

# Robustness of Strong Stability of Discrete Semigroups

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## Abstract

In this paper we study the robustness of strong stability of a discrete semigroup on a Hilbert space under bounded perturbations. As the main result we present classes of perturbations preserving the strong stability of the semigroup.

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## 1. Introduction

Due to the high level of generality and the many forms of strong stability, finding conditions for preservation of strong stability of a semigroup under perturbations of its generator is a challenging research problem. However, recent advances in the theory of nonuniform stability of semigroups [1, 2, 3, 4] have made it possible to study robustness of stability of semigroups that are not exponentially stable [5, 6]. While general strongly stable semigroups may have no intrinsic robustness properties, the theory of nonuniform stability of semigroups opens doors for research on robustness properties for many important subclasses of strongly stable semigroups.

In this short paper we consider the preservation of strong stability of a discrete semigroup  $(A^n)_{n \in \mathbb{N}}$  with  $A \in \mathcal{L}(X)$  under additive perturbations  $A + BC$  with  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  for some separable Hilbert space  $Y$ . In particular, we assume that the unperturbed semigroup  $(A^n)_{n \in \mathbb{N}}$  is strongly stable in such a way that  $A$  has a finite number of spectral points on the unit circle  $\mathbb{T}$ , and the growth of its resolvent operator is polynomially bounded near these points.

The main result of this paper is a discrete analogue of the set of conditions for preservation of strong stability of strongly continuous semigroups presented in [6]. The techniques employed here are similar to those used in [6], but in many situations the proofs can be greatly simplified due to the fact that the operator  $A$  is bounded. The discrete proofs also require several modifications, mainly in estimating the behaviour of the resolvent operator near the unit disk  $\mathbb{D}$ . To the author's knowledge, the preservation of strong stability of discrete semigroups has not been studied previously in the literature. Moreover, the resolvent estimates presented in this paper generalize the results found in the literature by allowing  $A$  to have multiple spectral points on  $\mathbb{T}$ .

Assumption 1 below states the standing assumptions on the semigroup  $(A^n)_{n \in \mathbb{N}}$ . The strong stability of  $(A^n)_{n \in \mathbb{N}}$  implies that  $\sigma_p(A) \cap \mathbb{T} = \emptyset$ . Since  $X$  is a Hilbert space, Theorem I.2.9 and Corollary I.2.11 in [7] imply that for all  $\varphi \in [0, 2\pi]$

$$X = \mathcal{N}(A - e^{i\varphi}) \oplus \overline{\mathcal{R}(A - e^{i\varphi})} = \overline{\mathcal{R}(A - e^{i\varphi})}.$$

Because of this, all spectral points of  $A$  on the unit circle belong to  $\sigma_c(A)$ .

**Assumption 1.** *Let  $X$  be a Hilbert space and assume the discrete semigroup  $(A^n)_{n \in \mathbb{N}}$  with  $A \in \mathcal{L}(X)$  is strongly stable in such a way that  $\sigma(A) \cap \mathbb{T} = \{e^{i\varphi_k}\}_{k=1}^N$  for some  $N \in \mathbb{N}$  and  $d_A = \min_{k \neq l} |\varphi_k - \varphi_l| > 0$ . Assume further that for some  $\alpha \geq 1$ ,  $M_A \geq 1$  and  $0 < \varepsilon_A \leq \min\{\pi/8, d_A/3\}$*

$$\sup_{0 < |\varphi - \varphi_k| \leq \varepsilon_A} |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A)\| \leq M_A, \quad (1)$$

for all  $k \in \{1, \dots, N\}$  and  $\|R(e^{i\varphi}, A)\| \leq M_A$  whenever  $|\varphi - \varphi_k| > \varepsilon_A$  for all  $k$ .

We assume that for some  $\beta, \gamma \geq 0$  the operators  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  of the perturbation satisfy

$$\mathcal{R}(B) \subset \mathcal{R}((1 - e^{-i\varphi_k} A)^\beta) \quad (2a)$$

$$\mathcal{R}(C^*) \subset \mathcal{R}((1 - e^{i\varphi_k} A^*)^\gamma) \quad (2b)$$

for every  $k \in \{1, \dots, N\}$  and

$$(1 - e^{-i\varphi_k} A)^{-\beta} B, \quad (1 - e^{i\varphi_k} A^*)^{-\gamma} C^* \quad (3a)$$

$$\text{are Hilbert–Schmidt operators for all } k. \quad (3b)$$

We recall that if  $(e_k)_{k=1}^\infty$  is an orthonormal basis of  $Y$ , then  $T \in \mathcal{L}(Y, X)$  is called Hilbert–Schmidt if  $(Te_k)_{k=1}^\infty \in \ell^2(X)$ . The condition (2) combined with the Closed Graph Theorem implies  $(1 - e^{-i\varphi_k} A)^{-\beta} B \in \mathcal{L}(Y, X)$  and  $(1 - e^{i\varphi_k} A^*)^{-\gamma} C^* \in \mathcal{L}(X, Y)$ . Since  $A$  is bounded, also  $B$  and  $C^*$  are necessarily Hilbert–Schmidt operators whenever (3) is satisfied. Finally, if  $\dim Y < \infty$ , i.e., if the perturbation  $BC$  is of finite rank, then (3) follows immediately from (2).

The following theorem presenting conditions for preservation of strong stability is the main result of this paper.

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**Theorem 2.** *Let Assumption 1 be satisfied for some  $\alpha \geq 1$  and let  $\beta, \gamma \geq 0$  be such that  $\beta + \gamma \geq \alpha$ . There exists  $\delta > 0$  such that if  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  satisfy (2) and (3) and*

$$\|(1 - e^{-i\varphi_k} A)^{-\beta} B\| < \delta, \quad \text{and} \quad \|(1 - e^{i\varphi_k} A^*)^{-\gamma} C^*\| < \delta$$

for all  $k \in \{1, \dots, N\}$ , then the discrete semigroup  $((A + BC)^n)_{n \in \mathbb{N}}$  is strongly stable. Moreover, we then have  $\sigma(A + BC) \cap \mathbb{T} = \sigma_c(A + BC) \cap \mathbb{T} = \{e^{i\varphi_k}\}_{k=1}^N$ , and for all  $k$

$$\sup_{0 < |\varphi - \varphi_k| \leq \varepsilon_A} |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A + BC)\| < \infty.$$

We begin the paper by studying the behaviour of the resolvent operator  $R(\lambda, A)$  near the unit disk  $\mathbb{D}$  in Section 2. These results are required in the proof of Theorem 2, which is presented subsequently in Section 3.

