# Robustness of Strong Stability of Semigroups

L. Paunonen

*Department of Mathematics, Tampere University of Technology. PO. Box 553, 33101 Tampere, Finland.*

**Abstract**

In this paper we study the preservation of strong stability of strongly continuous semigroups on Hilbert spaces. In particular, we study a situation where the generator of the semigroup has a finite number of spectral points on the imaginary axis and the norm of its resolvent operator is polynomially bounded near these points. We characterize classes of perturbations preserving the strong stability of the semigroup. In addition, we improve recent results on preservation of polynomial stability of a semigroup under perturbations of its generator. Theoretic results are illustrated with an example where we consider the preservation of the strong stability of a multiplication semigroup.

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## 1. Introduction

The properties of a linear abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X \tag{1}$$

on a Hilbert space $X$ can be studied using the theory of strongly continuous semigroups [8, 2]. If the operator $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$ on $X$, then the initial value problem (1) has a unique solution given by $x(t) = T(t)x_0$ for all $t \geq 0$. In particular, the asymptotic behaviour and different types of stability of the solutions of (1) can be studied by analyzing the stability properties of the semigroup $T(t)$. The use of semigroups provides a unified approach to developing theory for — for example — classes of linear differential, partial differential, and integral equations that can be written in the form (1).

In this paper we are interested in robustness of the stability properties of the semigroup $T(t)$ in the situation where its infinitesimal generator $A$ is perturbed. It is a well-known fact that the exponential stability of a strongly continuous semigroup is preserved under all bounded perturbations whose operator norms are sufficiently small. However, in a situation where $T(t)$ is not exponentially stable, but merely *strongly stable*, i.e.,

$$\lim_{t\to\infty} \|T(t)x\| = 0, \qquad \forall x \in X,$$

no general conditions for the preservation the stability of $T(t)$ are known. On the contrary, it is acknowledged that strong stability may be extremely sensitive to even arbitrarily small perturbations of its infinitesimal generator.

Recently in [13, 14] it was shown that a subclass of strongly stable semigroups, the so-called *polynomially stable semigroups*, do indeed possess good robustness properties. The key observation was that in the case of polynomial stability, the size of the perturbation $A+BC$ should not be measured using the regular operator

*Email address:* lassi.paunonen@tut.fi (L. Paunonen)

norms $\|B\|$ and $\|C\|$, but instead using graph norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$ for suitable exponents $\beta$ and $\gamma$. The polynomially stable semigroups have a characteristic property that their generators have no spectrum on the imaginary axis $i\mathbb{R}$. Therefore, many of the strongly stable semigroups encountered in applications are beyond the scope of the perturbation results in [13, 14]. In this paper we study the robustness properties of semigroups whose generators do have spectrum on the imaginary axis. In particular, we consider a situation where $A$ has a finite number of spectral points on the imaginary axis, and the norm of the resolvent operator of $A$ is polynomially bounded near these points. We show that the semigroups of this type have surprising robustness properties.

The results presented in this paper again demonstrate that for a strongly stable semigroup $T(t)$, the size of the perturbation should not be measured using the regular operator norm, but instead using suitable graph norms related to the generator $A$. Our main results reveal large and easily characterizable classes perturbations that preserve the strong stability of $T(t)$. The results can be applied, for example, in the study of linear partial differential equations, and in robust control of infinite-dimensional linear systems [15].

To the author's knowledge, robustness properties of strong stability of semigroups with spectrum on the imaginary axis have not been studied previously in the literature. Some results on preservation of strong stability of compact semigroups can be found in [7]. However, any strongly stable compact semigroup is actually exponentially stable [8, Ex. V.1.6(4)].

To illustrate our conditions for the preservation of stability, we begin by stating our main result in a situation where $A$ has a single imaginary spectral point $\sigma(A) \cap i\mathbb{R} = \{0\}$ belonging to the continuous spectrum of $A$ and the perturbing operator is of finite rank. We further assume that there exists $\alpha \geq 1$ such that

$$\sup_{0<|\omega|\leq 1} |\omega|^\alpha \|R(i\omega, A)\| < \infty \qquad \text{and} \qquad \sup_{|\omega|>1} \|R(i\omega, A)\| < \infty. \tag{2}$$

These assumptions are satisfied, for example, if $A$ generates a strongly stable analytic semigroup with $0 \in \sigma_c(A)$, and $|\lambda|\|R(\lambda, A)\| \leq M$ outside some sector in $\mathbb{C}^-$. Our assumptions on the behaviour of the resolvent operator $R(i\omega, A)$ are closely related to the rates of decay of the orbits $T(t)x$ of the semigroup. Indeed, recently in [4] it was shown that the resolvent operator satisfies (2) if and only if there exists a constant $M \geq 1$ such that

$$\|T(t)Ax\| \leq \frac{M}{t^{1/\alpha}}(\|x\| + \|Ax\|), \qquad \forall x \in \mathcal{D}(A) \tag{3}$$

for all $x \in \mathcal{D}(A)$ and $t > 0$ [4, Thm. 7.5].

Since $A$ is injective and $\mathcal{R}(A)$ is dense, the operator $-A$ has a densely defined inverse $(-A)^{-1}$. Furthermore, $(-A)^{-1}$ and $(-A^*)^{-1}$ are sectorial operators, and thus for $\beta, \gamma \geq 0$ the fractional powers $(-A)^{-\beta} : \mathcal{R}((-A)^\beta) \subset X \to X$ and $(-A^*)^{-\gamma} : \mathcal{R}((-A^*)^\gamma) \subset X \to X$ are well-defined (see e.g. [10, 17] for details). We consider finite rank perturbations of the form $A + BC$, where $B \in \mathcal{L}(\mathbb{C}^p, X)$, and $C \in \mathcal{L}(X, \mathbb{C}^p)$ satisfy

$$\mathcal{R}(B) \subset \mathcal{R}((-A)^\beta), \qquad \mathcal{R}(C^*) \subset \mathcal{R}((-A^*)^\gamma) \tag{4}$$

for some $\beta, \gamma \geq 0$. We choose to measure the size of the perturbation $BC$ using the graph norms $\|B\| + \|(-A)^{-\beta}B\|$ and $\|C\| + \|(-A^*)^{-\gamma}C^*\|$. Theorem 1 shows that this is exactly the right choice for the purposes of studying the preservation of the strong stability of $T(t)$.

**Theorem 1.** *Let $\beta + \gamma \geq \alpha$. There exists $\delta > 0$ such that if $B \in \mathcal{L}(\mathbb{C}^p, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^p)$ satisfy* (4) *and*

$$\|B\| + \|(-A)^{-\beta}B\| < \delta, \quad \textit{and} \quad \|C\| + \|(-A^*)^{-\gamma}C^*\| < \delta,$$

*then the semigroup generated by $A + BC$ is strongly stable.*

In Section 2 we state Theorem 1 in a more general situation where $\sigma(A) \cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$ for some $N \in \mathbb{N}$. In this case, the preservation of stability requires that for all $k \in \{1, \dots, N\}$ the graph norms $\|B\| + \|(i\omega_k - A)^{-\beta} B\|$ and $\|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\|$ are sufficiently small. Moreover, in our main result presented in Section 2 the perturbations $A+BC$ are such that $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X, Y)$ for a separable Hilbert space $Y$, and $B$ and $C^*$ are Hilbert–Schmidt operators.

As was already mentioned, in the case $\sigma(A) \cap i\mathbb{R} = \{0\}$ the boundedness property (2) of the resolvent is equivalent to the nonuniform polynomial decay (3) of the semigroup [4, Thm. 7.5]. In Section 5 we use the results and the theory developed in [5, 6, 4] to connect the behaviour of the resolvent operator on $i\mathbb{R}$ to the asymptotic behaviour of the semigroup in the situation where $A$ has multiple spectral points on the imaginary axis.

In addition to studying the preservation of strong stability, we also improve the results concerning robustness of polynomial stability presented in [13, 14]. In these references it was shown that the polynomial stability of a semigroup generated by $A$ is preserved under a finite rank perturbation $A + BC$ if for some $\beta, \gamma \geq 0$ satisfying $\beta + \gamma \geq \alpha$ we have

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta), \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma), \tag{5}$$

and if the graph norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$ are small enough. However, in these results one of the associated exponents $\beta, \gamma \geq 0$ was required to be an integer, or alternatively, larger than or equal to $\alpha$. The techniques used in this paper allow us to remove these restrictions on the exponents. In particular, we show that for arbitrary exponents $\beta, \gamma \geq 0$ satisfying $\beta + \gamma \geq \alpha$ the polynomial stability of a semigroup generated by $A$ is preserved provided that the perturbation satisfies (5) and the corresponding graph norms are small enough. In addition, in this paper the results in [13, 14] are extended from finite rank perturbations to perturbations $A + BC$ where $B$ and $C^*$ are Hilbert–Schmidt operators.

The paper is organized as follows. In Section 2 we state our main results on the preservation of strong and polynomial stability. The result on robustness of strong stability is proved in parts throughout Sections 3 and 4. In Section 5 we connect the polynomial boundedness of the resolvent operator to the asymptotic behaviour of the semigroup. Section 6 contains the proof of the result on the preservation of polynomial stability. In Section 7 we illustrate the theoretic results with an example where we study the robustness properties of a strongly stable multiplication semigroup. Section 8 contains concluding remarks.

We conclude this section by applying Theorem 1 to study the preservation of the strong stability of a semigroup generated by a bounded diagonal operator.

**Example 2.** Let $X = \ell^2(\mathbb{C})$ and define $A \in \mathcal{L}(X)$ by

$$A = \sum_{k=1}^\infty -\frac{1}{k} \langle \cdot, e_k \rangle e_k$$

where $e_k$ are the natural basis vectors. The operator generates a strongly stable semigroup $T(t)$ and satisfies $\sigma(A) \cap i\mathbb{R} = \{0\} \subset \sigma_c(A)$. Since for $\omega \neq 0$ we have $\|R(i\omega, A)\| = \operatorname{dist}(i\omega, \sigma(A))^{-1} = |\omega|^{-1}$, the assumptions of Theorem 1 are satisfied for $\alpha = 1$. The operator $-A$ has an unbounded self-adjoint inverse, and for $\beta \geq 0$ its fractional powers are given by

$$(-A)^{-\beta} x = \sum_{k=1}^\infty k^\beta \langle x, e_k \rangle e_k, \qquad x \in \mathcal{R}((-A)^\beta) = \Big\{ x \in X \;\Big|\; \sum_{k=1}^\infty k^{2\beta} |\langle x, e_k \rangle|^2 < \infty \Big\}.$$

If we consider a rank one perturbation $A + \langle \cdot, c \rangle b$ with $b, c \in X$, then Theorem 1 in particular states that the semigroup generated by the perturbed operator is strongly stable if $\|b\|$ and $\|c\|$ are small, and for some $\beta, \gamma \geq 0$ satisfying $\beta + \gamma = 1$ the norms

$$\|(-A)^{-\beta} b\|^2 = \sum_{k=1}^\infty k^{2\beta} |\langle b, e_k \rangle|^2 \qquad \text{and} \qquad \|(-A^*)^{-\gamma} c\|^2 = \sum_{k=1}^\infty k^{2\gamma} |\langle c, e_k \rangle|^2$$

are finite and small.

If $X$ and $Y$ are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ the domain, the range, and the kernel of $A$, respectively. The space of bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$. If $A : \mathcal{D}(A) \subset X \to X$, then $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$ and $\rho(A)$ denote the spectrum, the point spectrum, the continuous spectrum and the resolvent set of $A$, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. If $Y$ is a separable Hilbert space with an orthonormal basis $(e_k)_{k=1}^\infty$, then $B \in \mathcal{L}(Y,X)$ is a Hilbert–Schmidt operator if $(Be_k)_{k=1}^\infty \in \ell^2(X)$.

For a function $f : \mathbb{R} \to \mathbb{R}$ and for $\alpha \geq 0$ we use the notation

$$f(\omega) = \mathcal{O}\left(|\omega|^\alpha\right)$$

if there exist constants $M > 0$ and $\omega_0 \geq 0$ such that $|f(\omega)| \leq M|\omega|^\alpha$ for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$.

## 2. Main Results

In this section we present our main results. It is well-known that if the semigroup generated by $A$ is strongly stable, then $A$ has no eigenvalues on the imaginary axis, and therefore operators $A - i\omega$ are injective for all $\omega \in \mathbb{R}$. Moreover, since $X$ is a Hilbert space, the Mean Ergodic Theorem [2, Sec. 4.3] applied to operators $A - i\omega$ shows that

$$X = \mathcal{N}(A - i\omega) \oplus \overline{\mathcal{R}(A - i\omega)} = \overline{\mathcal{R}(A - i\omega)}.$$

Therefore, the part of the spectrum of $A$ that is on the imaginary axis belongs to the continuous spectrum.

In the following we formulate our assumptions on the unperturbed operator $A$ as well as on the components $B$ and $C$ of the perturbing operator. The main assumption is that the intersection $\sigma(A) \cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$ is finite, and the norm of the resolvent operator is polynomially bounded near the points $i\omega_k$.

**Assumption 3.** Assume $A : \mathcal{D}(A) \subset X \to X$ generates a strongly stable semigroup $T(t)$ on a Hilbert space $X$, $\sigma(A) \cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$ for some $N \in \mathbb{N}$, and $d_A = \min_{k \neq l} |\omega_k - \omega_l| > 0$. Moreover, assume that for some constants $\alpha \geq 1$, $M_A > 0$, and $0 < \varepsilon_A \leq \max\{1, d_A/3\}$ we have

$$\sup_{0 < |\omega - \omega_k| \leq \varepsilon_A} |\omega - \omega_k|^\alpha \|R(i\omega, A)\| \leq M_A, \tag{6}$$

for all $k \in \{1, \dots, N\}$ and $\|R(i\omega, A)\| \leq M_A$ whenever $|\omega - \omega_k| > \varepsilon_A$ for all $k \in \{1, \dots, N\}$.

We consider perturbations of the form $A + BC$ where $B \in \mathcal{L}(Y,X)$ and $C \in \mathcal{L}(X,Y)$ for a separable Hilbert space $Y$. We assume that for some exponents $\beta, \gamma \geq 0$ the operators $B$ and $C$ satisfy the conditions

$$\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\beta) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\gamma) \qquad \text{for all } k \in \{1, \dots, N\} \tag{7}$$

and for all $k \in \{1, \dots, N\}$

$$B, \quad C^*, \quad (i\omega_k - A)^{-\beta}B, \quad \text{and} \quad (-i\omega_k - A^*)^{-\gamma}C^*, \quad \text{are Hilbert–Schmidt operators.} \tag{8}$$

If $Y$ is finite-dimensional, i.e., if the perturbing operator $BC$ is of finite rank, then the condition (8) follows immediately from (7). Since $(i\omega_k - A)^{-\beta}$ for $k \in \{1, \dots, N\}$ are sectorial, we have that $(i\omega_k - A)^{-\beta}B$ are closed operators. Since $\mathcal{D}((i\omega_k - A)^{-\beta}B) = Y$ by (7), the Closed Graph Theorem implies $(i\omega_k - A)^{-\beta}B \in \mathcal{L}(Y,X)$ for every $k \in \{1, \dots, N\}$. Similarly, we have that $(-i\omega_k - A^*)^{-\gamma}C^* \in \mathcal{L}(Y,X)$ for all $k \in \{1, \dots, N\}$.

Our first main result concerns the preservation of strong stability.

**Theorem 4.** *Let Assumption* 3 *be satisfied for some $\alpha \geq 1$, and let $\beta, \gamma \geq 0$ be such that $\beta + \gamma \geq \alpha$. There exists $\delta > 0$ such that if $B \in \mathcal{L}(Y,X)$ and $C \in \mathcal{L}(X,Y)$ satisfy conditions* (7) *and* (8) *and*

$$\|B\| + \|(i\omega_k - A)^{-\beta}B\| < \delta, \quad \text{and} \quad \|C\| + \|(-i\omega_k - A^*)^{-\gamma}C^*\| < \delta$$

*for all $k \in \{1, \dots, N\}$, then the semigroup generated by $A + BC$ is strongly stable.*

*In particular, the spectrum of $A + BC$ satisfies $\sigma(A + BC) \cap i\mathbb{R} = \sigma_c(A + BC) \cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$, for all $k \in \{1, \dots, N\}$*

$$\sup_{0<|\omega-\omega_k|\le\varepsilon_A} |\omega - \omega_k|^\alpha \|R(i\omega, A + BC)\| < \infty,$$

*and $\sup_\omega \|R(i\omega, A + BC)\| < \infty$ for $\omega \in \mathbb{R}$ such that $|\omega - \omega_k| > \varepsilon_A$ for all $k \in \{1, \dots, N\}$.*

It should also be noted that if the exponents in Theorem 4 satisfy $\beta, \gamma \ge \alpha$, then condition (8) becomes redundant and the strong stability is preserved even if $B$ and $C^*$ are not Hilbert–Schmidt operators. See Remark 20 for details.

