Polynomial Stability of Semigroups Generated by Operator Matrices

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Abstract. In this paper we study the stability properties of strongly continuous semigroups generated by block operator matrices. We consider triangular and full operator matrices whose diagonal operator blocks generate polynomially stable semigroups. As our main results, we present conditions under which also the semigroup generated by the operator matrix is polynomially stable. The theoretical results are used to derive conditions for the polynomial stability of a system consisting of a two-dimensional and a one-dimensional damped wave equation.

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1. Introduction

The main topic of this paper is the nonuniform stability of strongly continuous semigroups generated by 2×2 block operator matrices. In particular, we are interested in the asymptotic behaviour of semigroups generated by operators of the form

$$A = \begin{pmatrix} A_1 & BC \\ 0 & A_2 \end{pmatrix}, \qquad \text{and} \qquad A = \begin{pmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix} \tag{1}$$

where $A_2 : \mathcal{D}(A_2) \subset X_2 \to X_2$ and $A_2 : \mathcal{D}(A_2) \subset X_2 \to X_2$ generate strongly continuous semigroups $T_1(t)$ and $T_2(t)$, respectively, and where X_1 and X_2 are Hilbert spaces. The remaining operators are assumed to be bounded. In both of the cases in (1) we denote by $T(t)$ the semigroup generated by A on the Hilbert space $X = X_1 \times X_2$. We concentrate on a situation where the semigroups $T_1(t)$ and $T_2(t)$ are polynomially stable [1, 3, 4], that is, the semigroups are uniformly bounded, $\sigma(A_1) \subset \mathbb{C}^-$ and $\sigma(A_2) \subset \mathbb{C}^-$, and there exist $\alpha_1, \alpha_2 > 0$ and $M_1, M_2 \geq 1$ such that for all $t > 0$

$$\|T_1(t)A_1^{-1}\| \leq \frac{M_1}{t^{1/\alpha_1}} \qquad \text{and} \qquad \|T_2(t)A_2^{-1}\| \leq \frac{M_2}{t^{1/\alpha_2}}.$$

As the main results of this paper, we present conditions under which also the semigroup $T(t)$ generated by A is polynomially stable.

If the semigroups $T_1(t)$ and $T_2(t)$ are exponentially stable, the operator A can be seen as a bounded perturbation of an operator

$$A_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

which generates an exponentially stable semigroup. The perturbation theory for exponentially stable semigroups then states that also the semigroup generated by A is exponentially stable provided that the norms of the operators BC, or B_1C_2 and B_2C_1, are sufficiently small [7, Thm. III.1.3]. In fact, the semigroup generated by the block triangular operator A in (1) is exponentially stable regardless of the size of $\|BC\|$. However, if the stability of $T_1(t)$ and $T_2(t)$ is not exponential, the situation becomes more complicated, as is illustrated by the following example.

Example 1. If $A_1 : \mathcal{D}(A_1) \subset X_1 \to X_1$ generates a semigroup $T_1(t)$ on X_1 and if $\varepsilon > 0$, then the block operator matrix

$$A = \begin{pmatrix} A_1 & \varepsilon I \\ 0 & A_1 \end{pmatrix}, \qquad \mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_1)$$

generates a semigroup

$$T(t) = \begin{pmatrix} T_1(t) & \varepsilon t T_1(t) \\ 0 & T_1(t) \end{pmatrix}$$

on $X = X_1 \times X_1$. In order for this semigroup to be uniformly bounded, it is necessary that

$$\sup_{t>0} \varepsilon t \|T_1(t)\| < \infty,$$

which implies $\|T_1(t)\| \to 0$ as $t \to \infty$. However, this is only possible if the semigroup $T_1(t)$ is exponentially stable [7, Prop. V.1.7]. This implies that the semigroup $T(t)$ is unstable whenever the semigroup $T_1(t)$ is not exponentially stable.

In this paper we show that if $T_1(t)$ and $T_2(t)$ are not exponentially stable, then the stability of the semigroup $T(t)$ also depends on other properties of BC, B_1C_2, and B_2C_1 besides their norms. In fact, if $T_1(t)$ and $T_2(t)$ are polynomially stable, it is necessary to impose smoothness conditions on these operators in order to guarantee the stability of $T(t)$. In particular, we assume the bounded operators in A satisfy range conditions of the form

$$\mathcal{R}(B) \subset \mathcal{D}((-A_1)^\beta) \qquad \text{and} \qquad \mathcal{R}(C^*) \subset \mathcal{D}((-A_2^*)^\gamma)$$

for some $\beta, \gamma \geq 0$, or

$$\begin{aligned} \mathcal{R}(B_1) &\subset \mathcal{D}((-A_1)^{\beta_1}), & \mathcal{R}(C_1^*) &\subset \mathcal{D}((-A_1^*)^{\gamma_1}) \\ \mathcal{R}(B_2) &\subset \mathcal{D}((-A_2)^{\beta_2}), & \mathcal{R}(C_2^*) &\subset \mathcal{D}((-A_2^*)^{\gamma_2}) \end{aligned}$$

for some $\beta_k, \gamma_k \geq 0$ and $k = 1, 2$. We will show that the semigroup generated by a triangular A is polynomially stable provided that the exponents β and γ

are sufficiently large. In the case of the semigroup generated by a full operator matrix A, it is in addition required that the graph norms

$$\|(-A_1)^{\beta_1}B_1\|, \quad \|(-A_1^*)^{\gamma_1}C_1^*\|, \quad \|(-A_2)^{\beta_2}B_2\|, \quad \text{and} \quad \|(-A_2^*)^{\gamma_2}C_2^*\|$$

are small enough. The results presented in this paper are based on the recent characterization of polynomial stability of a semigroup on a Hilbert space in terms of the behaviour of the resolvent operator of its generator [3, 4, 2].

In addition to our main focus, which is the case where both $T_1(t)$ and $T_2(t)$ are polynomially stable, we separately consider the situations where one of $T_1(t)$ and $T_2(t)$ is exponentially stable and the other is polynomially stable. We show that in such a situation it is possible to completely omit the conditions on the operator BC, and relax those on operators B_2C_1 and B_1C_2 in the stability results. In fact, we will see that these conditions agree with the interpretation of exponential stability as the "limit case" of polynomial stability with the exponent $\alpha = 0$.

To the author's knowledge, the polynomial stability of semigroups generated by block operator matrices has not been studied previously in the literature. One known result regarding nonuniform stability of triangular systems states that if one of $T_1(t)$ and $T_2(t)$ is exponentially stable and the other is strongly stable, the semigroup generated a triangular A is also strongly stable, see, for example, [10, Lem. 20]. The result only applies to triangular systems, and in the corresponding situation for a full operator matrix the stability can in general be destroyed even by operators B_1C_2 and B_2C_1 with arbitrarily small norms. Example 19 in Section 6 demonstrates this situation.

The results presented in this paper can be used in studying the asymptotic behaviour of linear partial differential equations. In addition, they have applications in the control of infinite-dimensional linear systems. The procedure for stabilizing a linear system using an observer-based dynamic feedback controller requires studying the stability of semigroups generated by block operator matrices, see for example [18, 10, 17, 16], and [6, Sec. 5.3]. If the controlled system is only strongly or polynomially stabilizable, determining the stability of the closed-loop requires results on operators of the form (1) where both of $T_1(t)$ and $T_2(t)$ are strongly or polynomially stable. In particular, since the systems under consideration usually have finite numbers of inputs and outputs, the interconnections corresponding to the operator blocks BC, B_1C_2, and B_2C_2 in (1) are very often finite rank operators.