If  $X$  and  $Y$  are Banach spaces and  $A : X \rightarrow Y$  is a linear operator, we denote by  $\mathcal{D}(A)$ ,  $\mathcal{R}(A)$ , and  $\mathcal{N}(A)$  the domain, the range, and the kernel of  $A$ , respectively. The space of bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . If  $A : \mathcal{D}(A) \subset X \rightarrow X$ , then  $\sigma(A)$ ,  $\sigma_p(A)$ ,  $\sigma_c(A)$  and  $\rho(A)$  denote the spectrum, the point spectrum, the continuous spectrum and the resolvent set of  $A$ , respectively. For  $\lambda \in \rho(A)$  the resolvent operator is given by  $R(\lambda, A) = (\lambda - A)^{-1}$ . The inner product on a Hilbert space is denoted by  $\langle \cdot, \cdot \rangle$ . We denote  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ ,  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ ,  $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ .

## 2. Resolvent Estimates

In this section we study the behaviour of the resolvent operator  $R(\lambda, A)$  near the unit disk  $\mathbb{D}$ . In particular, the proof of Theorem 2 is based on the property that the polynomial growth of the resolvent operator near the points  $e^{i\varphi_k}$  can be cancelled by a suitable operator. The general form of the resolvent estimates follows the recent results for strongly continuous semigroups that have appeared in [3, 8, 4], and the results in this section can be seen as straightforward discrete reformulations of corresponding results in the previous references. The main difference compared to the previous references is that we allow the operator  $A$  to have multiple spectral points on the unit circle  $\mathbb{T}$ .

Define  $\Lambda_k = 1 - e^{-i\varphi_k} A$  for  $k \in \{1, \dots, N\}$ . The operators  $\Lambda_k$  and  $\Lambda_l$  commute for every  $k, l \in \{1, \dots, N\}$ , we have  $\Lambda_k^* = 1 - e^{i\varphi_k} A^*$ , and the families  $(\Lambda_k)_{k=1}^N$  and  $(\Lambda_k^*)_{k=1}^N$  are uniformly sectorial [9, Sec. 2.1]. Indeed, since the operator  $A$  is power bounded, the strong Kreiss resolvent condition [7, Sec. II.1.2] implies  $\|R(\lambda, e^{-i\varphi_k} A)\| \leq M/(|\lambda| - 1)$  for all  $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , where  $M = \sup_{n \in \mathbb{N}} \|A^n\| = \sup_{n \in \mathbb{N}} \|(e^{-i\varphi_k} A)^n\|$ . This implies that for every  $\lambda > 0$  we have

$$\|\lambda(\lambda + 1 - e^{-i\varphi_k} A)^{-1}\| \leq \lambda \frac{M}{|\lambda + 1| - 1} = M.$$

Since the bound is independent of  $\varphi_k \in [0, 2\pi]$ , by [9, Prop. 2.1.1] the family  $(\Lambda_k)_{k=1}^N$  is uniformly sectorial. Since  $\sigma_p(A) \cap \mathbb{T} = \emptyset$ , the operators  $\Lambda_k$  are injective and have sectorial inverses  $\Lambda_k^{-1} : \mathcal{R}(\Lambda_k) \subset X \rightarrow X$  [9, Prop. 2.1.1(b)]. The same conclusions are true for the operators  $\Lambda_k^* = 1 - e^{i\varphi_k} A^*$ . The fractional powers  $\Lambda_k^\beta$  and  $(\Lambda_k^*)^\gamma$  are therefore defined for all  $\beta, \gamma \in \mathbb{R}$ .

Consider regions  $\Omega_k \subset \mathbb{C} \setminus \mathbb{D}$  defined as (see Figure 1)

$$\Omega_k = \{\lambda \in \mathbb{C} \mid |\lambda| \geq 1, 0 < |\lambda - e^{i\varphi_k}| \leq r_A\},$$

where  $r_A = |1 - e^{i\varepsilon_A}|$ . We have  $0 < r_A \leq 1$  and  $|e^{i\varphi_k} - e^{i(\varphi_k \pm \varepsilon_A)}| = r_A$  for all  $k$ .

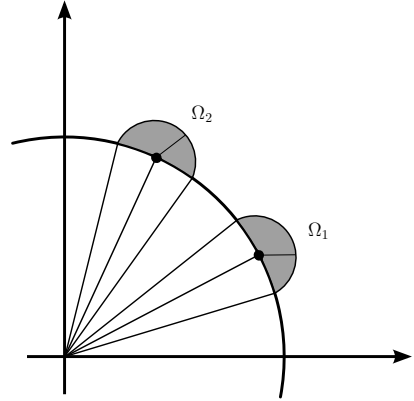


Figure 1: The domains  $\Omega_k$ .

The following is the main resolvent estimate required in the proof of Theorem 2.

**Theorem 3.** *If Assumption 1 is satisfied, there exists  $M_1 \geq 1$  such that*

$$\sup_{\lambda \in \Omega_k} \|R(\lambda, A) \Lambda_k^\alpha\| \leq M_1 \quad (4)$$

for all  $k \in \{1, \dots, N\}$ .

The proof of the theorem is based on the following two lemmas. The Moment Inequality in Lemma 4 is an essential tool used several times during the course of the paper.

**Lemma 4.** *Let  $0 < \tilde{\theta} < \theta$ . There exists  $M_{\tilde{\theta}} \geq 1$  such that for all  $k \in \{1, \dots, N\}$*

$$\|\Lambda_k^{\tilde{\theta}} x\| \leq M_{\tilde{\theta}} \|x\|^{1-\tilde{\theta}/\theta} \|\Lambda_k^\theta x\|^{\tilde{\theta}/\theta} \quad \forall x \in X.$$

If  $Y$  is a Banach space and  $R \in \mathcal{L}(Y, X)$ , then

$$\|\Lambda_k^{\tilde{\theta}} R\| \leq M_{\tilde{\theta}} \|R\|^{1-\tilde{\theta}/\theta} \|\Lambda_k^\theta R\|^{\tilde{\theta}/\theta}$$

for all  $k$ . The corresponding results are valid for  $(\Lambda_k^*)_k$ .

*Proof.* For a fixed  $k$  the properties follow from [9, Prop. 6.6.4]. However, by [9, Prop. 2.6.11] and the uniform sectoriality of the operator family  $(\Lambda_k)_k$  it is possible to choose  $M_{\tilde{\theta}}$  to be independent of  $k$ .  $\square$

**Lemma 5.** *If Assumption 1 is satisfied, then there exists  $M_0 \geq 1$  such that for all  $k$*

$$\sup_{\lambda \in \Omega_k} |\lambda - e^{i\varphi_k}|^\alpha \|R(\lambda, A)\| \leq M_0.$$

*Proof.* Let  $M = \sup_{n \in \mathbb{N}} \|A^n\|$ . From Assumption 1 we have

$$\sup_{0 < |\varphi - \varphi_k| \leq \varepsilon_A} |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A)\| \leq M_A.$$

The strong Kreiss resolvent condition implies  $(|\lambda| - 1)\|R(\lambda, A)\| \leq M$  whenever  $|\lambda| > 1$ .