The proof of Theorem 4 is divided into two parts. In Section 3 we study the change of the spectrum of $A$. In Section 4 we complete the proof of Theorem 4 by showing that the uniform boundedness of $T(t)$ is preserved under the perturbations.

We remark that the polynomial growth condition for the resolvent was assumed to be satisfied for $\alpha \ge 1$. The following lemma shows that this assumption does not result in any loss of generality.

**Lemma 5.** *If $\sigma(A) \cap i\mathbb{R} \ne \varnothing$, then $\alpha \ge 1$ in the condition* (6).

*Proof.* Let $k \in \{1, \dots, N\}$. Since $i\omega_k \in \sigma(A)$, for $i\omega$ near $i\omega_k$ we have $\operatorname{dist}(i\omega, \sigma(A)) \le |\omega - \omega_k|$. Thus for all such $\omega$ satisfying $0 < |\omega - \omega_k| \le \varepsilon_A$ the standard estimate $\|R(\omega, A)\| \ge \operatorname{dist}(\lambda, \sigma(A))^{-1}$ (see [8, Cor. IV.1.14]) implies

$$\frac{1}{|\omega - \omega_k|} \le \frac{1}{\operatorname{dist}(i\omega, \sigma(A))} \le \|R(i\omega, A)\| \le \frac{M_A}{|\omega - \omega_k|^\alpha},$$

which further implies $|\omega - \omega_k|^{\alpha-1} \le M_A$. However, for small $|\omega - \omega_k|$ this is only possible if $\alpha \ge 1$. □

Our second main result concerns the preservation of polynomial stability of a semigroup. The semigroup $T(t)$ generated by $A$ on the Hilbert space $X$ is called *polynomially stable* if $T(t)$ is uniformly bounded, if $\sigma(A) \cap i\mathbb{R} = \varnothing$, and if there exists $\alpha > 0$ and $M \ge 1$ such that [3, 5, 6]

$$\|T(t)A^{-1}\| \le \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

The following theorem gives conditions for the preservation of the polynomial stability under perturbations. The theorem extends the results in [13, 14] by removing all restrictions on the exponents $\beta \ge 0$ and $\gamma \ge 0$, and by allowing perturbations that are not of finite rank.

**Theorem 6.** *Assume $T(t)$ generated by $A$ is polynomially stable with exponent $\alpha > 0$, and $\beta, \gamma \ge 0$ are such that $\beta + \gamma \ge \alpha$. There exists $\delta > 0$ such that if $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X, Y)$ satisfy*

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta), \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma), \tag{9}$$

*if $(-A)^\beta B$ and $(-A^*)^\gamma C^*$ are Hilbert–Schmidt operators and if $\|(-A)^\beta B\| < \delta$ and $\|(-A^*)^\gamma C^*\| < \delta$, then the semigroup generated by $A + BC$ is polynomially stable with the same exponent $\alpha$.*

## 3. Perturbation of the Spectrum

In this section we show that under the conditions of Theorem 4 the spectrum of the perturbed operator satisfies $\sigma(A + BC) \subset \overline{\mathbb{C}^+}$, and that $A + BC$ does not have eigenvalues on the imaginary axis. On its own, this result is valid under weaker assumptions than those in Theorem 4. In particular, the results remains valid if $\sigma(A) \cap i\mathbb{R} = \{i\omega_k\}_{k\in I_A}$ for a countable set $I_A$ of indices if the points $\omega_k$ have a uniform gap $d_A = \inf_{k\ne l} |\omega_k - \omega_l| > 0$. Moreover, the perturbation does not need to satisfy condition (8), but instead for some Banach space $Y$ we can consider any operators $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X, Y)$ that satisfy $\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\beta)$ and $\mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\gamma)$ for all $k \in I_A$.

**Theorem 7.** *Let Assumption 3 be satisfied for some $\alpha \geq 1$ and let $\beta, \gamma \geq 0$ be such that $\beta + \gamma \geq \alpha$. There exists $\delta > 0$ such that if $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X, Y)$ satisfy $\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\beta)$ and $\mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\gamma)$ and*

$$\|B\| + \|(i\omega_k - A)^{-\beta}B\| < \delta, \quad \text{and} \quad \|C\| + \|(-i\omega_k - A^*)^{-\gamma}C^*\| < \delta,$$

*for all $k \in \{1, \dots, N\}$, then $\overline{\mathbb{C}^+} \setminus \{i\omega_k\}_{k=1}^N \subset \rho(A + BC)$ and $i\omega_k \in \sigma(A + BC) \setminus \sigma_p(A + BC)$ for all $k \in \{1, \dots, N\}$. In particular, under the above conditions we have*

$$\sup_{\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_k\}_k} \|(I - CR(\lambda, A)B)^{-1}\| < \infty.$$

We prove the theorem in parts. For the study of the change of the spectrum of $A$ we use the well-known Sherman–Morrison–Woodbury formula given in the following lemma.

**Lemma 8.** *Let $\lambda \in \rho(A)$, $B \in \mathcal{L}(Y, X)$, $C \in \mathcal{L}(X, Y)$. If $1 \in \rho(CR(\lambda, A)B)$, then $\lambda \in \rho(A + BC)$ and*

$$R(\lambda, A + BC) = R(\lambda, A) + R(\lambda, A)B(I - CR(\lambda, A)B)^{-1}CR(\lambda, A).$$

Throughout the paper we make use of the operators $\Lambda_k = (i\omega_k - A)(1 + i\omega_k - A)^{-1}$ for $k \in \{1, \dots, N\}$. The following lemma states some of the most important properties of the operators $\Lambda_k$ [4, 10]. Since $\Lambda_k^* = (-i\omega_k - A^*)(1 - i\omega_k - A^*)^{-1}$, the family of operators $(\Lambda_k^*)_k$ has the same propeties as $(\Lambda_k)_k$.

**Lemma 9.** *Define $\Lambda_k = (i\omega_k - A)(1 + i\omega_k - A)^{-1}$ for $k \in \{1, \dots, N\}$. Then $(\Lambda_k)_{k=1}^N$ is a uniformly sectorial family of injective operators.*

*For every $\beta > 0$ we have $\Lambda_k^\beta = (i\omega_k - A)^\beta (1 + i\omega_k - A)^{-\beta}$ with range $\mathcal{R}(\Lambda_k^\beta) = \mathcal{R}((i\omega_k - A)^\beta)$. Moreover, $\Lambda_k^{-\beta} = (1 + i\omega_k - A)^\beta (i\omega_k - A)^{-\beta}$ with domain $\mathcal{D}(\Lambda_k^{-\beta}) = \mathcal{R}((i\omega_k - A)^\beta)$.*

*Proof.* For every $\lambda > 0$ we have the identity

$$\lambda(\lambda + \Lambda_k)^{-1} = \frac{\lambda}{\lambda + 1} + \frac{\lambda}{(\lambda + 1)^2} \left( \frac{\lambda}{\lambda + 1} + i\omega_k - A \right)^{-1}.$$

If $M = \sup_{t>0} \|T(t)\| = \sup_{t>0} \|e^{i\omega_k t} T(t)\|$, then the Hille–Yosida Theorem [8, Thm. II.3.8] implies that $\mu \|R(\mu, A - i\omega_k)\| \leq M$ for all $\mu > 0$ and $k \in \{1, \dots, N\}$. We have

$$\sup_{\lambda > 0} \|\lambda(\lambda + \Lambda_k)^{-1}\| \leq \sup_{\lambda > 0} \left( \frac{\lambda}{\lambda + 1} + \frac{1}{\lambda + 1} \cdot \frac{\lambda}{\lambda + 1} \left\| \left( \frac{\lambda}{\lambda + 1} + i\omega_k - A \right)^{-1} \right\| \right) \leq 1 + M.$$

Since the bound is independent of $k \in \{1, \dots, N\}$, we have from [10, Prop. 2.1.1] that $(\Lambda_k)_{k=1}^N$ is a uniformly sectorial family of operators. The operators $\Lambda_k$ are injective since $i\omega_k - A$ and $(1 + i\omega_k - A)^{-1}$ are injective.

Since $i\omega_k - A$ and $(i\omega_k - A)^{-1}$ commute, we have $\Lambda_k^\beta = (i\omega_k - A)^\beta (1 + i\omega_k - A)^{-\beta}$, and $\mathcal{R}(\Lambda_k^\beta) = \mathcal{R}((i\omega_k - A)^\beta)$ follows from the fact that $\mathcal{R}((1 + i\omega_k - A)^{-\beta}) = \mathcal{D}((1 + i\omega_k - A)^\beta) = \mathcal{D}((i\omega_k - A)^\beta)$ [10, Prop. 3.1.9].

Since $\Lambda_k^\beta$ is injective, it has an inverse $\Lambda_k^{-\beta}$ with domain $\mathcal{D}(\Lambda_k^{-\beta}) = \mathcal{R}(\Lambda_k^\beta) = \mathcal{R}((i\omega_k - A)^\beta)$. Finally, we have $\Lambda_k^{-\beta} = (1 + i\omega_k - A)^\beta (i\omega_k - A)^{-\beta}$ by [10, Prop. 3.1.9]. □

The Moment Inequality [10, Prop. 6.6.4] is one of our most important tools in dealing with non-integer exponents. We use it in the form given in the next lemma for the families $(\Lambda_k)_{k=1}^N$, $(\Lambda_k^*)_{k=1}^N$, $((i\omega_k - A)^{-1})_{k=1}^N$, $((-i\omega_k - A^*)^{-1})_{k=1}^N$, and for the operator $(-A)^{-1}$.

**Lemma 10.** *Assume $(\Pi_k)_k$ is a uniformly sectorial family of operators, and let $0 < \tilde{\theta} < \theta$. There exists $M_{\tilde{\theta}/\theta} \geq 1$ such that for all $k$ we have*

$$\|\Pi_k^{\tilde{\theta}} x\| \leq M_{\tilde{\theta}/\theta} \|x\|^{1 - \tilde{\theta}/\theta} \|\Pi_k^\theta x\|^{\tilde{\theta}/\theta} \qquad \forall x \in \mathcal{D}(\Pi_k^\theta)$$

*If $Y$ is a Banach space and $R \in \mathcal{L}(Y, X)$ is such that $\mathcal{R}(R) \subset \mathcal{D}(\Pi^\theta)$, then*

$$\|\Pi_k^{\tilde{\theta}} R\| \le M_{\tilde{\theta}/\theta} \|R\|^{1-\tilde{\theta}/\theta} \|\Pi_k^\theta R\|^{\tilde{\theta}/\theta}$$

*for all $k$.*

*Proof.* For a fixed $k$ the properties follow from [10, Prop. 6.6.4]. However, by [10, Prop. 2.6.11] and the uniform sectoriality of the operator family $(\Pi_k)_k$ it is possible to choose $M_{\tilde{\theta}/\theta}$ to be independent of $k$. □

**Lemma 11.** *Let $\beta, \gamma \ge 0$. There exists $M_\Lambda \ge 1$ such that for all $k \in \{1, \dots, N\}$ we have*

$$\|\Lambda_k^{-\beta} x\| \le M_\Lambda \left( \|x\| + \|(i\omega_k - A)^{-\beta} x\| \right)$$
$$\|(\Lambda_k^*)^{-\gamma} y\| \le M_\Lambda \left( \|y\| + \|(-i\omega_k - A^*)^{-\gamma} y\| \right)$$

*for all $x \in \mathcal{R}((i\omega_k - A)^\beta)$ and $y \in \mathcal{R}((-i\omega_k - A^*)^\gamma)$.*

*Proof.* Let $\beta > 0$ and denote $A_k = A - i\omega_k$ for $k \in \{1, \dots, N\}$. We have from [10, Prop. 3.1.9] that $\mathcal{D}((-A_k)^\beta) = \mathcal{D}((1 - A_k)^\beta)$. The operator $(1 - A_k)^\beta$ is a closed operator from the Banach space $X_A = (\mathcal{D}((-A_k)^\beta), \|\cdot\| + \|(-A_k)^\beta \cdot\|)$ to $X$. Since $(1 - A_k)^\beta$ is defined on all of $X_A$, we have from the Closed Graph Theorem [8, Thm. B.6] that $(1 - A_k)^\beta \in \mathcal{L}(X_A, X)$, which implies that there exists $M_k \ge 1$ such that

$$\|(1 - A_k)^\beta x\| \le M_k \left( \|x\| + \|(-A_k)^\beta x\| \right), \qquad \forall x \in \mathcal{D}((-A_k)^\beta).$$

Since the assumptions on all of the operators $A_k$ for $k \in \{1, \dots, N\}$ are identical (i.e., the family $(-A_k)_{k=1}^N$ of operators is uniformly sectorial), we can choose $M_\Lambda \ge 1$ such that $M_k \le M_\Lambda$ for all $k \in \{1, \dots, N\}$. This immediately implies that for all $x \in \mathcal{D}(\Lambda_k^{-\beta}) = \mathcal{R}((-A_k)^\beta)$ we have

$$\begin{aligned}\|\Lambda_k^{-\beta} x\| &= \|(1 - A_k)^\beta (-A_k)^{-\beta} x\| \le M_\Lambda \left( \|(-A_k)^{-\beta} x\| + \|(-A_k)^\beta (-A_k)^{-\beta} x\| \right) \\ &= M_\Lambda \left( \|x\| + \|(i\omega_k - A)^{-\beta} x\| \right).\end{aligned}$$

The claim concerning operators $-i\omega_k - A^*$ can be shown analogously. □

We begin the proof of Theorem 7 by showing that we can choose $\delta > 0$ in such a way that $\|CR(\lambda, A)B\| \le c < 1$ for all $\lambda \in \bigcup_k \Omega_k$, where $\Omega_k = \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \ge 0,\ 0 < |\lambda - i\omega_k| \le \varepsilon_A \}$ (see Figure 1).

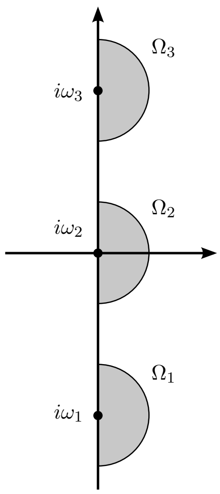


Figure 1: The domains $\Omega_k$.

**Lemma 12.** *If Assumption 3 is satisfied, then there exists $M_0 \geq 1$ such that*

$$\sup_{\lambda\in\Omega_k} |\lambda - i\omega_k|^\alpha \|R(\lambda, A)\| \leq M_0$$

*for every $k \in \{1, \ldots, N\}$.*

*Proof.* Let $M > 0$ be such that $\|T(t)\| \leq M$. From Assumption 3 we have

$$\sup_{0<|\omega-\omega_k|\leq\varepsilon_A} |\omega - \omega_k|^\alpha \|R(i\omega, A)\| \leq M_A.$$

The Hille–Yosida Theorem implies that $\operatorname{Re}\lambda\|R(\lambda, A)\| \leq M$ whenever $\operatorname{Re}\lambda > 0$.