The operators in (1) can be seen as perturbations of the block-diagonal operator $A_0 = \operatorname{diag}(A_1, A_2)$. Therefore, the perturbation results in [13, 14, 15] could be used to derive conditions for the stability of the semigroup generated by A. During the course of this paper we will see that taking into account the structure of the operator matrices yields considerably better results. In particular, the general perturbation results in the above references require that the exponents β and γ are sufficiently large, and the corresponding graph norms of the perturbing operators are small enough. The results in this paper show that in the case of the triangular block operator matrix, the conditions on the graph norms can be omitted completely. Moreover, for both triangular and full operator matrices the conditions on the exponents $\beta,\gamma \geq 0$, and $\beta_1,\gamma_1,\beta_2,\gamma_2 \geq 0$ are weaker than the conditions achievable by a direct application of the perturbation results in [13, 14, 15].

We illustrate the applicability of the theoretical results by studying a system consisting of two damped wave equations — one two-dimensional and the other one-dimensional. Both of the wave equations are polynomially stable, and they are coupled in one direction. We use our results on triangular systems to derive conditions under which the full coupled system is polynomially stable. In addition, in Section 6 we present two shorter examples demonstrating that the conditions on the exponents $\beta,\gamma \geq 0$, and $\beta_1,\gamma_1,\beta_2,\gamma_2 \geq 0$ in our main results are, in a certain sense, optimal.

The paper is organized as follows. In Section 2 we introduce notation and collect some essential results on polynomially stable semigroups. The main results of the paper are presented in Section 3. The results concerning the stability of semigroups generated by triangular and full systems are proved in Sections 4 and 5, respectively. Section 6 contains two examples illustrating the optimality of our results. In Section 7 we apply the theoretical results to determining the stability of two coupled wave equations.

2. Background on Polynomially Stable Semigroups

In this section we introduce the notation used throughout the paper, and review the definition and some of the most important properties of polynomially stable semigroups. If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, then we denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$ the domain and the range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : \mathcal{D}(A) \subset X \to X$ is a closed operator, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda,A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space and the dual pairing on a Banach space are both denoted by $\langle \cdot,\cdot \rangle$.

For a function $f : \mathbb{R} \to \mathbb{R}$ and for $\alpha \geq 0$ we use the notation $f(\omega) = \mathcal{O}\left(|\omega|^\alpha\right)$ if there exist constants $M > 0$ and $\omega_0 \geq 0$ such that $|f(\omega)| \leq M|\omega|^\alpha$ for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$.

Definition 2. Let $\alpha > 0$. A semigroup $T(t)$ on a Banach space X generated by $A : \mathcal{D}(A) \subset X \to X$ is *polynomially stable with power* α, if $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$, and if there exists $M \geq 1$ such that

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

For a polynomially stable semigroup $T(t)$ generated by A, the operators operators $-A$ and $-A^*$ are sectorial in the sense of [9, Ch. 2] due to the fact that $T(t)$ is uniformly bounded. Therefore, the fractional powers $(-A)^\beta$ and $(-A^*)^\beta$ are well-defined for all $\beta \geq 0$.

The theory presented in this paper is based on the following characterizations for polynomial stability of a semigroup on a Hilbert space. For proofs of the equivalences, see [2, Lem. 4.2], [4, Lem. 2.3, Thm. 2.4], and [11, Lem. 3.2].

Theorem 3. *Assume A generates a uniformly bounded semigroup $T(t)$ on a Hilbert space X, and $i\mathbb{R} \subset \rho(A)$. For fixed $\alpha, \beta > 0$ the following are equivalent.*

$$\text{(a)} \quad \|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0$$

$$\text{(a}'\text{)} \quad \|T(t)(-A)^{-\beta}\| \leq \frac{M}{t^{\beta/\alpha}}, \qquad \forall t > 0$$

$$\text{(b)} \quad \|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$$

$$\text{(c)} \quad \sup_{\operatorname{Re}\lambda \geq 0} \|R(\lambda, A)(-A)^{-\alpha}\| < \infty.$$

Lemma 4. *Assume $T(t)$ generated by A on a Hilbert space X is polynomially stable with power $\alpha > 0$, assume $\beta, \gamma \geq 0$ are such that $\beta + \gamma \geq \alpha$, and let Y be a Banach space. There exists a constant $M \geq 1$ such that if $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X, Y)$ satisfy $\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta)$ and $\mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma)$, then*

$$\|CR(\lambda, A)B\| \leq M\|(-A)^\beta B\|\|(-A^*)^\gamma C^*\|$$

for all $\lambda \in \overline{\mathbb{C}^+}$.

Proof. Since $(-A)^\beta$ has a bounded inverse, $(-A)^\beta$ and $(-A)^\beta B$ are closed operators. Since $\mathcal{D}((-A)^\beta B) = Y$, the Closed Graph Theorem implies $(-A)^\beta B \in \mathcal{L}(Y, X)$. Similarly, we have $(-A^*)^\gamma C^* \in \mathcal{L}(Y, X)$ and $C(-A)^\gamma$ extends to a bounded operator $C_\gamma \in \mathcal{L}(X, Y)$ with norm $\|C_\gamma\| \leq \|(-A^*)^\gamma C^*\|$. If we choose $M = \|(-A)^{\alpha-\beta-\gamma}\| \cdot \sup_{\lambda \in \overline{\mathbb{C}^+}} \|R(\lambda, A)(-A)^{-\alpha}\|$, then for all $\lambda \in \overline{\mathbb{C}^+}$

$$\begin{aligned}
\|CR(\lambda, A)B\| &= \|C(-A)^\gamma R(\lambda, A)(-A)^{-\alpha}(-A)^{\alpha-\beta-\gamma}(-A)^\beta B\| \\
&\leq \|C_\gamma\|\|R(\lambda, A)(-A)^{-\alpha}\|\|(-A)^{\alpha-\beta-\gamma}\|\|(-A)^\beta B\| \\
&\leq M\|(-A)^\beta B\|\|(-A^*)^\gamma C^*\|.
\end{aligned}$$

□

Remark 5. The proof of Lemma 4 shows that the assumption $\mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma)$ could be replaced with the condition that $C(-A)^\gamma : \mathcal{D}((-A)^\gamma) \subset X \to Y$ has a bounded extension $C_\gamma \in \mathcal{L}(X, Y)$. In this version of the result the estimate on the operator $CR(\lambda, A)B$ would become

$$\|CR(\lambda, A)B\| \leq M\|(-A)^\beta B\|\|C_\gamma\|.$$

Lemma 6. *Let A generate a semigroup $T(t)$ on a Hilbert space X and let $\sigma(A) \subset \overline{\mathbb{C}^-}$. The semigroup $T(t)$ is uniformly bounded if and only if for all $x, y \in X$ we have*

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \left(\|R(\xi + i\eta, A)x\|^2 + \|R(\xi + i\eta, A)^* y\|^2\right) d\eta < \infty.$$