Let  $\lambda = re^{i\varphi} \in \Omega_k$ . Since  $|\varphi - \varphi_k| \leq \varepsilon_A \leq \pi/8$ , and since  $|\varphi - \varphi_k|$  is equal to the arc length between points  $e^{i\varphi} \in \mathbb{T}$  and  $e^{i\varphi_k} \in \mathbb{T}$ , we have  $|e^{i\varphi} - e^{i\varphi_k}| \leq |\varphi - \varphi_k|$ . For  $r = 1$  the bound  $|\lambda - e^{i\varphi_k}|^\alpha \|R(\lambda, A)\| \leq |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A)\| \leq M_A$  follows from (1). On the other hand, if  $\varphi = \varphi_k$ ,  $1 < r \leq 1 + r_A$  and  $\lambda = re^{i\varphi_k}$ , then the strong Kreiss resolvent condition implies

$$\begin{aligned} |\lambda - e^{i\varphi_k}|^\alpha \|R(\lambda, A)\| &= (r-1)^\alpha \|R(\lambda, A)\| \\ &\leq (r-1) \|R(\lambda, A)\| \leq M \end{aligned}$$

since  $(r-1)^\alpha \leq r-1$  due to the fact that  $\alpha \geq 1$  and  $0 < r-1 \leq r_A \leq 1$ . It remains to consider the case  $\lambda = re^{i\varphi} \in \Omega_k$  with  $r > 1$  and  $\varphi \neq \varphi_k$ . We can estimate

$$\begin{aligned} |re^{i\varphi} - e^{i\varphi_k}| &\leq |re^{i\varphi} - e^{i\varphi}| + |e^{i\varphi} - e^{i\varphi_k}| \\ &\leq r-1 + |\varphi - \varphi_k|. \end{aligned}$$

Since  $\alpha \geq 1$  and  $1 < r \leq 2$ , we have  $(r-1)^\alpha \leq r-1$  and (using the scalar inequality  $(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$  for  $a, b \geq 0$ ) we get

$$|\lambda - e^{i\varphi_k}|^\alpha \leq 2^\alpha (r-1 + |\varphi - \varphi_k|)^\alpha$$

and the resolvent identity  $R(re^{i\varphi}, A) = R(e^{i\varphi}, A)(1 - (r-1)e^{i\varphi}R(re^{i\varphi}, A))$  implies

$$\begin{aligned} \|R(re^{i\varphi}, A)\| &\leq \|R(e^{i\varphi}, A)\|(1 + (r-1)\|R(re^{i\varphi}, A)\|) \\ &\leq \|R(e^{i\varphi}, A)\|(1 + M) \end{aligned}$$

and

$$\begin{aligned} |\lambda - e^{i\varphi_k}|^\alpha \|R(\lambda, A)\| &\leq 2^\alpha (r-1 + |\varphi - \varphi_k|)^\alpha \|R(\lambda, A)\| \\ &\leq 2^\alpha (M + |\varphi - \varphi_k|)^\alpha \|R(e^{i\varphi}, A)\|(1 + M) \\ &\leq 2^\alpha (M + M_A(1 + M)). \end{aligned}$$

Since in each of the situations the bound for  $|\lambda - e^{i\varphi_k}|^\alpha \|R(\lambda, A)\|$  is independent of  $k \in \{1, \dots, N\}$ , the proof is complete.  $\square$

*Proof of Theorem 3.* Let  $k \in \{1, \dots, N\}$ ,  $\lambda \in \Omega_k$ , and denote  $R_\lambda = R(\lambda, A)$  and  $\lambda_k = \lambda - e^{i\varphi_k}$  for brevity.

We begin by showing that if  $\alpha = n + \tilde{\alpha}$  with  $n \in \mathbb{N}$  and  $0 \leq \tilde{\alpha} < 1$ , then there exists  $\tilde{M} \geq 1$  (independent of  $k$ ) such that

$$\sup_{\lambda \in \Omega_k} |\lambda_k|^n \|R(\lambda, A)\Lambda_k^{\tilde{\alpha}}\| \leq \tilde{M}. \quad (5)$$

By Lemma 5 there exists  $M_0 \geq 1$  such that  $|\lambda - e^{i\varphi_k}|^\alpha \|R(\lambda, A)\| \leq M_0$  for all  $k$ . If  $\alpha = n$  and  $\tilde{\alpha} = 0$ , we have  $|\lambda_k|^n \|R_\lambda \Lambda_k^{\tilde{\alpha}}\| = |\lambda_k|^n \|R_\lambda\| \leq M_0$ . Thus the claim is satisfied with  $\tilde{M} = M_0$ , which is independent of  $k$ .

If  $0 < \tilde{\alpha} < 1$ , then by Lemma 4 there exists a constant  $M_{\tilde{\alpha}}$  independent of  $k$  and  $\lambda$  such that  $\|R_\lambda \Lambda_k^{\tilde{\alpha}}\| \leq M_{\tilde{\alpha}} \|R_\lambda\|^{1-\tilde{\alpha}} \|R_\lambda \Lambda_k\|^{\tilde{\alpha}}$ . Using

$$e^{i\varphi_k} R_\lambda \Lambda_k = R_\lambda (e^{i\varphi_k} - A) = 1 - \lambda_k R_\lambda \quad (6)$$

and the scalar inequality  $(a+b)^{\tilde{\alpha}} \leq 2^{\tilde{\alpha}}(a^{\tilde{\alpha}} + b^{\tilde{\alpha}})$  we get

$$\begin{aligned} |\lambda_k|^n \|R_\lambda \Lambda_k^{\tilde{\alpha}}\| &\leq M_{\tilde{\alpha}} |\lambda_k|^n \|R_\lambda\|^{1-\tilde{\alpha}} \|R_\lambda \Lambda_k\|^{\tilde{\alpha}} \\ &= M_{\tilde{\alpha}} |\lambda_k|^n \|R_\lambda\|^{1-\tilde{\alpha}} \|1 - \lambda_k R_\lambda\|^{\tilde{\alpha}} \\ &\leq 2^{\tilde{\alpha}} M_{\tilde{\alpha}} |\lambda_k|^n \|R_\lambda\|^{1-\tilde{\alpha}} (1 + |\lambda_k|^{\tilde{\alpha}} \|R_\lambda\|^{\tilde{\alpha}}) \\ &\leq 2^{\tilde{\alpha}} M_{\tilde{\alpha}} [(\|\lambda_k\|^{\frac{n}{1-\tilde{\alpha}}} \|R_\lambda\|)^{1-\tilde{\alpha}} + |\lambda_k|^{n+\tilde{\alpha}} \|R_\lambda\|]. \end{aligned}$$