Let $\lambda = \mu + i\omega \in \Omega_k$. For $\mu = 0$ the bound $|\lambda - i\omega_k|^\alpha \|R(\lambda, A)\| = |\omega - \omega_k|^\alpha \|R(i\omega, A)\| \leq M_A$ follows directly from (6). On the other hand, if $\omega = \omega_k$ and $\lambda = \mu > 0$, then the Hille–Yosida Theorem implies

$$|\lambda - i\omega_k|^\alpha \|R(\lambda, A)\| = \mu^\alpha \|R(\lambda, A)\| \leq \mu\|R(\lambda, A)\| \leq M$$

since $\mu^\alpha \leq \mu$ due to the fact that $\alpha \geq 1$ and $0 < \mu \leq \varepsilon_A \leq 1$. It remains to consider the case $\lambda = \mu+i\omega \in \Omega_k$ with $\mu > 0$ and $\omega \neq \omega_k$. In particular, we then have $0 < |\omega - \omega_k| \leq \varepsilon_A$ and $0 < \mu \leq \varepsilon_A \leq 1$. Since $\alpha \geq 1$ and $0 < \mu \leq 1$, we have $\mu^\alpha \leq \mu$ and

$$\begin{aligned}|\lambda - i\omega_k|^\alpha &= (\mu^2 + (\omega - \omega_k)^2)^{\alpha/2} \leq (2\max\{\mu^2, (\omega - \omega_k)^2\})^{\alpha/2} = 2^{\alpha/2}\max\{\mu^\alpha, |\omega - \omega_k|^\alpha\}\\ &\leq 2^{\alpha/2}(\mu^\alpha + |\omega - \omega_k|^\alpha) \leq 2^{\alpha/2}(\mu + |\omega - \omega_k|^\alpha),\end{aligned}$$

and thus using the resolvent identity $R(\lambda, A) = R(i\omega, A) - \mu R(\lambda, A)R(i\omega, A)$ we get

$$\begin{aligned}|\lambda - i\omega_k|^\alpha \|R(\lambda, A)\| &\leq 2^{\alpha/2}(\mu + |\omega - \omega_k|^\alpha)\|R(\lambda, A)\|\\ &= 2^{\alpha/2}\mu\|R(\lambda, A)\| + 2^{\alpha/2}|\omega - \omega_k|^\alpha \|R(i\omega, A) - \mu R(\lambda, A)R(i\omega, A)\|\\ &\leq 2^{\alpha/2}M + 2^{\alpha/2}|\omega - \omega_k|^\alpha \|R(i\omega, A)\|(1 + \mu\|R(\lambda, A)\|)\\ &\leq 2^{\alpha/2}\left(M + M_A(1 + M)\right).\end{aligned}$$

Since in each of the situations the bound for $|\lambda - i\omega_k|^\alpha \|R(\lambda, A)\|$ is independent of $k \in \{1, \ldots, N\}$, the proof is complete. □

A property similar to the one in the following lemma was first presented in [4, Cor. 7.5], where the estimate was shown for $A$ with a single spectral point on $i\mathbb{R}$. In this paper we need a slightly modified version of [4, Cor. 7.5] to accomodate for multiple spectral points on $i\mathbb{R}$.

**Lemma 13.** *If Assumption 3 is satisfied, then there exists $M_1 \geq 1$ such that*

$$\sup_{\lambda\in\Omega_k} \|R(\lambda, A)\Lambda_k^\alpha\| \leq M_1$$

*for all $k \in \{1, \ldots, N\}$.*

*Proof.* Let $k \in \{1, \ldots, N\}$, $\lambda \in \Omega_k$, and denote $R_\lambda = R(\lambda, A)$, $A_k = A - i\omega_k$, and $\lambda_k = \lambda - i\omega_k$ for brevity.

We begin by showing that if $\alpha = n + \tilde{\alpha}$ with $n \in \mathbb{N}$ and $0 \leq \tilde{\alpha} < 1$, then there exists $\tilde{M} \geq 1$ (not depending on $k \in \{1, \ldots, N\}$) such that

$$\sup_{\lambda\in\Omega_k} |\lambda_k|^n \|(-A_k)^{\tilde{\alpha}} R(\lambda, A)\| \leq \tilde{M}. \tag{10}$$

By Lemma 12 there exists $M_0 \geq 1$ such that $|\lambda - i\omega_k|^\alpha \|R(\lambda, A)\| \leq M_0$ for all $k \in \{1, \ldots, N\}$. If $\alpha = n$ and $\tilde{\alpha} = 0$, we have

$$|\lambda_k|^n \|(-A_k)^{\tilde{\alpha}} R_\lambda\| = |\lambda_k|^\alpha \|R_\lambda\| \leq M_0.$$

Thus the claim is satisfied with $\tilde{M} = M_0$, which is independent of $k \in \{1, \ldots, N\}$.

If $0 < \tilde{\alpha} < 1$, then by Lemma 10 there exists a constant $M_{\tilde{\alpha}}$ independent of $k \in \{1, \ldots, N\}$ such that $\|(-A_k)^{\tilde{\alpha}}x\| \leq M_{\tilde{\alpha}}\|x\|^{1-\tilde{\alpha}}\|(-A_k)x\|^{\tilde{\alpha}}$ for all $x \in \mathcal{D}(A)$. This further implies that $\|(-A_k)^{\tilde{\alpha}}R_\lambda\| \leq M_{\tilde{\alpha}}\|R_\lambda\|^{1-\tilde{\alpha}}\|(-A_k)R_\lambda\|^{\tilde{\alpha}}$. Using

$$(-A_k)R_\lambda = (i\omega_k - A)R_\lambda = (i\omega_k - \lambda + \lambda - A)R_\lambda = -\lambda_k R_\lambda + I \tag{11}$$

and the scalar inequality $(a+b)^{\tilde{\alpha}} \leq 2^{\tilde{\alpha}}(a^{\tilde{\alpha}} + b^{\tilde{\alpha}})$ we get

$$\begin{aligned}|\lambda_k|^n\|(-A_k)^{\tilde{\alpha}}R_\lambda\| &\leq M_{\tilde{\alpha}}|\lambda_k|^n\|R_\lambda\|^{1-\tilde{\alpha}}\|(-A_k)R_\lambda\|^{\tilde{\alpha}} \leq M_{\tilde{\alpha}}|\lambda_k|^n\|R_\lambda\|^{1-\tilde{\alpha}}(1 + |\lambda_k|\|R_\lambda\|)^{\tilde{\alpha}} \\ &\leq 2^{\tilde{\alpha}}M_{\tilde{\alpha}}|\lambda_k|^n\|R_\lambda\|^{1-\tilde{\alpha}}(1 + |\lambda_k|^{\tilde{\alpha}}\|R_\lambda\|^{\tilde{\alpha}}) \leq 2^{\tilde{\alpha}}M_{\tilde{\alpha}}\left[(|\lambda_k|^{\frac{n}{1-\tilde{\alpha}}}\|R_\lambda\|)^{1-\tilde{\alpha}} + |\lambda_k|^{n+\tilde{\alpha}}\|R_\lambda\|\right].\end{aligned}$$

Since $n = \lfloor \alpha \rfloor \geq 1$ we have

$$\frac{n}{1-\tilde{\alpha}} = \frac{n(n+\tilde{\alpha})}{(1-\tilde{\alpha})(n+\tilde{\alpha})} = \frac{n(n+\tilde{\alpha})}{n - \tilde{\alpha}(n-1) - \tilde{\alpha}^2} \geq \frac{n(n+\tilde{\alpha})}{n} = n + \tilde{\alpha} = \alpha.$$

Since $\lambda \in \Omega_k$, we have $|\lambda_k| \leq \varepsilon_A \leq 1$, and thus $|\lambda_k|^{\frac{n}{1-\tilde{\alpha}}} \leq |\lambda_k|^\alpha$, and

$$\begin{aligned}|\lambda_k|^n\|(-A_k)^{\tilde{\alpha}}R_\lambda\| &\leq 2^{\tilde{\alpha}}M_{\tilde{\alpha}}\left[(|\lambda_k|^{\frac{n}{1-\tilde{\alpha}}}\|R_\lambda\|)^{1-\tilde{\alpha}} + |\lambda_k|^{n+\tilde{\alpha}}\|R_\lambda\|\right] \\ &\leq 2^{\tilde{\alpha}}M_{\tilde{\alpha}}\left[(|\lambda_k|^{\alpha}\|R_\lambda\|)^{1-\tilde{\alpha}} + |\lambda_k|^{\alpha}\|R_\lambda\|\right] \leq 2^{\tilde{\alpha}}M_{\tilde{\alpha}}\left[M_0^{1-\tilde{\alpha}} + M_0\right] \leq 2^{\tilde{\alpha}+1}M_{\tilde{\alpha}}M_0,\end{aligned}$$

since it was assumed that $M_0 \geq 1$. Therefore the claim holds with $\tilde{M} = 2^{\tilde{\alpha}+1}M_{\tilde{\alpha}}M_0$, which is independent of $k \in \{1, \ldots, N\}$.

We can now turn to showing that there exists $M_1 \geq 1$ such that (10) is satisfied for all $k \in \{1, \ldots, N\}$. First of all, since $(-A_k)_k$ is a uniformly sectorial family of operators, by [10, Cor. 3.1.13] there exists $K > 0$ such that

$$\|(1 - A_k)^{-r}\| \leq K, \qquad \text{and} \qquad \|\Lambda_k^r\| \leq K,$$

for all $0 \leq r \leq \alpha$. Using (11) repeatedly, we get for $\alpha = n + \tilde{\alpha}$

$$\begin{aligned}R(\lambda, A)\Lambda_k^n(1 - A_k)^{-\tilde{\alpha}} &= R(\lambda, A)(-A_k)^n(1 - A_k)^{-\alpha} \\ &= (-\lambda_k)^n R(\lambda, A)(1 - A_k)^{-\alpha} + \sum_{j=0}^{n-1}(-\lambda_k)^{n-1-j}(-A_k)^j(1 - A_k)^{-\alpha}\end{aligned}$$

and thus

$$\begin{aligned}\|R(\lambda, A)\Lambda_k^\alpha\| &= \|(-A_k)^{\tilde{\alpha}}R(\lambda, A)\Lambda_k^n(1 - A_k)^{-\tilde{\alpha}}\| \\ &= \left\|(-\lambda_k)^n(-A_k)^{\tilde{\alpha}}R(\lambda, A)(1 - A_k)^{-\alpha} + \sum_{j=0}^{n-1}(-\lambda_k)^{n-1-j}(-A_k)^{j+\tilde{\alpha}}(1 - A_k)^{-\alpha}\right\| \\ &\leq |\lambda_k|^n\|(-A_k)^{\tilde{\alpha}}R(\lambda, A)\|\|(1 - A_k)^{-\alpha}\| + \sum_{j=0}^{n-1}|\lambda_k|^{n-1-j}\|\Lambda_k^{j+\tilde{\alpha}}\|\|(1 - A_k)^{-(m-j)}\| \\ &\leq \tilde{M}K + \sum_{j=0}^{n-1}\varepsilon_A^{n-1-j}K^2.\end{aligned}$$

Since the bound is indpendent of both $\lambda \in \Omega_k$ and $k \in \{1, \ldots, N\}$, this concludes the proof. □

**Lemma 14.** *Let Assumption* 3 *be satisfied for some* $\alpha \geq 1$*, let* $\beta, \gamma \geq 0$ *be such that* $\beta + \gamma \geq \alpha$*, and let* $0 < c < 1$*. There exists* $\delta > 0$ *such that if* $B \in \mathcal{L}(Y, X)$ *and* $C \in \mathcal{L}(X, Y)$ *satisfy* $\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\beta)$ *and* $\mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\gamma)$ *and*

$$\|B\| + \|(i\omega_k - A)^{-\beta} B\| < \delta, \quad \text{and} \quad \|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\| < \delta,$$

*for every* $k \in \{1, \dots, N\}$*, then* $\|CR(\lambda, A)B\| \leq c < 1$ *for all* $\lambda \in \bigcup_{k=1}^N \Omega_k$*.*

*Proof.* Because $(i\omega_k - A)_k$ is a uniformly sectorial family of operators and since $\beta + \gamma - \alpha \geq 0$, by [10, Cor. 3.1.13] we can choose $K > 0$ such that $\|\Lambda_k^{\beta+\gamma-\alpha}\| \leq K$, for all $k$. Let $M_\Lambda \geq 1$ and $M_1 \geq 1$ be as in Lemmas 11 and 13, respectively, and choose

$$\delta = \frac{\sqrt{c}}{\sqrt{M_1 K M_\Lambda}} > 0.$$

Let $k$ be arbitrary, and let $x, y \in Y$ be such that $\|x\| = \|y\| = 1$. If $B$ and $C$ satisfy the assumptions in the lemma, then $Bx \in \mathcal{D}(\Lambda_k^{-\beta}) = \mathcal{R}((i\omega_k - A)^\beta)$ and $C^* y \in \mathcal{D}((\Lambda_k^*)^{-\gamma}) = \mathcal{R}((-i\omega_k - A^*)^\gamma)$, and for all $\lambda \in \Omega_k$

$$\begin{aligned}|\langle CR(\lambda, A)Bx, y\rangle| &= |\langle \Lambda_k^\gamma R(\lambda, A)\Lambda_k^\beta \Lambda_k^{-\beta} Bx, (\Lambda_k^*)^{-\gamma} C^* y\rangle| \leq \|R(\lambda, A)\Lambda_k^\alpha\| \|\Lambda_k^{\beta+\gamma-\alpha}\| \|\Lambda_k^{-\beta} Bx\| \|(\Lambda_k^*)^{-\gamma} C^* y\| \\ &\leq M_1 K M_\Lambda^2 \|x\| \|y\| \left(\|B\| + \|(-A_k)^{-\beta} B\|\right)\left(\|C\| + \|(-A_k^*)^{-\gamma} C^*\|\right) \leq M_1 K M_\Lambda^2 \delta^2 = c.\end{aligned}$$

This shows that for all $\lambda \in \Omega_k$ we have

$$\|CR(\lambda, A)B\| = \sup_{\|x\|=\|y\|=1} |\langle CR(\lambda, A)Bx, y\rangle| \leq c < 1.$$

Since $k$ was arbitrary, the proof is complete. □

**Lemma 15.** *Let Assumption* 3 *be satisfied. There exists* $M_2 \geq 1$ *such that*

$$\sup_{\lambda \in \overline{\mathbb{C}^+} \setminus (\cup_k \Omega_k)} \|R(\lambda, A)\| \leq M_2.$$

*Proof.* Let $\lambda \in \overline{\mathbb{C}^+} \setminus (\bigcup_k \Omega_k)$ and let $\lambda_0$ be such that $0 \leq \operatorname{Re} \lambda_0 \leq \operatorname{Re} \lambda$, $\operatorname{Im} \lambda_0 = \operatorname{Im} \lambda$ and $\lambda_0$ lies on the boundary of $\overline{\mathbb{C}^+} \setminus (\bigcup_k \Omega_k)$. Then either $\lambda_0 \in i\mathbb{R}$, which implies $\|R(\lambda_0, A)\| \leq M_A$ by Assumption 3, or otherwise $\lambda_0 \in \Omega_k$ and $|\lambda_0 - i\omega_k| = \varepsilon_A$ for some $k \in \{1, \dots, N\}$. By Lemma 12 we have that there exists $M_0$ (independent of $k$) such that in the latter case we have

$$|\lambda_0 - i\omega_k|^\alpha \|R(\lambda_0, A)\| \leq M_0 \qquad \Leftrightarrow \qquad \|R(\lambda_0, A)\| \leq \frac{M_0}{\varepsilon_A^\alpha}.$$

Now, if $M \geq 1$ is such that $\|T(t)\| \leq M$, then $\operatorname{Re} \lambda \|R(\lambda, A)\| \leq M$ by the Hille–Yosida Theorem. Using the resolvent identity $R(\lambda, A) = R(\lambda_0, A) - (\lambda - \lambda_0)R(\lambda_0, A)R(\lambda, A)$ we get

$$\begin{aligned}\|R(\lambda, A)\| &\leq \|R(\lambda_0, A)\|(1 + |\lambda - \lambda_0|\|R(\lambda, A)\|) \\ &\leq \max\{M_A, M_0/\varepsilon_A^\alpha\}(1 + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_0)\|R(\lambda, A)\|) \\ &\leq \max\{M_A, M_0/\varepsilon_A^\alpha\}(1 + \operatorname{Re} \lambda \|R(\lambda, A)\|) \leq \max\{M_A, M_0/\varepsilon_A^\alpha\}(1 + M) =: M_2.\end{aligned}$$

□

**Lemma 16.** *Let Assumption* 3 *be satisfied for some* $\alpha \geq 1$ *and let* $\beta, \gamma \geq 0$ *be such that* $\beta + \gamma \geq \alpha$*. There exists* $\delta > 0$ *such that if* $B \in \mathcal{L}(Y, X)$ *and* $C \in \mathcal{L}(X, Y)$ *satisfy* $\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\beta)$ *and* $\mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\gamma)$ *and*

$$\|B\| + \|(i\omega_k - A)^{-\beta} B\| < \delta, \quad \text{and} \quad \|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\| < \delta$$

*for all* $k \in \{1, \dots, N\}$*, then* $i\omega_k \in \sigma(A + BC) \setminus \sigma_p(A + BC)$ *for every* $k$*.*

*Proof.* Choose $0 \le \beta_1 \le \beta$ and $0 \le \gamma_1 \le \gamma$ in such a way that $\beta_1 + \gamma_1 = 1$. Let $k \in \{1, \dots, N\}$ be arbitrary and assume $\|(i\omega_k - A)^{-\beta_1} B\| < 1$ and $\|(-i\omega_k - A^*)^{-\gamma_1} C^*\| < 1$. Since $0 \le \gamma_1 \le 1$, we have $\mathcal{R}(i\omega_k - A) \subset \mathcal{R}((i\omega_k - A)^{\gamma_1}) \subset X$, which implies $\overline{\mathcal{D}((i\omega_k - A)^{-\gamma_1})} = X$ due to the fact that $i\omega_k \in \sigma_c(A)$. Because of this, the operator $C(i\omega_k - A)^{-\gamma_1}$ has a unique bounded extension $C_{\gamma_1} \in \mathcal{L}(X, Y)$ with norm $\|C_{\gamma_1}\| = \|(-i\omega_k - A^*)^{-\gamma_1} C^*\| < 1$.