Moreover, if $\tilde{B} \in \mathcal{L}(Y, X)$ *where* $\dim Y < \infty$, *then*

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)\tilde{B}\|^2 d\eta < \infty, \qquad \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* \tilde{B}\|^2 d\eta < \infty.$$

Proof. The proof of the first part can be found in [8, Thm. 2]. The second part follows from the first part together with the estimate $\|R\tilde{B}\|^2 \leq \sum_{j=1}^{p} \|Rb_j\|^2$ for $R \in \mathcal{L}(X)$ when $\tilde{B}u = \sum_{j=1}^{p} u_j b_j$ for $u \in \mathbb{C}^p$. □

3. Stability of Semigroups Generated by Operator Matrices

In this section we present our main results. The proofs of the theorems are given in Sections 4 and 5. Throughout the paper we assume $T_1(t)$ and $T_2(t)$ are strongly continuous semigroups generated by $A_1 : \mathcal{D}(A_1) \subset X_1 \to X_1$ and $A_2 : \mathcal{D}(A_2) \subset X_2 \to X_2$, respectively. Most of our results concern the case where both X_1 and X_2 are Hilbert spaces, and we specifically point out the results that are also valid for Banach spaces. Unless otherwise stated, we assume $T_1(t)$ is polynomially stable with power $\alpha_1 > 0$, and $T_2(t)$ is polynomially stable with power $\alpha_2 > 0$.

Our first main interest is in the stability of the semigroup $T(t)$ generated by

$$A = \begin{pmatrix} A_1 & BC \\ 0 & A_2 \end{pmatrix}, \qquad \mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2) \tag{2}$$

where $B \in \mathcal{L}(Y, X_1)$ and $C \in \mathcal{L}(X_2, Y)$ for some Banach space Y. Since the operator BC is bounded, the semigroup $T(t)$ has the form [6, Lem. 3.2.2]

$$T(t) = \begin{pmatrix} T_1(t) & S(t) \\ 0 & T_2(t) \end{pmatrix},$$

where $S(t) \in \mathcal{L}(X_2, X_1)$ is such that

$$S(t)x_2 = \int_0^t T_1(t-s)BCT_2(s)x_2 ds \qquad \forall x_2 \in X_2.$$

We assume the operators B and C satisfy

$$\mathcal{R}(B) \subset \mathcal{D}((-A_1)^\beta) \qquad \text{and} \qquad \mathcal{R}(C^*) \subset \mathcal{D}((-A_2^*)^\gamma) \tag{3}$$

for some $\beta, \gamma \geq 0$. As seen in the proof of Lemma 4, these conditions imply $(-A_1)^\beta B \in \mathcal{L}(Y, X_1)$ and $(-A_2^*)^\gamma C^* \in \mathcal{L}(Y, X_2)$.

The first two results provide sufficient conditions for the stability of the semigroup $T(t)$ on Hilbert and Banach spaces, respectively.

Theorem 7. *Assume* X_1 *and* X_2 *are Hilbert spaces. If* $\beta/\alpha_1 + \gamma/\alpha_2 > 1$, *then the semigroup generated by* A *in* (2) *is polynomially stable with power* $\alpha = \max\{\alpha_1, \alpha_2\}$. *If* $\dim Y < \infty$, *then it is sufficient that* $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$.

Theorem 8. *Assume X_1, X_2, and Y are Banach spaces. If $\beta/\alpha_1 + \gamma/\alpha_2 > 1$, then the semigroup generated by A in (2) is strongly stable. If $\alpha = \alpha_1 + \alpha_2$, then there exists $M \geq 1$ such that*

$$\|T(t)A^{-1}\| \leq M \left(\frac{\ln t}{t}\right)^{1/\alpha} \qquad \forall t > 0.$$

If one of the subsystems is exponentially stable, then the requirements on the exponents β and γ can be omitted completely.

Theorem 9. *If $T_1(t)$ is exponentially stable, then the semigroup $T(t)$ generated by A in (2) is polynomially stable with power $\alpha = \alpha_2$. Similarly, if $T_2(t)$ is exponentially stable, then $T(t)$ is polynomially stable with power $\alpha = \alpha_1$.*

The above results are stated for upper triangular systems, but the analogous results are also valid for lower triangular systems. Indeed, any lower triangular block operator matrix can be transformed into an upper triangular one with a similarity transformation

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A_1 & BC \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} A_2 & 0 \\ BC & A_1 \end{pmatrix}.$$

Since the stability properties considered in this paper are invariant under similarity transformations, Theorems 7, 8, and 9 also provide conditions for stability of semigroups generated by lower triangular block operator matrices.

The remaining results in this section concern the stability of the semigroup generated by an operator of the form

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix}, \qquad \mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2) \tag{4}$$

where $B_1 \in \mathcal{L}(Y_1, X_1)$, $B_2 \in \mathcal{L}(Y_2, X_2)$, $C_1 \in \mathcal{L}(X_1, Y_2)$, and $C_2 \in \mathcal{L}(X_2, Y_1)$ satisfy

$$\mathcal{R}(B_1) \subset \mathcal{D}((-A_1)^{\beta_1}), \quad \mathcal{R}(C_1^*) \subset \mathcal{D}((-A_1^*)^{\gamma_1}) \tag{5a}$$

$$\mathcal{R}(B_2) \subset \mathcal{D}((-A_2)^{\beta_2}), \quad \mathcal{R}(C_2^*) \subset \mathcal{D}((-A_2^*)^{\gamma_2}) \tag{5b}$$

for some $\beta_1, \beta_2, \gamma_1, \gamma_2 \geq 0$.

Theorem 10. *Assume X_1 and X_2 are Hilbert spaces and let one of the following conditions be satisfied:*

(i) $\beta_1, \gamma_1 \geq \alpha_1$ *and* $\beta_2, \gamma_2 \geq \alpha_2$
(ii) $\dim Y_1 < \infty$, $\beta_1 + \gamma_1 \geq \alpha_1$, *and* $\beta_2, \gamma_2 \geq \alpha_2$
(iii) $\dim Y_2 < \infty$, $\beta_1, \gamma_1 \geq \alpha_1$, *and* $\beta_2 + \gamma_2 \geq \alpha_2$
(iv) $\dim Y_1 < \infty$, *and* $\dim Y_2 < \infty$ *and* $\beta_k/\alpha_k + \gamma_l/\alpha_l \geq 1$ *for every* $k, l \in \{1, 2\}$.

Then there exists $\delta > 0$ such that if B_1, C_1, B_2, and C_2 satisfy (5) and

$$\|(-A_1)^{\beta_1}B_1\| \cdot \|(-A_1^*)^{\gamma_1}C_1^*\| \cdot \|(-A_2)^{\beta_2}B_2\| \cdot \|(-A_2^*)^{\gamma_2}C_2^*\| < \delta, \tag{6}$$

then the semigroup generated by A in (4) is polynomially stable with power $\alpha = \max\{\alpha_1, \alpha_2\}$.

Written out explicitly, the conditions (iv) in Theorem 10 become

$$\beta_1 + \gamma_1 \geq \alpha_1, \qquad \beta_1/\alpha_1 + \gamma_2/\alpha_2 \geq 1,$$
$$\beta_2 + \gamma_2 \geq \alpha_2, \qquad \beta_2/\alpha_2 + \gamma_1/\alpha_1 \geq 1.$$

If the semigroup $T_1(t)$ is exponentially stable, it is possible to remove the requirements on the exponents β_1 and γ_1 from the assumptions.

