\| + \|Q(i\omega)\|.$$

Here $\|G(i\omega)Q(i\omega)\| \leq \eta(|\omega|)^{-1}$ by Lemma A.1(b). We claim that $\|Q(i\omega)\| \lesssim \eta(|\omega|)^{-1}$ for $\omega \in \Omega_\varepsilon \setminus \{\omega_k\}_{k\in\mathcal{I}}$. If this is not true, then (considering $Q(i\omega)^*$) there exist sequences $(s_n)_n \subset \Omega_\varepsilon \setminus \{\omega_k\}_k$ and $(u_n)_n \subset Y$ with $\|u_n\| = 1$ such that $\eta(|s_n|)^{-1}\|(I + G(is_n)^* P(is_n)^*)u_n\| \to 0$ as $n \to \infty$. Since $\sup_{\omega\in\mathbb{R}}\|P(i\omega)\| < \infty$, we have that also

$$0 \leftarrow \frac{1}{\eta(|s_n|)} \operatorname{Re}\langle (I + G(is_n)^* P(is_n)^*)u_n, P(is_n)^* u_n\rangle \geq \frac{\operatorname{Re}\langle P(is_n)u_n, u_n\rangle}{\eta(|s_n|)}$$

as $n \to \infty$, which is impossible since $\eta(|s_n|)^{-1} \operatorname{Re}\langle P(is_n)u_n, u_n\rangle \geq 1$ by assumption. This contradiction shows that the claim holds. Thus $\|C_{c\Lambda} S_A(i\omega)^{-1} B_c\| \lesssim \eta(|\omega|)^{-1} \leq M_R(|\omega|)$ for some $M_0 > 0$ and for all $\omega \in \Omega_\varepsilon \setminus \{\omega_k\}_k$, and by continuity the same estimate holds for every $\omega \in \Omega_\varepsilon$.

To estimate the norms $\|S_A(i\omega)^{-1}\|$, $\|S_A(i\omega)^{-1}B_c\|$, $\|C_{c\Lambda} S_A(i\omega)^{-1}\|$, let $\omega \in \Omega_\varepsilon$ with $|\omega| \geq 1$ and define $P_{\omega,\delta} = \operatorname{diag}(\beta_k I_{Z_k})_{k\in\mathcal{I}} \in \mathcal{L}(Z)$, where $\beta_k = 1$ for those $k \in \mathcal{I}$ for which $|\omega - \omega_k| < \delta(|\omega|)$ and $\beta_k = 0$ otherwise. The operator $P_{\omega,\delta}$ is a spectral projection of A_c associated to the part $\{i\omega_k\}_k \cap (i\omega - i\delta(|\omega|), i\omega + i\delta(|\omega|))$ of its spectrum and $P_{\omega,\delta} z$ is a $(\omega, \delta(|\omega|))$-wavepacket of A_c for every $z \in Z$. Let $u \in Y$ and $y \in Z$ be arbitrary and define $z = S_A(i\omega)^{-1}(B_c u + y) \in Z_{B_c}$, i.e., $(i\omega - A_c + B_c P_{cl}(i\omega) C_{c\Lambda})z = B_c u + y$.

Define $z_0 = P_{\omega,\delta} z$, $z_c = z - z_0$, $y_c = P_{\omega,\delta} y$, $y_c = y - y_0$. Similarly decompose $A_c = A_c^0 + A_c^c$, $B_c = B_c^0 + B_c^c$, and $C_{c\Lambda} = C_c^0 + C_{c\Lambda}^c$, where $A_c^0 = A_c P_{\omega,\delta}$, $B_c^0 = P_{\omega,\delta} B_c$ and $C_c^0 = C_c P_{\omega,\delta}$. The diagonal structure of A_c and the decompositions imply

$$\begin{aligned}
&(i\omega - A_c^c)z_c = y_c + B_c^c(u - P_{cl}(i\omega)C_{c\Lambda} z) \\
\Rightarrow \quad & z_c = R(i\omega, A_c^c)y_c + R(i\omega, A_c^c)B_c^c(u - P_{cl}(i\omega)C_{c\Lambda} z) \\
\Rightarrow \quad & C_{c\Lambda} z_c = C_{c\Lambda} R(i\omega, A_c^c)y_c + G_{0c}(i\omega)(u - P_{cl}(i\omega)C_{c\Lambda} z),
\end{aligned}$$

where we have denoted $G_{0c}(i\omega) = C_{c\Lambda}^c R(i\omega, A_c^c)B_c^c$. The system $(A_c^c, B_c^c, C_{c\Lambda}^c)$ is regular and due to the diagonal structure of A_c we have $\|R(i\omega, A_c^c)\| \lesssim \delta(|\omega|)^{-1}$. The resolvent identity $R(i\omega, A_c^c) = R(i\omega + 1, A_c^c) + R(i\omega, A_c^c)R(i\omega + 1, A_c^c)$ and the admissibility of B_c^c and C_c^c further imply

$$\|R(i\omega, A_c^c)B_c^c\| \lesssim \delta(|\omega|)^{-1}, \quad \|C_{c\Lambda} R(i\omega, A_c^c)\| \lesssim \delta(|\omega|)^{-1}, \quad \|G_{0c}(i\omega)\| \lesssim \delta(|\omega|)^{-1}.$$

Since z_0 is a $(\omega, \delta(|\omega|))$-wavepacket, we have also $\|z_0\| \leq \gamma(|\omega|)^{-1}\|C_c z_0\|$. The above expressions for z_c and $C_{c\Lambda} z_c$ together with $C_c z_0 = C_{c\Lambda} z - C_{c\Lambda} z_c$ and $\sup_{s\in\mathbb{R}}\|P_{cl}(is)\| < \infty$ (Lemma A.2) therefore imply

$$\begin{aligned}
\|z\|^2 &= \|z_c\|^2 + \|z_0\|^2 \leq \|z_c\|^2 + \gamma(|\omega|)^{-2}\|C_c z_0\|^2 \\
&\lesssim \|z_c\|^2 + \gamma(|\omega|)^{-2}\|C_{c\Lambda} z\|^2 + \gamma(|\omega|)^{-2}\|C_{c\Lambda}^c z_c\|^2 \\
&\lesssim \big(\|R(i\omega, A_c^c)\|^2 + \gamma(|\omega|)^{-2}\|C_{c\Lambda} R(i\omega, A_c^c)\|^2\big)\|y_c\|^2 + \gamma(|\omega|)^{-2}\|C_{c\Lambda} z\|^2 \\
&\quad + \big(\|R(i\omega, A_c^c)B_c^c\|^2 + \gamma(|\omega|)^{-2}\|G_{0c}(i\omega)\|^2\big)\big(\|u\|^2 + \|P_{cl}(i\omega)\|^2\|C_{c\Lambda} z\|^2\big) \\
&\lesssim \gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\big(\|y\|^2 + \|u\|^2 + \|C_{c\Lambda} z\|^2\big).
\end{aligned}$$

First let $u=0$ to estimate $\|S_A(i\omega)^{-1}\|$ and $\|C_{c\Lambda}S_A(i\omega)^{-1}\|$. Then $z=S_A(i\omega)y\in D(S_A(i\omega))$. The passivity of $(A_c,B_c,C_{c\Lambda},D_c)$ implies

$$\begin{aligned}\mathrm{Re}\langle y,x\rangle &= -\mathrm{Re}\langle A_cz+B_c(-P_{cl}(i\omega)C_{c\Lambda}z),z\rangle\\ &\geq \mathrm{Re}\langle C_{c\Lambda}z-D_cP_{cl}(i\omega)C_{c\Lambda}z,P_{cl}(i\omega)C_{c\Lambda}z\rangle\\ &= \mathrm{Re}\langle (I+D_cP(i\omega))^{-1}C_{c\Lambda}z,P(i\omega)(I+D_cP(i\omega))^{-1}C_{c\Lambda}z\rangle\\ &\geq \eta(|\omega|)\|I+D_cP(i\omega)\|^{-2}\|C_{c\Lambda}z\|^2\geq \frac{\eta(|\omega|)}{M_P^2}\|C_{c\Lambda}z\|^2,\end{aligned}$$

where $M_P=1+\|D_c\|\sup_{\omega\in\mathbb{R}}\|P(i\omega)\|<\infty$, and thus $\|C_{c\Lambda}z\|^2\lesssim\eta(|\omega|)^{-1}\|z\|\|y\|$. The above estimate for $\|z\|^2$ (again with $u=0$) together with the scalar inequality $2ab\leq\varepsilon a^2+b^2/\varepsilon$ for $\varepsilon>0$ implies

$$\begin{aligned}\|z\|^2 &\lesssim \gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\left(\|y\|^2+\|C_{c\Lambda}z\|^2\right)\\ &\lesssim \gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\|y\|^2+\eta(|\omega|)^{-1}\gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\|z\|\|y\|\\ &\leq \gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\|y\|^2+\frac{\varepsilon}{2}\|z\|^2+\frac{1}{2\varepsilon}\eta(|\omega|)^{-2}\gamma(|\omega|)^{-4}\delta(|\omega|)^{-4}\|y\|^2.\end{aligned}$$

Letting $\varepsilon>0$ be small shows that $\|z\|\lesssim\eta(|\omega|)^{-1}\gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\|y\|$. Since $y\in Z$ was arbitrary, we have that $\|S_A(i\omega)^{-1}\|\leq M_R(|\omega|)$ for some $M_0>0$. Moreover, our earlier estimate $\|C_{c\Lambda}z\|^2\lesssim\eta(|\omega|)^{-1}\|z\|\|y\|$ further implies

$$\|C_{c\Lambda}S_A(i\omega)^{-1}y\|^2=\|C_{c\Lambda}z\|^2\lesssim\eta(|\omega|)^{-1}\|z\|\|y\|\lesssim\eta(|\omega|)^{-2}\gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\|y\|^2,$$

and thus $\|C_{c\Lambda}S_A(i\omega)^{-1}\|\lesssim\eta(|\omega|)^{-1}\gamma(|\omega|)^{-1}\delta(|\omega|)^{-1}\leq M_R(|\omega|)$ for some $M_0>0$.

Finally, to estimate $\|S_A(i\omega)^{-1}B_c\|$, let $y=0$ and let $u\in Y$ be arbitrary. Now we have $z=S_A(i\omega)^{-1}B_cu$, and thus $\|C_{c\Lambda}z\|=\|C_{c\Lambda}S_A(i\omega)B_cu\|\lesssim\eta(|\omega|)^{-1}\|u\|$ due to our earlier estimate. Because of this, we also have

$$\begin{aligned}\|S_A(i\omega)^{-1}B_cu\|^2=\|z\|^2 &\lesssim \gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\left(\|u\|^2+\|C_{c\Lambda}z\|^2\right)\\ &\lesssim \gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}(1+\eta(|\omega|)^{-2})\|u\|^2\end{aligned}$$

and thus $\|S_A(i\omega)^{-1}B_c\|\lesssim\eta(|\omega|)^{-1}\gamma(|\omega|)^{-1}\delta(|\omega|)^{-1}\leq M_R(|\omega|)$ for some $M_0>0$. ◻

In the case where $X=\{0\}$, $A=0\in\mathcal{L}(X)$, $B=0\in\mathcal{L}(U,X)$, $C=0\in\mathcal{L}(X,U)$, and $D=I\in\mathcal{L}(U)$ the operator $S_A(i\omega)$ reduces to $i\omega-A_c+B_c(I+D_c)^{-1}C_{c\Lambda}$. This way Theorem 3.7 can also be used to study the nonuniform stability of semigroups generated by operators of the form $A_c-B_cB_c^*$ and $A_c-B_c(I+D_c)^{-1}C_{c\Lambda}$. This topic is considered in detail in [12].