Because $\|(i\omega_k - A)^{-\beta_1} B C_{\gamma_1}\| \le \|(i\omega_k - A)^{-\beta_1} B\| \|C_{\gamma_1}\| < 1$, the operator $I - (i\omega_k - A)^{-\beta_1} B C_{\gamma_1}$ is boundedly invertible, and

$$(i\omega_k - A - BC)x = (i\omega_k - A)^{\beta_1}(I - (i\omega_k - A)^{-\beta_1} B C_{\gamma_1})(i\omega_k - A)^{\gamma_1} x$$

for all $x \in \mathcal{D}(i\omega_k - A - BC) = \mathcal{D}(A)$. Since $(i\omega_k - A)^{\beta_1}$ and $(i\omega_k - A)^{\gamma_1}$ are injective and at least one of them is not surjective, the operator $i\omega_k - A - BC$ is injective but not surjective. This implies $i\omega_k \in \sigma(A + BC) \setminus \sigma_p(A + BC)$.

Finally, Lemma 10 can be used to conclude that there exists $\delta > 0$ such that the condition $\|(i\omega_k - A)^{-\beta_1} B\| < 1$ and $\|(-i\omega_k - A^*)^{-\gamma_1} C^*\| < 1$ is satisfied for all $k \in \{1, \dots, N\}$ whenever $\|B\| + \|(i\omega_k - A)^{-\beta} B\| < \delta$, and $\|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\| < \delta$ for all $k \in \{1, \dots, N\}$. □

*Proof of Theorem 7.* Let $0 < c < 1$ and let $M_2 \ge 1$ be as in Lemma 15. Choose $\delta_1 > 0$ as in Lemma 14, and $\delta_2 > 0$ as in Lemma 16. We will show that the claims of the theorem are satisfied with the choice $\delta = \min\{\delta_1, \delta_2, \sqrt{c/M_2}\}$. To this end, for the rest of the proof, we assume that the operators $B$ and $C$ satisfy $\|B\| + \|(i\omega_k - A)^{-\beta} B\| < \delta$, and $\|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\| < \delta$ for all $k \in \{1, \dots, N\}$.

Since $\|B\|, \|C\| < \delta \le \sqrt{c/M_2}$, for all $\lambda \in \overline{\mathbb{C}^+} \setminus (\bigcup_k \Omega_k)$ we have

$$\|CR(\lambda, A)B\| \le \|C\| \|B\| \|R(\lambda, A)\| < \frac{\sqrt{c}}{\sqrt{M_2}} \cdot \frac{\sqrt{c}}{\sqrt{M_2}} \cdot M_2 = c < 1.$$

Furthermore, since $\delta \le \delta_1$, we have from Lemma 14 that $\|CR(\lambda, A)B\| \le c < 1$ also for $\lambda \in \bigcup_k \Omega_k$. Combining these estimates, we can see that $\|CR(\lambda, A)B\| \le c < 1$ and $1 \in \rho(CR(\lambda, A)B)$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_k\}_{k=1}^N$. The Sherman–Morrison–Woodbury formula in Lemma 8 therefore implies that $\overline{\mathbb{C}^+} \setminus \{i\omega_k\}_{k=1}^N \subset \rho(A + BC)$. Finally, since $\delta \le \delta_2$, we have from Lemma 16 that $i\omega_k \in \sigma(A + BC) \setminus \sigma_p(A + BC)$ for all $k$.

If $\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_k\}_k$, then $\|CR(\lambda, A)B\| \le c < 1$ implies

$$\|(I - CR(\lambda, A)B)^{-1}\| = \left\| \sum_{k=0}^{\infty} (CR(\lambda, A)B)^n \right\| \le \sum_{k=0}^{\infty} \|CR(\lambda, A)B\|^n \le \sum_{k=0}^{\infty} c^n = \frac{1}{1 - c},$$

which concludes the final claim of the lemma. □

## 4. Preservation of Strong Stability

In this section we complete the proof of Theorem 4. In particular, this requires showing that under the stated conditions the perturbed semigroup is uniformly bounded. For this we use the following condition on the resolvent operators. The proof of Theorem 17 can be found in [9, Thm. 2].

**Theorem 17.** *Let $A$ generate a semigroup $T(t)$ on a Hilbert space $X$ and let $\sigma(A) \subset \overline{\mathbb{C}^-}$. The semigroup $T(t)$ is uniformly bounded if and only if for all $x, y \in X$ we have*

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \left( \|R(\xi + i\eta, A)x\|^2 + \|R(\xi + i\eta, A)^* y\|^2 \right) d\eta < \infty.$$

We begin with two auxiliary lemmata used in proving the uniform boundedness of the perturbed semigroup, as well as in showing the polynomial boundedness of the perturbed resolvent operator near the points $i\omega_k$.

**Lemma 18.** *Assume $A$ generates a uniformly bounded semigroup on a Hilbert space $X$. If $Y$ is a separable Hilbert space and if $\tilde{B} \in \mathcal{L}(Y,X)$ is a Hilbert–Schmidt operator, then*

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}\|^2 d\eta < \infty, \qquad \text{and} \qquad \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)^*\tilde{B}\|^2 d\eta < \infty.$$

*Proof.* By [16, Rem. 3.2] there exists $M>0$ such that

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)x\|^2 d\eta \le M\|x\|^2, \qquad \text{and} \qquad \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)^*x\|^2 d\eta \le M\|x\|^2$$

for all $x \in X$. If $Y$ is a Hilbert space with an orthonormal basis $(e_l)_{l=1}^\infty \subset Y$ and if $(\tilde{B}e_l)_{l=1}^\infty \in \ell^2(X)$, then

$$\sum_{l=1}^\infty \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}e_l\|^2 d\eta \le M \sum_{l=1}^\infty \|\tilde{B}e_l\|^2 < \infty.$$

Moreover, for every $R \in \mathcal{L}(X)$ we have $\|R\tilde{B}\|^2 \le \sum_{l=1}^\infty \|R\tilde{B}e_l\|^2$. Together these properties imply

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}\|^2 d\eta \le \sum_{l=1}^\infty \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}e_l\|^2 d\eta < \infty.$$

The second claim can be shown analogously. □

The following lemma contains the most important estimates used in the proof of Theorem 4.

**Lemma 19.** *Let $\beta+\gamma \ge \alpha$, and let $\delta>0$ be chosen as in Theorem 7. Assume that $B \in \mathcal{L}(Y,X)$ and $C \in \mathcal{L}(X,Y)$, and for all $k \in \{1,\dots,N\}$ we have $\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\beta)$ and $\mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\gamma)$, the operators $B$, $C^*$, $(i\omega_k - A)^{-\beta}B$, and $(-i\omega_k - A^*)^{-\gamma}C^*$ are Hilbert–Schmidt, and $\|B\|+\|(i\omega_k-A)^{-\beta}B\| < \delta$, and $\|C\|+\|(-i\omega_k - A^*)^{-\gamma}C^*\| < \delta$. Then for every $k \in \{1,\dots,N\}$ there exists a function $f_k : \overline{\mathbb{C}^+} \setminus \{i\omega_l\}_{l=1}^N \to \mathbb{R}^+$ such that*

$$\|R(\lambda,A)B\|\|CR(\lambda,A)\| \le f_k(\lambda) \qquad \forall \lambda \in \Omega_k,$$

*and $f_k(\cdot)$ has the properties $\sup_{0<|\omega-\omega_k|\le\varepsilon_A} |\omega-\omega_k|^\alpha f_k(i\omega) < \infty$ and*

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} f_k(\xi+i\eta)^2 d\eta < \infty.$$

*Proof.* Let $k \in \{1,\dots,N\}$. We begin by considering the case where $\beta,\gamma>0$. Choose $0<\beta_1\le\beta$ and $0<\gamma_1\le\gamma$ in such a way that $\beta_1+\gamma_1=\alpha$. Since $(\Lambda_k)_k$ and $(\Lambda_k^*)_k$ are uniformly sectorial, we have from [10, Cor. 3.1.13] that there exists $K_0>0$ such that $\|\Lambda_l^{\beta-\beta_1}\| \le K_0$ and $\|(\Lambda_l^*)^{\gamma-\gamma_1}\| \le K_0$ for all $l \in \{1,\dots,N\}$. Denote $B_{\beta_1} = \Lambda_k^{-\beta_1}B \in \mathcal{L}(Y,X)$ and $\tilde{C}_{\gamma_1} = (\Lambda_k^*)^{-\gamma_1}C^* \in \mathcal{L}(Y,X)$. By Lemma 11 the norms of these operators satisfy

$$\|B_{\beta_1}\| = \|\Lambda_k^{\beta-\beta_1}\Lambda_k^{-\beta}B\| \le \|\Lambda_k^{\beta-\beta_1}\| \cdot M_\Lambda \left(\|B\| + \|(i\omega_k - A)^{-\beta}B\|\right) < K_0 M_\Lambda \delta$$
$$\|\tilde{C}_{\gamma_1}\| = \|(\Lambda_k^*)^{\gamma-\gamma_1}(\Lambda_k^*)^{-\gamma}C^*\| \le \|(\Lambda_k^*)^{\gamma-\gamma_1}\| \cdot M_\Lambda \left(\|C\| + \|(-i\omega_k - A^*)^{-\gamma}C^*\|\right) < K_0 M_\Lambda \delta.$$

Moreover, if $(e_l)_{l=1}^\infty \subset Y$ is an orthonormal basis of $Y$, then we also have from Lemma 11 that for all $l \in \mathbb{N}$

$$\|B_{\beta_1}e_l\|^2 = \|\Lambda_k^{\beta-\beta_1}\Lambda_k^{-\beta}Be_l\|^2 \le 2K_0^2 M_\Lambda^2 \left(\|Be_l\|^2 + \|(i\omega_k - A)^{-\beta}Be_l\|^2\right)$$
$$\|\tilde{C}_{\gamma_1}e_l\|^2 = \|(\Lambda_k^*)^{\gamma-\gamma_1}(\Lambda_k^*)^{-\gamma}C^*e_l\|^2 \le 2K_0^2 M_\Lambda^2 \left(\|C^*e_l\|^2 + \|(-i\omega_k - A^*)^{-\gamma}C^*e_l\|^2\right).$$

Since $B$, $(i\omega_k - A)^{-\beta}B$, $C^*$, and $(-i\omega_k - A^*)^{-\gamma}C^*$ are Hilbert–Schmidt by assumption, the above estimates imply that also $B_{\beta_1}$ and $\tilde{C}_{\gamma_1}$ are Hilbert–Schmidt operators. Let $M_1 \ge 1$ be as in Lemma 13, and choose constants $M_{\beta_1/\alpha} \geq 1$ and $M_{\gamma_1/\alpha} \geq 1$ as in Lemma 10 corresponding to the families $(\Lambda_k)_k$ and $(\Lambda_k^*)_k$ of operators, respectively. For every $\lambda \in \Omega_k$ and $y \in Y$ with $\|y\| = 1$ we have (denote $R_\lambda = R(\lambda, A)$ for brevity)

$$\begin{aligned}\|R_\lambda By\| &= \|\Lambda_k^{\beta_1} R_\lambda B_{\beta_1} y\| \leq M_{\beta_1/\alpha} \|R_\lambda B_{\beta_1} y\|^{1-\beta_1/\alpha} \|\Lambda_k^\alpha R_\lambda B_{\beta_1} y\|^{\beta_1/\alpha} \\ &\leq M_{\beta_1/\alpha} \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|\Lambda_k^\alpha R_\lambda\|^{\beta_1/\alpha} \|B_{\beta_1}\|^{\beta_1/\alpha} \leq M_{\beta_1/\alpha} M_1^{\beta_1/\alpha} (K_0 M_\Lambda \delta)^{\beta_1/\alpha} \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha}\end{aligned}$$

and

$$\begin{aligned}\|R_\lambda^* C^* y\| &= \|(\Lambda_k^*)^{\gamma_1} R_\lambda^* \tilde{C}_{\gamma_1} y\| \leq M_{\gamma_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1} y\|^{1-\gamma_1/\alpha} \|(\Lambda_k^*)^\alpha R_\lambda^* \tilde{C}_{\gamma_1} y\|^{\gamma_1/\alpha} \\ &\leq M_{\gamma_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \|R_\lambda \Lambda_k^\alpha\|^{\gamma_1/\alpha} \|\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha} \leq M_{\gamma_1/\alpha} M_1^{\gamma_1/\alpha} (K_0 M_\Lambda \delta)^{\gamma_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}.\end{aligned}$$

Therefore, since $\beta_1 + \gamma_1 = \alpha$, for all $\lambda \in \Omega_k$ we have

$$\|R_\lambda B\| \|CR_\lambda\| \leq M_{\beta_1/\alpha} M_{\gamma_1/\alpha} M_1 K_0 M_\Lambda \delta \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} = K \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}$$

for $K = M_{\beta_1/\alpha} M_{\gamma_1/\alpha} M_1 K_0 M_\Lambda \delta$. We choose $f_k(\cdot)$ such that $f_k(\lambda) = K \|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha} \|R_\lambda^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_l\}_{l=1}^N$. It remains to show that $f_k(\cdot)$ has the desired properties.

Since $1 - \beta_1/\alpha + 1 - \gamma_1/\alpha = 1$, for all $\omega \in \mathbb{R}$ with $0 < |\omega - \omega_k| \leq \varepsilon_A$ we have from Assumption 3 that

$$\begin{aligned}|\omega - \omega_k|^\alpha f_k(i\omega) &= |\omega - \omega_k|^\alpha K \|R(i\omega, A) B_{\beta_1}\|^{1-\beta_1/\alpha} \|R(i\omega, A)^* \tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \\ &\leq |\omega - \omega_k|^\alpha \|R(i\omega, A)\| K \|B_{\beta_1}\|^{1-\beta_1/\alpha} \|\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \leq M_A K K_0 M_\Lambda \delta.\end{aligned}$$

This concludes that $\sup_{0<|\omega-\omega_k|\leq\varepsilon_A} |\omega - \omega_k|^\alpha f_k(i\omega) < \infty$.

If we denote $q = 1/(1 - \beta_1/\alpha) > 1$, $r = 1/(1 - \gamma_1/\alpha) > 1$, then $1/q + 1/r = 1$ and the Hölder inequality implies

$$\begin{aligned}&\sup_{\xi>0} \xi \int_{-\infty}^\infty f_k(\xi + i\eta)^2 d\eta \\ &= K^2 \sup_{\xi>0} \xi \int_{-\infty}^\infty \|R(\xi + i\eta, A) B_{\beta_1}\|^{2(1-\beta_1/\alpha)} \|R(\xi + i\eta, A)^* \tilde{C}_{\gamma_1}\|^{2(1-\gamma_1/\alpha)} d\eta \\ &\leq K^2 \left( \sup_{\xi>0} \xi \int_{-\infty}^\infty \|R(\xi + i\eta, A) B_{\beta_1}\|^2 d\eta \right)^{\frac{1}{q}} \left( \sup_{\xi>0} \xi \int_{-\infty}^\infty \|R(\xi + i\eta, A)^* \tilde{C}_{\gamma_1}\|^2 d\eta \right)^{\frac{1}{r}} < \infty\end{aligned}$$

by Lemma 18, since $B_{\beta_1}$ and $\tilde{C}_{\gamma_1}$ are Hilbert–Schmidt operators.