Theorem 11. *Assume $T_1(t)$ is exponentially stable and $\beta_2, \gamma_2 \geq \alpha_2$. There exists $\delta > 0$ such that if B_2, and C_2 satisfy* (5b) *and*

$$\|B_1\| \cdot \|C_1\| \cdot \|(-A_2)^{\beta_2} B_2\| \cdot \|(-A_2^*)^{\gamma_2} C_2^*\| < \delta,$$

then the semigroup generated by A in (4) *is polynomially stable with power $\alpha = \alpha_2$. If $\dim Y_2 < \infty$, it is sufficient that the exponents satisfy $\beta_2 + \gamma_2 \geq \alpha_2$.*

Applying a similarity transformation QAQ^{-1} with $Q = Q^{-1} = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ yields the following analogue of Theorem 11 in the case where $T_2(t)$ is exponentially stable.

Corollary 12. *Assume $T_2(t)$ is exponentially stable and $\beta_1, \gamma_1 \geq \alpha_1$. There exists $\delta > 0$ such that if B_1, and C_1 satisfy* (5a) *and*

$$\|(-A_1)^{\beta_1} B_1\| \cdot \|(-A_1^*)^{\gamma_1} C_1^*\| \cdot \|B_2\| \cdot \|C_2\| < \delta,$$

then the semigroup generated by A in (4) *is polynomially stable with power $\alpha = \alpha_1$. If $\dim Y_1 < \infty$, it is sufficient that the exponents satisfy $\beta_1 + \gamma_1 \geq \alpha_1$.*

We begin by presenting the proofs for the results concerning semigroup generated by triangular operator matrices. The results on the stability of semigroup generated by full operator matrices are proved in Section 5.

4. Semigroups Generated By Triangular Operator Matrices

In this section we present the proofs for Theorems 7, 8, and 9 concerning the stability of the semigroup generated by the triangular operator matrix

$$A = \begin{pmatrix} A_1 & BC \\ 0 & A_2 \end{pmatrix}.$$

The main standing assumptions are that the semigroup $T_1(t)$ generated by A_1 is polynomially stable with power α_1, the semigroup $T_2(t)$ generated by A_2 is polynomially stable with power α_2, and that B and C satisfy

$$\mathcal{R}(B) \subset \mathcal{D}((-A_1)^\beta) \qquad \text{and} \qquad \mathcal{R}(C^*) \subset \mathcal{D}((-A_2^*)^\gamma)$$

for some $\beta, \gamma \geq 0$. For a triangular operator matrix, the spectral properties of A are determined by those of the operators A_1 and A_2.

Lemma 13. *Assume X_1, X_2, and Y are Banach spaces. The spectrum of A satisfies $\sigma(A) \subset \mathbb{C}^-$ and*

$$R(\lambda, A) = \begin{pmatrix} R(\lambda, A_1) & R(\lambda, A_1) BC R(\lambda, A_2) \\ 0 & R(\lambda, A_2) \end{pmatrix} \tag{7}$$

for every $\lambda \in \overline{\mathbb{C}^+}$.

Proof. Let $\lambda \in \overline{\mathbb{C}^+}$ be arbitrary. Since $\lambda \in \rho(A_1)$ and $\lambda \in \rho(A_2)$, a direct computation shows that $\lambda - A$ has a bounded inverse given by the right-hand side of (7). This immediately implies $\lambda \in \rho(A)$. □

Lemma 14. *If X_1, X_2, and Y are Banach spaces and $\beta/\alpha_1 + \gamma/\alpha_2 > 1$, then the semigroup $T(t)$ is uniformly bounded.*

Proof. Since the semigroups $T_1(t)$ and $T_2(t)$ are uniformly bounded, the semigroup $T(t)$ is uniformly bounded if (and only if) the operators $S(t)$ are uniformly bounded with respect to $t \geq 0$. Denote $M_1 = \sup_{t\geq 0}\|T_1(t)\|$ and $M_2 = \sup_{t\geq 0}\|T_2(t)\|$. Moreover, let $M_\beta, M_\gamma \geq 1$ be such that

$$\|T_1(t)(-A_1)^{-\beta}\| \leq \frac{M_\beta}{t^{\beta/\alpha_1}}, \qquad \text{and} \qquad \|T_2(t)(-A_2)^{-\gamma}\| \leq \frac{M_\gamma}{t^{\gamma/\alpha_2}},$$

for all $t > 0$. Let $x \in X_2$ and $t \geq 2$. If we denote $B_\beta = (-A_1)^\beta B \in \mathcal{L}(Y, X_1)$ and $C_\gamma = \overline{C(-A_2)^\gamma} \in \mathcal{L}(X_2, Y)$, then for $s \in [1, t-1]$ we have

$$\begin{aligned}
&\|T_1(t-s)BCT_2(s)x\| \\
&\quad = \|T_1(t-s)(-A_1)^{-\beta}(-A_1)^\beta BC(-A_2)^\gamma T_2(s)(-A_2)^{-\gamma}x\| \\
&\quad \leq \|T_1(t-s)(-A_1)^{-\beta}\|\|B_\beta\|\|C_\gamma\|\|T_2(s)(-A_2)^{-\gamma}\|\|x\| \\
&\quad \leq M_\beta M_\gamma \|B_\beta\|\|C_\gamma\|\|x\|(t-s)^{-\beta/\alpha_1}s^{-\gamma/\alpha_2}
\end{aligned}$$

and thus

$$\begin{aligned}
\|S(t)x\| &\leq \int_0^t \|T_1(t-s)BCT_2(s)x\|ds \\
&\leq \int_0^1 \|T_1(t-s)\|\|BC\|\|T_2(s)\|\|x\|ds + \int_1^{t-1} \|T_1(t-s)BCT_2(s)x\|ds \\
&\quad + \int_{t-1}^t \|T_1(t-s)\|\|BC\|\|T_2(s)\|\|x\|ds \\
&\leq 2M_1M_2\|BC\|\|x\| + M_\beta M_\gamma\|B_\beta\|\|C_\gamma\|\|x\| \int_1^{t-1}(t-s)^{-\beta/\alpha_1}s^{-\gamma/\alpha_2}ds.
\end{aligned}$$

Since $x \in X_2$ was arbitrary, we have that $\sup_{t\geq 0}\|S(t)\| < \infty$ if

$$\sup_{t\geq 2} \int_1^{t-1} (t-s)^{-\beta/\alpha_1}s^{-\gamma/\alpha_2}ds < \infty. \tag{8}$$

If $\gamma/\alpha_2 > 1$, then

$$\int_1^{t-1}(t-s)^{-\beta/\alpha_1}s^{-\gamma/\alpha_2}ds \leq \int_1^{t-1} s^{-\gamma/\alpha_2}ds \leq \int_1^\infty s^{-\gamma/\alpha_2}ds < \infty,$$

and if $\beta/\alpha_1 > 1$, then similarly

$$\int_1^{t-1}(t-s)^{-\beta/\alpha_1}s^{-\gamma/\alpha_2}ds \leq \int_1^{t-1}(t-s)^{-\beta/\alpha_1}ds \leq \int_1^\infty r^{-\beta/\alpha_1}dr < \infty.$$