Remark 3.8. Assume $\{\omega_k\}_{k\in\mathcal{I}}$ has a uniform gap, i.e., $\inf_{k\neq l}|\omega_k-\omega_l|>0$, and $\tilde{\gamma}:\mathbb{R}_+\to(0,1]$ is a continuous nonincreasing function such that $\inf_{\omega>0}\tilde{\gamma}(\omega+\delta_0)/\tilde{\gamma}(\omega)>0$ for some $0<\delta_0<\min\{1,\frac{1}{2}\inf_{k\neq l}|\omega_k-\omega_l|\}$ (so that $\tilde{\gamma}(\cdot)$ does not decrease too rapidly). If $\|B_{ck}^*z_k\|\geq\tilde{\gamma}(|\omega_k|)\|z_k\|$ for all $k\in\mathcal{I}$ and $z_k\in Z_k$, then there exists a constant $0<c\leq 1$ for which the functions $\gamma(\cdot)=c\tilde{\gamma}(\cdot)$ and $\delta(\cdot)\equiv\delta_0>0$ are such that $\|C_cz\|\geq\gamma(|\omega|)\|z\|$ for every $\omega\in\mathbb{R}$ and every $(\omega,\delta(|\omega|))$-wavepacket z of A_c.

4. The robust output regulation problem. We will now turn our attention to constructing passive controllers of the form (1.4) to achieve robust output tracking and disturbance rejection for a passive regular linear system (2.1). We assume the reference signal $y_{ref}(t)$ and the disturbance signal $w_{dist}(t)$ are of the form

$$y_{ref}(t)=\sum_{k\in\mathcal{I}}y_{ref}^ke^{i\omega_kt}\quad\text{and}\quad w_{dist}(t)=\sum_{k\in\mathcal{I}}w_{dist}^ke^{i\omega_kt}\tag{4.1}$$

with a given set $\{\omega_k\}_{k\in\mathcal{I}} \subset \mathbb{R}$ of distinct frequencies with no finite accumulation points, and $\{y_{ref}^k\}_{k\in\mathcal{I}} \subset Y$ and $\{w_{dist}^k\}_{k\in\mathcal{I}} \subset U_d$. We use the notation $w_{ext}(t) = (w_{dist}(t), y_{ref}(t))^T$ and $w_{ext}^k = (w_{dist}^k, y_{ref}^k)^T$. We consider $y_{ref}(t)$ and $w_{dist}(t)$ with both finite and infinite number of frequency components, and these two classes of signals are treated separately. The latter situation is encountered in tracking and rejection of nonsmooth periodic signals [24]. If $\mathcal{I}$ is infinite, we assume $(y_{ref}^k)_{k\in\mathcal{I}} \in \ell^1(\mathcal{I}; Y)$ and $(w_{dist}^k)_{k\in\mathcal{I}} \in \ell^1(\mathcal{I}; U_d)$, which imply that $y_{ref}(t)$ and $w_{dist}(t)$ are uniformly continuous *almost periodic functions* [5, Def. 4.5.6]. In the case of real-valued $y_{ref}(t)$ and $w_{dist}(t)$ we have $\pm\omega_n \in \{\omega_k\}_{k\in\mathcal{I}}$ for all $n \in \mathcal{I}$.

We make the following standing assumption on the system (2.1). Here $P_S(\lambda)$ is the transfer function of the system $(A^S, B^S, C_\Lambda^S, D^S)$ obtained from (2.1) with admissible output feedback $u(t) = -D_{c2}y(t)$ with $D_{c2} \geq 0$. It should be noted that Assumption 4.1 is satisfied for some $D_{c2} \geq 0$ for which $\{i\omega_k\}_k \subset \rho(A^S)$ if and only if it is satisfied for all $D_{c2} \geq 0$ with this property. In particular, if $i\omega_k \in \rho(A)$ for some $k \in \mathcal{I}$, then $P_S(i\omega_k)$ is invertible if and only if $P(i\omega_k)$ is invertible.

Assumption 4.1. There exists $D_{c2} \geq 0$ such that $i\omega_k \in \rho(A^S)$ and $P_S(i\omega_k)$ is boundedly invertible for all $k \in \mathcal{I}$.

We define the *regulation error* as $e(t) = y_{ref}(t) - y(t)$. Our aim is to choose $(A_c, B_c, C_{c\Lambda}, D_c)$ in such a way that $e(t)$ converges to zero in a suitable sense as $t \to \infty$. The closed-loop system consisting of (2.1) and the controller (1.4) with state $x_e(t) = (x(t), z(t))^T$ on $X_e = X \times Z$ is of the form

$$\dot{x}_e(t) = A_e x_e(t) + B_e w_{ext}(t), \qquad x_e(0) = x_{e0} = (x_0, z_0)^T \in X_e, \tag{4.2a}$$

$$e(t) = C_e x_e(t) + D_e w_{ext}(t), \tag{4.2b}$$

where $w_{ext}(t) = (w_{dist}(t), y_{ref}(t))^T$. If we denote $Q_1 = (I + DD_c)^{-1}$ and $Q_2 = (I + D_c D)^{-1}$, then A_e and $D(A_e)$ are as in (3.1) and

$$B_e = \begin{bmatrix} B_d & BD_cQ_1 \\ 0 & B_cQ_1 \end{bmatrix}, \quad C_e = \begin{bmatrix} -Q_1C_\Lambda & -Q_1DC_{c\Lambda} \end{bmatrix}, \quad D_e = \begin{bmatrix} 0 & Q_1 \end{bmatrix}.$$

The following result shows that the closed-loop system is a regular linear system. The result also holds whenever $\operatorname{Re} D_c \geq 0$ and $I + DD_c$ is invertible.

LEMMA 4.2. *The closed-loop system* (4.2) *is regular and* A_e *in* (3.1) *generates a contraction semigroup.*

Proof. Consider the regular linear system

$$\left(\begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix}, \begin{bmatrix} B & B_d & 0 \\ 0 & 0 & B_c \end{bmatrix}, \begin{bmatrix} C_\Lambda & 0 \\ 0 & C_{c\Lambda} \end{bmatrix}, \begin{bmatrix} D & 0 & 0 \\ 0 & 0 & D_c \end{bmatrix} \right).$$

The closed-loop system (4.2) is obtained from the above system with output feedback with $\hat{K} = \left[\begin{smallmatrix} 0 & I \\ 0 & 0 \\ -I & 0 \end{smallmatrix}\right]$, which is an admissible feedback operator since $I + DD_c$ is boundedly invertible by Lemma A.1(d). Thus (4.2) is regular [43].

Since A_e generates a semigroup $T_e(t)$ on X_e, the Lumer–Phillips Theorem implies that $T_e(t)$ is contactive if A_e is dissipative. The estimates $\operatorname{Re}\langle Ax + Bu, x\rangle \leq \operatorname{Re}\langle C_\Lambda x + Du, u\rangle$ and $\operatorname{Re}\langle A_c z + B_c y, z\rangle \leq \operatorname{Re}\langle C_{c\Lambda} z + D_c y, y\rangle$ and a direct computation show that for any $x_e = (x, z)^T \in D(A_e)$ we have

$$\begin{aligned}\mathrm{Re}\langle A_e x_e, x_e\rangle &= \mathrm{Re}\langle Ax + BQ_2(-D_c C_\Lambda x + C_{c\Lambda} z), x\rangle \\ &\quad + \mathrm{Re}\langle A_c z + B_c Q_1(-C_\Lambda x - DC_{c\Lambda} z), z\rangle \\ &\le \mathrm{Re}\langle C_\Lambda x + DQ_2(-D_c C_\Lambda x + C_{c\Lambda} z), Q_2(-D_c C_\Lambda x + C_{c\Lambda} z)\rangle \\ &\quad + \mathrm{Re}\langle C_{c\Lambda} z + D_c Q_1(-C_\Lambda x - DC_{c\Lambda} z), Q_1(-C_\Lambda x - DC_{c\Lambda} z)\rangle = 0,\end{aligned}$$

and thus A_e is dissipative. ∎

In the following we define the robust output regulation problem for the regular linear system (2.1). In the problem we consider perturbations for which the perturbed system $(\tilde{A}, [\tilde{B}, \tilde{B}_d], \tilde{C}_\Lambda, \tilde{D})$ and the perturbed closed-loop system remain regular. The robustness of the controller also implies that output tracking and disturbance rejection are achieved even if the operators B_c, C_c, and D_c of the controller are perturbed or approximated in such a way that the closed-loop stability is preserved and the additional conditions on the perturbations stated in section 5 are satisfied.

The robust output regulation problem. *Choose $(A_c, B_c, C_{c\Lambda}, D_c)$ in such a way that the following are satisfied:*

(a) *The semigroup $T_e(t)$ generated by A_e is strongly stable.*

(b) *For the reference and disturbance signals of the form* (4.1) *and for all initial states $x_{e0} \in X_e$ the regulation error satisfies*

$$\int_t^{t+1} \|e(s)\| ds \to 0 \qquad \text{as} \quad t \to \infty. \tag{4.3}$$

(c) *If $(A, B, B_d, C_\Lambda, D)$ are perturbed to $(\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}_\Lambda, \tilde{D})$ in such a way that the perturbed closed-loop system is strongly stable, then for the signals* (4.1) *and for all initial states $x_{e0} \in X_e$ the regulation error satisfies* (4.3).

It follows from the results in [30, sec. 3] that if the closed-loop system is exponentially stable, then convergence in (4.3) is uniformly exponentially fast, i.e., there exist $M_e, \alpha > 0$ such that $\int_t^{t+1} \|e(s)\| ds \le M_e e^{-\alpha t}(\|x_{e0}\| + 1)$ for all $x_{e0} \in X_e$. If the input and output operators of the system and the controller are bounded, then the error convergences pointwise, i.e., $\|y(t) - y_{ref}(t)\| \to 0$ as $t \to \infty$, and the rate is exponential if $T_e(t)$ is exponentially stable.

5. Passive controllers for robust output regulation.

The controller constructions in this section are based on the internal model principle [17, 31, 32], which implies that a controller solves the robust output regulation problem provided that its dynamics contain a suitable number of copies of the frequencies $\{\omega_k\}_{k\in\mathcal{I}}$ of the signals (4.1) and the closed-loop system is stable. If $\dim Y < \infty$, then $(A_c, B_c, C_{c\Lambda}, D_c)$ contains an internal model of the signals (4.1) if [30, Thm. 13]

$$\dim \mathcal{N}(i\omega_k - A_c) \ge \dim Y \qquad \forall k \in \mathcal{I}.$$

In the case of an infinite-dimensional output space, the controller contains an internal model if [30, Thm. 13]

$$\mathcal{R}(i\omega_k - A_c) \cap \mathcal{R}(B_c) = \{0\} \qquad \forall k \in \mathcal{I}, \tag{5.1a}$$

$$\mathcal{N}(B_c) = \{0\}. \tag{5.1b}$$

We consider three different situations: In section 5.1 we construct a finite-dimensional robust controller for a strongly stabilizable system (2.1). If (A, B, C_Λ, D)

is exponentially stabilizable, then the convergence of the error is exponentially fast. In section 5.2 we design a robust controller to track and reject nonsmooth τ-periodic reference signals. The controller is based on a periodic transport equation and achieves exponential closed-loop stability if the system (2.1) is exponentially stabilizable and satisfies $\operatorname{Re} P(i\omega) \geq \eta > 0$ for some constant $\eta > 0$ near the points $\omega_k = \frac{2\pi k}{\tau}$ for $k \in \mathbb{Z}$. In section 5.3 we design an infinite-dimensional robust controller for nonsmooth signals (4.1) with a general set of frequencies $\{\omega_k\}_{k\in\mathcal{I}}$. In general, the closed-loop system cannot be stabilized exponentially, and we introduce conditions for nonuniform subexponential rates of convergence of the output.