It remains to show that the claims are true if $\beta = 0$ or $\gamma = 0$, or equivalently, whenever either $\gamma \geq \alpha$ or $\beta \geq \alpha$. Let $M_1 \geq 1$ be as in Lemma 13. If $\beta \geq \alpha$, then we have $\mathcal{R}(B) \subset \mathcal{R}((i\omega_k - A)^\alpha)$. Choose $K_0 > 0$ so that $\|\Lambda_l^{\beta-\alpha}\| \leq K_0$ for all $l \in \{1, \dots, N\}$ [10, Cor. 3.1.13]. Using Lemma 11 and $\|\Lambda_k^{-\beta} B\| \leq M_\Lambda \left( \|B\| + \|(i\omega_k - A)^{-\beta} B\| \right) < M_\Lambda \delta$ we have that for all $\lambda \in \Omega_k$

$$\|R_\lambda B\| \|CR_\lambda\| \leq \|R_\lambda \Lambda_k^\alpha\| \|\Lambda_k^{\beta-\alpha}\| \|\Lambda_k^{-\beta} B\| \|R_\lambda^* C^*\| < K \|R_\lambda^* C^*\|,$$

where $K = M_1 K_0 M_\Lambda \delta$. We choose $f_k(\cdot)$ in such a way that $f_k(\lambda) = K \|R_\lambda^* C^*\|$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_l\}_{l=1}^N$. Then $|\omega - \omega_k|^\alpha f_k(i\omega_k) \leq |\omega - \omega_k|^\alpha K \|R_\lambda\| \|C\| \leq K M_A \|C\|$ whenever $0 < |\omega - \omega_k| \leq \varepsilon_A$, and

$$\sup_{\xi>0} \xi \int_{-\infty}^\infty f_k(\xi + i\eta)^2 d\eta = K^2 \sup_{\xi>0} \xi \int_{-\infty}^\infty \|R(\xi + i\eta, A)^* C^*\|^2 d\eta < \infty$$

by Lemma 18, because $C^*$ is Hilbert–Schmidt. Similarly, if $\gamma \geq \alpha$, then $\mathcal{R}(C^*) \subset \mathcal{R}((-i\omega_k - A^*)^\alpha)$, and we choose $K_0 > 0$ so that $\|\Lambda_l^{\gamma-\alpha}\| \leq K_0$ for all $l \in \{1, \dots, N\}$ [10, Cor. 3.1.13]. Using Lemma 11 and $\|(\Lambda_k^*)^{-\gamma} C^*\| \leq M_\Lambda (\|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\|) < M_\Lambda \delta$, we have that for all $\lambda \in \Omega_k$

$$\|R_\lambda B\| \|CR_\lambda\| \leq \|R_\lambda B\| \|R_\lambda \Lambda_k^\alpha\| \|\Lambda_k^{\gamma-\alpha}\| \|(\Lambda_k^*)^{-\gamma} C^*\| < K \|R_\lambda B\|,$$

where $K = M_1 K_0 M_\Lambda \delta$. We now choose $f_k(\cdot)$ so that $f_k(\lambda) = K\|R_\lambda B\|$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_l\}_{l=1}^N$. Then we again have $|\omega - \omega_k|^\alpha f_k(i\omega_k) \le |\omega - \omega_k|^\alpha K\|R_\lambda\|\|B\| \le K M_A \|B\|$ whenever $0 < |\omega - \omega_k| \le \varepsilon_A$, and

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} f_k(\xi + i\eta)^2 d\eta = K^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)B\|^2 d\eta < \infty$$

by Lemma 18, because $B$ is Hilbert–Schmidt. □

*Proof of Theorem* 4. Let $\delta > 0$ be chosen as in Theorem 7 and assume $\|B\| + \|(i\omega_k - A)^{-\beta} B\| < \delta$, and $\|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\| < \delta$ for all $k \in \{1, \dots, N\}$. By Theorem 7 there exists $M_D \ge 1$ such that $\|(I - CR(\lambda, A)B)^{-1}\| \le M_D$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{i\omega_k\}_{k=1}^N$. We begin the proof by showing that the semigroup generated by $A + BC$ is uniformly bounded.

Let $x \in X$ and for brevity denote $\lambda = \xi + i\eta$, $R_\lambda = R(\xi + i\eta, A)$ and $D_\lambda = I - CR(\xi + i\eta, A)B$. Using the Sherman–Morrison–Woodbury formula in Lemma 8 and the scalar inequality $(a + b)^2 \le 2(a^2 + b^2)$ for $a, b \ge 0$ we get

$$\begin{aligned}
\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A + BC)x\|^2 d\eta &= \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda x + R_\lambda B D_\lambda^{-1} C R_\lambda x\|^2 d\eta \\
&\le 2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \left( \|R_\lambda x\|^2 + \|R_\lambda B\|^2 \|D_\lambda^{-1}\|^2 \|C R_\lambda\|^2 \|x\|^2 \right) d\eta \\
&\le 2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda x\|^2 d\eta + 2 M_D^2 \|x\|^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2 \|C R_\lambda\|^2 d\eta.
\end{aligned}$$

Similarly, using $\|(R_\lambda B D_\lambda^{-1} C R_\lambda)^*\| = \|R_\lambda B D_\lambda^{-1} C R_\lambda\| \le M_D \|R_\lambda B\| \|C R_\lambda\|$ we get

$$\begin{aligned}
\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A + BC)^* x\|^2 d\eta &= \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda^* x + (R_\lambda B D_\lambda^{-1} C R_\lambda)^* x\|^2 d\eta \\
&\le 2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda^* x\|^2 d\eta + 2 M_D^2 \|x\|^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2 \|C R_\lambda\|^2 d\eta.
\end{aligned}$$

In both cases the first supremums are finite by Theorem 17. Therefore, Theorem 17 implies that the semigroup generated by $A + BC$ is uniformly bounded if

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2 \|C R_\lambda\|^2 d\eta < \infty. \tag{12}$$

For all $k \in \{1, \dots, N\}$ let $f_k(\cdot)$ be the functions in Lemma 19. By Lemma 15 we can choose $M_2 \ge 1$ such that $\|R(\lambda, A)\| \le M_2$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus (\bigcup_k \Omega_k)$.

Let $\xi > 0$. For each $k \in \{1, \dots, N\}$ denote by $E_k^\xi \subset \mathbb{R}$ the interval such that $\xi + i\eta \in \Omega_k$ if and only if $\eta \in E_k^\xi$. Finally, denote $E^\xi = \mathbb{R} \setminus \left( \bigcup_k E_k^\xi \right)$. Now

$$\begin{aligned}
\xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2 \|C R_\lambda\|^2 d\eta &= \xi \int_{E^\xi} \|R_\lambda B\|^2 \|C R_\lambda\|^2 d\eta + \sum_{k=1}^N \xi \int_{E_k^\xi} \|R_\lambda B\|^2 \|C R_\lambda\|^2 d\eta \\
&\le \xi \int_{E^\xi} \|R_\lambda B\|^2 \|C\|^2 M_2^2 d\eta + \sum_{k=1}^N \xi \int_{E_k^\xi} f_k(\xi + i\eta)^2 d\eta \\
&\le M_2^2 \|C\|^2 \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2 d\eta + \sum_{k=1}^N \xi \int_{-\infty}^{\infty} f_k(\xi + i\eta)^2 d\eta \\
&\le M_2^2 \|C\|^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2 d\eta + \sum_{k=1}^N \sup_{\xi>0} \xi \int_{-\infty}^{\infty} f_k(\xi + i\eta)^2 d\eta < \infty
\end{aligned}$$

by Lemmas 18 and 19. Since the bound is independent of $\xi > 0$, this shows that (12) is satisfied, and thus concludes that the semigroup generated by $A + BC$ is uniformly bounded.

Since the perturbed semigroup is uniformly bounded and $X$ is a Hilbert space, the Mean Ergodic Theorem [2, Sec. 4.3] implies that $\sigma(A + BC) \cap i\mathbb{R} \subset \sigma_p(A + BC) \cup \sigma_c(A + BC)$. However, by Theorem 7 we have that $\sigma_p(A + BC) \cap i\mathbb{R} = \varnothing$ and $i\mathbb{R} \cap \sigma(A + BC) = \{i\omega_k\}_{k=1}^N$. This concludes that $\sigma(A + BC) \cap i\mathbb{R} = \sigma_c(A + BC) \cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$.

Theorem 7 shows that $\sigma(A + BC) \cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$ is countable and $\sigma_p(A + BC) \cap i\mathbb{R} = \varnothing$. The Arent–Batty–Lyubich–Vũ Theorem [1, 11] therefore concludes that the semigroup generated by $A + BC$ is strongly stable.

It remains to show that for all $k \in \{1, \ldots, N\}$ the resolvent operator $R(\lambda, A + BC)$ satisfies

$$\sup_{0<|\omega-\omega_k|\leq\varepsilon_A} |\omega - \omega_k|^\alpha \|R(i\omega, A + BC)\| < \infty. \tag{13}$$

To this end, let $k \in \{1, \ldots, N\}$ be arbitary. By Lemma 19 there exists $M_k \geq 1$ such that $|\omega - \omega_k|^\alpha f_k(i\omega) \leq M_k$ whenever $0 < |\omega - \omega_k| \leq \varepsilon_A$. The Sherman–Morrison–Woodbury formula in Lemma 8 implies that for all $\omega \in \mathbb{R}$ satisfying $0 < |\omega - \omega_k| \leq \varepsilon_A$ we have

$$\begin{aligned}\|R(i\omega, A + BC)\| &= \|R(i\omega, A) + R(i\omega, A)B(I - CR(i\omega, A)B)^{-1}CR(i\omega, A)\| \\ &\leq \|R(i\omega, A)\| + \|R(i\omega, A)B\|\|(I - CR(i\omega, A)B)^{-1}\|\|CR(i\omega, A)\| \\ &\leq \|R(i\omega, A)\| + M_D f_k(i\omega),\end{aligned}$$

and thus

$$|\omega - \omega_k|^\alpha \|R(i\omega, A + BC)\| \leq |\omega - \omega_k|^\alpha \|R(i\omega, A)\| + M_D |\omega - \omega_k|^\alpha f_k(i\omega) \leq M_A + M_D M_k.$$

This concludes that (13) is satisfied. On the other hand, if $|\omega - \omega_k| > \varepsilon_A$ for all $k \in \{1, \ldots, N\}$, then

$$\begin{aligned}\|R(i\omega, A + BC)\| &\leq \|R(i\omega, A)\| + \|R(i\omega, A)B\|\|(I - CR(i\omega, A)B)^{-1}\|\|CR(i\omega, A)\| \\ &\leq \|R(i\omega, A)\| + M_D\|B\|\|C\|\|R(i\omega, A)\|^2 \leq M_A + M_D\|B\|\|C\|M_A^2,\end{aligned}$$

and thus $\|R(i\omega, A + BC)\|$ is uniformly bounded for $\omega \in \mathbb{R}$ satisfying $|\omega - \omega_k| > \varepsilon_A$ for all $k \in \{1, \ldots, N\}$. This concludes the proof. □

**Remark 20.** If the exponents in Theorem 4 satisfy $\beta, \gamma \geq \alpha$, then the strong stability is preserved even without the assumption that $B$, $C^*$, $(i\omega_k - A)^{-\beta}B$, and $(-i\omega_k - A^*)^{-\gamma}C^*$ are Hilbert–Schmidt operators for all $k \in \{1, \ldots, N\}$. Indeed, in the proof of Theorem 4 this assumption was only required in showing that perturbed semigroup is uniformly bounded. If $\beta, \gamma \geq \alpha$, then the uniform boundedness of the perturbed semigroup can be verified similarly as in the proof of [13, Thm. 5], due to the fact that we then have

$$\sup_{\lambda\in\mathbb{C}^+} \|R(\lambda, A)B\| < \infty \qquad \text{and} \qquad \sup_{\lambda\in\mathbb{C}^+} \|CR(\lambda, A)\| < \infty.$$

## 5. Polynomial Decay Rates for the Semigroup $T(t)$

Our main goal in this section is to study the connection between Assumption 3 and the asymptotic behaviour of the semigroup $T(t)$ using the theory developed in [5, 6, 4]. In this study we use the operators

$$\Lambda_k(\xi) = (i\omega_k - A)(\xi + i\omega_k - A)^{-1} = I - \xi(\xi + i\omega_k - A)^{-1}$$

for $0 < \xi \leq 1$ and $k \in \{1, \ldots, N\}$. The motivation for introducing the parameter $\xi$ is that if the distance between the points $\omega_k$ and $\omega_l$ is small, the product operator $\Lambda_1 \cdots \Lambda_N$ will not be sectorial. The parameter

$\xi$ gives us control over the properties of $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$. It is clear that for a fixed $0<\xi\leq 1$ the operators $\Lambda_k(\xi)$ have all of the properties of $\Lambda_k$ in Lemma 9. From [10, Cor. 3.1.13] we also have

$$\|\Lambda_k(\xi)^\alpha\| \leq \sup_{\lambda>0} \left\| \left( (i\omega_k - A)(\lambda + i\omega_k - A)^{-1} \right)^\alpha \right\| \leq (M+1)^{\lceil \alpha \rceil}$$

for $M = \sup_{t>0}\|T(t)\|$.

The following theorem outlines the connection between the behaviour of the resolvent operator on $i\mathbb{R}$ and the asymptotic behaviour of the semigroup. We will see that the implication from part (i) to part (ii) is a direct corollary of Theorem 4.7 in [4]. The implication from (iii) to (i) is shown with the technique used in the proof of Theorem 6.10 in [4] with appropriate modifications to accomodate for multiple spectral points on $i\mathbb{R}$. The theorem presents a partial generalization of Theorem 7.6 in [4] to the situation where $A$ may have more than one (but still finitely many) spectral points on the imaginary axis and the norms $\|R(i\omega,A)\|$ are uniformly bounded for large $|\omega|$. Theorem 21 in particular shows that in the case of a finite intersection $\sigma(A)\cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$, the operator $\Lambda_0 = (i\omega_0 - A)(1+i\omega_0 - A)^{-1}$ that was used in cancelling the resolvent growth in [4, Thm. 7.6] should be replaced with a product of operators of the same form, one for each spectral point $i\omega_k \in \sigma(A)$. The generalization is not complete due to the fact that the scaling property from part (ii) to (iii) requires additional assumptions. The decay of semigroups for which the intersection $\sigma(A)\cap i\mathbb{R}$ is finite was also recently studied in [12, Sec. 3.2] (in addition, in [12] the norm $\|R(i\omega,A)\|$ was allowed to grow polynomially for large $|\omega|$). In particular, in [12, Thm. 3.4] it was shown that the growth rates of the resolvent imply nonuniform decay of the semigroup on a Banach space. Theorem 21 illustrates that on a Hilbert space, the logarithmic terms of the decay rates in [12, Thm. 3.4] can be removed.

**Theorem 21.** *Assume $A$ generates a uniformly bounded semigroup $T(t)$ on the Hilbert space $X$ and let $\alpha\geq 1$ and $0<\xi\leq 1$. Consider the following conditions.*

(i) *The operator $A$ satisfies the conditions of Assumption 3, i.e., $\sigma(A)\cap i\mathbb{R}\subset\{i\omega_k\}_{k=1}^N$,*

$$\sup_{0<|\omega-\omega_k|\leq\varepsilon_A} |\omega-\omega_k|^\alpha \|R(i\omega,A)\| < \infty, \qquad \textit{and} \qquad \sup_{|\omega| \text{ large}} \|R(i\omega,A)\| < \infty.$$

(ii) *There exists $M_T\geq 1$ such that*

$$\|T(t)\Lambda_1(\xi)^\alpha\Lambda_2(\xi)^\alpha\cdots\Lambda_N(\xi)^\alpha\| \leq \frac{M_T}{t} \qquad \forall t>0. \tag{14}$$

(iii) *There exists $\tilde{M}_T\geq 1$ such that*

$$\|T(t)\Lambda_1(\xi)\Lambda_2(\xi)\cdots\Lambda_N(\xi)\| \leq \frac{\tilde{M}_T}{t^{1/\alpha}} \qquad \forall t>0. \tag{15}$$

*For every fixed $0<\xi\leq 1$ the condition* (i) *implies* (ii)*, and* (iii) *implies* (i)*. Moreover, if $\alpha\in\mathbb{N}$ and if $0<\xi\leq 1$ is such that $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$ is a sectorial operator, then for this fixed $\xi$ the conditions* (i)–(iii) *are equivalent.*

As we will see in the proof of Theorem 21, the additional assumptions for the implication from (ii) to (iii) are required to first use The Moment Inequality for the operator $\Lambda_1(\xi)\Lambda_2(\xi)\cdots\Lambda_N(\xi)$, and subsequently to show that $(\Lambda_1(\xi)\Lambda_2(\xi)\cdots\Lambda_N(\xi))^\alpha = \Lambda_1(\xi)^\alpha\Lambda_2(\xi)^\alpha\cdots\Lambda_N(\xi)^\alpha$. It is an important open problem to study whether the implication from (ii) to (iii) remains valid without the additional conditions, or under less strict assumptions. Another related open problem is to find sufficient conditions for the sectoriality of the operator $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$. Later in this section we show that if $A$ is a normal operator, then $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$ will be sectorial provided that $0<\xi\leq 1$ is small enough.