In both of these cases (8) is satisfied. It remains to consider the case where $0 < \beta/\alpha_1 \leq 1$ and $0 < \gamma/\alpha_2 \leq 1$ satisfy $\beta/\alpha_1 + \gamma/\alpha_2 > 1$. Choose $c = \beta/\alpha_1 + \gamma/\alpha_2 > 1$, $p = c\alpha_1/\beta$, and $q = c\alpha_2/\gamma$. Then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{c}(\beta/\alpha_1 + \gamma/\alpha_2) = 1.$$

Since $p > \alpha_1/\beta \geq 1$, and $q > \alpha_2/\gamma \geq 1$, and since $p \cdot \beta/\alpha_1 = c > 1$ and $q \cdot \gamma/\alpha_2 = c > 1$, the Hölder inequality with exponents p and q shows that

$$\begin{aligned} \int_1^{t-1} (t-s)^{-\beta/\alpha_1} \cdot s^{-\gamma/\alpha_2} ds &\leq \left(\int_1^{t-1} (t-s)^{-p\cdot\beta/\alpha_1} ds \right)^{\frac{1}{p}} \left(\int_1^{t-1} s^{-q\cdot\gamma/\alpha_2} ds \right)^{\frac{1}{q}} \\ &= \left(\int_1^{t-1} r^{-c} dr \right)^{1/p} \left(\int_1^{t-1} s^{-c} ds \right)^{1/q} \leq \left(\int_1^{\infty} r^{-c} dr \right)^{1/p} \left(\int_1^{\infty} s^{-c} ds \right)^{1/q} < \infty. \end{aligned}$$

This shows that (8) holds also in the case where $0 < \beta/\alpha_1 \leq 1$, $0 < \gamma/\alpha_2 \leq 1$, and $\beta/\alpha_1 + \gamma/\alpha_2 > 1$. This finally implies that $\sup_{t\geq 0}\|S(t)\| < \infty$, and thus $T(t)$ is uniformly bounded. □

Lemma 15. *Assume X_1, X_2 are Hilbert spaces and Y_1 and Y_2 are Banach spaces, and that $\tilde{B} \in \mathcal{L}(Y_1, X_1)$ and $\tilde{C} \in \mathcal{L}(X_2, Y_2)$ satisfy $\mathcal{R}(\tilde{B}) \subset \mathcal{D}((-A_1)^\beta)$ and $\mathcal{R}(\tilde{C}^*) \subset \mathcal{D}((-A_2^*)^\gamma)$ for some $\beta, \gamma \geq 0$. If $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$, then $\|R(i\omega, A_1)\tilde{B}\|\|\tilde{C}R(i\omega, A_2)\| = \mathcal{O}(|\omega|^\alpha)$ with $\alpha = \max\{\alpha_1, \alpha_2\}$. Moreover, if $\beta \geq \alpha_1$, we then have*

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_1)\tilde{B}\|^2 \|\tilde{C}R(\xi + i\eta, A_2)x\|^2 d\eta < \infty \qquad \forall x \in X_2. \quad (9)$$

If $\dim Y_1 < \infty$, *then* (9) *is satisfied whenever* $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$.

Proof. By Theorem 3 we can choose $M_0 \geq 1$ such that $\|R(\lambda, A_1)(-A_1)^{-\alpha_1}\| \leq M_0$ and $\|R(\lambda, A_2)(-A_2)^{-\alpha_2}\| \leq M_0$ for all $\lambda \in \overline{\mathbb{C}^+}$.

If $\beta/\alpha_1 \geq 1$, then for every $\lambda \in \overline{\mathbb{C}^+}$ and $x \in X_2$ we can estimate

$$\begin{aligned} \|R(\lambda, A_1)\tilde{B}\|\|\tilde{C}R(\lambda, A_2)x\| &= \|R(\lambda, A_1)(-A_1)^{-\alpha_1}(-A_1)^{\alpha_1}\tilde{B}\|\|\tilde{C}R(\lambda, A_2)x\| \\ &\leq \|R(\lambda, A_1)(-A_1)^{-\alpha_1}\|\|(-A_1)^{\alpha_1}\tilde{B}\|\|\tilde{C}\|\|R(\lambda, A_2)x\| \\ &\leq M_0\|(-A_1)^{\alpha_1}\tilde{B}\|\|\tilde{C}\|\|R(\lambda, A_2)x\|, \end{aligned}$$

which in particular implies $\|R(i\omega, A_1)\tilde{B}\|\|\tilde{C}R(i\omega, A_2)\| = \mathcal{O}(|\omega|^\alpha)$ due to the fact that $\|R(i\omega, A_2)\| = \mathcal{O}(|\omega|^\alpha)$. Moreover, we then have

$$\begin{aligned} &\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_1)\tilde{B}\|^2 \|\tilde{C}R(\xi + i\eta, A_2)x\|^2 d\eta \\ &\qquad \leq M_0^2 \|(-A_1)^{\alpha_1}\tilde{B}\|^2 \|\tilde{C}\|^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_2)x\|^2 d\eta < \infty \end{aligned}$$

by Lemma 6. Since $x \in X_2$ was arbitrary, this shows that (9) is satisfied.

If $\gamma/\alpha_2 \geq 1$, then for every $\lambda \in \overline{\mathbb{C}^+}$

$$
\begin{aligned}
\|R(\lambda, A_1)\tilde{B}\|\|\tilde{C}R(\lambda, A_2)x\| &= \|R(\lambda, A_1)\tilde{B}\|\|\tilde{C}(-A_2)^{\alpha_2}(-A_2)^{-\alpha_2}R(\lambda, A_2)x\| \\
&\leq \|R(\lambda, A_1)\tilde{B}\|\|\tilde{C}_{\alpha_2}\|\|R(\lambda, A_2)(-A_2)^{-\alpha_2}x\| \\
&\leq M_0\|\tilde{C}_{\alpha_2}\|\|x\|\|R(\lambda, A_1)\tilde{B}\|
\end{aligned}
$$

where $\tilde{C}_{\alpha_2}$ is the bounded extension of $\tilde{C}(-A_2)^{\alpha_2}$ to X_2. Since $\|R(i\omega, A_1)\| = \mathcal{O}(|\omega|^\alpha)$, this again implies $\|R(i\omega, A_1)\tilde{B}\|\|\tilde{C}R(i\omega, A_2)\| = \mathcal{O}(|\omega|^\alpha)$. If in addition $\dim Y_1 < \infty$, we have

$$
\begin{aligned}
&\sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_1)\tilde{B}\|^2\|\tilde{C}R(\xi + i\eta, A_2)x\|^2 d\eta \\
&\qquad \leq M_0^2\|\tilde{C}_{\alpha_2}\|^2 \sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_2)\tilde{B}\|^2 d\eta < \infty
\end{aligned}
$$

again by Lemma 6. This shows that (9) holds if $\beta = 0$.

