Designing Controllers with Reduced Order Internal Models

Lassi Paunonen

In this paper we study robust output tracking for autonomous linear systems. We introduce a new approach to designing robust controllers using a recent observation that a full internal model is not always necessary for robustness. Especially this may be the case if the control law is only required to be robust with respect to a specific predetermined class of uncertainties in the parameters of the plant. The results are illustrated with an example on robust output tracking for coupled harmonic oscillators.

***Index Terms*—Output tracking, linear systems, robustness.**

I. Introduction

In this paper we design a controller to achieve robust output tracking for an autonomous linear system. This control objective (also known as the robust output regulation problem) has been studied extensively for both finite-dimensional [1], [2], [3] and infinite-dimensional plants [4], [5], [6], [7], [8], [9], [10]. In particular, it is well-known that to achieve robustness with respect to perturbations in the parameters of the plant it is both necessary and sufficient that the controller incorporates a so-called p-copy internal model [1], [2], [10]. In this paper we design controllers using a new approach, which arises from a recent observation that if the class of admissible perturbations is restricted, robustness may be achievable without a full p-copy internal model in the controller [11]. Instead, in such situations a smaller internal model may be sufficient.

The results in this paper are presented for infinite-dimensional linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

on a Banach space X. This class of systems can be used in modelling the behaviour of, in particular, heat and diffusion processes, deformations and vibrations, and time delays. Moreover, the class also includes all autonomous finite-dimensional systems, as they can be studied by choosing a state space $X = \mathbb{C}^n$ or $X = \mathbb{R}^n$, and letting A, B, C, and D be matrices of appropriate sizes. The results presented in this paper — in particular the method for constructing the robust controller — are also new for finite-dimensional linear systems. Because of this, we make periodic remarks on the terminology and on how the main results are represented for finite-dimensional systems.

In our main control problem, the controller is to be chosen in such a way that the output $y(\cdot)$ of (1) asymptotically tracks a given reference signal $y_{ref}(\cdot)$, i.e., $\|y(t) - y_{ref}(t)\| \to 0$ as $t \to \infty$. Moreover, for a predetermined class $\mathcal{O}$ of perturbations of the plant (1), the control law is required to be robust with respect to all perturbations $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ for which the closed-loop stability is preserved. The control problem is a modification of the standard robust output regulation problem [3], [9], where the class of considered perturbations consists of all $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ for which the perturbed closed-loop system is stable. The main contribution of this paper is a method for solving the robust output tracking problem for a given class $\mathcal{O}$ of perturbations using an observer-based controller that incorporates a reduced order internal model.

The robust output regulation problem for a specific class $\mathcal{O}$ of perturbations was studied recently in [11]. The results in [11] concentrated on the properties of controllers that solve the control problem, while methods for designing controllers were not discussed. The purpose of this paper is to use the results in [11] to construct a controller that solves the robust output regulation problem for a given class $\mathcal{O}$ of perturbations.

The construction of the controller is completed in two parts. In the first part we introduce the general controller structure, and design a reduced order internal model that guarantees robustness with respect to the class $\mathcal{O}$ of perturbations. In the second part the construction is completed by fixing the free parameters of the controller to achieve closed-loop stability.

The solution of the robust output tracking problem with respect to perturbations in $\mathcal{O}$ is not achievable by the controller structure used in, for example, [3], [9], [12]. Instead, we introduce a new controller structure that makes the inclusion of a reduced order internal model possible. The new controller can also be used in solving the standard robust output regulation problem. In particular, if $\mathcal{O}$ is chosen to consist of all possible perturbations to the plant (1), our reduced order internal model becomes the full p-copy internal model. In these regards, the theory presented in this paper extends the results in [3], [9], [12].

The results are illustrated with an example on control of two interconnected driven harmonic oscillators. The controller is designed to achieve output tracking of a sinusoidal signal and to be robust with respect to small variations in the damping coefficients of the harmonic oscillators.

The paper is organized as follows. In Section II we state the fundamental assumptions on the plant, the reference signals, the controller, and the closed-loop system. In Section III we formulate our main control objective, and recall the characterization of controllers solving the problem. The controller is constructed in Section IV, where we design the reduced order internal model, and in Section V where the closed-loop is stabilized. In Section VI we control a system of two harmonic oscillators. Section VII contains concluding remarks.

If X and Y are Banach spaces and $A : X \to Y$ is a linear

Tampere University of Technology, PO.Box 553, 33101 Tampere, Finland, lassi.paunonen@tut.fi

operator, then $\mathcal{D}(A)$ and $\mathcal{N}(A)$ denote the domain and the null space of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. (If $X=\mathbb{C}^n$ and $Y=\mathbb{C}^m$, then $\mathcal{L}(X,Y)=\mathbb{C}^{m\times n}$.) If $A:X\to X$, then $\sigma(A)$ and $\rho(A)=\mathbb{C}\setminus\sigma(A)$ are its spectrum (the set of eigenvalues, if $A\in\mathbb{C}^{n\times n}$) and its resolvent set, respectively. For $\lambda\in\rho(A)$ the resolvent operator is given by $R(\lambda,A)=(\lambda I-A)^{-1}$. The inner product is denoted by $\langle\cdot,\cdot\rangle$.

If the plant is finite-dimensional, the assumption "A generates a strongly continuous semigroup $T(t)$ on X" is always satisfied, and the semigroup is precisely the matrix exponential function, i.e., $T(t)=e^{At}$ for all $t\geq 0$ [13]. The exponential stability of the semigroup corresponds to the asymptotic stability of the matrix exponential function e^{At}, and e^{At} is stable if and only if $\sigma(A)\subset\mathbb{C}^-$ (i.e., if the matrix A is *Hurwitz*).