Since  $n = \lfloor \alpha \rfloor \geq 1$  we have

$$\frac{n}{1-\tilde{\alpha}} = \frac{n\alpha}{(1-\tilde{\alpha})(n+\tilde{\alpha})} = \frac{n\alpha}{n-\tilde{\alpha}(n-1)-\tilde{\alpha}^2} \geq \alpha.$$

Since  $\lambda \in \Omega_k$ , we have  $|\lambda_k| \leq r_A \leq 1$ , and thus  $|\lambda_k|^{\frac{n}{1-\tilde{\alpha}}} \leq |\lambda_k|^\alpha$ , and

$$\begin{aligned} |\lambda_k|^n \|R_\lambda \Lambda_k^{\tilde{\alpha}}\| &\leq 2^{\tilde{\alpha}} M_{\tilde{\alpha}} [(\|\lambda_k\|^\alpha \|R_\lambda\|)^{1-\tilde{\alpha}} + |\lambda_k|^\alpha \|R_\lambda\|] \\ &\leq 2^{\tilde{\alpha}} M_{\tilde{\alpha}} [M_0^{1-\tilde{\alpha}} + M_0] \leq 2^{\tilde{\alpha}+1} M_{\tilde{\alpha}} M_0, \end{aligned}$$

since  $M_0 \geq 1$ . Therefore the claim holds with  $\tilde{M} = 2^{\tilde{\alpha}+1} M_{\tilde{\alpha}} M_0$ , which is independent of  $k$ .

We can now show that there exists  $M_1 \geq 1$  such that (4) is satisfied for all  $k$ . Again denote  $\alpha = n + \tilde{\alpha}$  and let  $K > 0$  be such that  $\|\Lambda_k^{l+\tilde{\alpha}}\| \leq K$  for all  $0 \leq l \leq n-1$  and all  $k$ . Using the identity (6) repeatedly, we obtain

$$e^{in\varphi_k} R(\lambda, A) \Lambda_k^n = (-\lambda_k)^n R(\lambda, A) + \sum_{l=0}^{n-1} (-\lambda_k)^{n-1-l} e^{il\varphi_k} \Lambda_k^l$$

and thus (using  $|\lambda_k| \leq r_A \leq 1$ )

$$\begin{aligned} \|R(\lambda, A) \Lambda_k^\alpha\| &= \|e^{in\varphi_k} R(\lambda, A) \Lambda_k^n \Lambda_k^{\tilde{\alpha}}\| \\ &\leq |\lambda_k|^n \|R(\lambda, A) \Lambda_k^{\tilde{\alpha}}\| + \sum_{l=0}^{n-1} |\lambda_k|^{n-1-l} \|\Lambda_k^{l+\tilde{\alpha}}\| \leq \tilde{M} + nK. \end{aligned}$$

Since the bound is independent of both  $\lambda \in \Omega_k$  and  $k$ , the proof is complete.  $\square$

**Lemma 6.** *Let Assumption 1 be satisfied. There exists  $M_2 \geq 1$  such that*

$$\sup_{\lambda \notin \mathbb{D} \cup \bigcup_k \Omega_k} \|R(\lambda, A)\| \leq M_2.$$

*Proof.* Let  $\lambda = re^{i\varphi} \in \mathbb{C} \setminus (\mathbb{D} \cup \bigcup_k \Omega_k)$  and let  $\lambda_0 = r_0 e^{i\varphi}$  be such that  $1 \leq r_0 \leq r$  and  $\lambda_0$  lies on the boundary of  $\mathbb{D} \cup \bigcup_k \Omega_k$ . Then either  $\lambda_0 \in \mathbb{T}$ , which implies  $\|R(\lambda_0, A)\| \leq M_A$  by Assumption 1, or otherwise  $\lambda_0 \in \Omega_k$  and  $|\lambda_0 -$

$e^{i\varphi_k} = r_A$  for some  $k \in \{1, \dots, N\}$ . By Lemma 5 we have that there exists  $M_0$  (independent of  $k$ ) such that in the latter case

$$|\lambda_0 - e^{i\varphi_k}|^\alpha \|R(\lambda_0, A)\| \leq M_0 \quad \Leftrightarrow \quad \|R(\lambda_0, A)\| \leq \frac{M_0}{r_A^\alpha}.$$

Now, if  $M = \sup_{n \in \mathbb{N}} \|A^n\|$ , then  $(|\lambda| - 1) \|R(\lambda, A)\| \leq M$  by the strong Kreiss resolvent condition. Using the resolvent identity  $R(\lambda, A) = R(\lambda_0, A) + (\lambda_0 - \lambda)R(\lambda_0, A)R(\lambda, A)$  and  $|\lambda - \lambda_0| = r - r_0 \leq r - 1 = |\lambda| - 1$  we have

$$\begin{aligned} \|R(\lambda, A)\| &\leq \|R(\lambda_0, A)\|(1 + |\lambda - \lambda_0| \|R(\lambda, A)\|) \\ &\leq \max\{M_A, M_0/r_A^\alpha\}(1 + (|\lambda| - 1)\|R(\lambda, A)\|) \\ &\leq \max\{M_A, M_0/r_A^\alpha\}(1 + M). \end{aligned}$$

Since the bound is independent of  $\lambda$ , this concludes the proof.  $\square$

Combining the above results shows that the growth of the resolvent operator  $R(\lambda, A)$  near the unit disk  $\mathbb{D}$  is cancelled by the operator  $\Lambda_1^\alpha \cdots \Lambda_N^\alpha$ .

**Corollary 7.** *If Assumption 1 is satisfied, then*

$$\sup_{\lambda \notin \mathbb{D} \cup \{e^{i\varphi_k}\}_k} \|R(\lambda, A)\Lambda_1^\alpha \cdots \Lambda_N^\alpha\| < \infty.$$

### 3. The Preservation of Strong Stability

In this section we present the proof of Theorem 2. We begin by studying the change of the spectrum of  $A$  under the perturbations.