In the constructions we choose the feedthrough of the controller to have the form $D_c = D_{c1} + D_{c2}$, where $D_{c2} \geq 0$ is used to prestabilize the system (A, B, C_Λ, D). We assume that the system $(A^S, B^S, C_\Lambda^S, D^S) = (A - BD_{c2}Q_1^S C_\Lambda, BQ_2^S, Q_1^S C_\Lambda, Q_1^S D)$, where $Q_1^S = (I + DD_{c2})^{-1}$ and $Q_2^S = (I + D_{c2}D)^{-1}$ obtained from (2.1) with the output feedback $u(t) = -D_{c2}y(t)$, is either strongly or exponentially stable. Its transfer function is denoted by $P_S(\lambda)$. The passivity of (A, B, C_Λ, D) implies that also $(A^S, B^S, C_\Lambda^S, D^S)$ is passive.

5.1. A robust finite-dimensional controller. In this section we assume the signals (4.1) contain a finite number of frequencies $\{\omega_k\}_{k=1}^q$, i.e., $\mathcal{I} = \{1, \dots, q\}$. The controller parameters are chosen in the following way.

DEFINITION 5.1. *Choose $Z = Y^q$ and*

$$A_c = \operatorname{diag}(i\omega_1 I_Y, \dots, i\omega_q I_Y) \in \mathcal{L}(Z),$$

where I_Y is the identity operator on Y. Choose $C_c \in \mathcal{L}(Z, Y)$ of the form $C_c z = \sum_{k=1}^q C_{ck} z_k$ for $z = (z_k)_{k=1}^q \in Z$ so that $C_{ck} \in \mathcal{L}(Y)$ are boundedly invertible for all k, choose $B_c = C_c^*$, and choose $D_c = D_{c1} + D_{c2}$ with $D_{c1} > 0$. Finally, choose $D_{c2} \geq 0$ in such a way that $(A^S, B^S, C_\Lambda^S, D^S)$ is passive and strongly stable with $i\mathbb{R} \subset \rho(A^S)$.

In the case where Y and U_d are real spaces and $w_{dist}(\cdot)$ and $y_{ref}(\cdot)$ real-valued functions we have $\{\omega_k\}_{k=1}^q = \{0, \pm\omega_1, \dots, \pm\omega_{q'}\}$ or $\{\omega_k\}_{k=1}^q = \{\pm\omega_1, \dots, \pm\omega_{q'}\}$ for some $\omega_1, \dots, \omega_{q'} > 0$. In this case the controller can be chosen to be real by choosing (J_0 is omitted if $0 \notin \{\omega_k\}_{k=1}^q$)

$$A_c = \operatorname{diag}(J_0, J_1, \dots, J_{q'}), \quad J_0 = 0 \in \mathcal{L}(Y), \quad J_k = \begin{bmatrix} 0 & \omega_k I_Y \\ -\omega_k I_Y & 0 \end{bmatrix},$$

and $C_c = C_{c0}z_0 + \sum_{k=1}^{q'} C_{ck} z_k^1$ for $z = (z_0, z_1^1, z_1^2, \dots, z_{q'}^1, z_{q'}^2) \in Z = Y^{2q'+1}$, where $C_{ck} \in \mathcal{L}(Y)$ are boundedly invertible for $0 \leq k \leq q'$, $B_c = C_c^*$, and $D_c > 0$ is as in Definition 5.1. This controller is passive and it will achieve robust output regulation by Theorem 5.2 due to the fact that under the similarity transform

$$V = \operatorname{diag}(I_Y, V_1, \dots, V_{q'}), \qquad V_k = \frac{1}{\sqrt{2}} \begin{bmatrix} I_Y & I_Y \\ iI_Y & -iI_Y \end{bmatrix}$$

the system $(V^* A_c V, V^* B_c, C_c V, D_c)$ is of the form given in Definition 5.1.

THEOREM 5.2. *The controller in Definition* 5.1 *solves the robust output regulation problem. The closed-loop system is strongly stable and $i\mathbb{R} \subset \rho(A_e)$.*

If $(A^S, B^S, C_\Lambda^S, D^S)$ is exponentially stable, then also the closed-loop system is exponentially stable and for any $y_{ref}(t)$ and $w_{dist}(t)$ there exist $M_e, \alpha > 0$ such that

$$\int_t^{t+1} \|e(s)\| ds \leq M_e e^{-\alpha t}(\|x_{e0}\| + 1) \qquad \forall x_{e0} \in X_e.$$

In both cases the controller is robust with respect to all perturbations that preserve the stability of the closed-loop system and for which $i\mathbb{R} \subset \rho(\tilde{A}_e)$.

Proof. The controller (A_c, B_c, C_c, D_{c1}) is passive and its transfer function $G(\lambda)$ satisfies $\operatorname{Re} G(i\omega) = D_{c1} > 0$ for all $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k=1}^q$. The operators (A_c, B_c) satisfy (5.1). Indeed, the injectivity of B_c in (5.1b) follows directly from the fact that the components C_{ck}^* of B_c are boundedly invertible by assumption. Condition (5.1a) can be verified using the diagonal structure of A_c and the invertibility of C_{ck}^*.

To prove closed-loop stability, we apply Theorem 3.2 to $(A^S, B^S, C_\Lambda^S, D^S)$ and (A_c, B_c, C_c, D_{c1}). Condition (2) of the theorem is satisfied since for any $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k=1}^q$ we have $\operatorname{Re} G(i\omega) = \operatorname{Re}(C_c R(i\omega, A_c) B_c + D_{c1}) = D_{c1} > 0$, and condition (3) is satisfied by Lemma 3.4 since C_{ck} are invertible. Thus the strong and exponential closed-loop stabilities follow from Theorem 3.2. Finally, the conclusion that the controller solves the robust output regulation problem follows from [30, Thm. 13]. The results in [30] are presented for controllers with $D_c = 0$, but they are applicable since $D_c \geq 0$ can be written as an output feedback for the system (2.1) without changing the properties of the closed-loop system. Moreover, the results are presented for an infinite set $\{\omega_k\}_{k\in\mathcal{I}}$, but they also apply trivially when $\mathcal{I}$ is finite. ∎

PROPOSITION 5.3. *The regulation error in Theorem* 5.2 *converges pointwise, i.e.,* $\|e(t)\| \to 0$ *as* $t \to \infty$ *for all initial states* $x_{e0} \in X_e$ *satisfying* $A_e x_{e0} + B_e w_{ext}(0) \in X_e$. *If the closed-loop system is exponentially stable, then for all* $y_{ref}(t)$ *and* $w_{dist}(t)$ *there exist* $M_e, \alpha > 0$ *such that*

$$\|e(t)\| \leq M_e e^{-\alpha t}(\|A_e x_{e0} + B_e w_{ext}(0)\| + 1)$$

for all $x_{e0} \in X_e$ *satisfying* $A_e x_{e0} + B_e w_{ext}(0) \in X_e$.

The proof of Proposition 5.3 is based on the following technical lemma, which is also used later in the following sections. The assumptions on H are automatically satisfied if $\mathcal{I}$ is finite, or if the closed-loop system is exponentially stable. In the latter case the property $Hv \in D(C_{e\Lambda})$ can be verified similarly as in the proof of Theorem 5.11.

LEMMA 5.4. *Assume the controller solves the robust output regulation problem and* $y_{ref}(t)$ *and* $w_{dist}(t)$ *are such that for some fixed* $(f_k)_k \in \ell^2(\mathbb{C})$ *the operator* $H : D(H) \subset \ell^2(\mathbb{C}) \to X_e$ *defined by* $Hv = \sum_{k\in\mathcal{I}} f_k^{-1} R(i\omega_k, A_e) B_e w_{ext}^k v_k$ *for* $v = (v_k)_k$ *satisfies* $H \in \mathcal{L}(\ell^2(\mathbb{C}), X_e)$ *and* $Hv \in D(C_{e\Lambda})$ *for all* $v \in \ell^2(\mathbb{C})$. *If* $y_{ref}(t)$ *and* $w_{dist}(t)$ *are such that the series*

$$q_{ext} = \sum_{k\in\mathcal{I}} i\omega_k R(i\omega_k, A_e) B_e w_{ext}^k \tag{5.2}$$

converges in X_e, *then for all* $x_{e0} \in X_e$ *satisfying* $A_e x_{e0} + B_e w_{ext}(0) \in X_e$ *and for almost all* $t > 0$ *we have*

$$e(t) = C_{e\Lambda} T_e(t) A_e^{-1}(A_e x_{e0} + B_e w_{ext}(0) - q_{ext}).$$

Proof. It follows from the properties of H and the results in [30] that for every $x_{e0} \in X_e$ and almost all $t > 0$ the regulation error is given by

$$e(t) = C_{e\Lambda} T_e(t) \left(x_{e0} - \sum_{k\in\mathcal{I}} R(i\omega_k, A_e) B_e w_{ext}^k \right).$$

If $A_e x_{e0} + B_e w_{ext}(0) \in X_e$, then a direct computation and $q_{ext} \in X_e$ show

$$A_e \sum_{k\in\mathcal{I}} R(i\omega_k, A_e) B_e w_{ext}^k = \sum_{k\in\mathcal{I}} i\omega_k R(i\omega_k, A_e) B_e w_{ext}^k - B_e w_{ext}(0),$$

which implies the claim. ☐

Proof of Proposition 5.3. Since $\mathcal{I}$ is finite, the conditions of Lemma 5.4 are satisfied. If $x_{e0} \in X_e$ is such that $A_e x_{e0} + B_e w_{ext}(0) \in X_e$, then the estimate $\|e(t)\| \leq \|C_{e\Lambda} A_e^{-1}\| \|T_e(t)\| \|A_e x_{e0} + B_e w_{ext}(0) - q_{ext}\|$ implies both claims of the proposition. ☐

The following sufficient condition for $A_e x_{e0} + B_e w_{ext}(0) \in X_e$ follows directly from the structures of A_e and B_e. Later in section 5.4 the same condition implies a nonuniform decay rate for the regulation error.

LEMMA 5.5. *If $B_c \in \mathcal{L}(U, X)$, $C_c \in \mathcal{L}(X, Y)$, and $w_{dist}(0) = 0$, then $A_e x_{e0} + B_e w_{ext}(0) \in X_e$ is satisfied for $x_{e0} = (x_0, z_0)^T \in D(A) \times D(A_c)$ if $C_c z_0 = D_c(Cx_0 - y_{ref}(0))$.*

5.2. A robust controller for τ-periodic signals. In this section we will construct a regular linear controller that achieves exponentially fast output regulation of τ-periodic reference and disturbance signals. The controller structure is based on a shift semigroup with periodic boundary conditions and is related to controllers constructed in [21, 45, 23]. We assume that $\dim Y = p < \infty$ and that $y_{ref}(t)$ and $w_{dist}(t)$ are τ-periodic functions, i.e., $\mathcal{I} = \mathbb{Z}$ and $\{\omega_k\}_{k\in\mathbb{Z}} = \{\frac{2\pi k}{\tau}\}_{k\in\mathbb{Z}}$.

DEFINITION 5.6. *Choose the controller as*

$$z_t(\xi, t) = z_\xi(\xi, t), \qquad \xi \in (0, \tau), \quad t \geq 0, \tag{5.3a}$$

$$z(\cdot, 0) = z_0(\cdot) \in L^2(0, \tau; \mathbb{C}^p), \tag{5.3b}$$

$$e(t) = 2^{-1/2}(z(\tau, t) - z(0, t)), \tag{5.3c}$$

$$u(t) = 2^{-1/2}(z(\tau, t) + z(0, t)) + (D_{c1} + D_{c2})e(t), \tag{5.3d}$$

where $z(\xi, t) = (z_1(\xi, t), \ldots, z_p(\xi, t))^T$ and $D_{c1} > 0$. Choose $D_{c2} \geq 0$ in such a way that $(A^S, B^S, C_\Lambda^S, D^S)$ is passive and exponentially stable.