II. Assumptions on the Plant and the Controller

The operators of the plant (1) on a Banach space X are such that $A:\mathcal{D}(A)\subset X\to X$ generates a strongly continuous semigroup $T(t)$ on X [13]. The input, output, and feedthrough operators satisfy $B\in\mathcal{L}(U,X)$, $C\in\mathcal{L}(X,Y)$, and $D\in\mathcal{L}(U,Y)$, where $U=Y=\mathbb{C}^p$ are the input and output spaces (in particular, the plant is assumed to have an equal number of inputs and outputs). We assume that the pair (A,B) is stabilizable and the pair (C,A) is detectable. The transfer function of the plant is denoted by

$$P(\lambda)=CR(\lambda,A)B+D,\qquad \lambda\in\rho(A). \tag{2}$$

If the system (1) is finite-dimensional, then the state space is $X=\mathbb{C}^n$ for some $n\in\mathbb{N}$, $A\in\mathbb{C}^{n\times n}$ is a square matrix, and the input, output, and feedthrough matrices satisfy $B\in\mathbb{C}^{n\times p}$, $C\in\mathbb{C}^{p\times n}$, and $D\in\mathbb{C}^{p\times p}$. The transfer function is defined by (2) whenever $\lambda\in\mathbb{C}$ is not an eigenvalue of A.

As is customary in the study of the output regulation problem, instead of a single reference signal $y_{\mathit{ref}}(\cdot)$ we consider a class of reference signals generated by an exosystem

$$\dot v(t)=Sv(t),\qquad v(0)=v_0\in W \tag{3a}$$
$$y_{\mathit{ref}}(t)=Fv(t) \tag{3b}$$

on the space $W=\mathbb{C}^q$, where $S=\operatorname{diag}(i\omega_1,\ldots,i\omega_q)$, $(\omega_k)_{k=1}^q\subset\mathbb{R}$ are distinct frequencies, and $F\in\mathcal{L}(W,Y)=\mathbb{C}^{p\times q}$. The signals generated by (3) are of the form $y_{\mathit{ref}}(t)=\sum_{k=1}^q e^{i\omega_k t}\langle v_0,e_k\rangle Fe_k$, where $\{e_k\}_{k=1}^q$ are the Euclidean basis vectors. We assume that $i\omega_k\in\rho(A)$ and that $P(i\omega_k)$ is invertible for every $k\in\{1,\ldots,q\}$.

The regulation error is defined as $e(t)=y(t)-y_{\mathit{ref}}(t)$. We consider a dynamic error feedback controller

$$\dot z(t)=\mathcal{G}_1z(t)+\mathcal{G}_2e(t),\qquad z(0)=z_0\in Z \tag{4a}$$
$$u(t)=Kz(t) \tag{4b}$$

on a Banach space Z, where $\mathcal{G}_1:\mathcal{D}(\mathcal{G}_1)\subset Z\to Z$ generates a strongly continuous semigroup, $\mathcal{G}_2\in\mathcal{L}(Y,Z)$, and $K\in\mathcal{L}(Z,U)$. If the controller is finite-dimensional, then its state space is $Z=\mathbb{C}^r$ for some $r\in\mathbb{N}$, and the matrices $\mathcal{G}_1$, $\mathcal{G}_2$, and K have dimensions $\mathcal{G}_1\in\mathbb{C}^{r\times r}$, $\mathcal{G}_2\in\mathbb{C}^{r\times p}$, and $K\in\mathbb{C}^{p\times r}$.

The plant and the controller can be written together as a closed-loop system on the product space $X_e=X\times Z$ as

$$\dot x_e(t)=A_ex_e(t)+B_ev(t),\qquad x_e(0)=x_{e0}=\left[\begin{smallmatrix}x_0\\ z_0\end{smallmatrix}\right] \tag{5a}$$
$$e(t)=C_ex_e(t)+D_ev(t), \tag{5b}$$

where $C_e=[C\quad DK]$, $D_e=-F$,

$$A_e=\begin{bmatrix}A & BK\\ \mathcal{G}_2C & \mathcal{G}_1+\mathcal{G}_2DK\end{bmatrix}\quad\text{and}\quad B_e=\begin{bmatrix}0\\ -\mathcal{G}_2F\end{bmatrix}.$$

Due to our assumptions, the operator A_e generates a strongly continuous semigroup $T_e(t)$ on X_e. If the plant (1) and the controller are both finite-dimensional systems, also the closed-loop system is finite-dimensional and $X_e=\mathbb{C}^{n+r}$. The closed-loop system is then stable precisely if $\sigma(A_e)\subset\mathbb{C}^-$.

III. The Robust Output Regulation Problem

In this section we define the main control problem studied in this paper. We consider situations where the operators (or matrices, if (1) is finite-dimensional) A, B, C, and D are perturbed to $\tilde A:\mathcal{D}(\tilde A)\subset X\to X$, $\tilde B\in\mathcal{L}(U,X)$, $\tilde C\in\mathcal{L}(X,Y)$, and $\tilde D\in\mathcal{L}(U,Y)$, respectively. For every $\lambda\in\rho(\tilde A)$ we denote the transfer function of the perturbed plant by $\tilde P(\lambda)=\tilde CR(\lambda,\tilde A)\tilde B+\tilde D$. Moreover, the parameters of the closed-loop system consisting of the perturbed plant and the controller are denoted by $\tilde C_e=\begin{bmatrix}\tilde C & \tilde DK\end{bmatrix}$, and

$$\tilde A_e=\begin{bmatrix}\tilde A & \tilde BK\\ \mathcal{G}_2\tilde C & \mathcal{G}_1+\mathcal{G}_2\tilde DK\end{bmatrix}.$$

The perturbations in A, B, C, and D do not affect B_e or D_e.