**Theorem 8.** *Let Assumption 1 be satisfied for some  $\alpha \geq 1$  and let  $\beta, \gamma \geq 0$  be such that  $\beta + \gamma \geq \alpha$ . There exists  $\delta > 0$  such that if  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  satisfy  $\mathcal{R}(B) \subset \mathcal{R}(\Lambda_k^\beta)$  and  $\mathcal{R}(C^*) \subset \mathcal{R}((\Lambda_k^*)^\gamma)$  and*

$$\|\Lambda_k^{-\beta} B\| < \delta, \quad \text{and} \quad \|(\Lambda_k^*)^{-\gamma} C^*\| < \delta,$$

for every  $k$ , then  $\sigma(A+BC) \subset \mathbb{D} \cup \{e^{i\varphi_k}\}_{k=1}^N$  and  $\{e^{i\varphi_k}\}_k \subset \sigma(A+BC) \setminus \sigma_p(A+BC)$ . In particular, under the above conditions we have

$$\sup_{\lambda \notin \mathbb{D} \cup \{e^{i\varphi_k}\}_k} \|(1 - CR(\lambda, A)B)^{-1}\| < \infty. \quad (7)$$

The proof of Theorem 8 is based on the following two lemmas.

**Lemma 9.** *Let Assumption 1 be satisfied for some  $\alpha \geq 1$  and let  $\beta, \gamma \geq 0$  be such that  $\beta + \gamma \geq \alpha$ . There exists  $M_R \geq 1$  such that if  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  satisfy  $\mathcal{R}(B) \subset \mathcal{R}(\Lambda_k^\beta)$  and  $\mathcal{R}(C^*) \subset \mathcal{R}((\Lambda_k^*)^\gamma)$  for some  $k$ , then*

$$\|CR(\lambda, A)B\| \leq M_R \|\Lambda_k^{-\beta} B\| \|(\Lambda_k^*)^{-\gamma} C^*\|$$

for all  $\lambda \in \Omega_k$ .

*Proof.* Since  $\Lambda_k^\beta \in \mathcal{L}(X)$ , the operators  $\Lambda_k^{-\beta}$  and  $\Lambda_k^{-\beta} B$  are closed. Since  $\mathcal{D}(\Lambda_k^{-\beta} B) = Y$ , the Closed Graph Theorem implies  $\Lambda_k^{-\beta} B \in \mathcal{L}(Y, X)$ . Similarly  $(\Lambda_k^*)^{-\gamma} C^* \in \mathcal{L}(Y, X)$  and  $C\Lambda_k^{-\gamma}$  extends to a bounded operator  $C_\gamma \in \mathcal{L}(X, Y)$  with norm  $\|C_\gamma\| \leq \|(\Lambda_k^*)^{-\gamma} C^*\|$ . Choose  $M_k = \|\Lambda_k^{\beta+\gamma-\alpha}\| \cdot \sup_{\lambda \in \Omega_k} \|R(\lambda, A)\Lambda_k^\alpha\|$ . Then for all  $\lambda \in \Omega_k$

$$\begin{aligned} \|CR(\lambda, A)B\| &= \|C\Lambda_k^{-\gamma} R(\lambda, A)\Lambda_k^\alpha \Lambda_k^{\beta+\gamma-\alpha} \Lambda_k^{-\beta} B\| \\ &\leq \|C_\gamma\| \|R(\lambda, A)\Lambda_k^\alpha\| \|\Lambda_k^{\beta+\gamma-\alpha}\| \|\Lambda_k^{-\beta} B\| \\ &\leq M_k \|\Lambda_k^{-\beta} B\| \|(\Lambda_k^*)^{-\gamma} C^*\|. \end{aligned}$$

Finally, we can choose  $M_R = \max\{M_1, \dots, M_N\}$ .  $\square$

**Lemma 10.** *Let Assumption 1 be satisfied for some  $\alpha \geq 1$  and let  $\beta, \gamma \geq 0$  be such that  $\beta + \gamma \geq \alpha$ . There exists  $\delta_0 > 0$  such that if  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  satisfy  $\mathcal{R}(B) \subset \mathcal{R}(\Lambda_k^\beta)$  and  $\mathcal{R}(C^*) \subset \mathcal{R}((\Lambda_k^*)^\gamma)$  and  $\|\Lambda_k^{-\beta} B\| < \delta_0$  and  $\|(\Lambda_k^*)^{-\gamma} C^*\| < \delta_0$  for all  $k$ , then  $\{e^{i\varphi_k}\}_k \subset \sigma(A+BC) \setminus \sigma_p(A+BC)$ .*

*Proof.* Choose  $0 \leq \beta_1 \leq \beta$  and  $0 \leq \gamma_1 \leq \gamma$  such that  $\beta_1 + \gamma_1 = 1$ . Let  $k \in \{1, \dots, N\}$  and assume  $\|\Lambda_k^{-\beta_1} B\| < 1$  and  $\|(\Lambda_k^*)^{-\gamma_1} C^*\| < 1$ . The condition  $0 \leq \gamma_1 \leq 1$  implies  $\mathcal{R}(\Lambda_k) \subset \mathcal{R}(\Lambda_k^{\gamma_1}) \subset X$ , and thus  $\mathcal{D}(\Lambda_k^{-\gamma_1}) = X$  due to the fact that  $e^{i\varphi_k} \in \sigma_c(A)$ . The operator  $C\Lambda_k^{-\gamma_1}$  has a unique bounded extension  $C_{\gamma_1}$  with norm  $\|C_{\gamma_1}\| = \|(\Lambda_k^*)^{-\gamma_1} C^*\| < 1$ .

Since  $\|e^{i\varphi_k} \Lambda_k^{-\beta_1} B C_{\gamma_1}\| \leq \|\Lambda_k^{-\beta_1} B\| \|C_{\gamma_1}\| < 1$ , the operator  $1 - e^{i\varphi_k} \Lambda_k^{-\beta_1} B C_{\gamma_1}$  is boundedly invertible, and

$$e^{i\varphi_k} - A - BC = e^{i\varphi_k} \Lambda_k^{\beta_1} (1 - e^{-i\varphi_k} \Lambda_k^{-\beta_1} B C_{\gamma_1}) \Lambda_k^{\gamma_1}.$$

Since  $\Lambda_k^{\beta_1}$  and  $\Lambda_k^{\gamma_1}$  are injective and at least one of them is not surjective, the operator  $e^{i\varphi_k} - A - BC$  is injective but not surjective. This implies  $e^{i\varphi_k} \in \sigma(A+BC) \setminus \sigma_p(A+BC)$ .