To achieve closed-loop stability, we also assume that $\operatorname{Re} P_S(i\omega_k) \geq \eta > 0$ for some constant $\eta > 0$ and for all $k \in \mathbb{Z}$. If this condition is not satisfied, then exponential closed-loop stability is unachievable, but strong closed-loop stability can be studied using Theorem 5.11 in the next section.

THEOREM 5.7. *Let $y_{ref}(t)$ and $w_{dist}(t)$ be as in* (4.1) *with $\omega_k = \frac{2\pi k}{\tau}$ for some $\tau > 0$. Assume there exist $\eta, \varepsilon > 0$ such that $\operatorname{Re} P_S(i\omega) \geq \eta > 0$ for $\omega \in \Omega_\varepsilon = \{\, \omega \in \mathbb{R} \mid \exists k \in \mathbb{Z} : |\omega - \omega_k| < \varepsilon \,\}$, and $\operatorname{Re} D > 0$. Then the controller in Definition* 5.6 *solves the robust output regulation problem in such a way that the closed-loop system is exponentially stable, and there exist $M_e, \alpha > 0$ such that*

$$\int_t^{t+1} \|e(s)\| ds \leq M_e e^{-\alpha t}(\|x_{e0}\| + 1) \qquad \forall x_{e0} \in X_e.$$

The controller is robust with respect to all perturbations that preserve the exponential closed-loop stability, and for which $u(t) = -D_{c2}y(t)$ remains an admissible output feedback and $\{i\omega_k\}_{k\in\mathbb{Z}} \subset \rho(\tilde{A}^S)$.

Proof. The controller in Definition 5.6 consists of $p = \dim Y$ independent one-dimensional periodic transport equations with boundary control and observation and an additional feedthrough $(D_{c1} + D_{c2})e(t)$. The system (5.3) defines a regular linear system with state $z(t) = z(\cdot, t)$ on $Z = L^2(0, \tau; \mathbb{C}^p)$ [51, Thm. 2.4], and a direct computation shows that its transfer function from $e(t)$ to $u(t)$ is

$$G_0(\lambda) = \frac{1 + e^{-\lambda\tau}}{1 - e^{-\lambda\tau}} I + D_{c1} + D_{c2}, \qquad \lambda \notin \left\{ i\frac{2\pi k}{\tau} \right\}_{k\in\mathbb{Z}}.$$

Thus the controller can be written as a system $(A_c, B_c, C_{c\Lambda}, D_c)$ on Z, where A_c satisfying $A_c f = f'$ for $f \in D(A_c) = \{ f \in H^1(0, \tau; \mathbb{C}^p) \mid f(0) = f(\tau) \}$ generates a unitary group with spectrum $\sigma(A_c) = \{i\frac{2\pi k}{\tau}\}_{k\in\mathbb{Z}}$. We also have $\dim \mathcal{N}(i\omega_k - A_c) = \dim Y$ for every $k \in \mathbb{Z}$, and thus A_c contains an internal model of the signals (4.1). By [30, Thm. 13] the controller solves the robust output regulation problem if the closed-loop system is exponentially stable.

To show closed-loop stability, we will verify the conditions of Theorem 3.5 for the systems $(A^S, B^S, C^S_\Lambda, D^S)$ and $(A_c, B_c, C_{c\Lambda}, D_{c1})$ with $\Omega = \Omega_\varepsilon$. For this we will consider the controller with inputs and outputs

$$\begin{aligned} u_c(t) &= 2^{-1/2}(z(\tau, t) - z(0, t)), \\ y_c(t) &= 2^{-1/2}(z(\tau, t) + z(0, t)) + (D_{c1} + D_{c2})u_c(t). \end{aligned}$$

The feedthrough operator of the controller is given by $D_c = \lim_{\lambda\to\infty} G_0(\lambda) = I + D_{c1} + D_{c2}$. Without the component $(D_{c1} + D_{c2})u_c(t)$ of the feedthrough the solutions of (5.3) satisfy $\frac{d}{dt}\|z(t)\|^2_{L^2} = 2\operatorname{Re}\langle u_c(t), y_c(t)\rangle$, and thus the controller is passive by [38, Thm. 4.2]. Let $d_c > 0$ be such that $D_{c1} \geq d_c > 0$. The transfer function $G(\lambda)$ of $(A_c, B_c, C_{c\Lambda}, I + D_{c1})$ satisfies $\operatorname{Re} G(i\omega) = D_{c1} \geq d_c > 0$ for all $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathbb{Z}}$, and thus condition (2) of Theorem 3.5 is satisfied. To show that condition (3) of Theorem 3.5 is satisfied, it is sufficient to show that for any $D_0 \in \mathcal{L}(U)$ with $\operatorname{Re} D_0 > 0$ the system $(A_c, B_c, C_{c\Lambda}, I + D_{c1})$ is stabilized exponentially with feedback $u_c(t) = -D_0 y_c(t)$. The feedback leads to a partial differential equation

$$\begin{aligned} z_t(\xi, t) &= z_\xi(\xi, t), \qquad \xi \in (0, \tau), \quad t \geq 0, \\ (I + D_{tot})z(\tau, t) &= (I - D_{tot})z(0, t), \end{aligned}$$

where $D_{tot} = D_0(I + D_{c1}D_0)^{-1}$. The exponential stability of this system follows from a straightforward application of [41, Thm. III.2], since $\operatorname{Re} D_{tot} > 0$ by Lemma A.1(c). Thus Theorem 3.5 shows that the closed-loop system is exponentially stable. ∎

Remark 5.8. The results in [30] also show that if $(y^k_{ref})_k = (a_k y_k)_k$ and $(w^k_{dist})_k = (a_k w_k)_k$, where $(y_k)_k \in \ell^2(Y)$, $(w_k)_k \in \ell^2(U_d)$ are fixed, and $(a_k)_k \in \ell^2(\mathbb{C})$, then there exist $M_e, \alpha > 0$ such that $\int_t^{t+1} \|e(s)\| ds \leq M_e e^{-\alpha t}(\|x_{e0}\| + \|(a_k)_k\|_{\ell^2})$ for all $x_{e0} \in X_e$ and $(a_k)_k \in \ell^2(\mathbb{C})$.

Lemma 5.4 implies the following result on the pointwise convergence of $\|e(t)\|$. The conditions require that $y_{ref}(t)$ and $w_{dist}(t)$ have a sufficient levels of smoothness.

Corollary 5.9. *If the signals* (4.1) *are such that* $(ky^k_{ref})_k \in \ell^1(Y)$ *and* $(kw^k_{dist})_k \in \ell^1(U_d)$*, then in Theorem* 5.7 *there exist* $M_e, \alpha > 0$ *such that for all* $x_{e0} \in X_e$ *satisfying* $A_e x_{e0} + B_e w_{ext}(0) \in X_e$ *we have*

$$\|e(t)\| \leq M_e e^{-\alpha t}(\|A_e x_{e0} + B_e w_{ext}(0)\| + 1).$$

If $P(i\mu_j)$ is not invertible for some $\{i\mu_j\}_{j=1}^N \subset \{i\frac{2\pi k}{\tau}\}_{k\in\mathbb{Z}}$, for example, for $\mu_j = 0$, then the robust output regulation problem is not solvable for signals $y_{ref}(t)$ and $w_{dist}(t)$ containing these frequencies. In this situation we can modify the controller in Definition 5.6 by replacing (5.3a) with

$$z_t(\xi,t) = z_\xi(\xi,t) - \frac{1}{\tau}\sum_{j=1}^N \sum_{k=1}^p e_k \cdot e^{i\mu_j \xi} \int_0^\tau z_k(s,t) e^{-i\mu_j s} ds, \quad \xi \in (0,\tau),$$

where $\{e_k\}_{k=1}^p$ are the Euclidean basis vectors of $\mathbb{C}^p$. This corresponds to stabilizing the eigenvalues $\{i\mu_j\}_{j=1}^N$ of the transport system (5.3), and the resulting controller has the property $\sigma(A_c) \cap i\mathbb{R} = \{i\frac{2\pi k}{\tau}\}_{k\in\mathbb{Z}} \setminus \{i\mu_j\}_{j=1}^N$. With this modification the system operator of the controller is of the form $A_c = A_c^0 - B_0 B_0^*$ with $B_0 \in \mathcal{L}(\mathbb{C}^{Np}, Z)$. The controller is again passive and is stabilized exponentially with feedback $u_c(t) = -D_0 y_c(t)$ with $\operatorname{Re} D_0 > 0$, and the exponential closed-loop stability follows from Theorem 3.5.

5.3. A robust controller for nonsmooth signals. In this section we construct an infinite-dimensional diagonal controller for signals (4.1) with a general set $\{\omega_k\}_{k\in\mathbb{Z}}$ of distinct frequencies with no finite accumulation points. The controller can also be used for systems with an infinite-dimensional output space Y. If $y_{ref}(t)$ and $w_{dist}(t)$ are τ-periodic and $\dim Y < \infty$, then the controller is of similar form as in Definition 5.6.

DEFINITION 5.10. *Choose $Z = \ell^2(\mathcal{I}; Y)$ and*

$$A_c = \operatorname{diag}(i\omega_k I_Y)_{k\in\mathcal{I}}, \quad D(A_c) = \left\{ (z_k)_k \in Z \;\middle|\; (|\omega_k| \|z_k\|)_k \in \ell^2(\mathbb{C}) \right\},$$

where I_Y is the identity operator on Y. Let $D_c = D_{c1} + D_{c2}$ with $D_{c1} > 0$ and $D_{c2} \geq 0$. Choose admissible $B_c \in \mathcal{L}(Y, Z_{-1})$ and $C_c \in \mathcal{L}(Z_1, Y)$ as

$$B_c y = (B_{ck} y)_k \quad \forall y \in Y, \qquad C_c z = \sum_{k\in\mathcal{I}} B_{ck}^* z_k \quad \forall z \in D(A_c)$$

with boundedly invertible $B_{ck} \in \mathcal{L}(Y)$ so that $(A_c, B_c, C_{c\Lambda}, D_{c1})$ is a regular linear system whose transfer function $G(\lambda)$ satisfies $\operatorname{Re} G(i\omega) \geq d_c > 0$ for some constant $d_c > 0$ and for all $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k\in\mathcal{I}}$. Finally, choose $D_{c2} \geq 0$ in such a way that $(A^S, B^S, C_\Lambda^S, D^S)$ is passive and strongly stable with $i\mathbb{R} \subset \rho(A^S)$.

If $\dim Y < \infty$ and $\{\omega_k\}_{k\in\mathcal{I}}$ has a uniform gap, i.e., $\inf_{k\neq l} |\omega_k - \omega_l| > 0$, then [39, Cor. 5.2.5, Prop. 5.3.5] imply that B_c and C_c are admissible with respect to A_c if $(\|B_{ck}\|)_{k\in\mathcal{I}} \in \ell^\infty(\mathbb{C})$ and $(\|C_{ck}\|)_{k\in\mathcal{I}} \in \ell^\infty(\mathbb{C})$. For more general conditions for admissibility, see [39, sec. 5.3]. The system $(A_c, B_c, C_{c\Lambda}, D_{c1})$ is regular whenever B_c and C_c are admissible and there exists $\varepsilon > 0$ such that $((1+|\omega_k|)^{-1/2+\varepsilon} \|B_{ck}\|)_k \in \ell^2(\mathbb{C})$ [14, Prop. 4.1]. However, there are also regular linear systems, such as the controller in Definition 5.6, for which neither of these conditions is satisfied. If $\{\omega_k\}_{k\in\mathbb{Z}}$ has a uniform gap, $(|\omega_k|^\varepsilon \|B_{ck}\|)_k \in \ell^\infty(\mathbb{C})$ for some $\varepsilon > 0$ and $D_{c1} > 0$, then $(A_c, B_c, C_{c\Lambda}, D_{c1})$ satisfies the conditions of Definition 5.10.