Definition 1. The class of all considered perturbations of (A,B,C,D) is denoted by $\mathcal{O}$. The class has the property $(A,B,C,D)\in\mathcal{O}$, and all $(\tilde A,\tilde B,\tilde C,\tilde D)\in\mathcal{O}$ satisfy the following conditions.

(a) $\tilde A$ generates a strongly continuous semigroup on X.
(b) For every $k\in\{1,\ldots,q\}$ we have that $i\omega_k\in\rho(\tilde A)$ and $\tilde P(i\omega_k)$ is invertible.

If the plant (1) is finite-dimensional, condition (a) in Definition 1 is trivially satisfied for any perturbation of the matrix A. Moreover, if a controller stabilizes the unperturbed closed-loop system exponentially, then the conditions of Definition 1 are satisfied and the closed-loop system remains stable for any bounded perturbations with sufficiently small norms.

The main control problem studied in this paper is defined in the following.

The Robust Output Regulation Problem for $\mathcal{O}$. Choose $(\mathcal{G}_1,\mathcal{G}_2,K)$ in such a way that the following are satisfied:

1. The closed-loop system is stable, i.e., the semigroup $T_e(t)$ generated by A_e is exponentially stable.
2. For all initial states $v_0\in W$ and $x_{e0}\in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty}e(t)=0$.
3. If (A,B,C,D) are perturbed to $(\tilde A,\tilde B,\tilde C,\tilde D)\in\mathcal{O}$ in such a way that the perturbed closed-loop system is exponentially stable, then $\lim_{t\to\infty}e(t)=0$ for all $v_0\in W$ and $x_{e0}\in X_e$.

The largest possible class $\mathcal{O}$ contains all the perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ that satisfy the two conditions in Definition 1. On the other hand, the problem statement with $\mathcal{O}$ allows the study of perturbations of very particular types. This is illustrated in Section VI. Finally, in the other extreme case, the class $\mathcal{O}$ can be chosen to only contain the operators (A,B,C,D) of the nominal plant. The control problem then reduces to output tracking without the robustness requirement.

The following theorem presented in [11, Thm. 4] characterizes the controllers solving the robust output regulation problem for the class $\mathcal{O}$ of perturbations.

Theorem 2. *A controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ stabilizing the closed-loop system solves the robust output regulation problem for $\mathcal{O}$ if and only if for every $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})\in\mathcal{O}$ for which the perturbed closed-loop system is stable the equations*

$$\tilde{P}(i\omega_k)Kz^k = -Fe_k \tag{6a}$$

$$(i\omega_k I-\mathcal{G}_1)z^k = 0 \tag{6b}$$

have solutions $z^k\in\mathcal{D}(\mathcal{G}_1)$ for all $k\in\{1,\dots,q\}$.

IV. The Controller with a Reduced Order Internal Model

In this section we introduce a controller structure that is suitable for solving the robust output regulation problem for the class $\mathcal{O}$ of perturbations. The internal model principle states that a stabilizing controller is robust with respect to *all* perturbations of (A,B,C,D) (as long as closed-loop stability is preserved) if the internal model contains $p=\dim Y$ copies of every frequency $i\omega_k$ of the exosystem. However, if $\mathcal{O}$ is a smaller class, a smaller number of copies of some of the frequencies $i\omega_k$ may already be enough to guarantee robustness with respect to perturbations in $\mathcal{O}$.

As the state space of the controller we choose $Z=Z_1\times Z_2$, where $Z_1=\mathbb{C}^{nz}$ and Z_2 is a Banach space. The parameters $(\mathcal{G}_1,\mathcal{G}_2,K)$ of the controller are of the form

$$\mathcal{G}_1=\begin{bmatrix}G_1 & R_1\\ 0 & R_2\end{bmatrix},\qquad \mathcal{G}_2=\begin{bmatrix}G_2\\ R_3\end{bmatrix},\quad K=\begin{bmatrix}K_1, & -K_2\end{bmatrix},$$

where $G_1\in\mathcal{L}(Z_1)$, $R_1\in\mathcal{L}(Z_2,Z_1)$, $R_2:\mathcal{D}(R_2)\subset Z_2\to Z_2$ generates a semigroup on Z_2, $G_2\in\mathcal{L}(Y,Z_1)$, $R_3\in\mathcal{L}(Y,Z_2)$, $K_1\in\mathcal{L}(Z_1,U)$, and $K_2\in\mathcal{L}(Z_2,U)$.

The matrix G_1 is called the *internal model* of the exosystem (3), and it will be chosen to contain a suitable number of copies of the exosystem's frequencies.

For every $k\in\{1,\dots,q\}$ define

$$\mathcal{S}_k=\operatorname{span}\left\{\,\tilde{P}(i\omega_k)^{-1}Fe_k \;\middle|\; (\tilde{A},\tilde{B},\tilde{C},\tilde{D})\in\mathcal{O}\,\right\}\subset U$$

and $p_k=\dim\mathcal{S}_k$. We will show that the robust output regulation problem for $\mathcal{O}$ can be solved with an internal model G_1 containing exactly p_k copies of every frequency $i\omega_k$ of the exosystem. In particular, we choose G_1 to be a diagonal matrix

$$G_1=\begin{bmatrix}i\omega_1 I_{p_1} & & \\ & \ddots & \\ & & i\omega_q I_{p_q}\end{bmatrix},$$

where the sizes of the diagonal blocks satisfy $i\omega_k I_{p_k}\in\mathbb{C}^{p_k\times p_k}$. Since the frequencies $i\omega_k$ are distinct, we immediately have $\dim\mathcal{N}(i\omega_k I-G_1)=p_k$, which together with the structure of $\mathcal{G}_1$ further implies $\dim\mathcal{N}(i\omega_k I-\mathcal{G}_1)\geq p_k$. We have $G_1\in\mathbb{C}^{n_Z\times n_Z}$, where $n_Z=p_1+p_2+\cdots+p_q$. Label the Euclidean basis vectors as $(e_1,e_2,\dots,e_{n_Z})=(\varphi_1^1,\dots,\varphi_{p_1}^1,\varphi_1^2,\dots,\varphi_{p_2}^2,\dots,\varphi_1^q,\dots,\varphi_{p_q}^q)$. Then for every $k\in\{1,\dots,q\}$ the vectors $\{\varphi_1^k,\dots,\varphi_{p_k}^k\}$ form an orthonormal basis of $\mathcal{N}(i\omega_k I-G_1)$.