It remains to consider the case where $0 < \beta < \alpha_1$ and $0 < \gamma < \alpha_2$ satisfy $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$. We can choose $0 < \beta_0 \leq \beta$ and $0 < \gamma_0 \leq \gamma$ such that $\beta_0/\alpha_1 + \gamma_0/\alpha_2 = 1$. By the Moment Inequality [9, Prop. 6.6.2] there exist $M_{\beta_0/\alpha_1}, M_{\gamma_0/\alpha_2} \geq 1$ such that

$$
\begin{aligned}
\|(-A_1)^{-\beta_0}R\| &\leq M_{\beta_0/\alpha_1}\|R\|^{1-\beta_0/\alpha_1}\|(-A_1)^{-\alpha_1}R\|^{\beta_0/\alpha_1} \\
\|(-A_2)^{-\gamma_0}Q\| &\leq M_{\gamma_0/\alpha_2}\|Q\|^{1-\gamma_0/\alpha_2}\|(-A_2)^{-\alpha_2}Q\|^{\gamma_0/\alpha_2}
\end{aligned}
$$

for any $R \in \mathcal{L}(Y, X_1)$ and $Q \in \mathcal{L}(Y, X_2)$. Let $\lambda \in \overline{\mathbb{C}^+}$ and for brevity denote $R_1 = R(\lambda, A_1)$ and $R_2 = R(\lambda, A_2)$. If $\tilde{C}_{\gamma_0} \in \mathcal{L}(X_2, Y)$ is the bounded extension of $\tilde{C}(-A_2)^{\gamma_0}$ to X_2, and $\tilde{B}_{\beta_0} = (-A_1)^{\beta_0}\tilde{B} \in \mathcal{L}(Y, X_1)$, then for every $\lambda \in \overline{\mathbb{C}^+}$

$$
\begin{aligned}
\|R_1\tilde{B}\|\|\tilde{C}R_2x\| &= \|(-A_1)^{-\beta_0}R_1(-A_1)^{\beta_0}\tilde{B}\|\|\tilde{C}(-A_2)^{\gamma_0}(-A_2)^{-\gamma_0}R_2x\| \\
&\leq \|(-A_1)^{-\beta_0}R_1\tilde{B}_{\beta_0}\|\|\tilde{C}_{\gamma_0}\|\|(-A_2)^{-\gamma_0}R_2x\| \\
&\leq M_{\beta_0/\alpha_1}\|R_1\tilde{B}_{\beta_0}\|^{1-\beta_0/\alpha_1}\|(-A_1)^{-\alpha_1}R_1\tilde{B}_{\beta_0}\|^{\beta_0/\alpha_1} \\
&\qquad \times \|\tilde{C}_{\gamma_0}\|M_{\gamma_0/\alpha_2}\|R_2x\|^{1-\gamma_0/\alpha_2}\|(-A_2)^{-\alpha_2}R_2x\|^{\gamma_0/\alpha_2} \\
&\leq M_{\beta_0/\alpha_1}M_{\gamma_0/\alpha_2}M_0^2\|\tilde{B}_{\beta_0}\|^{\beta_0/\alpha_1}\|\tilde{C}_{\gamma_0}\|\|x\|^{\gamma_0/\alpha_2} \\
&\qquad \times \|R(\lambda, A_1)\tilde{B}_{\beta_0}\|^{1-\beta_0/\alpha_1}\|R(\lambda, A_2)x\|^{1-\gamma_0/\alpha_2}
\end{aligned}
$$

which implies $\|R(i\omega, A_1)\tilde{B}\|\|\tilde{C}R(i\omega, A_2)\| = \mathcal{O}(|\omega|^\alpha)$ since $\|R(i\omega, A_1)\| = \mathcal{O}(|\omega|^\alpha)$, $\|R(i\omega, A_2)\| = \mathcal{O}(|\omega|^\alpha)$, and $1-\beta_0/\alpha_1+1-\gamma_0/\alpha_2 = 1$. If in addition $\dim Y_1 < \infty$, using the Hölder inequality for $p = 1/(1-\beta_0/\alpha_1)$ and $q = 1/(1-\gamma/\alpha_2)$ (which satisfy $1/p + 1/q = 1-\beta_0/\alpha_1 + 1-\gamma_0/\alpha_2 = 1$) and denoting $\tilde{M} = M_{\beta_0/\alpha_1} M_{\gamma_0/\alpha_2} M_0^2 \|\tilde{B}_{\beta_0}\|^{\beta_0/\alpha_1} \|\tilde{C}_{\gamma_0}\| \|x\|^{\gamma_0/\alpha_2}$, we have

$$\begin{aligned}
&\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_1)\tilde{B}\|^2 \|\tilde{C}R(\xi+i\eta,A_2)x\|^2 d\eta \\
&\quad \leq \tilde{M}^2 \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_1)\tilde{B}_{\beta_0}\|^{2(1-\frac{\beta_0}{\alpha_1})} \|R(\xi+i\eta,A_2)x\|^{2(1-\frac{\gamma_0}{\alpha_2})} d\eta \\
&\quad \leq \tilde{M}^2 \sup_{\xi>0} \left(\xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_1)\tilde{B}_{\beta_0}\|^2 d\eta \right)^{\frac{1}{p}} \left(\xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_2)x\|^2 d\eta \right)^{\frac{1}{q}} \\
&\quad \leq \tilde{M}^2 \left(\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_1)\tilde{B}_{\beta_0}\|^2 d\eta \right)^{\frac{1}{p}} \\
&\qquad \times \left(\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_2)x\|^2 d\eta \right)^{\frac{1}{q}} < \infty
\end{aligned}$$

by Lemma 6. This shows that (9) is true if $\beta,\gamma > 0$, and thus concludes the proof. □

Proof of Theorem 7. The claim of the theorem is that if $\beta/\alpha_1 + \gamma/\alpha_2 > 1$, then the semigroup $T(t)$ is polynomially stable with power $\alpha = \max\{\alpha_1,\alpha_2\}$. Moreover, if $\dim Y < \infty$, then the condition $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$ is sufficient.

We have from Lemma 13 that $\sigma(A) \subset \mathbb{C}^-$. In order to prove that $T(t)$ is polynomially stable with power $\alpha = \max\{\alpha_1,\alpha_2\}$, we need to show that $T(t)$ is uniformly bounded and $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$. Assume first that $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$. By Lemma 13 the resolvent operator is of the form

$$R(\lambda,A) = \begin{pmatrix} R(\lambda,A_1) & R(\lambda,A_1)BCR(\lambda,A_2) \\ 0 & R(\lambda,A_2) \end{pmatrix}$$

for every $\lambda \in \overline{\mathbb{C}^+}$. We have $\|R(i\omega,A_1)\| = \mathcal{O}(|\omega|^\alpha)$, and $\|R(i\omega,A_2)\| = \mathcal{O}(|\omega|^\alpha)$ by assumption, and

$$\|R(i\omega,A_1)BCR(i\omega,A_2)\| \leq \|R(i\omega,A_1)B\| \|CR(i\omega,A_2)\| = \mathcal{O}(|\omega|^\alpha)$$

by Lemma 15. Together these properties imply that $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$.

If $\beta/\alpha_1 + \gamma/\alpha_2 > 1$, the uniform boundedness of $T(t)$ follows directly from Lemma 14.