Due to the lack of exponential closed-loop stability, the solvability of the robust output regulation problem requires additional conditions on the reference and disturbance signals. These conditions relate the behavior of the coefficients y_{ref}^k and w_{dist}^k to the behavior of the transfer functions $P(\lambda)$ and $P_d(\lambda)$ on the frequencies $\{\omega_k\}_{k\in\mathcal{I}}$.

We pose conditions on the sequences $\Pi_{ext} = (\Pi_{ext}(k))_{k\in\mathcal{I}} \subset X_{B,B_d} \times Y$ consisting of the elements $\Pi_{ext}(k) = (\Pi^1_{ext}(k), \Pi^2_{ext}(k))$ with

$$\Pi^1_{ext}(k) = R(i\omega_k, A^S)B^S u_k + R(i\omega_k, A^S)B_d w^k_{dist}, \quad \Pi^2_{ext}(k) = (B^*_{ck})^{-1}(u_k - D_{c2}y^k_{ref}),$$

where $u_k = P_S(i\omega_k)^{-1}y^k_{ref} - P_S(i\omega_k)^{-1}C^S_\Lambda R(i\omega_k, A^S)B_d w^k_{dist}$. In the case of a perturbed system, we define $\tilde{\Pi}_{ext} = (\tilde{\Pi}_{ext}(k))_{k\in\mathcal{I}}$ analogously. Alternate ways of expressing $\Pi_{ext}(k)$ are presented in Lemma 5.12. Note in particular that if $(A^S, B^S, C^S_\Lambda, D^S)$ is exponentially stable, then (5.4) are satisfied provided that $(\|u_k\|)_k \in \ell^1(\mathbb{C})$ and $(\|B^{-1}_{ck}\|\|u_k - D_{c2}y^k_{ref}\|)_k \in \ell^2(\mathbb{C})$.

Theorem 5.11. *Assume* $\operatorname{Re} P_S(i\omega_k) > 0$ *for all* $k \in \mathcal{I}$. *The controller in Definition* 5.10 *solves the robust output regulation problem for all* $y_{ref}(t)$ *and* $w_{dist}(t)$ *whose coefficients satisfy*

$$(\Pi^1_{ext}(k))_k \in \ell^1(X), \qquad (\Pi^2_{ext}(k))_k \in \ell^2(Y), \qquad (u_k)_k \in \ell^1(U). \tag{5.4}$$

The closed-loop system is strongly stable and $i\mathbb{R} \subset \rho(A_e)$.

The controller is robust with respect to all perturbations $(\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}_\Lambda, \tilde{D})$ *for which* $u(t) = -D_{c2}y(t)$ *remains an admissible output feedback, the strong closed-loop stability is preserved,* $\{i\omega_k\}_{k\in\mathcal{I}} \subset \rho(\tilde{A}_e) \cap \rho(\tilde{A}^S)$, $\tilde{P}_S(i\omega_k)$ *are invertible for* $k \in \mathcal{I}$, *and* $(\tilde{\Pi}_{ext}(k))_{k\in\mathcal{I}}$ *satisfies* (5.4).

If the closed-loop system is exponentially stable, then (5.4) *are satisfied automatically, and there exist* $M_e, \alpha > 0$ *such that* $\int_t^{t+1}\|e(s)\|ds \le M_e e^{-\alpha t}(\|x_{e0}\| + 1)$ *for all* $x_{e0} \in X_e$.

Proof. The proof is based on the application of [30, Thm. 13]. The diagonal structure of the controller and the invertibility of B_{ck} imply that A_c and B_c satisfy the conditions (5.1). To show that the closed-loop system is strongly stable, we apply Theorem 3.2 for the systems $(A^S, B^S, C^S_\Lambda, D^S)$ and $(A_c, B_c, C_{c\Lambda}, D_{c1})$. Conditions (1) and (2) are satisfied due to the construction in Definition 5.10, and condition (3) is satisfied by Lemma 3.4 since $C_{ck} = B^*_{ck}$ are invertible. Thus by Theorem 3.2 the closed-loop system is strongly stable and $i\mathbb{R} \subset \rho(A_e)$.

To apply [30, Thm. 13] directly, we would need $R(i\omega_k, A_e)B_e w^k_{ext} \in \ell^1(X_e)$. However, in [30] this property is used as a sufficient condition for the existence of $(f_k)_k \in \ell^2(\mathbb{C})$ such that the operator $H : D(H) \subset \ell^2(\mathbb{C}) \to X_e$ in Lemma 5.4 satisfies $H \in \mathcal{L}(\ell^2(\mathbb{C}), X_e)$ and $\mathcal{R}(H) \subset D(C_{e\Lambda})$. Here we will verify that the sequence $(f_k)_k \in \ell^2(\mathbb{C})$ with

$$f_k = \begin{cases} \|\Pi^2_{ext}(k)\| + (\|w^k_{ext}\| + \|\Pi^1_{ext}(k)\| + \|u_k\|)^{1/2} & \text{if } w^k_{ext} \neq 0, \\ 2^{-|k|} & \text{if } w^k_{ext} = 0 \end{cases}$$

has this property. If $k \in \mathcal{I}$ and $x^k_e = (\Pi^1_{ext}(k), z_k) \in X_{B,B_d} \times Z_{B_c}$ where

$$z_k = (z^j_k)_{j\in\mathcal{I}}, \quad z^k_k = \Pi^2_{ext}(k), \quad z^j_k = 0, \ j \neq k,$$

then it is straightforward to verify that $(i\omega_k - A_e)x^k_e = B_e w^k_{ext}$, and thus we have $R(i\omega_k, A_e)B_e w^k_{ext} = (\Pi^1_{ext}(k), z_k)$. Now $(f_k^{-1}(\|w^k_{ext}\| + \|\Pi^1_{ext}(k)\| + \|u_k\|))_k \in \ell^2(\mathbb{C})$ and $(f_k^{-1}\Pi^2_{ext}(k))_k \in \ell^\infty(Y)$. These properties and the structure of $R(i\omega_k, A_e)B_e w^k_{ext}$ imply that Hv is well-defined for every $v \in \ell^2(\mathbb{C})$, and

$$\begin{aligned} \|Hv\|^2 &= \Big\|\sum_{k\in\mathcal{I}} f_k^{-1}\Pi^1_{ext}(k)v_k\Big\|_X^2 + \big\|\big(f_k^{-1}\Pi^2_{ext}(k)v_k\big)_k\big\|^2_{\ell^2(Y)} \\ &\le \|v\|^2\|(f_k^{-1}\Pi^1_{ext}(k))_k\|^2_{\ell^2(X)} + \|v\|^2\|(f_k^{-1}\Pi^2_{ext}(k))_k\|^2_{\ell^\infty(Y)} \end{aligned}$$

implies $H \in \mathcal{L}(\ell^2(\mathbb{C}), X_e)$. It remains to show $\mathcal{R}(\Sigma) \subset D(C_{e\Lambda})$. If we denote $P_{e0}(\lambda) = C_{e\Lambda}R(\lambda, A_e)B_e$, then $P_{e0}(i\omega_k)w_{ext}^k = -Q_1(C_\Lambda \Pi_{ext}^1(k) + D(u_k - D_{c2}y_{ref}^k))$ for every $k \in \mathcal{I}$. The regularity of $(A^S, B^S, C_\Lambda^S, D^S)$ and (5.4) imply $(f_k^{-1}P_{e0}(i\omega_k)w_{ext}^k)_k \in \ell^2(Y)$. If $v \in \ell^2(\mathbb{C})$ and $\lambda > 0$, the resolvent identity implies

$$\begin{aligned}\lambda C_{e\Lambda}R(\lambda, A_e)Hv &= \sum_{k\in\mathcal{I}} \frac{\lambda f_k^{-1}v_k}{\lambda - i\omega_k} P_{e0}(i\omega_k)w_{ext}^k - P_{e0}(\lambda)\sum_{k\in\mathcal{I}} \frac{\lambda f_k^{-1}v_k}{\lambda - i\omega_k} w_{ext}^k \\ &\longrightarrow \sum_{k\in\mathcal{I}} f_k^{-1}P_{e0}(i\omega_k)w_{ext}^k v_k\end{aligned}$$

as $\lambda \to \infty$ since (A_e, B_e, C_e) is regular and since $(f_k^{-1}P_{e0}(i\omega_k)w_{ext}^k v_k)_k \in \ell^1(Y)$ and $(f_k^{-1}w_{ext}^k v_k) \in \ell^1(U_d \times Y)$. Thus $Hv \in D(C_{e\Lambda})$ by definition. An analogous argument shows that for perturbed systems $(\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}_\Lambda, \tilde{D})$ the sequence $(f_k)_k$ can again be chosen so that $\tilde{H}$ has the required properties. Thus the claims of the theorem follow from [30, Thm. 13]. If the closed-loop system is exponentially stable, then $(\Pi_{ext}^1(k), z_k) = R(i\omega_k, A_e)B_e w_{ext}^k$ implies $(\Pi_{ext}(k))_k \in \ell^1(X \times Y)$, which also shows $(\|u_k\|)_k \in \ell^1(\mathbb{C})$. ∎

The following alternate expressions for $\Pi_{ext}(k)$ can be verified using standard operator identities and Lemma 2.1.

LEMMA 5.12. *If $i\omega_k \in \rho(A)$ for some $k \in \mathcal{I}$, then*

$$\begin{aligned}\Pi_{ext}^1(k) &= R(i\omega_k, A)B_d w_{dist}^k + R(i\omega_k, A)B\tilde{u}_k, \\ \Pi_{ext}^2(k) &= (B_{ck}^*)^{-1}\tilde{u}_k, \qquad u_k = \tilde{u}_k + D_{c2}y_{ref}^k,\end{aligned}$$

where $\tilde{u}_k = P(i\omega_k)^{-1}y_{ref}^k - P(i\omega_k)^{-1}P_d(i\omega_k)w_{dist}^k$. If D is boundedly invertible, then $\Pi_{ext}^1(k) = R_k^D B_d w_{dist}^k + R(i\omega_k, A^S)B^S P_S(i\omega_k)^{-1}y_{ref}^k$ for all $k \in \mathcal{I}$, where $R_k^D = R(i\omega_k, A^S - B^S(D^S)^{-1}C_\Lambda^S)$.

The following result shows that pointwise convergence is achieved for sufficiently smooth signals $y_{ref}(t)$ and $w_{dist}(t)$ and for suitable intial states.

PROPOSITION 5.13. *Assume $y_{ref}(t)$ and $w_{dist}(t)$ are such that $(\omega_k \Pi_{ext}^1(k))_k \in \ell^1(X)$ and $(\omega_k \Pi_{ext}^2(k))_k \in \ell^2(Y)$. If $x_{e0} \in X_e$ and $A_e x_{e0} + B_e w_{ext}(0) \in X_e$, then the regulation error in Theorem 5.11 satisfies $\|e(t)\| \to 0$ as $t \to \infty$. If the closed-loop system is exponentially stable, then there exist $M_e, \alpha > 0$ such that*

$$\|e(t)\| \le M_e e^{-\alpha t}(\|A_e x_{e0} + B_e w_{ext}(0)\| + 1)$$

for all $x_{e0} \in X_e$ satisfying $A_e x_{e0} + B_e w_{ext}(0) \in X_e$.