Finally, let  $K > 0$  be such that  $\|\Lambda_k^{\beta-\beta_1}\| \leq K$  and  $\|(\Lambda_k^*)^{\gamma-\gamma_1}\| \leq K$  for all  $k$ . Then  $\|\Lambda_k^{-\beta} B\| \leq K \|\Lambda_k^{-\beta_1} B\|$  and  $\|(\Lambda_k^*)^{-\gamma} C^*\| \leq K \|(\Lambda_k^*)^{-\gamma_1} C^*\|$ . This concludes that  $\|\Lambda_k^{-\beta} B\| < 1$  and  $\|(\Lambda_k^*)^{-\gamma} C^*\| < 1$  can be achieved by choosing a small enough  $\delta_0 > 0$ .  $\square$

*Proof of Theorem 8.* Let  $\beta + \gamma \geq \alpha$  and choose  $K > 0$  such that  $\|\Lambda_k^\beta\| \leq K$  and  $\|(\Lambda_k^*)^\gamma\| \leq K$  for all  $k$ . We then have  $\|B\| \leq K \|\Lambda_k^{-\beta} B\|$  and  $\|C\| \leq K \|(\Lambda_k^*)^{-\gamma} C^*\|$ . Lemmas 6, 9, and 10 now imply that it is possible to choose  $\delta > 0$  in such a way that if

$$\|\Lambda_k^{-\beta} B\| < \delta, \quad \text{and} \quad \|(\Lambda_k^*)^{-\gamma} C^*\| < \delta,$$

for all  $k$ , then  $\|CR(\lambda, A)B\| \leq c < 1$  for every  $\lambda \notin \mathbb{D} \cup \{e^{i\varphi_k}\}_k$ , and  $\{e^{i\varphi_k}\}_k \subset \sigma(A+BC) \setminus \sigma_p(A+BC)$ . The Sherman–Morrison–Woodbury formula

$$R(\lambda, A+BC) = R(\lambda, A) \quad (8a)$$

$$+ R(\lambda, A)B(1 - CR(\lambda, A)B)^{-1}CR(\lambda, A) \quad (8b)$$

now implies that  $\sigma(A + BC) \subset \mathbb{D} \cup \{e^{i\varphi_k}\}_k$ . Moreover, a standard Neumann series argument shows that  $\|(1 - CR(\lambda, A)B)^{-1}\| \leq 1/(1 - c)$  for every  $\lambda \notin \mathbb{D} \cup \{e^{i\varphi_k}\}_k$ , which in turn concludes that (7) is satisfied.  $\square$

The following theorem characterizes uniform boundedness of a discrete semigroup on a Hilbert space [7, Thm. II.1.12].

**Theorem 11.** *Let  $A \in \mathcal{L}(X)$  on a Hilbert space  $X$  be such that  $\sigma(A) \subset \overline{\mathbb{D}}$ . The discrete semigroup  $(A^n)_{n \in \mathbb{N}}$  is bounded if and only if for all  $x, y \in X$*

$$\sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} (\|R(re^{i\varphi}, A)x\|^2 + \|R(re^{i\varphi}, A)^*y\|^2) d\varphi < \infty.$$

**Lemma 12.** *Assume that  $(A^n)_{n \in \mathbb{N}}$  is bounded. If  $Y$  is a separable Hilbert space and if  $\tilde{B} \in \mathcal{L}(Y, X)$  is a Hilbert–Schmidt operator, then*

$$\begin{aligned} \sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} \|R(re^{i\varphi}, A)\tilde{B}\|^2 d\varphi &< \infty, \\ \sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} \|R(re^{i\varphi}, A)^*\tilde{B}\|^2 d\varphi &< \infty. \end{aligned}$$

*Proof.* Let  $M = \sup_{n \in \mathbb{N}} \|A^n\|$ . As in the proof of [10, Thm. 3.11] and in [10, Rem. 3.2] the Parseval’s equality shows that for every  $1 < r \leq 2$  and  $x \in X$

$$\begin{aligned} (r-1) \int_0^{2\pi} \|R(re^{i\varphi}, A)x\|^2 d\varphi &= 2\pi \frac{r-1}{r^2} \sum_{n=0}^{\infty} \frac{\|A^n x\|^2}{r^{2n}} \\ &\leq 2\pi M^2 \|x\|^2 \frac{r-1}{r^2} \sum_{n=0}^{\infty} \frac{1}{r^{2n}} = \frac{2\pi M^2 \|x\|^2}{r+1} \leq \pi M^2 \|x\|^2. \end{aligned}$$

If  $(e_l)_{l=1}^{\infty} \subset Y$  is an orthonormal basis of  $Y$ , then

$$\sum_{l=1}^{\infty} \sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} \|R(re^{i\varphi}, A)\tilde{B}e_l\|^2 d\varphi \leq \pi M^2 \sum_{l=1}^{\infty} \|\tilde{B}e_l\|^2.$$

For every  $R \in \mathcal{L}(X)$  we have  $\|R\tilde{B}\|^2 \leq \sum_{l=1}^{\infty} \|R\tilde{B}e_l\|^2$ . Combining these properties we get

$$\begin{aligned} \sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} \|R(re^{i\varphi}, A)\tilde{B}\|^2 d\varphi \\ \leq \sum_{l=1}^{\infty} \sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} \|R(re^{i\varphi}, A)\tilde{B}e_l\|^2 d\varphi \\ \leq \pi M^2 \sum_{l=1}^{\infty} \|\tilde{B}e_l\|^2 < \infty. \end{aligned}$$

The second claim can be shown analogously.  $\square$

**Lemma 13.** *Let Assumption 1 be satisfied for some  $\alpha \geq 1$ , let  $\beta, \gamma \geq 0$  be such that  $\beta + \gamma \geq \alpha$ , and let  $k \in \{1, \dots, N\}$ . If  $B \in \mathcal{L}(Y, X)$  and  $C \in \mathcal{L}(X, Y)$  satisfy the conditions (2) and (3), then there exists a function  $f_k : \mathbb{C} \setminus (\mathbb{D} \cup \{e^{i\varphi_l}\}_{l=1}^N) \rightarrow \mathbb{R}^+$  such that*

$$\|R(\lambda, A)B\| \|CR(\lambda, A)\| \leq f_k(\lambda) \quad \forall \lambda \in \Omega_k,$$

and  $f_k(\cdot)$  has the properties  $\sup_{0 < |\varphi - \varphi_k| \leq \varepsilon_A} |\varphi - \varphi_k|^\alpha f_k(e^{i\varphi}) < \infty$  and

$$\sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} f_k(re^{i\varphi})^2 d\varphi < \infty. \quad (9)$$

*Proof.* We begin by considering the case where  $\beta > 0$  and  $\gamma > 0$ . Choose  $0 < \beta_1 \leq \beta$  and  $0 < \gamma_1 \leq \gamma$  such that  $\beta_1 + \gamma_1 = \alpha$ . For brevity, denote  $R_\lambda = R(\lambda, A)$  and  $\lambda_k = \lambda - e^{i\varphi_k}$ . Moreover, denote  $B_{\beta_1} = \Lambda_k^{-\beta_1} B \in \mathcal{L}(Y, X)$  and  $\tilde{C}_{\gamma_1} = (\Lambda_k^*)^{-\gamma_1} C^* \in \mathcal{L}(Y, X)$ . Since  $B_{\beta_1} = \Lambda_k^{\beta-\beta_1} \Lambda_k^{-\beta} B$  and  $\tilde{C}_{\gamma_1} = (\Lambda_k^*)^{\gamma-\gamma_1} (\Lambda_k^*)^{-\gamma} C^*$ , condition (3) implies that also  $B_{\beta_1}$  and  $\tilde{C}_{\gamma_1}$  are Hilbert–Schmidt operators.