It remains to show that if $\dim Y < \infty$, then the semigroup $T(t)$ is uniformly bounded whenever $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$. Since we already showed that $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$, the polynomial stability will then follow from Theorem 3. For any $x = (x_1,x_2) \in X$ and $\lambda \in \mathbb{C}^+$ we have (denoting $R_1 = R(\lambda,A_1)$ and $R_2 = R(\lambda,A_2)$ for brevity)

$$\begin{aligned}
\|R(\lambda,A)x\|^2 &= \|R_1x_1 + R_1BCR_2x_2\|^2 + \|R_2x_2\|^2 \\
&\leq 2\|R_1x_1\|^2 + 2\|R_1B\|^2\|CR_2x_2\|^2 + \|R_2x_2\|^2 \\
\|R(\lambda,A)^*x\|^2 &= \|R_1^*x_1\|^2 + \|(R_1BCR_2)^*\,x_1 + R_2^*x_2\|^2 \\
&\leq \|R_1^*x_1\|^2 + 2\|R_2^*C^*\|^2\|B^*R_1^*x_1\|^2 + 2\|R_2^*x_2\|^2 \\
&\leq \|R_1^*x_1\|^2 + 2\|R(\overline{\lambda},A_2^*)C^*\|^2\|B^*R(\overline{\lambda},A_1^*)x_1\|^2 + 2\|R_2^*x_2\|^2.
\end{aligned}$$

Now

$$
\begin{aligned}
&\sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \left(\|R(\xi+i\eta,A)x\|^2 + \|R(\xi+i\eta,A)^*x\|^2\right) d\eta \\
&\quad \leq 2\sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \left(\|R(\xi+i\eta,A_1)x_1\|^2 + \|R(\xi+i\eta,A_1)^*x_1\|^2\right) d\eta \\
&\qquad + 2\sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \left(\|R(\xi+i\eta,A_2)x_2\|^2 + \|R(\xi+i\eta,A_2)^*x_2\|^2\right) d\eta \\
&\qquad + 2\sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_1)B\|^2\|CR(\xi+i\eta,A_2)x_2\|^2 d\eta \\
&\qquad + 2\sup_{\xi>0}\ \xi \int_{-\infty}^{\infty} \|R(\xi-i\eta,A_2^*)C^*\|^2\|B^*R(\xi-i\eta,A_1^*)x_1\|^2 d\eta < \infty,
\end{aligned}
$$

where the first two suprema on the right hand side are finite by Lemma 6 since $T_1(t)$ and $T_2(t)$ are uniformly bounded. The third and the fourth suprema are finite by Lemma 15 since $\dim Y < \infty$. Now Lemma 6 shows that the semigroup $T(t)$ is uniformly bounded, and it is therefore polynomially stable with power $\alpha = \max\{\alpha_1, \alpha_2\}$. □

Proof of Theorem 8. We want to show that if $\beta/\alpha_1 + \gamma/\alpha_2 > 1$, then $T(t)$ is strongly stable, and there exists a constant $M \geq 1$ such that $\|T(t)A^{-1}\| \leq M\left(\frac{\ln t}{t}\right)^{1/(\alpha_1+\alpha_2)}$ for all $t > 0$.

We have from Lemmas 13 and 14 that $\sigma(A) \subset \mathbb{C}^-$ and that the semigroup $T(t)$ is uniformly bounded. We therefore have from [5, Cor. 4.2] that $T(t)$ is strongly stable.

Since the semigroups $T_1(t)$ and $T_2(t)$ are polynomially stable, we have from [3, Prop 1.3 & Ex. 1.4] that $\|R(i\omega, A_1)\| = \mathcal{O}(|\omega|^{\alpha_1})$ and $\|R(i\omega, A_2)\| = \mathcal{O}(|\omega|^{\alpha_2})$. If $\alpha = \alpha_1 + \alpha_2$, then

$$
\|R(i\omega,A_1)B\|\|CR(i\omega,A_2)\| \leq \|R(i\omega,A_1)\|\|B\|\|C\|\|R(i\omega,A_2)\| = \mathcal{O}(|\omega|^{\alpha}),
$$

which together with Lemma 13 further implies that $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^{\alpha})$. We now have from [3, Thm. 1.5 & Ex. 1.7] that there exists $M \geq 1$ such that $\|T(t)A^{-1}\| \leq M\left(\frac{\ln t}{t}\right)^{1/\alpha}$ for all $t > 0$. □

Proof of Theorem 9. The claim of the theorem is that if $T_1(t)$ is exponentially stable and $T_2(t)$ is polynomially stable with power α_2, then the semigroup $T(t)$ is polynomially stable with power $\alpha = \alpha_2$. Moreover, we also want to show that if $T_2(t)$ is exponentially stable and $T_1(t)$ is polynomially stable with power α_1, then the semigroup $T(t)$ is polynomially stable with power $\alpha = \alpha_1$.

Since by Definition 2 a polynomially stable semigroup is also strongly stable, we have from [10, Lem. 20] that the semigroup $T(t)$ is strongly stable. In particular this implies that $T(t)$ is uniformly bounded. By Theorem 3 it remains to show that $i\mathbb{R} \subset \rho(A)$ and $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^{\alpha})$.

Let $x = (x_1, x_2)^T \in X$ be such that $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 = 1$. For brevity denote $R_1 = R(i\omega, A_1)$ and $R_2 = R(i\omega, A_2)$. Now

$$\begin{aligned}\|R(i\omega, A)x\|^2 &= \|R_1x_1 + R_1BCR_2x_2\|^2 + \|R_2x_2\|^2 \\ &\le 2\left(\|R_1\|^2\|x_1\|^2 + \|R_1\|^2\|BC\|^2\|R_2\|^2\|x_2\|^2\right) + \|R_2\|^2\|x_2\|^2 \\ &\le 2\left(\|x_1\|^2 + \|x_2\|^2\right)\left(\|R_1\|^2 + \|R_1\|^2\|BC\|^2\|R_2\|^2 + \|R_2\|^2\right) \\ &\le 2\max\left\{\|BC\|^2, 1\right\}\left(\|R_1\|^2(1 + \|R_2\|^2) + \|R_2\|^2\right) \\ &\le 2(\|BC\|^2 + 1)(\|R(i\omega, A_1)\|^2 + 1)(\|R(i\omega, A_2)\|^2 + 1).\end{aligned}$$

Due to the assumptions and Theorem 3 one of the norms $\|R(i\omega, A_1)\|$ and $\|R(i\omega, A_2)\|$ is of order $\mathcal{O}(|\omega|^\alpha)$, and the other is uniformly bounded. This together with the above estimate shows that $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$. □

5. Semigroups Generated By Full Operator Matrices

In this section we prove the results concerning the semigroup generated by the block operator matrix

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix}$$

where A_1 and A_2 generate polynomially stable semigroups. The operators $B_1 \in \mathcal{L}(Y_1, X_1)$, $B_2 \in \mathcal{L}(Y_2, X_2)$, $C_1 \in \mathcal{L}(X_1, Y_2)$, and $C_2 \in \mathcal{L}(X_2, Y_1)$ satisfy

$$\mathcal{R}(B_1) \subset \mathcal{D}((-A_1)^{\beta_1}), \quad \mathcal{R}(C_1^*) \subset \mathcal{D}((-A_1^*)^{\gamma_1})$$
$$\mathcal{R}(B_2) \subset \mathcal{D}((-A_2)^{\beta_2}), \quad \mathcal{R}(C_2^*) \subset \mathcal{D}((-A_2^*)^{\gamma_2})$$

for some $\beta_1, \beta_2, \gamma_1, \gamma_2 \ge 0$.