Proof. As in the proof of Theorem 5.11, $R(i\omega_k, A_e)B_e w_{ext}^k = (\Pi_{ext}^1(k), z_k)$, where $z_k = (z_k^j)_j$ is such that $z_k^k = \Pi_{ext}^2(k)$ and $z_k^j = 0$ for $j \neq k$. This structure, $(\omega_k \Pi_{ext}^1(k))_k \in \ell^1(X)$, and $(\omega_k \Pi_{ext}^2(k))_k \in \ell^2(Y)$ imply that q_{ext} in (5.2) satisfies $q_{ext} \in X_e$. Since the required properties of H were verified in the proof of Theorem 5.11, the claims follow from Lemma 5.4. ∎

5.4. Nonuniform convergence rates of the regulation error. We will now use Theorem 3.7 to derive convergence rates for the regulation error in Theorem 5.11. The estimates are valid for reference and disturbance signals with sufficient levels of smoothness. In particular, we assume $\{\omega_k\}_{k\in\mathcal{I}}$ has a uniform gap and the coefficients of $y_{ref}(t)$ and $w_{dist}(t)$ satisfy

$$(5.5)\qquad \big(\omega_k \Pi_{ext}^1(k)\big)_{k\in\mathcal{I}} \in \ell^1(X), \qquad \big(\omega_k \Pi_{ext}^2(k)\big)_{k\in\mathcal{I}} \in \ell^2(Y),$$

which is a strictly stronger condition than the first two parts of (5.4).

THEOREM 5.14. *Assume* $(A^S, B^S, C_\Lambda^S, D^S)$ *is passive and exponentially stable, the controller is as in Definition* 5.10*, and the conditions of Theorem* 5.11 *are satisfied. Assume there exists* $0 < \varepsilon < \frac{1}{2}\inf_{k\neq l}|\omega_k - \omega_l|$ *such that* $\operatorname{Re} P_S(i\omega) > 0$ *for all* $\omega \in \Omega_\varepsilon = \{\,\omega \in \mathbb{R} \mid \exists k \in \mathcal{I} : |\omega - \omega_k| < \varepsilon\}$*. Let* $\eta(\cdot), \gamma(\cdot) : \mathbb{R}_+ \to (0,1]$ *be continuous nonincreasing functions with the property* $\inf_{\omega>0} \gamma(\omega+\delta_0)/\gamma(\omega) > 0$ *for some* $0 < \delta_0 < \min\{1, \varepsilon\}$ *such that the following hold:*

- $\operatorname{Re} P_S(i\omega) \geq \eta(|\omega|)$ *for all* $\omega \in \Omega_\varepsilon$*.*
- $\|B_{ck}^* y\| \geq \gamma(|\omega_k|)\|y\|$ *for all* $k \in \mathcal{I}$ *and* $y \in Y$*.*

Then the controller solves the robust output regulation problem and there exists $M_0 > 0$ *such that* $\|R(i\omega, A_e)\| \leq M_R(|\omega|)$ *with* $M_R(\cdot) = M_0\eta(\cdot)^{-1}\gamma(\cdot)^{-2}$*. If* $\sup_{\omega>0} M_R(\omega) < \infty$*, then the closed-loop system is exponentially stable. More generally, there exist* $M_e^e, t_0 \geq 1$ *such that if* (5.5) *hold, then for all* $x_{e0} \in X_e$ *satisfying* $A_e x_{e0} + B_e w_{ext}(0) \in X_e$ *we have*

$$(5.6)\qquad \int_t^{t+1} \|e(s)\| ds \leq \frac{M_e^e}{M_T(t)} \left(\|A_e x_{e0} + B_e w_{ext}(0)\| + M_{ext}\right), \qquad t \geq t_0,$$

where $M_T(t)$ *is determined by* (b)–(c) *in Theorem* 3.7 *and* $M_{ext}^2 = \|(\omega_k \Pi_{ext}^1(k))\|_{\ell^1}^2 + \|(\omega_k \Pi_{ext}^2(k))_k\|_{\ell^2}^2$*. In particular, if* $\eta(\omega)^{-1}\gamma(\omega)^{-2} = O(\omega^\alpha)$ *for some* $\alpha > 0$*, then* (5.6) *holds with* $M_T(t) = t^{1/\alpha}$*.*

Proof. Theorem 5.11 shows that the controller solves the robust output regulation problem, and $\|R(i\omega, A_e)\| \leq M_R(|\omega|)$ follows from Theorem 3.7 and Remark 3.8. Thus (3.4) holds $M_T(\cdot)$ and for some $M_e, t_0 > 0$. As shown in the proofs of Theorem 5.11 and Lemma 5.13, the conditions of Lemma 5.4 are satisfied whenever $y_{ref}(t)$ and $w_{dist}(t)$ are such that (5.4) and (5.5) hold. If $x_{e0} \in X_e$ is such that $A_e x_{e0} + B_e w_{ext}(0) \in X_e$, then $e(t) = C_{e\Lambda} T_e(t) A_e^{-1}(A_e x_{e0} + B_e w_{ext}(0) - q_{ext})$. The admissibility of $C_{e\Lambda}$ and (3.4) imply

$$\begin{aligned}\int_t^{t+1} \|e(s)\| ds &\lesssim \|T_e(t) A_e^{-1}(A_e x_{e0} + B_e w_{ext}(0) - q_{ext})\| \\ &\leq \frac{M_e^e}{M_T(t)} \left(\|A_e x_{e0} + B_e w_{ext}(0)\| + \|q_{ext}\|\right),\end{aligned}$$

which implies the claim since $\|q_{ext}\|^2 \leq M_{ext}^2$. ∎

If $C \in \mathcal{L}(X, Y)$ and $C_c \in \mathcal{L}(Z, U)$ in Theorem 5.14, then (5.6) can be replaced with a pointwise rate $\|e(t)\| \leq \frac{M_e^e}{M_T(t)} (\|A_e x_{e0} + B_e w_{ext}(0)\| + M_{ext})$ for $t \geq t_0$. If $w_{dist}(0) = 0$ and $B_c \in \mathcal{L}(Z, U)$, then Lemma 5.5 gives a sufficient condition for initial states $z_0 \in Z$ that achieve the convergence rate (5.6).

The following result presents necessary conditions for exponential closed-loop stability with controllers satisfying the conditions (5.1), which in turn are necessary for robustness by [30, Thm. 13].

PROPOSITION 5.15. *Assume* $(A^S, B^S, C_\Lambda^S, D^S)$ *is strongly stable,* $\{i\omega_k\}_{k\in\mathcal{I}} \subset \rho(A^S)$*, and* $(A_c, B_c, C_{c\Lambda}, D_c)$ *satisfies* (5.1)*. If the closed-loop system is exponentially stable, then* $\sup_{k\in\mathcal{I}} \|P_S(i\omega_k)^{-1}\| < \infty$*.*

Proof. It follows from the proof of Lemma 4.2 that $B_e^0 = \begin{bmatrix} 0 \\ B_c \end{bmatrix}$ and $C_e^0 = [0, C_{c\Lambda}]$ are admissible with respect to A_e. The proof of Theorem 3.2 implies $C_e^0 R(i\omega_k, A_e) B_e^0 = C_{c\Lambda} S_A(i\omega_k)^{-1} B_c$, where $S_A(i\omega_k) = i\omega_k - A_c + B_c P_{cl}(i\omega_k) C_{c\Lambda}$ and $P_{cl}(i\omega_k) = P_S(i\omega_k)(I + D_{c1} P_S(i\omega_k))^{-1}$. Since the closed-loop system is exponentially stable, we must have

$$\sup_{k\in\mathcal{I}} \|C_{c\Lambda} S_A(i\omega_k)^{-1} B_c\| < \infty. \tag{5.7}$$

Let $y \in Y$ and denote $z = S_A(i\omega_k)^{-1} B_c y \in Z_{B_c}$, which implies $(i\omega_k - A_c)z = B_c(y - P_{cl}(i\omega_k) C_{c\Lambda} z)$. The conditions (5.1) show that we must have $y = P_{cl}(i\omega_k) C_{c\Lambda} z$. Thus $C_{c\Lambda} S_A(i\omega_k)^{-1} B_c y = P_{cl}(i\omega_k)^{-1} y = (P_S(i\omega_k)^{-1} + D_{c1}) y$ for all $y \in Y$, and the claim follows from (5.7). ∎

6. Examples.

6.1. A wave equation with boundary control. We consider a one-dimensional undamped wave equation with boundary control and observation,

$$w_{tt}(\xi,t) = w_{\xi\xi}(\xi,t), \qquad \xi \in (0,1), \tag{6.1a}$$
$$w_\xi(\xi,0) = w_0(\xi), \quad w_t(\xi,0) = w_1(\xi), \tag{6.1b}$$
$$u(t) = -w_\xi(0,t), \quad w_\xi(1,t) = 0, \tag{6.1c}$$
$$y(t) = w_t(0,t). \tag{6.1d}$$

The results in [51] show that (6.1) defines a regular linear system with state $x(t) = (w_\xi(\cdot,t), w_t(\cdot,t))^T$ on $X = L^2(0,1) \times L^2(0,1)$. Its transfer function is given by

$$P(\lambda) = \frac{1 + e^{-2\lambda}}{1 - e^{-2\lambda}}, \qquad \lambda \neq i\pi k, \qquad k \in \mathbb{Z},$$

and $D = 1$. In particular, we have $\operatorname{Re} P(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$. We will construct a controller that achieves exponential closed-loop stability and robust output regulation for 1-periodic signals of the form $y_{ref}(t) = \sum_{k\in\mathbb{Z}} y_{ref}^k e^{i2\pi kt}$ with $(y_{ref}^k)_k \in \ell^1(\mathbb{C})$. For this we will use a controller based on the transport equation presented in section 5.2 with $\tau = 1$.

The system (6.1) can be stabilized exponentially with negative output feedback $u(t) = -D_{c2} y(t)$ with $D_{c2} > 0$. For $\lambda \in \mathbb{C}_+$ the transfer function $P_S(\lambda)$ of the stabilized system $(A^S, B^S, C_\Lambda^S, D^S)$ is given by

$$P_S(\lambda) = P(\lambda)(I + D_{c2} P(\lambda))^{-1} = \frac{1 + e^{-2\lambda}}{1 + D_{c2} + (D_{c2} - 1)e^{-2\lambda}}$$

and $\operatorname{Re} P_S(i\omega) = \frac{D_{c2} \cos(\omega)^2}{1 + (D_{c2}^2 - 1)\cos(\omega)^2}$. Now $\operatorname{Re} P_S(i\omega) = 0$ if and only if $\omega = (k + 1/2)\pi$ for some $k \in \mathbb{Z}$. Therefore for any fixed $0 < \varepsilon < \pi/2$ there exists $\eta > 0$ such that $\operatorname{Re} P_S(i\omega) \geq \eta > 0$ for all $\omega \in \Omega_\varepsilon = \{\, \omega \in \mathbb{R} \mid \exists k \in \mathcal{I} : |\omega - 2\pi k| < \varepsilon \,\}$.

The conditions of Theorem 5.7 are satisfied, and thus the controller in Definition 5.6 solves the robust output regulation problem for all 1-periodic reference signals with $(y_{ref}^k)_k \in \ell^1(\mathbb{C})$ and the output of the controlled system converges to $y_{ref}(t)$ at an exponential rate. The closed-loop system consisting of (6.1) and the controller (without the reference signal) becomes

$$\begin{aligned}
w_{tt}(\xi,t) &= w_{\xi\xi}(\xi,t), \qquad \xi \in (0,1),\\
z_t(\xi,t) &= z_\xi(\xi,t), \qquad \xi \in (0,1),\\
w_\xi(\xi,0) &= w_0(\xi), \quad w_t(\xi,0) = w_1(\xi), \quad z(\xi,t) = z_0(\xi),\\
w_\xi(0,t) &= (\beta - 2^{-1/2})z(0,t) - (\beta + 2^{-1/2})z(1,t),\\
w_t(0,t) &= 2^{-1/2}(z(0,t) - z(1,t)), \quad w_\xi(1,t) = 0,
\end{aligned}$$

where $\beta = D_{c1} + D_{c2} > 0$ is arbitrary. By Theorem 5.7 the semigroup $T_e(t)$ associated to this coupled system of partial differential equations is exponentially stable, and thus $\|w_\xi(\cdot,t)\|_{L^2}^2 + \|w_t(\cdot,t)\|_{L^2}^2 + \|z(\cdot,t)\|_{L^2}^2 \to 0$ at an exponential rate as $t \to \infty$.