Theorem 3. *For every $k\in\{1,\dots,q\}$ let $\{u_1^k,\dots,u_{p_k}^k\}\subset\mathcal{S}_k$ be a basis of $\mathcal{S}_k$. If the part K_1 in K is chosen as*

$$K_1=\sum_{k=1}^{q}\sum_{l=1}^{p_k}\langle\cdot,\varphi_l^k\rangle u_l^k, \tag{7}$$

then for any perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})\in\mathcal{O}$ and for all $k\in\{1,\dots,q\}$ the equations (6) *have a solution $z^k\in\mathcal{D}(\mathcal{G}_1)$.*

Proof. Let $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})\in\mathcal{O}$ and $k\in\{1,\dots,q\}$ be arbitrary. Since $-\tilde{P}(i\omega_k)^{-1}Fe_k\in\mathcal{S}_k$ and since $\{u_1^k,\dots,u_{p_k}^k\}$ is a basis of $\mathcal{S}_k$, there exist $\{\alpha_l^k\}_{l=1}^{p_k}\subset\mathbb{C}$ such that

$$-\tilde{P}(i\omega_k)^{-1}Fe_k=\sum_{l=1}^{p_k}\alpha_l^k u_l^k.$$

If we choose $z^k=(z_1^k,0)^T\in Z$, where $z_1^k=\sum_{l=1}^{p_k}\alpha_l^k\varphi_l^k\in Z_1$, then a direct computation yields

$$\begin{aligned}\tilde{P}(i\omega_k)Kz^k&=\tilde{P}(i\omega_k)K_1z_1^k=\tilde{P}(i\omega_k)\sum_{k'=1}^{q}\sum_{l=1}^{p_{k'}}\langle z_1^k,\varphi_l^{k'}\rangle u_l^{k'}\\&=\tilde{P}(i\omega_k)\sum_{l=1}^{p_k}\alpha_l^k u_l^k=\tilde{P}(i\omega_k)(-\tilde{P}(i\omega_k)^{-1}Fe_k)=-Fe_k,\end{aligned}$$

and thus $z^k\in\mathcal{D}(\mathcal{G}_1)$ is a solution of (6a). Furthermore, we have

$$\begin{aligned}(i\omega_k I-\mathcal{G}_1)z^k&=\begin{bmatrix}i\omega_k I-G_1 & -R_1\\ 0 & i\omega_k I-R_2\end{bmatrix}\begin{bmatrix}z_1^k\\ 0\end{bmatrix}\\&=\begin{bmatrix}\sum_{l=1}^{p_k}\alpha_l^k(i\omega_k I-G_1)\varphi_l^k\\ 0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\end{aligned}$$

since $\{\varphi_l^k\}_{l=1}^{p_k}\subset\mathcal{N}(i\omega_k I-G_1)$. This shows that z^k also satisfies (6b). Since $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})\in\mathcal{O}$ and $k\in\{1,\dots,q\}$ were arbitrary, the proof is complete. □

Remark 4. If $\mathcal{S}_k=U$ and $p_k=\dim\mathcal{S}_k=p$ for every $k\in\{1,\dots,q\}$, the internal model G_1 becomes the full p-copy internal model $G_1=\operatorname{block\,diag}(i\omega_1 I_p,\dots,i\omega_q I_p)$ where $I_p\in\mathbb{C}^{p\times p}$. In this case the bases $\{u_l^k\}_{l=1}^p$ can be chosen to consist of the Euclidean basis vectors, i.e., $u_l^k=e_l$ for every $k\in\{1,\dots,q\}$ and $l\in\{1,\dots,p\}$. The operator K_1 in (7) is then equal to $K_1=\begin{bmatrix}I_p & I_p & \dots & I_p\end{bmatrix}\in\mathbb{C}^{p\times pq}$. All the results presented in this paper remain valid also in this situation. Finally, if the class $\mathcal{O}$ of perturbations consists of all the perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ satisfying the two conditions in Definition 1, a full p-copy internal model is necessary for robustness. This follows from [11, Thm. 7] and the fact that the conditions of Definition 1 are satisfied and the closed-loop stability is preserved for all sufficiently small perturbations in D.

V. Stabilization of the Closed-Loop System

The following theorem fixes the remaining parameters of the controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ in such a way that the closed-loop system is stabilized exponentially.

Theorem 5. *Define $Z = Z_1 \times X$ and consider the controller* (4) *with*

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & G_2(C+DK_2) \\ 0 & A+BK_2+L(C+DK_2)\end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ L\end{bmatrix},$$

and $K = \begin{bmatrix} K_1 & -K_2\end{bmatrix}$, where the operators G_1, G_2, K_1, K_2, and L are chosen as follows.