We begin the proof of the theorem by showing the implications from (i) to (ii) and from (iii) to (i) separately in Lemmas 22 and 23, respectively. The first one of these implications follows from a more general result first presented in Theorem 4.7 of [4].

**Lemma 22.** *The implication from* (i) *to* (ii) *in Theorem* 21 *is true for any* $0 < \xi \le 1$.

*Proof.* Let $0 < \xi \le 1$ be fixed. By slightly modifying the proof of Lemma 13, it is easy to see that there exists $\tilde{M}_1 \ge 1$ such that for all $k \in \{1, \dots, N\}$ we have

$$\|R(\lambda, A)\Lambda_k(\xi)^\alpha\| \le \tilde{M}_1 \qquad \forall \lambda \in \Omega_k.$$

Therefore, for any $k \in \{1, \dots, N\}$ and $\lambda \in \Omega_k$ we have

$$\|R(\lambda, A)\Lambda_1(\xi)^\alpha \cdots \Lambda_N(\xi)^\alpha\| \le \|R(\lambda, A)\Lambda_k(\xi)^\alpha\| \cdot \prod_{\substack{l=1 \\ l \ne k}}^{N} \|\Lambda_l(\xi)^\alpha\| \le \tilde{M}_1 \prod_{\substack{l=1 \\ l \ne k}}^{N} (M+1)^{\lceil \alpha \rceil} = \tilde{M}_1 (M+1)^{(N-1)\lceil \alpha \rceil}.$$

Moreover, for all $\lambda \in \overline{\mathbb{C}^+} \setminus \left( \bigcup_{l=1}^N \Omega_l \right)$ we have

$$\|R(\lambda, A)\Lambda_1(\xi)^\alpha \cdots \Lambda_N(\xi)^\alpha\| \le \|R(\lambda, A)\| \|\Lambda_1(\xi)^\alpha\| \cdots \|\Lambda_N(\xi)^\alpha\| \le M_2 (M+1)^{N\lceil \alpha \rceil}$$

by Lemma 15. This concludes that $\sup_{\lambda \in \mathbb{C}^+} \|R(\lambda, A)\Lambda_1(\xi)^\alpha \cdots \Lambda_N(\xi)^\alpha\| < \infty$. Since $\Lambda_1(\xi)^\alpha \cdots \Lambda_N(\xi)^\alpha$ commutes with $T(t)$, we have from [4, Thm. 4.7] that there exists $M_T \ge 1$ such that (14) is satisfied. $\square$

The implication from (iii) to (i) can be shown using the technique developed in the proof of [4, Thm. 6.10] (see also [5, Sec. 2]) with modifications to accomodate for the multiple spectral points on the imaginary axis.

**Lemma 23.** *The implication from* (iii) *to* (i) *in Theorem* 21 *is true for any* $0 < \xi \le 1$.

*Proof.* Let $0 < \xi \le 1$ be fixed. Let $\omega \in \mathbb{R} \setminus \{\omega_k\}_{k=1}^N$, and denote $s_k = i\omega - i\omega_k$ and $A_k = A - i\omega_k$ for all $k \in \{1, \dots, N\}$. Then $s_k - A_k = i\omega - A$, and for any $x \in \mathcal{D}(A)$ and $k \in \{1, \dots, N\}$ we have

$$\begin{aligned} x &= \frac{s_k}{s_k}(\xi - A_k)^{-1}(\xi - A_k)x = \frac{1}{s_k}(\xi - A_k)^{-1}\left[\xi(s_k - A_k)x + (s_k - \xi)(-A_k)x\right] \\ &= \frac{\xi}{s_k}(\xi - A_k)^{-1}(i\omega - A)x + \frac{s_k - \xi}{s_k}\Lambda_k(\xi)x. \end{aligned}$$

Applying this identity for every $k$ from 1 to $N$ yields

$$\begin{aligned} x &= \frac{\xi}{s_1}(\xi - A_1)^{-1}(i\omega - A)x + \frac{s_1 - \xi}{s_1}\Lambda_1(\xi)x \\ &= \left[\frac{\xi}{s_1}(\xi - A_1)^{-1} + \frac{\xi(s_1 - \xi)}{s_1 s_2}(\xi - A_2)^{-1}\Lambda_1(\xi)\right](i\omega - A)x + \frac{(s_1 - \xi)(s_2 - \xi)}{s_1 s_2}\Lambda_1(\xi)\Lambda_2(\xi)x \\ &= \cdots = \left[\sum_{l=1}^{N} \frac{\xi^l \prod_{j=1}^{l-1}(s_j - \xi)}{\prod_{j=1}^{l} s_j}(\xi - A_l)^{-1}\prod_{j=1}^{l-1}\Lambda_j(\xi)\right](i\omega - A)x + \frac{\prod_{j=1}^{N}(s_j - \xi)}{\prod_{j=1}^{N} s_j}\Lambda_1(\xi)\cdots\Lambda_N(\xi)x. \end{aligned}$$

If $M = \sup_{t \ge 0}\|T(t)\|$, then the Hille–Yosida Theorem implies $\xi\|(\xi - A_k)^{-1}\| \le M$ and $\|\Lambda_k(\xi)\| = \|I - \xi(\xi - A_k)^{-1}\| \le 1 + M$ for every $k \in \{1, \dots, N\}$. We thus have

$$\begin{aligned} \|T(t)x\| &\le \|T(t)\| \left[\sum_{l=1}^{N} \frac{\xi^{l-1}\prod_{j=1}^{l-1}|s_j - \xi|}{\prod_{j=1}^{l}|s_j|} \cdot \xi\|(\xi - A_l)^{-1}\| \cdot \prod_{j=1}^{l-1}(M+1)^j\right] \|(i\omega - A)x\| \\ &\quad + \frac{\prod_{j=1}^{N}|s_j - \xi|}{\prod_{j=1}^{N}|s_j|}\|T(t)\Lambda_1(\xi)\cdots\Lambda_N(\xi)x\| \\ &\le M^2(M+1)^{N-1}\sum_{l=1}^{N}\frac{\prod_{j=1}^{l-1}|s_j - \xi|}{\prod_{j=1}^{l}|s_j|}\|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}}\frac{\prod_{j=1}^{N}|s_j - \xi|}{\prod_{j=1}^{N}|s_j|}\|x\|. \end{aligned}$$

Since for all $t > 0$

$$e^{i\omega t}x = e^{i\omega t}\int_0^t e^{-i\omega r}T(r)(i\omega - A)xdr + T(t)x$$

we can estimate

$$\begin{aligned}
\|x\| = \|e^{i\omega t}x\| &= \left\| e^{i\omega t}\int_0^t e^{-i\omega r}T(r)(i\omega - A)xdr + T(t)x \right\| \\
&\le \int_0^t \|e^{-i\omega r}T(r)\|\|(i\omega - A)x\|dr + \|T(t)x\| \le tM\|(i\omega - A)x\| + \|T(t)x\| \\
&\le \left( tM + M^2(M+1)^{N-1}\sum_{l=1}^N \frac{\prod_{j=1}^{l-1}|s_j - \xi|}{\prod_{j=1}^{l}|s_j|} \right)\|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}}\frac{\prod_{j=1}^{N}|s_j - \xi|}{\prod_{j=1}^{N}|s_j|}\|x\|.
\end{aligned}$$

If $\omega \in \mathbb{R}\setminus\{\omega_k\}_k$ is fixed, we can choose $t > 0$ to be large enough so that $\frac{\tilde{M}_T}{t^{1/\alpha}}(\prod_{j=1}^N|s_j - \xi|)/(\prod_{j=1}^N|s_j|) < 1$. Then the above estimate implies that there exists $c > 0$ such that

$$\|(i\omega - A)x\| \ge c\|x\|, \qquad x \in \mathcal{D}(A).$$

This means that $i\omega \notin \sigma_p(A)$ and $i\omega \notin \sigma_c(A)$. Since the semigroup $T(t)$ is uniformly bounded, we have from the Mean Ergodic Theorem [2, Sec. 4.3] that $\sigma(A) \cap i\mathbb{R} \subset \sigma_p(A) \cup \sigma_c(A)$, and thus $i\omega \in \rho(A)$. Since $i\omega$ was arbitrary, we have shown that $\sigma(A) \cap i\mathbb{R} \subset \{i\omega_k\}_{k=1}^N$.

Fix $k \in \{1, \dots, N\}$ and consider $\omega \in \mathbb{R}\setminus\{\omega_k\}_k$ such that $0 < |\omega - \omega_k| \le \varepsilon_A$. Then there exists $K \ge 1$ such that $|s_l - \xi| \le |\omega - \omega_l| + \xi \le K$ for every $l \in \{1, \dots, N\}$, and $1/|s_l| \le 1/\varepsilon_A$ for every $l \ne k$, due to the definition of $\varepsilon_A$. The estimate above implies

$$\begin{aligned}
\|x\| &\le \left( Mt + M^2(M+1)^{N-1}\sum_{l=1}^N \frac{\prod_{j=1}^{l-1}|s_j - \xi|}{\prod_{j=1}^{l}|s_j|} \right)\|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}}\frac{\prod_{j=1}^{N}|s_j - \xi|}{\prod_{j=1}^{N}|s_j|}\|x\| \\
&\le \left[ Mt + M^2(M+1)^{N-1}\left( \sum_{l=1}^{k-1}\frac{K^{l-1}}{\varepsilon_A^l} + \frac{1}{|s_k|}\sum_{l=k}^{N}\frac{K^{l-1}}{\varepsilon_A^{l-1}} \right) \right]\|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}}\frac{1}{|s_k|}\frac{K^N}{\varepsilon_A^{N-1}}\|x\| \\
&\le \left[ Mt + \frac{1}{|s_k|}M^2(M+1)^{N-1}N\frac{K^{N-1}}{\varepsilon_A^N} \right]\|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}}\frac{1}{|s_k|}\frac{K^N}{\varepsilon_A^{N-1}}\|x\|
\end{aligned}$$

since $1/|s_k| \ge 1/\varepsilon_A \ge 1$. Choosing $t = \left(2K^N/(\varepsilon_A^{N-1}|s_k|)\right)^\alpha$, we get

$$\|x\| \le \left( \frac{2^\alpha MK^{\alpha N}}{\varepsilon_A^{\alpha(N-1)}}\frac{1}{|s_k|^\alpha} + \frac{1}{|s_k|}NM^2\frac{K^{N-1}}{\varepsilon_A^N}(M+1)^{N-1} \right)\|(i\omega - A)x\| + \frac{1}{2}\|x\|.$$

This immediately implies

$$\begin{aligned}
|\omega - \omega_k|^\alpha\|R(i\omega, A)\| &\le \left( \frac{2^\alpha MK^{\alpha N}}{\varepsilon_A^{\alpha(N-1)}} + |\omega - \omega_k|^{\alpha-1}NM^2\frac{K^{N-1}}{\varepsilon_A^N}(M+1)^{N-1} \right) \\
&\le \left( \frac{2^\alpha MK^{\alpha N}}{\varepsilon_A^{\alpha(N-1)}} + \varepsilon_A^{\alpha-1}NM^2\frac{K^{N-1}}{\varepsilon_A^N}(M+1)^{N-1} \right),
\end{aligned}$$

which concludes that $\sup_{0<|\omega-\omega_k|\le\varepsilon_A}|\omega - \omega_k|^\alpha\|R(i\omega, A)\| < \infty$.

It remains to show that $\|R(i\omega, A)\|$ is uniformly bounded for $\omega \in \mathbb{R}$ with large $|\omega|$. If $\omega \in \mathbb{R}$ is such that $|\omega - \omega_k| \geq \varepsilon_A$ for all $k \in \{1, \dots, N\}$, then $|s_k| \geq \varepsilon_A$ and $|s_k - \xi|/|s_k| = |1 - \xi/s_k| \leq 1 + \xi/\varepsilon_A \leq 2/\varepsilon_A$.

$$\begin{aligned}\|x\| &\leq \left( tM + M^2(M+1)^{N-1} \sum_{l=1}^{N} \frac{\prod_{j=1}^{l-1}|s_j - \xi|}{\prod_{j=1}^{l}|s_j|} \right) \|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}} \frac{\prod_{j=1}^{N}|s_j - \xi|}{\prod_{j=1}^{N}|s_j|} \|x\| \\ &\leq \left( tM + M^2(M+1)^{N-1} \sum_{l=1}^{N} \frac{2^{l-1}}{\varepsilon_A^l} \right) \|(i\omega - A)x\| + \frac{\tilde{M}_T}{t^{1/\alpha}} \frac{2^N}{\varepsilon_A^N} \|x\|.\end{aligned}$$

If we choose $t = \left(2^{N+1}\tilde{M}_T/\varepsilon_A^N\right)^\alpha$, then $2^N\tilde{M}_T/(t^{1/\alpha}\varepsilon_A^N) = 1/2$, and the above estimate implies

$$\|R(i\omega, A)\| \leq 2M \left( \frac{2^{N+1}\tilde{M}_T}{\varepsilon_A^N} \right)^\alpha + 2M^2(M+1)^{N-1} \sum_{l=1}^{N} \frac{2^{l-1}}{\varepsilon_A^l}.$$

Since the bound is independent of $\omega$, this concludes that $\sup_{|\omega - \omega_k| \geq \varepsilon_A} \|R(i\omega, A)\| < \infty$. □

*Proof of Theorem* 21. Lemmas 22 and 23 show that (i) implies (ii) and that (iii) implies (i) for any fixed $0 < \xi \leq 1$. It remains to show that if $\alpha \in \mathbb{N}$ and $0 < \xi \leq 1$ is such that $\Lambda_1(\xi) \cdots \Lambda_N(\xi)$ is sectorial, then for this $\xi$ part (ii) implies part (iii). To this end, let $\alpha \in \mathbb{N}$ and fix $0 < \xi \leq 1$ in such a way that $\Lambda_1(\xi) \cdots \Lambda_N(\xi)$ is sectorial.