Let  $M_1 \geq 1$  be as in Theorem 3. By Lemma 4 there exist constants  $M_{\beta_1}, M_{\gamma_1} \geq 1$  such that for every  $\lambda \in \Omega_k$

$$\begin{aligned} \|R_\lambda B\| &= \|\Lambda_k^{\beta_1} R_\lambda B_{\beta_1}\| \\ &\leq M_{\beta_1} \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|\Lambda_k^\alpha R_\lambda B_{\beta_1}\|^{\beta_1/\alpha} \\ &\leq M_{\beta_1} \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|\Lambda_k^\alpha R_\lambda\|^{\beta_1/\alpha} \|B_{\beta_1}\|^{\beta_1/\alpha} \\ \|R_\lambda^* C^*\| &= \|(\Lambda_k^*)^{\gamma_1} R_\lambda^* \tilde{C}_{\gamma_1}\| \\ &\leq M_{\gamma_1} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \|(\Lambda_k^*)^\alpha R_\lambda^* \tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha} \\ &\leq M_{\gamma_1} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \|R_\lambda \Lambda_k^\alpha\|^{\gamma_1/\alpha} \|\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha}. \end{aligned}$$

Thus for  $K = M_{\beta_1} M_{\gamma_1} M_1 \|B_{\beta_1}\|^{\beta_1/\alpha} \|\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha}$  we have

$$\|R_\lambda B\| \|CR_\lambda\| \leq K \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}.$$

Define  $f_k(\cdot)$  by  $f_k(\lambda) = K \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}$  for all  $\lambda \in \mathbb{C} \setminus (\mathbb{D} \cup \{e^{i\varphi_l}\}_{l=1}^N)$ . We will now show that  $f_k(\cdot)$  has the desired properties.

Since  $1 - \beta_1/\alpha + 1 - \gamma_1/\alpha = 1$ , for all  $\varphi \in [0, 2\pi]$  with  $0 < |\varphi - \varphi_k| \leq \varepsilon_A$  we have from Assumption 1 that

$$\begin{aligned} |\varphi - \varphi_k|^\alpha f_k(e^{i\varphi}) \\ \leq |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A)\| K \|B_{\beta_1}\|^{1-\beta_1/\alpha} \|\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \\ \leq M_A K \|B_{\beta_1}\|^{1-\beta_1/\alpha} \|\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}. \end{aligned}$$

This concludes that  $\sup_{0 < |\varphi - \varphi_k| \leq \varepsilon_A} |\varphi - \varphi_k|^\alpha f_k(e^{i\varphi}) < \infty$ . Moreover, if we denote  $q = 1/(1 - \beta_1/\alpha)$ ,  $q' = 1/(1 - \gamma_1/\alpha)$ , then  $1/q + 1/q' = 1$  and the Hölder inequality implies

$$\begin{aligned} \int_0^{2\pi} f_k(re^{i\varphi})^2 d\varphi \\ = K^2 \int_0^{2\pi} \|R(re^{i\varphi}, A)B_{\beta_1}\|^{\frac{2}{q}} \|R(re^{i\varphi}, A)^* \tilde{C}_{\gamma_1}\|^{\frac{2}{q'}} d\varphi \\ \leq K^2 \left( \int_0^{2\pi} \|R(re^{i\varphi}, A)B_{\beta_1}\|^2 d\varphi \right)^{\frac{1}{q}} \\ \times \left( \int_0^{2\pi} \|R(re^{i\varphi}, A)^* \tilde{C}_{\gamma_1}\|^2 d\varphi \right)^{\frac{1}{q'}}. \end{aligned}$$

The property (9) now follows from Lemma 12 since  $B_{\beta_1}$  and  $\tilde{C}_{\gamma_1}$  are Hilbert–Schmidt.

It remains to show that the claims are true if  $\beta = 0$  or  $\gamma = 0$ , or equivalently, whenever either  $\gamma \geq \alpha$  or  $\beta \geq \alpha$ .

Let  $M_1 \geq 1$  be as in Theorem 3. If  $\beta \geq \alpha$ , then  $\mathcal{R}(B) \subset \mathcal{R}((1 - e^{-i\varphi_k} A)^\alpha)$ . Choose  $K_0 > 0$  so that  $\|\Lambda_l^{\beta-\alpha}\| \leq K_0$  for all  $l \in \{1, \dots, N\}$ . For all  $\lambda \in \Omega_k$

$$\begin{aligned} \|R_\lambda B\| \|CR_\lambda\| &\leq \|R_\lambda \Lambda_k^\alpha\| \|\Lambda_k^{\beta-\alpha}\| \|\Lambda_k^{-\beta} B\| \|R_\lambda^* C^*\| \\ &\leq K \|R_\lambda^* C^*\|, \end{aligned}$$

where  $K = M_1 K_0 M_\Lambda \delta$ . We choose  $f_k(\cdot)$  such that  $f_k(\lambda) = K \|R_\lambda^* C^*\|$  for all  $\lambda \in \mathbb{C} \setminus (\mathbb{D} \cup \{e^{i\varphi_l}\}_{l=1}^N)$ . Then  $|\varphi - \varphi_k|^\alpha f_k(e^{i\varphi}) \leq |\varphi - \varphi_k|^\alpha K \|R(e^{i\varphi}, A)\| \|C\| \leq K M_A \|C\|$  whenever  $0 < |\varphi - \varphi_k| \leq \varepsilon_A$ , and

$$\int_0^{2\pi} f_k(re^{i\varphi})^2 d\varphi \leq K^2 \int_0^{2\pi} \|R(re^{i\varphi}, A)^* C^*\|^2 d\varphi.$$

Since  $C^*$  is Hilbert–Schmidt, Lemma 12 shows that (9) is satisfied. The case with  $\beta = 0$  and  $\gamma \geq \alpha$  can be handled analogously.  $\square$

*Proof of Theorem 2.* Let  $\delta > 0$  be chosen as in Theorem 8 and assume  $\|\Lambda_k^{-\beta} B\| < \delta$ , and  $\|(\Lambda_k^*)^{-\gamma} C^*\| < \delta$  for all  $k$ . By Theorem 8 there exists  $M_D \geq 1$  such that  $\|(1 - CR(\lambda, A)B)^{-1}\| \leq M_D$  for all  $\lambda \notin \mathbb{D} \cup \{e^{i\varphi_k}\}_{k=1}^N$ . We begin the proof by showing that the semigroup  $((A + BC)^n)_{n \in \mathbb{N}}$  is bounded.