6.2. A strongly stabilizable wave equation. In this example we consider another one-dimensional wave equation, now with distributed control and observation,

$$w_{tt}(\xi,t) = w_{\xi\xi}(\xi,t) + b(\xi)u(t), \qquad \xi \in (0,1), \tag{6.2a}$$

$$w(0,t) = 0, \quad w(1,t) = 0, \tag{6.2b}$$

$$w(\xi,0) = w_0(\xi), \quad w_t(\xi,0) = w_1(\xi), \tag{6.2c}$$

$$y(t) = \int_0^1 b(\xi) w_t(\xi,t) d\xi, \tag{6.2d}$$

where $b(\xi) = 2(1-\xi)$. Equation (6.2) determines a passive linear system with state $x(t) = (w(\cdot,t), w_t(\cdot,t))^T$ on $X = H_0^1(0,1) \cap L^2(0,1)$ with bounded input and output operators satisfying $C = B^*$. The transfer function $P(\lambda)$ can be computed as in [13, sec. II]. Negative output feedback $u(t) = -D_{c2}y(t)$ stabilizes the system strongly for any $D_{c2} > 0$, but the system is not exponentially stabilizable. However, the semigroup generated by A^S is polynomially stable since $\int_0^1 b(\xi)\sin(k\pi\xi)d\xi = \frac{2}{k\pi}$ implies $\|R(i\omega, A - BD_{c2}C)\| = O(\omega^2)$ for $D_{c2} > 0$ by [37, Thm. 1].

Our aim is to design a controller to achieve robust output tracking of $y_{ref}(t) = \sin(\pi t) + \frac{1}{4}\cos(2\pi t)$. The frequencies of the signal $y_{ref}(t)$ are $\{\pm\pi, \pm 2\pi\}$. Due to robustness, the controller will be able to track any reference signal with these frequencies. Since $\dim Y = p = 1$, we can construct a passive feedback controller in Definition 5.1 on $Z = \mathbb{R}^4$ by choosing

$$A_c = \operatorname{blockdiag}(J_1, J_2), \quad J_1 = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix},$$

$C_c = [k_1, 0, k_2, 0]$, $B_c = C_c^*$, and $D_c > 0$. The values of $k_1, k_2 \in \mathbb{R}$ and D_c affect the stability properties of the closed-loop system. In this example we choose $k_1 = k_2 = 3$ and $D_c = 35$. By construction the controller is robust with respect to perturbations in the system provided that the strong stability of the closed-loop is preserved. Since B and C are bounded operators, Proposition 5.3 shows that $\|e(t)\| \to 0$ as $t \to \infty$ for all initial states $x_0 \in D(A)$ and $z_0 \in Z$.

For simulations, the system (6.2) was approximated with the finite element method with $N = 24$ points on $[0,1]$. Figure 1 depicts the behavior of the error $e(t)$ and the integrals $\int_t^{t+1}\|e(s)\|ds$ for $0 \le t \le 24$ for initial states $x_0(\xi) = \xi(1-\xi)(2-5\xi)$ and $z_0 = 0$. Figure 1 also plots the solution $w(\xi,t)$ of the controlled wave equation for $0 \le t \le 6$.

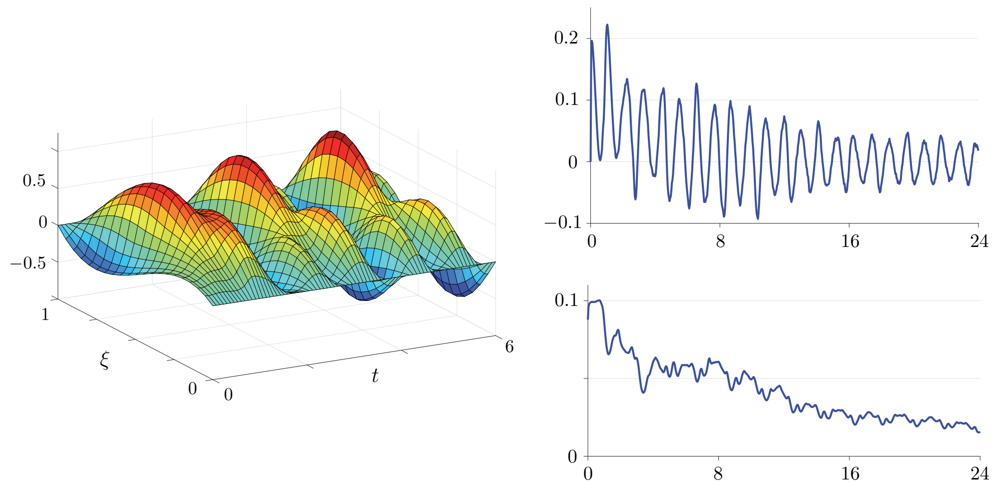

FIG. 1. *The solution $w(\xi,t)$ of controlled wave equation (left) and $e(t)$ (top right) and $\int_t^{t+1}\|e(s)\|ds$ (bottom right).*

6.3. Periodic output tracking for a heat equation. In the final example we consider a two-dimensional boundary controlled heat equation on $\Omega = [0,1]\times[0,1]$

$$x_t(\xi,t) = \Delta x(\xi,t), \qquad x(\xi,0) = x_0(\xi), \tag{6.3a}$$

$$\frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_1} = u(t), \qquad \frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_2} = w_{dist}(t), \qquad \frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_0} = 0, \tag{6.3b}$$

$$y(t) = \int_{\Gamma_1} x(\xi,t)d\xi, \tag{6.3c}$$

where the parts Γ_0, Γ_1, and Γ_2 of the boundary $\partial\Omega$ are defined so that $\Gamma_1 = \{\,\xi = (0,\xi_2) \mid 0 \le \xi_2 \le 1\,\}$, $\Gamma_2 = \{\,\xi = (\xi_1,1) \mid 1/2 \le \xi_1 \le 1\,\}$, $\Gamma_0 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$. By [11, Cor. 2] the heat equation defines a regular linear system with state $x(t) = x(\cdot,t)$ on $X = L^2(\Omega)$ with feedthrough $D = 0$. The system is passive,

$$P(\lambda) = \frac{\coth(\sqrt{\lambda})}{\sqrt{\lambda}}, \qquad \lambda \in \overline{\mathbb{C}_+} \setminus \{0\},$$

and $|P(i\omega)^{-1}| = O(|\sqrt{\omega}|)$ for $\omega \in \mathbb{R}$ with large $|\omega|$. The system (6.3) is exponentially stabilizable with feedback $u(t) = -D_{c2}y(t)$ for any $D_{c2} > 0$.

We will design an infinite-dimensional dynamic feedback controller that achieves robust output tracking of the 2-periodic nonsmooth reference signal $y_{ref}(t)$ in Figure 2 and rejects a suitable class of 2-periodic disturbance signals $w_{dist}(t)$. The frequencies of the signals are $\{\omega_k\}_{k\in\mathbb{Z}}$ with $\omega_k = \pi k$ for $k \in \mathbb{Z}$, and the Fourier coefficients of $y_{ref}(t)$ are such that $|y_{ref}^k| = O(|k|^{-3})$.

We can construct the controller as in Definition 5.10 by choosing $Z = \ell^2(\mathbb{C})$, $A_c = \operatorname{diag}(i\omega_k)_{k\in\mathcal{I}}$, $B_c = c((1+|k|)^{-1/2-\varepsilon})_{k\in\mathbb{Z}}$ for some small $\varepsilon > 0$, $C_c = B_c^*$, and $D_{c1} = 0$. The parameters $\varepsilon > 0$, $D_c = D_{c2} > 0$, and $c > 0$ affect the stability properties of the closed-loop system. Proposition 5.15 shows that since $P(\omega_k) \to 0$ as $|k| \to \infty$, the closed-loop system cannot be stabilized exponentially. However, by Theorem 3.7 the closed-loop system consisting of (2.1) and the controller with the above choices of parameters is polynomially stable. Indeed, since $\operatorname{Re} P_S(i\omega) = O(|\omega|^{-1/2})$ and $|B_{ck}^{-1}| = (1+|k|)^{1/2+\varepsilon} = O(|\omega_k|^{1/2+\varepsilon})$, we have from Theorem 5.14

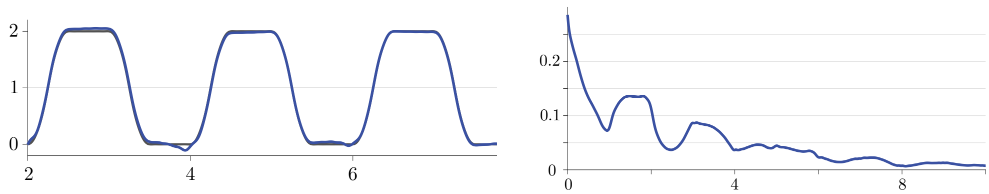

FIG. 2. *The reference $y_{ref}(t)$ (left, gray), the output $y(t)$ (left, blue), and $\int_t^{t+1}\|e(s)\|ds$ (right) for the heat equation.*

that $\|R(i\omega, A_e)\| = O(|\omega|^{3/2+2\varepsilon})$ and there exist $M_e, t_0 > 0$ such that

$$\|T_e(t)x_{e0}\| \leq \frac{M_e}{t^{1/\alpha}}\|A_e x_{e0}\|, \qquad x_{e0} \in D(A_e),\ t \geq t_0,$$

where $\alpha = 3/2 + 2\varepsilon$.

To verify that the controller is capable of regulating the given signals $y_{ref}(t)$ and $w_{dist}(t)$, we need to show that the conditions (5.4) are satisfied. The norms $\|R(i\omega, A)B\|$ and $\|R(i\omega, A)B_d\|$ are uniformly bounded for large $|\omega|$. Lemma 5.12 and $(B_{ck}^*)_k \in \ell^2(\mathbb{C})$ imply that it is sufficient to show

$$(|B_{ck}|^{-1}|P_S(i\omega_k)|^{-1}(|y_{ref}^k| + |P_d(i\omega_k)||w_{dist}^k|))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}).$$

The eigenfunction expansion of A can be used to show $|P_d(i\omega)| = O(|\omega|^{-1})$, and since $|P(i\omega)^{-1}| = O(|\omega|^{1/2})$, the above condition is satisfied for all $y_{ref}(t)$ and $w_{dist}(t)$ with

$$(|k|^{1+\varepsilon}|y_{ref}^k|)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}) \quad \text{and} \quad (|k|^{\varepsilon}|w_{dist}^k|)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}).$$

The condition on $(y_{ref}^k)_k$ in particular holds for $y_{ref}(t)$ in Figure 2.