Since $\alpha \in \mathbb{N}$ and $\Lambda_k(\xi)$ and $\Lambda_l(\xi)$ commute, we have $(\Lambda_1(\xi) \cdots \Lambda_N(\xi))^\alpha = \Lambda_1(\xi)^\alpha \cdots \Lambda_N(\xi)^\alpha$. Let $M = \sup_{t>0}\|T(t)\|$. If (ii) is satisfied, then by the Moment Inequality [10, Prop. 6.6.4] there exists $M_\alpha \geq 1$ such that for any $x \in X$ we have

$$\begin{aligned}\|T(t)\Lambda_1(\xi) \cdots \Lambda_N(\xi)x\| &= \|\Lambda_1(\xi) \cdots \Lambda_N(\xi)T(t)x\| \leq M_\alpha \|T(t)x\|^{1-1/\alpha} \|(\Lambda_1(\xi) \cdots \Lambda_N(\xi))^\alpha T(t)x\|^{1/\alpha} \\ &\leq M_\alpha M^{1-1/\alpha} \|x\|^{1-1/\alpha} \|T(t)\Lambda_1(\xi)^\alpha \cdots \Lambda_N(\xi)^\alpha x\|^{1/\alpha} \leq M_\alpha M^{1-1/\alpha} \|x\| \frac{M_T^{1/\alpha}}{t^{1/\alpha}}.\end{aligned}$$

This immediately implies (15) with the choice $\tilde{M}_T = M_\alpha M^{1-1/\alpha} M_T^{1/\alpha}$. □

We conclude this section by showing that if $A$ is a normal operator (or more generally, similar to a normal operator), then $\Lambda_1(\xi) \cdots \Lambda_N(\xi)$ is sectorial for all sufficiently small $0 < \xi \leq 1$. To show this, we define a function

$$f_\xi(z) = \prod_{k=1}^{N} \frac{i\omega_k + z}{\xi + i\omega_k + z}. \tag{16}$$

For all $\xi > 0$ the poles of $f_\xi(\cdot)$ are located at $-\xi - i\omega_k \in \mathbb{C}^-$, and $f_\xi(\cdot)$ is holomorphic in $\mathbb{C}^+$ and bounded in $\overline{\mathbb{C}^+}$. For an angle $\varphi \in (0, \pi)$ we denote a sector by

$$S_\varphi = \{\, z \in \mathbb{C} \setminus \{0\} \mid |\arg(z)| < \varphi \,\}.$$

**Lemma 24.** *Let $\varphi \in (\frac{\pi}{2}, \pi)$. There exists $\xi_0 > 0$ such that $f_\xi(\overline{\mathbb{C}^+}) \subset S_\varphi \cup \{0\}$ whenever $0 < \xi \leq \xi_0$.*

*Proof.* We can without loss of generality assume that the indexing of the set $\{i\omega_k\}_{k=1}^N$ is such that $\omega_1 < \omega_2 < \cdots < \omega_N$. Let $\varphi \in (\pi/2, \pi)$, denote $d = \min_{k \neq l}|\omega_k - \omega_l| > 0$ and let $\operatorname{Arg}(\cdot) : \mathbb{C} \to (-\pi, \pi]$ denote the principal argument. Let $z \in \overline{\mathbb{C}^+}$ be such that $z = \mu + i\omega$ (with $\mu \geq 0$). Then

$$\begin{aligned}\sum_{k=1}^{N} \operatorname{Arg}\left( \frac{\mu + i(\omega_k + \omega)}{\xi + \mu + i(\omega_k + \omega)} \right) &= \sum_{k=1}^{N} \operatorname{Arg}\left( \frac{(\mu + i(\omega_k + \omega))(\xi + \mu - i(\omega_k + \omega))}{|\xi + \mu + i(\omega_k + \omega)|^2} \right) \\ &= \sum_{k=1}^{N} \operatorname{Arg}\left( (\xi + \mu)\mu + (\omega_k + \omega)^2 + i\xi(\omega_k + \omega) \right).\end{aligned}$$

We will derive upper and lower bounds for the above sum of arguments. We start with the upper bound. If $\omega_k + \omega < 0$ for all $k \in \{1, \dots, N\}$, then the arguments in the last sum are all negative and the value of the sum is less than zero. If this is not the case, let $k_0 \in \{1, \dots, N\}$ be the smallest index such that $\omega_{k_0} + \omega \geq 0$, or equivalently $\omega \geq -\omega_{k_0}$. Then for indices $k \in \{1, \dots, k_0 - 1\}$ the arguments in the sum are negative, and we can estimate (using the fact that arctan is an increasing function)

$$\begin{aligned}\sum_{k=1}^{N} \operatorname{Arg}\left((\xi+\mu)\mu + (\omega_k+\omega)^2 + i\xi(\omega_k+\omega)\right) &\leq \sum_{k=k_0}^{N} \operatorname{Arg}\left((\xi+\mu)\mu + (\omega_k+\omega)^2 + i\xi(\omega_k+\omega)\right) \\ &= \operatorname{Arg}\left((\xi+\mu)\mu + (\omega_{k_0}+\omega)^2 + i\xi(\omega_{k_0}+\omega)\right) + \sum_{k=k_0+1}^{N} \arctan\left(\frac{\xi(\omega_k+\omega)}{(\xi+\mu)\mu + (\omega_k+\omega)^2}\right) \\ &\leq \frac{\pi}{2} + \sum_{k=k_0+1}^{N} \arctan\left(\frac{\xi(\omega_k+\omega)}{(\omega_k+\omega)^2}\right) = \frac{\pi}{2} + \sum_{k=k_0+1}^{N} \arctan\left(\frac{\xi}{\omega_k+\omega}\right) \\ &\leq \frac{\pi}{2} + \sum_{k=k_0+1}^{N} \arctan\left(\frac{\xi}{\omega_{k_0} + d(k-k_0) + \omega}\right) \leq \frac{\pi}{2} + \sum_{k=1}^{N-k_0} \arctan\left(\frac{\xi}{dk}\right) \\ &\leq \frac{\pi}{2} + \sum_{k=1}^{N} \arctan\left(\frac{\xi}{dk}\right).\end{aligned}$$

An analogous estimate can be used to show that for all $z \in \overline{\mathbb{C}^+}$

$$\sum_{k=1}^{N} \operatorname{Arg}\left((\xi+\mu)\mu + (\omega_k+\omega)^2 + i\xi(\omega_k+\omega)\right) \geq -\frac{\pi}{2} - \sum_{k=1}^{N} \arctan\left(\frac{\xi}{dk}\right).$$

Since $\arctan(\cdot)$ is a continuous and increasing function with $\arctan(0) = 0$, and since $\varphi > \pi/2$, it is possible to choose $\xi_0 > 0$ in such a way that $\sum_{k=1}^{N} \arctan(\frac{\xi_0}{dk}) < \varphi - \frac{\pi}{2}$. Then for all $0 < \xi \leq \xi_0$ and $z \in \overline{\mathbb{C}^+}$ we have

$$|\operatorname{Arg}(f_\xi(z))| = \left|\sum_{k=1}^{N} \operatorname{Arg}\left(\frac{\mu + i(\omega_k+\omega)}{\xi + \mu + i(\omega_k+\omega)}\right)\right| = \left|\sum_{k=1}^{N} \operatorname{Arg}\left((\xi+\mu)\mu + (\omega_k+\omega)^2 + i\xi(\omega_k+\omega)\right)\right| \tag{17a}$$

$$\leq \frac{\pi}{2} + \sum_{k=1}^{N} \arctan\left(\frac{\xi}{dk}\right) < \varphi. \tag{17b}$$

This concludes the proof. □

**Lemma 25.** *If $A$ is a normal operator, then there exists $\xi_0 > 0$ such that $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$ is a sectorial operator for every $0 < \xi \leq \xi_0$.*

*Proof.* Let $\varphi \in (\pi/2, \pi)$, choose $\xi_0 > 0$ as in Lemma 24, and let $0 < \xi \leq \xi_0$. Since $R(\lambda, A)$ is normal for one/all $\lambda \in \rho(A)$, using $\Lambda_k(\xi) = I - \xi R(\xi + i\omega_k, A)$ it is easy to see that $\Lambda_k(\xi)\Lambda_l(\xi)^* = \Lambda_l(\xi)^*\Lambda_k(\xi)$ for $k, l \in \{1, \dots, N\}$. This further implies that $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$ is a normal operator.

We have $f_\xi(\overline{\mathbb{C}^+}) \subset S_\varphi \cup \{0\}$ by Lemma 24, and since $\Lambda_1(\xi)\cdots\Lambda_N(\xi) = f_\xi(-A)$ and $\sigma(A) \subset \overline{\mathbb{C}^-}$, the Spectral Mapping Theorem [10, Thm. 2.7.8] implies $\sigma(\Lambda_1(\xi)\cdots\Lambda_N(\xi)) \subset S_\varphi \cup \{0\}$. Since $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$ is normal, for all $\lambda > 0$ we have

$$\begin{aligned}\|\lambda(\lambda + \Lambda_1(\xi)\cdots\Lambda_N(\xi))^{-1}\| &= \sup_{\mu \in \sigma(\Lambda_1(\xi)\cdots\Lambda_N(\xi))} \frac{\lambda}{|\lambda+\mu|} \leq \sup_{\mu \in S_\varphi \cup \{0\}} \frac{\lambda}{|\lambda+\mu|} = \frac{\lambda}{\operatorname{dist}(-\lambda, S_\varphi)} \\ &= \frac{\lambda}{\lambda \sin(\varphi)} = \frac{1}{\sin(\varphi)},\end{aligned}$$

since $\operatorname{dist}(-\lambda, S_\varphi) = \lambda \sin(\varphi)$. This implies

$$\sup_{\lambda>0} \|\lambda(\lambda + \Lambda_1(\xi)\cdots\Lambda_N(\xi))^{-1}\| \leq \frac{1}{\sin(\varphi)},$$

and thus by [10, Prop. 2.1.1] the operator $\Lambda_1(\xi)\cdots\Lambda_N(\xi)$ is sectorial. □

## 6. Preservation of Polynomial Stability

In this section we prove Theorem 6, which gives conditions for the preservation of polynomial stability of a semigroup.

*Proof of Theorem* 6. Choose $\delta = \sqrt{\delta_1} > 0$, where $\delta_1 > 0$ is chosen as in [14, Cor. 7]. Assume $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X, Y)$ satisfy (9), assume $(-A)^\beta B$ and $(-A^*)^\gamma C^*$ are Hilbert–Schmidt operators, and $\|(-A)^\beta B\| < \delta$ and $\|(-A^*)^\gamma C^*\| < \delta$. Since $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta_1$, we have from [14, Cor. 7] that $\sigma(A+BC) \subset \mathbb{C}^-$, $1 \in \rho(CR(\lambda, A)B)$ for all $\lambda \in \overline{\mathbb{C}^+}$, and if we denote $D_\lambda = I - CR(\lambda, A)B$, then there exists $M_D \geq 1$ such that

$$\sup_{\lambda\in\overline{\mathbb{C}^+}} \|D_\lambda^{-1}\| \leq M_D < \infty.$$

Since $A$ generates a polynomially stable semigroup, Theorem 2.4 and Lemma 2.3 in [6] show that we can choose $M_R \geq 1$ in such a way that $\|R(\lambda, A)(-A)^{-\alpha}\| \leq M_R$ for all $\lambda \in \overline{\mathbb{C}^+}$.

We begin by showing that

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)B\|^2 \|CR(\xi + i\eta, A)\|^2 d\eta < \infty \tag{18}$$

and

$$\|R(i\omega, A)B\|\|CR(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha). \tag{19}$$

We start by considering the case where $\beta, \gamma > 0$. Choose $0 < \beta_1 \leq \beta$ and $0 < \gamma_1 \leq \gamma$ such that $\beta_1 + \gamma_1 = \alpha$. Let $\lambda \in \overline{\mathbb{C}^+}$ and denote $R_\lambda = R(\lambda, A)$, $B_{\beta_1} = (-A)^{\beta_1} B$, and $\tilde{C}_{\gamma_1} = (-A^*)^{\gamma_1} C^*$. If $(e_l)_{l=1}^\infty \subset Y$ is an orthonormal basis of $Y$, then for all $l \in \mathbb{N}$

$$\|B_{\beta_1} e_l\| \leq \|(-A)^{\beta_1-\beta}\|\|(-A)^\beta B e_l\| \qquad \text{and} \qquad \|\tilde{C}_{\gamma_1} e_l\| \leq \|(-A^*)^{\gamma_1-\gamma}\|\|(-A^*)^\gamma C^* e_l\|,$$

where $(-A)^{\beta_1-\beta} \in \mathcal{L}(X)$ and $(-A^*)^{\gamma_1-\gamma} \in \mathcal{L}(X)$ since $\beta_1 - \beta \leq 0$ and $\gamma_1 - \gamma \leq 0$. Because $(-A)^\beta B$ and $(-A^*)^\gamma C^*$ are Hilbert–Schmidt by assumption, we have that $B_{\beta_1}$, and $\tilde{C}_{\gamma_1}$ are Hilbert–Schmidt operators as well. The Moment Inequality [10, Prop. 6.6.4] implies that there exist constants $M_{\beta_1/\alpha}$ and $M_{\gamma_1/\alpha}$ such that

$$\begin{aligned}\|R_\lambda B\| &= \|R_\lambda(-A)^{-\beta_1}(-A)^{\beta_1}B\| = \|(-A)^{-\beta_1}R_\lambda B_{\beta_1}\| \leq M_{\beta_1/\alpha}\|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha}\|(-A)^{-\alpha}R_\lambda B_{\beta_1}\|^{\beta_1/\alpha} \\ &\leq M_{\beta_1/\alpha}\|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha}\|(-A)^{-\alpha}R(\lambda,A)\|^{\beta_1/\alpha}\|B_{\beta_1}\|^{\beta_1/\alpha} \leq M_{\beta_1/\alpha}\|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha}M_R^{\beta_1/\alpha}\|B_{\beta_1}\|^{\beta_1/\alpha}\end{aligned}$$

and

$$\begin{aligned}\|CR_\lambda\| &= \|R_\lambda^* C^*\| = \|R_\lambda^*(-A^*)^{-\gamma_1}(-A^*)^{\gamma_1}C^*\| = \|(-A^*)^{-\gamma_1}R_\lambda^*\tilde{C}_{\gamma_1}\| \\ &\leq M_{\gamma_1/\alpha}\|R_\lambda^*\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}\|(-A^*)^{-\alpha}R_\lambda^*\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha} \\ &\leq M_{\gamma_1/\alpha}\|R_\lambda^*\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}\|(-A)^{-\alpha}R(\lambda,A)\|^{\gamma_1/\alpha}\|\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha} \leq M_{\gamma_1/\alpha}\|R_\lambda^*\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}M_R^{\gamma_1/\alpha}\|\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha}.\end{aligned}$$

If we choose $\tilde{M} = M_{\beta_1/\alpha}M_{\gamma_1/\alpha}M_R^{(\beta_1+\gamma_1)/\alpha}\|B_{\beta_1}\|^{\beta_1/\alpha}\|\tilde{C}_{\gamma_1}\|^{\gamma_1/\alpha}$, then

$$\|R_\lambda B\|\|CR_\lambda\| \leq \tilde{M}\|R_\lambda B_{\beta_1}\|^{1-\beta_1/\alpha}\|R_\lambda^*\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}.$$

Choose $q = 1/(1-\beta_1/\alpha) > 1$ and $r = 1/(1-\gamma_1/\alpha) > 1$. Then $1/q + 1/r = 2 - (\beta_1+\gamma_1)/\alpha = 1$, and using the Hölder inequality we get

$$\begin{aligned}
&\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B\|^2 \|CR(\xi+i\eta,A)\|^2 d\eta \\
&\quad \le \tilde{M}^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B_{\beta_1}\|^{2(1-\beta_1/\alpha)} \|R(\xi+i\eta,A)^*\tilde{C}_{\gamma_1}\|^{2(1-\gamma_1/\alpha)} d\eta \\
&\quad \le \tilde{M}^2 \left( \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B_{\beta_1}\|^2 d\eta \right)^q \left( \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)^*\tilde{C}_{\gamma_1}\|^2 d\eta \right)^r < \infty
\end{aligned}$$

by Lemma 18. This concludes (18). Moreover, for $\omega \in \mathbb{R}$ with large $|\omega|$ we have

$$\begin{aligned}
\|R(i\omega,A)B\|\|CR(i\omega,A)\| &\le \tilde{M}\|R(i\omega,A)B_{\beta_1}\|^{1-\beta_1/\alpha}\|R(i\omega,A)^*\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \\
&\le \tilde{M}\|R(i\omega,A)\|^{1-\beta_1/\alpha}\|B_{\beta_1}\|^{1-\beta_1/\alpha}\|R(i\omega,A)^*\|^{1-\gamma_1/\alpha}\|\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha} \\
&= \tilde{M}\|B_{\beta_1}\|^{1-\beta_1/\alpha}\|\tilde{C}_{\gamma_1}\|^{1-\gamma_1/\alpha}\|R(i\omega,A)\|^{1-\beta_1/\alpha+1-\gamma_1/\alpha} = \mathcal{O}(|\omega|^\alpha),
\end{aligned}$$

since $1-\beta_1/\alpha+1-\gamma_1/\alpha = 2-(\beta_1+\gamma_1)/\alpha = 1$ and $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$ by [6, Thm. 2.4]. This concludes (19).