(a) *G_1 and K_1 are as in Section* IV.
(b) *$K_2 \in \mathcal{L}(X,U)$ and $L_1 \in \mathcal{L}(Y,X)$ are chosen in such a way that the semigroups generated by $A+BK_2$ and $A+L_1C$ are exponentially stable.*
(c) *$H_{e1} \in \mathcal{L}(Z_1,X)$ is defined by*

$$H_{e1} = \sum_{k=1}^{q}\sum_{l=1}^{p_k} \langle \cdot, \varphi_l^k\rangle R(i\omega_k, A+L_1C)(B+L_1D)K_1\varphi_l^k$$

and it is the (unique) solution of the Sylvester equation

$$H_{e1}G_1 = (A+L_1C)H_{e1} + (B+L_1D)K_1. \tag{8}$$

(d) *With the choices in* (a)–(c)*, the pair $(CH_{e1}+DK_1, G_1)$ is detectable, and $G_2 \in \mathcal{L}(Y,Z_1)$ is chosen in such a way that $G_1 + G_2(CH_{e1}+DK_1)$ generates an exponentially stable semigroup (i.e., the matrix is Hurwitz).*
(e) *Finally, we choose $L = L_1 + H_{e1}G_2$.*

With these choices the closed-loop system (5) *is stable, and the controller* (4) *solves the robust output regulation problem for $\mathcal{O}$.*

Proof. Assume G_1 and K_1 are as in Section IV. We begin by verifying the properties given in parts (b)–(d) of the theorem.

(b) Due to the assumption that (A,B) and (C,A) are stabilizable and detectable, respectively, it is possible to choose K_2 and L_1 in such a way that $A+BK_2$ and $A+L_1C$ generate exponentially stable semigroups.

(c) Since $A+L_1C$ generates an exponentially stable semigroup and since G_1 is a diagonal matrix with $\sigma(G_1) = \{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$, we have from [14, Cor. 8] that the Sylvester equation (8) has a unique solution. If we choose H_{e1} as suggested and if $k \in \{1,\dots,q\}$ and $l \in \{1,\dots,p_k\}$, then a direct computation (using $G_1\varphi_l^k = i\omega_k\varphi_l^k$) yields

$$\begin{aligned}(H_{e1}G_1 - (A+L_1C)H_{e1})\varphi_l^k &= (i\omega_k I - A - L_1C)H_{e1}\varphi_l^k \\ &= (i\omega_k I - A - L_1C)R(i\omega_k, A+L_1C)(B+L_1D)K_1\varphi_l^k \\ &= (B+L_1D)K_1\varphi_l^k.\end{aligned}$$

Since $\{\,\varphi_l^k \mid k=1,\dots,q,\ l=1,\dots,p_k\,\}$ is a basis of Z_1, this concludes that H_{e1} is the unique solution of (8).

(d) Since $\sigma(G_1) = \{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$ and $\mathcal{N}(i\omega_k I - G_1) = \operatorname{span}\{\varphi_1^k,\dots,\varphi_{p_k}^k\}$ for each $k \in \{1,\dots,q\}$, the detectability of $(CH_{e1}+DK_1, G_1)$ can be verified by showing that $(CH_{e1}+DK_1)\varphi_l^k \neq 0$ for all $k \in \{1,\dots,q\}$ and $l \in \{1,\dots,p_k\}$ [15, Thm. 6.2-5]. To this end, let k and l be arbitrary. Using (7) and the formulas for H_{e1} and K_1 shows

$$\begin{aligned}&(CH_{e1}+DK_1)\varphi_l^k \\ &\quad = CR(i\omega_k, A+L_1C)(B+L_1D)K_1\varphi_l^k + DK_1\varphi_l^k \\ &\quad = P_L(i\omega_k)K_1\varphi_l^k = P_L(i\omega_k)u_l^k,\end{aligned}$$

where $P_L(\lambda) = CR(\lambda, A+L_1C)(B+L_1D) + D$ for $\lambda \in \overline{\mathbb{C}^+}$. It is straightforward to show (using the Woodbury formula) that for every $\lambda \in \rho(A)\cap\rho(A+L_1C)$ we have $1 \in \rho(CR(\lambda,A)L_1)$ and $P_L(\lambda) = (I - CR(\lambda,A)L_1)^{-1}P(\lambda)$. Since $P(i\omega_k)$ is invertible by assumption and $i\omega_k \in \rho(A)\cap\rho(A+L_1C)$, also $P_L(i\omega_k)$ is invertible. We therefore have $(CH_{e1}+DK_1)\varphi_l^k = P_L(i\omega_k)u_l^k \neq 0$, since $P_L(i\omega_k)$ is invertible and $u_l^k \neq 0$ (because $\{u_l^k\}_{l=1}^{p_k}$ is a basis of $\mathcal{S}_k$). This concludes that the pair $(CH_{e1}+DK_1, G_1)$ is detectable, and G_2 can be chosen in such a way that the matrix $G_1 + G_2(CH_{e1}+DK_1)$ is Hurwitz.

We have now shown that the parameters of the controller can be chosen as suggested in parts (a)–(e). It remains to show that the controller solves the robust output regulation problem for $\mathcal{O}$. We have from Theorem 3 that the equations (6) have solutions for all perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \in \mathcal{O}$ and all $k \in \{1,\dots,q\}$. Because of this, Theorem 2 implies that the controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ solves the control problem provided that the closed-loop system is stable. In the remaining part of the proof we will show that the semigroup $T_e(t)$ generated by A_e is exponentially stable.

If the controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ is chosen as in the statement of the theorem, the operator A_e is given by

$$A_e = \begin{bmatrix} A & BK_1 & -BK_2 \\ G_2C & G_1+G_2DK_1 & G_2C \\ LC & LDK_1 & A+BK_2+LC\end{bmatrix}.$$

If we choose a similarity transform $Q_e \in \mathcal{L}(X\times Z_1\times X)$ by

$$Q_e = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & I\end{bmatrix} \qquad \text{and} \qquad Q_e^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & 0 & I\end{bmatrix},$$

we can define $\hat{A}_e = Q_eA_eQ_e^{-1}$ on $X\times Z_1\times X$, and compute

$$\begin{aligned}\hat{A}_e &= Q_e\begin{bmatrix} A+BK_2 & BK_1 & -BK_2 \\ 0 & G_1+G_2DK_1 & G_2C \\ -A-BK_2 & LDK_1 & A+BK_2+LC\end{bmatrix} \\ &= \begin{bmatrix} A+BK_2 & BK_1 & -BK_2 \\ 0 & G_1+G_2DK_1 & G_2C \\ 0 & (B+LD)K_1 & A+LC\end{bmatrix}.\end{aligned}$$