Finally, we can study the rational rates of decay of $\|e(t)\|$ using Theorem 5.14. The conditions in (5.5) are both satisfied if

$$(|k|^{2+\varepsilon}|y_{ref}^k|)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}) \quad \text{and} \quad (|k|^{1+\varepsilon}|w_{dist}^k|)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}).$$

The first condition is satisfied for $y_{ref}(t)$ in Figure 2 whenever $0 < \varepsilon < 1/2$. Then for all $x_{e0} \in X_e$ such that $A_e x_{e0} + B_e v_0 \in X_e$ we have

$$\int_t^{t+1}\|e(s)\|ds \leq \frac{M_e^e}{t^{1/\alpha}}\left(\|A_e x_{e0} + B_e w_{ext}(0)\| + M_{ext}\right), \qquad t \geq t_0, \tag{6.4}$$

where $\alpha = 3/2 + 2\varepsilon$, and a direct estimate shows that for any fixed $\varepsilon > 0$

$$M_{ext} \lesssim \|(|k|^{2+\varepsilon}|y_{ref}^k| + |k|^{1+\varepsilon}|w_{dist}^k|)\|_{\ell^2}.$$

For disturbance signals satisfying $w_{dist}(0) = 0$, Lemma 5.5 shows that (6.4) holds whenever $x_0 \in D(A)$ and $z_0 \in D(A_c)$ are such that $C_c z_0 = D_c(C_\Lambda x_0 - y_{ref}(0))$. Moreover, by Proposition 5.13 the regulation error satisfies $\|e(t)\| \to 0$ as $t \to \infty$ for all such initial states.

For simulations the solution of the controlled heat equation (6.3) was approximated with finite differences using an $N \times N$ grid with $N = 20$. The free parameters of the controller were chosen as $\varepsilon = 1/10$, $c = 8$, and $D_c = 15$. The state of the controller was approximated by truncating the infinite matrix A_c to a 31×31 diagonal matrix with eigenvalues $\{i\pi k\}_{|k|\leq N_S}$ for $N_S = 15$. Figure 2 depicts the output of the controlled heat equation for $2 \leq t \leq 8$ and the behavior of the error integrals for $0 \leq t \leq 10$ for the initial state $x_0(\xi_1, \xi_2) = -(1 + \xi_1^2/4 - \xi_1^3/6)(\cos(\pi\xi_2)/10 + 2)$ such that $x_0 \in D(A)$ and an initial state $z_0 \in D(A_c)$ satisfying $C_c z_0 = D_c(Cx_0 - y_{ref}(0))$.

Appendix A.

LEMMA A.1. *Let X be a Hilbert space and let $T, S \in \mathcal{L}(X)$ be such that* $\operatorname{Re} T \geq c \geq 0$ *and* $\operatorname{Re} S \geq d \geq 0$.

(a) *If T is boundedly invertible, then* $\operatorname{Re} T^{-1} \geq c\|T\|^{-2}$. *If $c > 0$, then T^{-1} exists and $\|T^{-1}\| \leq \frac{1}{c}$.*

(b) *If $c > 0$ or $d > 0$, then $\|T(I + ST)^{-1}\| \leq \frac{\|T\|^2}{c+d\|T\|^2}$. If $c > 0$ and $d \geq 0$, then*

$$\operatorname{Re} T(I + ST)^{-1} \geq \frac{c^3 + c^2 d\|T\|^2}{\|T\|^2(1 + c\|S\|)^2}.$$

(c) *If T is invertible, $c \geq 0$, and $d > 0$, then* $\operatorname{Re} T(I+ST)^{-1} \geq d(\|T^{-1}\|+\|S\|)^{-2}$.

(d) *If $c \geq 0$ and $S \geq 0$, then $I + ST$ and $I + TS$ are boundely invertible, and* $\operatorname{Re} T(I + ST)^{-1} \geq 0$.

Proof. (a) The proof of the first part is elementary and latter claims follow from the estimate $\|Tx\|\|x\| \geq |\langle Tx, x\rangle| \geq \operatorname{Re}\langle Tx, x\rangle \geq c\|x\|^2$ for $x \in X$.

(b) If $c > 0$, we can use part (a) and $T(I + ST)^{-1} = (T^{-1} + S)^{-1}$. If $d > 0$, then an argument similar to the one used in [14, Lem. 2.3] shows that $\|T(I+ST)^{-1}\| \leq \frac{1}{d}$.

(c) The claim follows from $T(I + ST)^{-1} = (T^{-1} + S)^{-1}$ and part (a).

(d) Assume $\operatorname{Re} T \geq 0$ and $S \geq 0$. The invertibility of $I + ST$ implies that also $I+TS$ is invertible. It is straightforward to show that the range of $I+ST$ is dense in X. Thus it suffices to show that $I+ST$ is lower bounded. If this is not true there exists a sequence $(x_n)_n \subset X$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(I + ST)x_n\| \to 0$ as $n \to \infty$. Then $0 \leftarrow \operatorname{Re}\langle(I + ST)x_n, Tx_n\rangle \geq \|S^{1/2}Tx_n\|^2$, and further $\|STx_n\| \to 0$ as $n \to \infty$. However, since $\|x_n\| = 1$, we would then have $\|(I + ST)x_n\| \not\to 0$ as $n \to \infty$, which is a contradiction. Finally, the proof of $\operatorname{Re} T(I + ST)^{-1} \geq 0$ is elementary. ∎

LEMMA A.2. *Let $P(\cdot) : \overline{\mathbb{C}_+} \to \mathcal{L}(Y)$ be such that* $\operatorname{Re} P(\lambda) \geq 0$ *for all $\lambda \in \overline{\mathbb{C}_+}$ and let $D_c \geq 0$. Then $-1 \in \rho(D_cP(\lambda))$ for all $\lambda \in \overline{\mathbb{C}_+}$. If $\sup_{\lambda\in\overline{\mathbb{C}_+}}\|P(\lambda)\| < \infty$, then in addition $\sup_{\lambda\in\overline{\mathbb{C}_+}}\|(I + D_cP(\lambda))^{-1}\| < \infty$.*

Proof. The fact that $-1 \in \rho(D_cP(\lambda))$ for all $\lambda \in \overline{\mathbb{C}_+}$ follows from Lemma A.1(d). Assume $\sup_{\lambda\in\overline{\mathbb{C}_+}}\|P(\lambda)\| < \infty$. In order to show that $(I + D_cP(\lambda))^{-1}$ are uniformly bounded for $\lambda \in \overline{\mathbb{C}_+}$ it is sufficient to show that there exists a constant $r > 0$ such that $\|(I + D_cP(\lambda))u\| \geq r\|u\|$ for all $u \in U$ and $\lambda \in \overline{\mathbb{C}_+}$. If no such $r > 0$ exists, we can choose sequences $(\lambda_n)_n \subset \overline{\mathbb{C}_+}$ and $(u_n)_n \subset U$ with $\|u_n\| = 1$ for all $n \in \mathbb{N}$ such that $\|(I + D_cP(\lambda_n))u_n\| \to 0$ as $n \to \infty$. Then

$$0 \leftarrow \operatorname{Re}\langle(I + D_cP(\lambda_n))u_n, P(\lambda_n)u_n\rangle \geq \|D_c^{1/2}P(\lambda_n)u_n\|^2,$$

which implies $\|D_cP(\lambda_n)u_n\| \to 0$ as $n \to \infty$. However, since $\|u_n\| = 1$, we would then have $\|(I + D_cP(\lambda_n))u_n\| \not\to 0$ as $n \to \infty$, which is a contradiction. ∎

The last lemma concerns output feedback for passive systems. Several additional results on this topic can be found in [18].

LEMMA A.3. *Assume (A, B, C_Λ, D) is a passive regular linear system and $\sigma(A) \subset \mathbb{C}_-$. If $D_c \geq 0$, then the system $(A - BD_cQ_1C_\Lambda, BQ_2, Q_1C_\Lambda, Q_1D)$ with $Q_1 = (I + DD_c)^{-1}$ and $Q_2 = (I + D_cD)^{-1}$ is regular, passive, and strongly stable in such a way* *that $\sigma(A - BD_cQ_1C_\Lambda) \subset \mathbb{C}_-$. If A generates an exponentially stable semigroup, then the same is true for $A - BD_cQ_1C_\Lambda$.*

Proof. The system $(A - BD_cQ_1C_\Lambda, BQ_2, Q_1C_\Lambda, Q_1D)$ is obtained from (1.1) with output feedback $u(t) = -D_cy(t)$. The regularity follows from [43], since $-D_c$ is an admissible output feedback operator by Lemma A.1(d). Since $D_c \geq 0$, it is straightforward to verify that $(A - BD_cQ_1C_\Lambda, BQ_2, Q_1C_\Lambda, Q_1D)$ is passive. In particular $A - BD_cQ_1C_\Lambda$ generates a contraction semigroup, and the strong stability of the semigroup follows from the Arendt–Batty–Lyubich–Vũ Theorem [4, 27] once we have shown $i\mathbb{R} \subset \sigma(A - BD_cQ_1C_\Lambda)$. Let $\lambda \in \overline{\mathbb{C}_+}$. The transfer function $P(\lambda) = C_\Lambda R(\lambda, A)B + D$ satisfies $\operatorname{Re} P(\lambda) \geq 0$, and thus the operator $I + DD_c + C_\Lambda R(\lambda, A)BD_c = I + P(\lambda)D_c$ is boundedly invertible by Lemma A.1(d). Using Lemma 2.1 we therefore see that $\lambda \in \rho(A - BD_cQ_1C_\Lambda)$ and

$$R(\lambda, A - BD_cQ_1C_\Lambda) = R(\lambda, A) - R(\lambda, A)B(I + D_cP(\lambda))^{-1}D_cC_\Lambda R(\lambda, A).$$

Since $\lambda \in \overline{\mathbb{C}_+}$ was arbitrary, we have $\sigma(A - BD_cQ_1C_\Lambda) \subset \mathbb{C}_-$. If A generates an exponentially stable semigroup, then $\sup_{\lambda \in \overline{\mathbb{C}_+}} \|(I + D_cP(\lambda))^{-1}\| < \infty$ by Lemma A.2, and the regularity and exponential stability of (A, B, C_Λ, D) imply $\sup_{\lambda \in \overline{\mathbb{C}_+}} \|R(\lambda, A - BD_cQ_1C_\Lambda)\| < \infty$. Thus the semigroup generated by $A - BD_cQ_1C_\Lambda$ is exponentially stable. ∎

Proof of Lemma 2.1. Let $\lambda \in \rho(A)$ be such that $Q^{-1} + C_\Lambda R(\lambda, A)B$ has a bounded inverse. Denote $R_\lambda = R(\lambda, A)$ and $R(\lambda) = R_\lambda - R_\lambda B(Q^{-1} + C_\Lambda R_\lambda B)^{-1}C_\Lambda R_\lambda$. If $x \in X$, then $R(\lambda)x \in X_B$ and a computation on X_{-1} shows

$$\begin{aligned}&(\lambda - A + BQC_\Lambda)R(\lambda)x\\ &\qquad = x + B\left[Q - (I + QC_\Lambda R_\lambda B)(Q^{-1} + C_\Lambda R_\lambda B)^{-1}\right]C_\Lambda R_\lambda x = x \in X.\end{aligned}$$

Thus $R(\lambda)x \in D(A - BQC_\Lambda)$ and $(\lambda - A + BQC_\Lambda)R(\lambda) = I$. On the other hand, if $x \in D(A - BQC_\Lambda)$, then $x \in X_B$ and we can again compute on X_{-1} (considering $R(\lambda)$ as an operator $R(\lambda) : X + \mathcal{R}(B) \to X$)

$$\begin{aligned}&R(\lambda)(\lambda - A + BQC_\Lambda)x\\ &\qquad = x + R_\lambda B\left[Q - (Q^{-1} + C_\Lambda R_\lambda B)^{-1}(I + C_\Lambda R_\lambda BQ)\right]C_\Lambda x = x.\end{aligned}$$

Since $x \in D(A - BQC_\Lambda)$ was arbitrary, this completes the proof. ∎

Acknowledgments. The author is grateful to Reinhard Stahn for discussions regarding Theorem 3.7 and to Professor Charles Batty for helpful comments on non-uniform stability of semigroups.

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