Let  $x \in X$  and for brevity denote  $R_\lambda = R(re^{i\varphi}, A)$  and  $D_\lambda = 1 - CR(re^{i\varphi}, A)B$ . Using the Sherman–Morrison–Woodbury formula (8) and the scalar inequality  $(a+b)^2 \leq 2(a^2 + b^2)$  for  $a, b \geq 0$  we can estimate

$$\begin{aligned} &\int_0^{2\pi} \|R(re^{i\varphi}, A + BC)x\|^2 d\varphi \\ &= \int_0^{2\pi} \|R_\lambda x + R_\lambda B D_\lambda^{-1} C R_\lambda x\|^2 d\varphi \\ &\leq 2 \int_0^{2\pi} \|R_\lambda x\|^2 d\varphi + 2M_D^2 \|x\|^2 \int_0^{2\pi} \|R_\lambda B\|^2 \|CR_\lambda\|^2 d\varphi. \end{aligned}$$

Similarly, using  $\|(R_\lambda B D_\lambda^{-1} C R_\lambda)^*\| = \|R_\lambda B D_\lambda^{-1} C R_\lambda\| \leq M_D \|R_\lambda B\| \|CR_\lambda\|$  we get

$$\begin{aligned} &\int_0^{2\pi} \|R(re^{i\varphi}, A + BC)^* x\|^2 d\varphi \\ &= \int_0^{2\pi} \|R_\lambda^* x + (R_\lambda B D_\lambda^{-1} C R_\lambda)^* x\|^2 d\varphi \\ &\leq 2 \int_0^{2\pi} \|R_\lambda^* x\|^2 d\varphi + 2M_D^2 \|x\|^2 \int_0^{2\pi} \|R_\lambda B\|^2 \|CR_\lambda\|^2 d\varphi. \end{aligned}$$

The above estimates together with Theorem 11 imply that the semigroup  $((A + BC)^n)_{n \in \mathbb{N}}$  is uniformly bounded if

$$\sup_{1 < r \leq 2} (r-1) \int_0^{2\pi} \|R_\lambda B\|^2 \|CR_\lambda\|^2 d\varphi < \infty. \quad (10)$$

For all  $k \in \{1, \dots, N\}$  let  $f_k(\cdot)$  be the functions in Lemma 13. By Lemma 6 we can choose  $M_2 \geq 1$  such that  $\|R(\lambda, A)\| \leq M_2$  for all  $\lambda \notin \mathbb{D} \cup \bigcup_k \Omega_k$ . Let  $1 < r \leq 2$ . For each  $k \in \{1, \dots, N\}$  denote by  $E_k^r \subset [0, 2\pi]$  the interval

such that  $re^{i\varphi} \in \Omega_k$  if and only if  $\varphi \in E_k^r$ . Finally, denote  $E^r = [0, 2\pi] \setminus (\bigcup_k E_k^r)$ . Now

$$\begin{aligned} &\int_0^{2\pi} \|R_\lambda B\|^2 \|CR_\lambda\|^2 d\varphi \\ &= \int_{E^r} \|R_\lambda B\|^2 \|CR_\lambda\|^2 d\varphi + \sum_{k=1}^N \int_{E_k^r} \|R_\lambda B\|^2 \|CR_\lambda\|^2 d\varphi \\ &\leq \int_{E^r} M_2^2 \|B\|^2 \|C\|^2 M_2^2 d\varphi + \sum_{k=1}^N \int_{E_k^r} f_k(re^{i\varphi})^2 d\varphi \\ &\leq 2\pi M_2^4 \|B\|^2 \|C\|^2 + \sum_{k=1}^N \int_0^{2\pi} f_k(re^{i\varphi})^2 d\varphi, \end{aligned}$$

which immediately implies (10) by Lemmas 12 and 13, and therefore  $((A + BC)^n)_{n \in \mathbb{N}}$  is bounded.

Since the perturbed semigroup is bounded and  $X$  is a Hilbert space, Theorem I.2.9 and Corollary I.2.11 in [7] imply that  $\sigma(A + BC) \cap \mathbb{T} \subset \sigma_p(A + BC) \cup \sigma_c(A + BC)$ . However, by Theorem 8 we have that  $\{e^{i\varphi_k}\}_k \subset \sigma(A + BC) \setminus \sigma_p(A + BC)$ . Together these properties conclude that  $e^{i\varphi_k} \in \sigma_c(A + BC)$  for all  $k$ .

Theorem 8 shows that  $\sigma(A + BC) \cap \mathbb{T} = \{e^{i\varphi_k}\}_{k=1}^N$  is finite and  $\sigma_p(A + BC) \cap \mathbb{T} = \emptyset$ . The discrete Arendt–Batty–Lyubich–Vũ Theorem [7, Thm. II.2.18] thus concludes that the semigroup  $((A + BC)^n)_{n \in \mathbb{N}}$  is strongly stable.

It remains to show that for all  $k$  we have

$$\sup_{0 < |\varphi - \varphi_k| \leq \varepsilon_A} |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A + BC)\| < \infty. \quad (11)$$

Let  $k$  be arbitrary. By Lemma 13 there exists  $M_k \geq 1$  such that  $|\varphi - \varphi_k|^\alpha f_k(e^{i\varphi}) \leq M_k$  whenever  $0 < |\varphi - \varphi_k| \leq \varepsilon_A$ . The Sherman–Morrison–Woodbury formula (8) implies that for all  $\varphi \in [0, 2\pi]$  satisfying  $0 < |\varphi - \varphi_k| \leq \varepsilon_A$  we have

$$\begin{aligned} &\|R(e^{i\varphi}, A + BC)\| \\ &\leq \|R(e^{i\varphi}, A)\| + \|R(e^{i\varphi}, A)B\| M_D \|CR(e^{i\varphi}, A)\| \\ &\leq \|R(e^{i\varphi}, A)\| + M_D f_k(e^{i\varphi}), \end{aligned}$$

and thus

$$\begin{aligned} &|\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A + BC)\| \\ &\leq |\varphi - \varphi_k|^\alpha \|R(e^{i\varphi}, A)\| + M_D |\varphi - \varphi_k|^\alpha f_k(e^{i\varphi}) \\ &\leq M_A + M_D M_k. \end{aligned}$$

This concludes that (11) is satisfied.  $\square$

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