Denote

$$\hat{A}_{e1} = \begin{bmatrix} G_1+G_2DK_1 & G_2C \\ (B+LD)K_1 & A+LC\end{bmatrix}.$$

Since $A+BK_2$ is exponentially stable by assumption, the block triangular structure shows that $\hat{A}_e$ (and hence also A_e by similarity) is exponentially stable if $\hat{A}_{e1}$ is exponentially stable [13, Lem. 3.2.2]. Since $L = L_1 + H_{e1}G_2$, we have

$$\hat{A}_{e1} = \begin{bmatrix} G_1 & 0 \\ (B+L_1D)K_1 & A+L_1C\end{bmatrix} + \begin{bmatrix} G_2 \\ H_{e1}G_2\end{bmatrix}\begin{bmatrix} DK_1 & C\end{bmatrix}.$$

Define $Q_{e1} = \left[\begin{smallmatrix} I & 0 \\ H_{e1} & -I \end{smallmatrix}\right] \in \mathcal{L}(Z_1 \times X)$ with $Q_{e1}^{-1} = Q_{e1}$. Since H_{e1} satisfies the equation (8), a direct computation yields

$$Q_{e1}^{-1} \begin{bmatrix} G_1 & 0 \\ (B+L_1D)K_1 & A+L_1C \end{bmatrix} Q_{e1} = \begin{bmatrix} G_1 & 0 \\ 0 & A+L_1C \end{bmatrix}.$$

Therefore, if we define $A_{e1} = Q_{e1}^{-1} \hat{A}_{e1} Q_{e1}$, then

$$\begin{aligned} A_{e1} &= \begin{bmatrix} G_1 & 0 \\ 0 & A+L_1C \end{bmatrix} + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \begin{bmatrix} CH_{e1}+DK_1 & -C \end{bmatrix} \\ &= \begin{bmatrix} G_1 + G_2(CH_{e1}+DK_1) & -G_2C \\ 0 & A+L_1C \end{bmatrix}. \end{aligned}$$

Since $A+L_1C$ generates an exponentially stable semigroup and $G_1 + G_2(CH_{e1}+DK_1)$ is Hurwitz, also the semigroup generated by A_{e1} is exponentially stable by [13, Lem. 3.2.2]. This finally concludes that the closed-loop system is exponentially stable. □

The proof of Theorem 5 can be modified in a situation where the internal model (either full or of reduced order) is chosen in some other way than the one given in Section IV. Three properties need to be verified: (1) The pair (G_1, K_1) contains a suitable internal model, (2) the Sylvester equation (8) has a solution, and (3) the matrix G_2 can be chosen in such a way that $G_1 + G_2(CH_{e1}+DK_1)$ is exponentially stable.

VI. Driven Harmonic Oscillators

In this section we control a system consisting of two interconnected driven harmonic oscillators

$$\begin{aligned} \ddot{q}_1(t) + \alpha_1\dot{q}_1(t) + q_1(t) &= q_2(t) + F_1(t) \\ \ddot{q}_2(t) + \alpha_2\dot{q}_2(t) + 2q_2(t) &= -q_1(t) + F_2(t), \end{aligned}$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are the damping coefficients of the subsystems. In the nominal situation we have $\alpha_1 = 1$ and $\alpha_2 = 0$ (the first oscillator is damped and the second one is undamped). We design a controller that is robust with respect to small changes in the coefficients α_1 and α_2.

The control inputs to the system are the external forces $F_1(t)$ and $F_2(t)$, and the measurements are the positions $q_1(t)$ and $q_2(t)$ of the oscillators. With choices $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$, the system can be written in standard form (1) on the space $X = \mathbb{C}^4$, where $x = [x_1, x_2, x_3, x_4]^T$, $x_0 = [q_1(0), \dot{q}_1(0), q_2(0), \dot{q}_2(0)]^T$, $u(t) = [u_1(t), u_2(t)]^T = [F_1(t), F_2(t)]^T$, $y(t) = [y_1(t), y_2(t)]^T = [q_1(t), q_2(t)]^T$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\alpha_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -\alpha_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$C = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right]$, and $D = 0 \in \mathbb{C}^{2\times 2}$. In the nominal situation, when $\alpha_1 = 1$ and $\alpha_2 = 0$, the matrix A has two stable and two unstable eigenvalues. The pairs (A, B) and (C, A) are controllable and observable, respectively. The transfer function of the nominal plant is $P(\lambda) = CR(\lambda, A)B$. Since the matrices A, B, and C are real, we have $P(\overline{\lambda}) = \overline{P(\lambda)}$.

We consider output tracking of a reference signal

$$y_{ref}(t) = \begin{bmatrix} 1 + \cos(\pi t) \\ -1 + \cos(\pi t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos(\pi t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (9)$$

To generate $y_{ref}(\cdot)$, we choose an exosystem (3) on the space $W = \mathbb{C}^3$ with parameters

$$S = \operatorname{diag}(0, i\pi, -i\pi), \quad F = \left[\begin{smallmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{smallmatrix}\right].$$

The reference signal in (9) is then generated with an initial state $v_0 = (1, 1/2, 1/2)^T \in \mathbb{C}^3$. We have $q = 3$, $i\omega_1 = 0$, $i\omega_2 = i\pi$, and $i\omega_3 = -i\pi$. The transfer function $P(\lambda)$ of the plant exists and is invertible at $\lambda = 0, i\pi, -i\pi$.