It remains to show that (18) and (19) hold if $\beta = 0$ or $\gamma = 0$, or equivalently, whenever either $\beta \ge \alpha$ or $\gamma \ge \alpha$. If $\beta \ge \alpha$, then $\|R(\lambda,A)B\| \le \|R(\lambda,A)(-A)^{-\alpha}\|\|(-A)^\alpha B\| \le M_R\|(-A)^\alpha B\|$ for all $\lambda \in \overline{\mathbb{C}^+}$, since $\|R(\lambda,A)(-A)^{-\alpha}\| \le M_R$. This implies $\|R(i\omega,A)B\|\|CR(i\omega,A)\| \le M_R\|(-A)^\alpha B\|\|C\|\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$ by [6, Thm. 2.4], and

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta \le M_R^2\|(-A)^\alpha B\|^2 \cdot \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)^*C^*\|^2 d\eta < \infty$$

by Lemma 18, since $C^* = (-A^*)^{-\gamma}(-A^*)^\gamma C^*$ is a Hilbert–Schmidt operator. On the other hand, if $\gamma \ge \alpha$, we similarly have $\|CR(\lambda,A)\| \le \|R(\lambda,A)(-A)^{-\alpha}\|\|(-A^*)^\alpha C^*\| \le M_R\|(-A^*)^\alpha C^*\|$ for all $\lambda \in \overline{\mathbb{C}^+}$. This again implies $\|R(i\omega,A)B\|\|CR(i\omega,A)\| \le \|R(i\omega,A)\|\|B\|M_R\|(-A^*)^\alpha C^*\| = \mathcal{O}(|\omega|^\alpha)$, and

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta \le M_R^2\|(-A^*)^\alpha C^*\|^2 \cdot \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B\|^2 d\eta < \infty$$

by Lemma 18, since $B = (-A)^{-\beta}(-A)^\beta B$ is a Hilbert–Schmidt operator. This concludes that (18) and (19) hold if $\beta \ge \alpha$ or $\gamma \ge \alpha$.

We can now show that the semigroup generated by $A+BC$ is uniformly bounded. Let $x \in X$ and for brevity denote $\lambda = \xi + i\eta$, $R_\lambda = R(\xi+i\eta,A)$ and $D_\lambda = I - CR(\xi+i\eta,A)B$. Using the Sherman–Morrison–Woodbury formula in Lemma 8 we can estimate (exactly as in the proof of Theorem 4)

$$\begin{aligned}
\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A+BC)x\|^2 d\eta &= \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda x + R_\lambda B D_\lambda^{-1} CR_\lambda x\|^2 d\eta \\
&\le 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda x\|^2 + \|R_\lambda B\|^2\|D_\lambda^{-1}\|^2\|CR_\lambda\|^2\|x\|^2 d\eta \\
&\le 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda x\|^2 d\eta + 2M_D^2\|x\|^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2\|CR_\lambda\|^2 d\eta < \infty
\end{aligned}$$

due to Theorem 17 and (18). Analogously, using $\|(R_\lambda B D_\lambda^{-1} CR_\lambda)^*\| = \|R_\lambda B D_\lambda^{-1} CR_\lambda\|$ we get

$$\begin{aligned}
&\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A+BC)^*x\|^2 d\eta \\
&\quad \le 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda^* x\|^2 d\eta + 2M_D^2\|x\|^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R_\lambda B\|^2\|CR_\lambda\|^2 d\eta < \infty
\end{aligned}$$

again due to Theorem 17 and (18). Since $x \in X$ was arbitrary, Theorem 17 concludes that the semigroup generated by $A + BC$ is uniformly bounded.

Finally, the Sherman–Morrison–Woodbury formula in Lemma 8 together with (19) implies that for $\omega \in \mathbb{R}$ with large $|\omega|$ we have

$$\begin{aligned}\|R(i\omega, A+BC)\| &= \|R(i\omega, A) + R(i\omega, A)B(I - CR(i\omega, A)B)^{-1}CR(i\omega, A)\| \\ &\leq \|R(i\omega, A)\| + \|R(i\omega, A)B\|\|(I - CR(i\omega, A)B)^{-1}\|\|CR(i\omega, A)\| \\ &\leq \|R(i\omega, A)\| + M_D\|R(i\omega, A)B\|\|CR(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha).\end{aligned}$$

By Theorem 2.4 in [6] this concludes that the semigroup generated by $A + BC$ is polynomially stable with exponent $\alpha$. □

## 7. Perturbation of a Strongly Stable Multiplication Semigroup

In this section we apply our theoretic results in considering the preservation of strong stability of a multiplication semigroup [8, Par. II.2.9]

$$(T_A(t)f)(\mu) = e^{t\cdot\mu}f(\mu)$$

on $X = L^2(\Omega)$, where $\Omega = \{\, \lambda \mid |\lambda + 1| \leq 1 \,\}$ is a disk centered at $-1$ and with radius 1 (see Figure 2). The generator $A$ of the semigroup $T_A(t)$ is a bounded multiplication operator $(Af)(\mu) = \mu f(\mu)$.

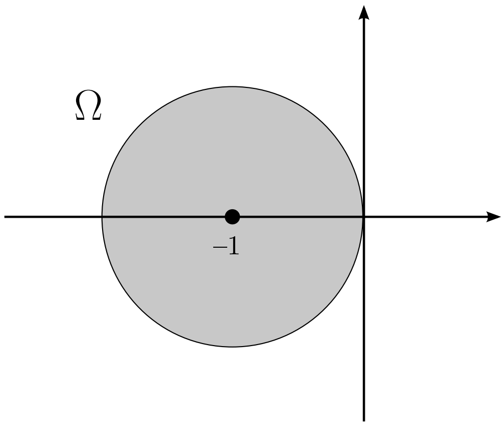


Figure 2: The domain Ω.

The spectrum of $A$ is given by $\sigma(A) = \sigma_c(A) = \Omega$, $\sigma(A) \cap i\mathbb{R} = \{0\} \subset \sigma_c(A)$, and the semigroup is uniformly bounded. Due to the Arendt–Batty–Lyubich–Vũ Theorem [1] the semigroup $T_A(t)$ is strongly stable. The operator $-A$ has an unbounded inverse $(-A)^{-1}$ with domain

$$\mathcal{D}((-A)^{-1}) = \left\{ f \in L^2(\Omega) \;\middle|\; \int_\Omega |\mu|^{-2}|f(\mu)|^2 d\mu < \infty \right\}.$$

We begin by finding a suitable value for $\alpha \geq 1$ in Assumption 3. Due to the geometry, for all $\omega \in \mathbb{R}$ with $0 < |\omega| \leq 1 =: \varepsilon_A$ we have

$$\begin{aligned}|\omega|^\alpha\|R(i\omega, A)\| &= \frac{|\omega|^\alpha}{\operatorname{dist}(i\omega, \Omega)} = \frac{|\omega|^\alpha}{\operatorname{dist}(i\omega, -1) - 1} = \frac{|\omega|^\alpha}{\sqrt{\omega^2+1} - 1} = \frac{|\omega|^\alpha(\sqrt{\omega^2+1}+1)}{\omega^2+1-1} \\ &= \frac{|\omega|^\alpha}{\omega^2}(\sqrt{\omega^2+1}+1) \leq M_A < \infty\end{aligned}$$

if and only if $\alpha \geq 2$. Thus we can choose $\alpha = 2$ in Assumption 3.

For $\beta \geq 0$ the domains of the operators $(-A)^{-\beta}$ and $(-A^*)^{-\beta}$ are given by

$$\mathcal{D}((-A)^{-\beta}) = \mathcal{D}((-A^*)^{-\beta}) = \left\{ f \in L^2(\Omega) \;\middle|\; \int_\Omega |\mu|^{-2\beta} |f(\mu)|^2 d\mu < \infty \right\}.$$

If $0 < \tilde{\beta} < \beta$ and $f \in \mathcal{D}((-A)^{-\beta})$, then the Hölder inequality with exponents $q = 1/(1 - \tilde{\beta}/\tilde{\beta})$ and $r = 1/(\tilde{\beta}/\beta)$ implies

$$\begin{aligned}\|(-A)^{-\tilde{\beta}} f\|^2 &= \int_\Omega |\mu|^{-2\tilde{\beta}} |f(\mu)|^2 d\mu = \int_\Omega |f(\mu)|^{2(1-\tilde{\beta}/\beta)} \cdot (|\mu|^{-2\beta}|f(\mu)|^2)^{\tilde{\beta}/\beta} d\mu \\ &\leq \left( \int_\Omega |f(\mu)|^2 d\mu \right)^{1-\tilde{\beta}/\beta} \left( \int_\Omega |\mu|^{-2\beta} |f(\mu)|^2 d\mu \right)^{\tilde{\beta}/\beta},\end{aligned}$$

or equivalently $\|(-A)^{-\tilde{\beta}} f\| \leq \|f\|^{1-\tilde{\beta}/\beta} \|(-A)^{-\beta} f\|^{\tilde{\beta}/\beta}$. In particular, this shows that if $\|B\| + \|(-A)^{-\beta} B\| < \delta$, then for all $0 < \tilde{\beta} < \beta$ we have

$$\|(-A)^{-\tilde{\beta}} B\| \leq \|B\|^{1-\tilde{\beta}/\beta} \|(-A)^{-\beta} B\|^{\tilde{\beta}/\beta} < \delta^{1-\tilde{\beta}/\beta} \delta^{\tilde{\beta}/\beta} = \delta, \tag{20}$$

and similarly for $\|(-A^*)^{-\tilde{\gamma}} C^*\|$ with $0 < \tilde{\gamma} < \gamma$.

We consider bounded finite rank perturbations $A + BC$ with $B \in \mathcal{L}(\mathbb{C}^p, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^p)$ for some $p \in \mathbb{N}$. Since the operator $A$ is bounded, we can approach the preservation of the strong stability of $T_A(t)$ more directly than in the proof of Theorem 4. Most notably, we can use the operators $-A$ and $-A^*$ in place of the operators $(-A)(1-A)^{-1}$ and $(-A^*)(1-A^*)^{-1}$, respectively. If $0 < c < 1$, then the theory presented in the earlier sections shows that the strong stability is preserved for all $B$ and $C$ for which

$$\|CR(\lambda, A)B\| \leq c < 1$$

for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$. In particular, if $\beta, \gamma \geq 0$ are such that $\beta + \gamma = 2$ and if $\mathcal{R}(B) \subset \mathcal{R}((-A)^\beta)$ and $\mathcal{R}(C^*) \subset \mathcal{R}((-A^*)^\gamma)$, then

$$\begin{aligned}\|CR(\lambda, A)B\| &= \sup_{\|x\|=\|y\|=1} |\langle R(\lambda, A)Bx, C^*y \rangle| = \sup_{\|x\|=\|y\|=1} |\langle (-A)^2 R(\lambda, A)(-A)^{-\beta} Bx, (-A^*)^{-\gamma} C^* y \rangle| \\ &\leq \|(-A)^2 R(\lambda, A)\| \|(-A)^{-\beta} B\| \|(-A^*)^{-\gamma} C^*\| \leq c < 1\end{aligned}$$

if $\|(-A)^{-\beta} B\| < \sqrt{c/M_1}$ and $\|(-A^*)^{-\gamma} C^*\| < \sqrt{c/M_1}$ where $M_1 \geq 1$ is such that $\|(-A)^2 R(\lambda, A)\| \leq M_1$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$. In the following we will search for a suitable constant $M_1 \geq 1$.

If $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ is such that $|\lambda| \geq 2$, then for all $f \in X$ with $\|f\| = 1$ we have

$$\begin{aligned}\|(-A)^2 R(\lambda, A) f\| &= \left( \int_\Omega \frac{|\mu|^4 |f(\mu)|^2}{|\lambda - \mu|^2} d\mu \right)^{1/2} \leq \sup_{\mu \in \Omega} \frac{|\mu|^2}{|\lambda - \mu|} \left( \int_\Omega |f(\mu)|^2 d\mu \right)^{1/2} \\ &= \sup_{\mu \in \Omega} \frac{|\mu|^2}{|\lambda - \mu|} \leq \sup_{\mu \in \Omega} \frac{4}{|\lambda - \mu|} = \frac{4}{\operatorname{dist}(\lambda, \Omega)} = \frac{4}{|\lambda + 1| - 1} \leq 4,\end{aligned}$$

since $|\lambda + 1| - 1 \geq |\lambda| - 1 \geq 2 - 1 = 1$. On the other hand, if $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ and $|\lambda| < 2$, then

$$\begin{aligned}\|(-A)^2 R(\lambda, A)\| &= \|(-A)(I - \lambda R(\lambda, A))\| \leq \|A\| + |\lambda| \|(-A) R(\lambda, A)\| \\ &\leq \|A\| + |\lambda| + |\lambda|^2 \|R(\lambda, A)\| \leq 2 + 2 + |\lambda|^2 \operatorname{dist}(\lambda, \Omega)^{-1}\end{aligned}$$

where

$$\begin{aligned}\frac{|\lambda|^2}{\operatorname{dist}(\lambda, \Omega)} &= \frac{|\lambda|^2}{|\lambda + 1| - 1} = \frac{|\lambda|^2 (|\lambda + 1| + 1)}{|\lambda + 1|^2 - 1} \leq \frac{|\lambda|^2 (|\lambda| + 2)}{(\operatorname{Re} \lambda + 1)^2 + (\operatorname{Im} \lambda)^2 - 1} \\ &\leq \frac{4|\lambda|^2}{(\operatorname{Re} \lambda)^2 + \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2} \leq \frac{4|\lambda|^2}{(\operatorname{Re} \lambda)^2 + (\operatorname{Im} \lambda)^2} = 4.\end{aligned}$$

Together these estimates imply that if we choose $M_1 = 8$, then $\|(-A)^2 R(\lambda, A)\| \le M_1$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$.

In order to guarantee $0 \notin \sigma_p(A+BC)$, the choice for $\delta > 0$ must satisfy the conditions of Lemma 16. The proof of the lemma shows that the appropriate condition for $B$ and $C$ is that $\|(-A)^{-\beta_1} B\| \cdot \|(-A^*)^{-\gamma_1} C^*\| < 1$ for some $0 \le \beta_1 \le \beta$ and $0 \le \gamma_1 \le \gamma$ satisfying $\beta_1 + \gamma_1 = 1$. Due to the property (20), this is true whenever $\delta \le 1$ and the perturbation satisfies $\|B\| + \|(-A)^{-\beta} B\| < \delta$ and $\|C\| + \|(-A^*)^{-\gamma} C^*\| < \delta$.

Together the above properties conclude that we can choose, for example, $c = 4/5 < 1$ and $\delta = \sqrt{c/M_1} = 1/\sqrt{10}$. In particular, the bound is independent of the values of $\beta$ and $\gamma$, as long as they satisfy $\beta+\gamma = 2$. As in Theorem 4 we can now conclude that if $B$ and $C$ are such that for $\beta+\gamma = 2$ we have $\|B\|+\|(-A)^{-\beta} B\| < \delta$ and $\|C\| + \|(-A^*)^{-\gamma} C^*\| < \delta$, then the semigroup generated by $A + BC$ is strongly stable. In particular, $\|CR(\lambda, A)B\| \le 4/5 < 1$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ and $\sup_{|\omega|\le 1} |\omega|^2 \|R(i\omega, A + BC)\| < \infty$.

For rank one perturbations we have $Cf = \langle f, c\rangle_{L^2}$ for a function $c \in L^2(\Omega)$ and $B = b \in L^2(\Omega)$. The perturbed semigroup is strongly stable if $\|b\|_{L^2} < 1/(2\sqrt{10})$, $\|c\|_{L^2} < 1/(2\sqrt{10})$,

$$\int_\Omega |\mu|^{-2\beta} |b(\mu)|^2 d\mu < \frac{1}{40}, \qquad \text{and} \qquad \int_\Omega |\mu|^{-2\gamma} |c(\mu)|^2 d\mu < \frac{1}{40}$$

for some $\beta, \gamma \ge 0$ satisfying $\beta + \gamma = 2$.

## 8. Conclusions

In this paper we have studied the preservation of strong stability of a semigroup whose generator has spectrum on the imaginary axis. We have shown that if the growth of the resolvent operator is polynomial near the spectral points $i\omega_k$, then the stability of the semigroup is indeed robust with respect to classes of bounded perturbations.

The results concerning the change of the spectrum of $A$ are also valid in the case where the operator $A$ has an infinite number of uniformly separated spectral points on the imaginary axis, and they can also be applied for more general bounded perturbations. However, the additional standing assumptions were required to show the preservation of the uniform boundedness of the semigroup. Therefore, generalizing the conditions on the preservation of uniform boundedness would also immediately improve the results on the preservation of strong stability.


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