The Class of Perturbations: The class $\mathcal{O}$ of perturbations consists of $(\tilde{A}, B, C)$ where the perturbed damping coefficients $\tilde{\alpha}_1, \tilde{\alpha}_2 \geq 0$ are such that $\tilde{P}(\lambda)$ exists and is invertible for $\lambda \in \{0, \pm i\pi\}$. The transfer function of the perturbed plant is $\tilde{P}(\lambda) = CR(\lambda, \tilde{A})B$. Since $\tilde{A}$, B, and C are real, we again have $\tilde{P}(\overline{\lambda}) = \overline{\tilde{P}(\lambda)}$.

Construction of the Reduced Order Internal Model: We need the spaces $\mathcal{S}_k$ corresponding to the frequencies $\{i\omega_k\}_{k=1}^3$ of the exosystem. We begin with $i\omega_1 = 0$. A direct computation shows that $\tilde{P}(i\omega_1)$ does not depend on the values of $\tilde{\alpha}_1 \geq 0$ and $\tilde{\alpha}_2 \geq 0$ (as long as $0 \in \rho(\tilde{A})$). Therefore $\tilde{P}(0) = P(0) = \frac{1}{3}\left[\begin{smallmatrix} 2 & 1 \\ -1 & 1 \end{smallmatrix}\right]$ for every $(\tilde{A}, B, C) \in \mathcal{O}$, and

$$\begin{aligned} \mathcal{S}_1 &= \operatorname{span}\{\, \tilde{P}(0)^{-1} F e_1 \mid (\tilde{A}, B, C) \in \mathcal{O} \,\} \\ &= \operatorname{span}\{P(0)^{-1} F e_1\} = \operatorname{span}\left\{ \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right] \right\}. \end{aligned}$$

Thus $p_1 = 1$, and we can choose the basis of $\mathcal{S}_1$ to be $\{u_1^1\}$ with $u_1^1 = \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$.

For $i\omega_2 = i\pi$ and $\tilde{\alpha}_1, \tilde{\alpha}_2 \geq 0$ for which $\{i\omega_k\}_{k=1}^3 \subset \rho(\tilde{A})$ and $\tilde{P}(i\omega_k)$ are invertible we have $\tilde{P}(i\omega_2)^{-1} F e_2 = \left[\begin{smallmatrix} -\pi^2 - i\pi\tilde{\alpha}_1 \\ 2 - \pi^2 - i\pi\tilde{\alpha}_2 \end{smallmatrix}\right]$. We want to show that the space

$$\mathcal{S}_2 = \operatorname{span}\{\, \tilde{P}(i\omega_2)^{-1} F e_2 \mid (\tilde{A}, B, C) \in \mathcal{O} \,\}$$

contains at least two linearly independent vectors. Since the plant with the nominal damping coefficients belongs to $\mathcal{O}$, it is sufficient to find perturbed coefficients $\tilde{\alpha}_1, \tilde{\alpha}_2 \geq 0$ such that $(\tilde{A}, B, C) \in \mathcal{O}$ and such that $P(i\omega_2)^{-1} F e_2$ and $\tilde{P}(i\omega_2)^{-1} F e_2$ are linearly independent. To this end, we can choose $\tilde{\alpha}_2 = \alpha_2 = 0$ and let $\tilde{\alpha}_1$ be such that $|\tilde{\alpha}_1 - \alpha_1|$ is small but nonzero (small enough so that $i\omega_k \in \rho(\tilde{A})$ and $\tilde{P}(i\omega_k)$ are invertible for $k = 1, 2, 3$). Then the two vectors

$$P(i\omega_2)^{-1} F e_2 = \left[\begin{smallmatrix} -\pi^2 - i\pi \\ 2 - \pi^2 \end{smallmatrix}\right], \quad \tilde{P}(i\omega_2)^{-1} F e_2 = \left[\begin{smallmatrix} -\pi^2 - i\pi\tilde{\alpha}_1 \\ 2 - \pi^2 \end{smallmatrix}\right]$$

are linearly independent. Since $\mathcal{S}_2 \subset \mathbb{C}^2$, this immediately concludes that $\mathcal{S}_2 = \mathbb{C}^2$ and $p_2 = \dim \mathcal{S}_2 = 2$. As the basis of $\mathcal{S}_2$ we can choose $\{u_1^2, u_2^2\}$ where $u_1^2 = e_1$ and $u_2^2 = e_2$.

Since $\tilde{P}(\overline{\lambda}) = \overline{\tilde{P}(\lambda)}$ and $F e_3 = F e_2$, it is easy to see that also $p_3 = \dim \mathcal{S}_3 = \dim \mathcal{S}_2 = 2$, and we can choose the basis $\{u_1^3, u_2^3\}$ of $\mathcal{S}_3$ such that $u_1^3 = e_1$ and $u_2^3 = e_2$.

Using the above information, we can see that the internal model can be chosen as $G_1 = \operatorname{diag}(0, i\pi, i\pi, -i\pi, -i\pi)$, and $(\varphi_1^1, \varphi_1^2, \varphi_2^2, \varphi_1^3, \varphi_2^3) = (e_1, e_2, e_3, e_4, e_5)$. The matrix K_1 in Theorem 3 is

$$K_1 = [u_1^1, u_1^2, u_2^2, u_1^3, u_2^3] = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Stabilization of the Closed-Loop System: The remaining parameters of the controller are chosen as in Theorem 5. We begin by using pole placement to choose matrices K_2 and L_1 so that $\sigma(A+BK_2)=\sigma(A+L_1C)=\{-2\pm i,-2\pm 1.3i\}$.

In the second part, we choose the matrix H_{e1} as in part (c) of Theorem 5. Furthermore, we need to choose G_2 in such a way that $G_1+G_2(CH_{e1}+DK_1)$ is Hurwitz. As we saw in the proof of Theorem 5, we have

$$CH_{e1}+DK_1=\sum_{k=1}^{3}\sum_{l=1}^{p_k}\langle\cdot,\varphi_l^k\rangle P_L(i\omega_k)\varphi_l^k$$

where $P_L(\lambda)=CR(\lambda,A+L_1C)B$ (since $D=0$). We use pole placement to choose G_2 so that $\sigma(G_1+G_2(CH_{e1}+DK_1))=\{-2,-2\pm 0.9\pi i,-2\pm 1.1\pi i\}$. Finally, we choose $L=L_1+H_{e1}G_2$. The parameters of the controller on $Z=\mathbb{C}^9$ are

$$\mathcal{G}_1=\begin{bmatrix}G_1 & G_2C\\ 0 & A+BK_2+LC\end{bmatrix}\in\mathbb{C}^{9\times 9},\ \mathcal{G}_2=\begin{bmatrix}G_2\\ L\end{bmatrix}\in\mathbb{C}^{2\times 9},$$

and $K=[K_1,\ -K_2]\in\mathbb{C}^{9\times 2}$.

The controlled system was simulated with initial states $x_0=[1,0,-2,0]^T$ for the plant, and $z_0=0\in\mathbb{C}^9$ for the controller. The output and the regulation error are depicted in Figures 1 and 2, respectively.

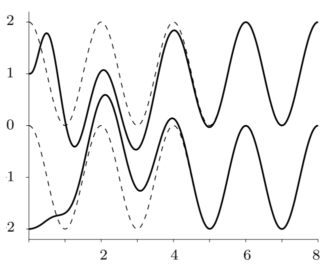

Fig. 1. y (solid) and y_{ref} (dashed)

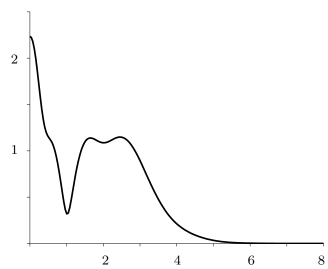

Fig. 2. Regulation error $e(t)$

By construction, the control law is robust with respect to small changes in α_1 and α_2. Figures 3 and 4 depict the behaviour of the output and the regulation error, respectively, with perturbed damping coefficients $\tilde{\alpha}_1=0.9$ and $\tilde{\alpha}_2=0.15$.

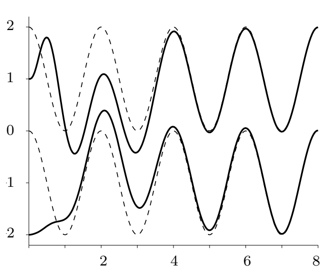

Fig. 3. y (solid) and y_{ref} (dashed)

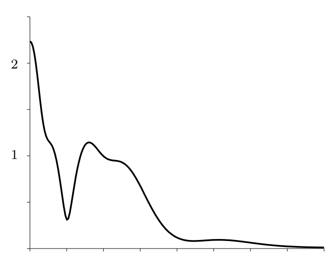

Fig. 4. Regulation error $e(t)$

VII. Conclusions

In this paper we have designed a controller to achieve robust output tracking for a linear system. We have studied a situation where the control law is only required to be robust with respect to a predetermined class $\mathcal{O}$ of perturbations. The results in this paper illustrate that in some situations the solution of this problem does not require a full p-copy internal model in the controller.

Further research topics include studying the use of other types of controllers besides the observer-based design used in this paper. In particular, if the original system is exponentially stable, the standard robust output regulation problem can be solved using a very simple controller [4], [6] that is finite-dimensional even for infinite-dimensional plants.

References

[1] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *Appl. Math. Optim.*, vol. 2, no. 2, pp. 170–194, 1975.

[2] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Automat. Control*, vol. 21, no. 1, pp. 25–34, 1976.

[3] J. Huang, *Nonlinear Output Regulation, Theory and Applications*. SIAM, Philadelphia, 2004.

[4] S. A. Pohjolainen, "Robust multivariable PI-controller for infinite-dimensional systems," *IEEE Trans. Automat. Control*, vol. 27, no. 1, pp. 17–31, 1982.

[5] H. Logemann and S. Townley, "Low-gain control of uncertain regular linear systems," *SIAM J. Control Optim.*, vol. 35, no. 1, pp. 78–116, 1997.

[6] T. Hämäläinen and S. Pohjolainen, "A finite-dimensional robust controller for systems in the CD-algebra," *IEEE Trans. Automat. Control*, vol. 45, no. 3, pp. 421–431, 2000.

[7] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica J. IFAC*, vol. 39, no. 9, pp. 1555–1569, 2003.

[8] E. Immonen, "On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control," *SIAM J. Control Optim.*, vol. 45, no. 6, pp. 2065–2093, 2007.

[9] T. Hämäläinen and S. Pohjolainen, "Robust regulation of distributed parameter systems with infinite-dimensional exosystems," *SIAM J. Control Optim.*, vol. 48, no. 8, pp. 4846–4873, 2010.

[10] L. Paunonen and S. Pohjolainen, "Internal model theory for distributed parameter systems," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4753–4775, 2010.

[11] ——, "Reduced order internal models in robust output regulation," *IEEE Trans. Automat. Control*, vol. 58, no. 9, pp. 2307–2318, 2013.

[12] ——, "Robust controller design for infinite-dimensional exosystems," *Internat. J. Robust Nonlinear Control*, vol. 24, no. 5, pp. 825–858, 2014, article first published online: 13 NOV 2012.

[13] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York, 1995.

[14] V. Q. Phông, "The operator equation $AX-XB=C$ with unbounded operators A and B and related abstract Cauchy problems," *Math. Z.*, vol. 208, pp. 567–588, 1991.

[15] T. Kailath, *Linear Systems*. Prentice-Hall, Englewood Cliffs, N.J., 1980.