Lemma 16. *If* $\lambda \in \overline{\mathbb{C}^+}$ *is such that* $1 \in \rho(C_2R(\lambda, A_2)B_2C_1R(\lambda, A_1)B_1)$*, then* $\lambda \in \rho(A)$ *and*

$$R(\lambda, A) = \begin{pmatrix} R_1 + R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1 & R_1B_1C_2S_1(\lambda)^{-1} \\ S_1(\lambda)^{-1}B_2C_1R_1 & S_1(\lambda)^{-1} \end{pmatrix}$$

where $R_1 = R(\lambda, A_1)$*,* $R_2 = R(\lambda, A_2)$*, and*

$$\begin{aligned} S_1(\lambda)^{-1} &= R(\lambda, A_2) + R(\lambda, A_2)B_2C_1R(\lambda, A_1)B_1D_\lambda^{-1}C_2R(\lambda, A_2), \\ D_\lambda &= I - C_2R(\lambda, A_2)B_2C_1R(\lambda, A_1)B_1. \end{aligned}$$

Proof. Let $\lambda \in \overline{\mathbb{C}^+}$ be such that $1 \in \rho(C_2R(\lambda, A_2)B_2C_1R(\lambda, A_1)B_1)$ and denote $R_1 = R(\lambda, A_1)$ and $R_2 = R(\lambda, A_2)$. The Schur complement $S_1(\lambda)$ of $\lambda - A_1$ in

$$\lambda - A = \begin{pmatrix} \lambda - A_1 & -B_1C_2 \\ -B_2C_1 & \lambda - A_2 \end{pmatrix}$$

is $S_1(\lambda) = \lambda - A_2 - B_2C_1R_1B_1C_2$. Since $1 \in \rho(C_2R_2B_2C_1R_1B_1)$, the Sherman–Morrison–Woodbury formula (see, e.g., [13, Lem. 10]) implies that $S_1(\lambda)$ is boundedly invertible and

$$S_1(\lambda)^{-1} = R_2 + R_2B_2C_1R_1B_1(I - C_2R_2B_2C_1R_1B_1)^{-1}C_2R_2.$$

Since the $\lambda - A_1$ and its Schur complement $S_1(\lambda)$ are boundedly invertible, we have that $\lambda \in \rho(A)$ and the resolvent operator $R(\lambda, A)$ is given by

$$\begin{aligned} R(\lambda, A) &= \begin{pmatrix} I & R_1B_1C_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ 0 & S_1(\lambda)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ B_2C_1R_1 & I \end{pmatrix} \\ &= \begin{pmatrix} R_1 + R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1 & R_1B_1C_2S_1(\lambda)^{-1} \\ S_1(\lambda)^{-1}B_2C_1R_1 & S_1(\lambda)^{-1} \end{pmatrix}. \end{aligned}$$

□

Lemma 17. *If any one of the conditions* (i)-(iv) *in Theorem* 10 *is satisfied, then* $\|R(i\omega, A_k)B_k\|\|C_lR(i\omega, A_l)\| = \mathcal{O}(|\omega|^\alpha)$ *with* $\alpha = \max\{\alpha_1, \alpha_2\}$, *and*

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_k)B_k\|^2 \|C_lR(\xi + i\eta, A_l)x_l\|^2 d\eta < \infty \tag{10a}$$

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_k)^*C_k^*\|^2 \|B_l^*R(\xi + i\eta, A_l)^*x_l\|^2 d\eta < \infty \tag{10b}$$

for every $k, l \in \{1, 2\}$.

Proof. The property $\|R(i\omega, A_k)B_k\|\|C_lR(i\omega, A_l)\| = \mathcal{O}(|\omega|^\alpha)$ follows from Lemma 15 since in each of the situations (i)-(iv) the exponents satisfy $\beta_k/\alpha_k + \gamma_l/\alpha_l \geq 1$.

We have from Lemma 15 that for fixed $k, l \in \{1, 2\}$ the condition (10a) is satisfied if either

(a) $\beta_k \geq \alpha_k$, or
(b) $\dim Y_k < \infty$ and $\beta_k/\alpha_k + \gamma_l/\alpha_l \geq 1$.

It is therefore sufficient to verify that in each of the situations (i)-(iv), for every $k, l \in \{1, 2\}$ either (a) or (b) is satisfied.

In the following we list the possible situations with respect to the conditions (i)-(iv), and the possible combinations of (k, l).

Condition (i): (a) is satisfied for $k = 1, 2$ (and consequently, for every $(k, l) \in \{1, 2\} \times \{1, 2\}$).

Condition (ii): For (k, l)

$(1, 1)$ (b) is satisfied since $\dim Y_1 < \infty$ and $\beta_1/\alpha_1 + \gamma_1/\alpha_1 \geq 1$
$(1, 2)$ (b) is satisfied since $\dim Y_1 < \infty$ and $\beta_1/\alpha_1 + \gamma_2/\alpha_2 \geq \beta_1/\alpha_1 + 1 \geq 1$
$(2, l)$ (a) is safisfied since $\beta_2 \geq \alpha_2$

Condition (iii): For (k, l)

$(1, l)$ (a) is satisfied since $\beta_1 \geq \alpha_1$
$(2, 1)$ (b) is satisfied since $\dim Y_2 < \infty$ and $\beta_2/\alpha_2 + \gamma_1/\alpha_1 \geq \beta_2/\alpha_2 + 1 \geq 1$
$(2, 2)$ (b) is satisfied since $\dim Y_2 < \infty$ and $\beta_2/\alpha_2 + \gamma_2/\alpha_2 \geq 1$

Condition (iv): (b) is satisfied for for every $(k, l) \in \{1, 2\} \times \{1, 2\}$ since $\dim Y_k < \infty$ and $\beta_k/\alpha_k + \gamma_l/\alpha_l \geq 1$ by assumption.

To show (10b), we note that

$$\begin{aligned}
&\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_k)^* C_k^*\|^2 \|B_l^* R(\xi+i\eta,A_l)^* x_l\|^2 d\eta \\
&\quad = \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi-i\eta,A_k^*)C_k^*\|^2 \|B_l^* R(\xi-i\eta,A_l^*)x_l\|^2 d\eta \\
&\quad = \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A_k^*)C_k^*\|^2 \|B_l^* R(\xi+i\eta,A_l^*)x_l\|^2 d\eta.
\end{aligned}$$

We can apply Lemma 15 to operators A^*, C_k^* and B_k^* for $k,l \in \{1,2\}$, and see that for fixed $k,l \in \{1,2\}$ the condition (10b) is satisfied if either

(a′) $\gamma_k \ge \alpha_k$, or
(b′) $\dim Y_k < \infty$ and $\beta_l/\alpha_l + \gamma_k/\alpha_k \ge 1$.

Similarly as above, it can be verified that in every situation (i)-(iv) either (a′) or (b′) is satisfied. □

Proof of Theorem 10. The assumption in the theorem is that the exponents $\beta_1,\beta_2,\gamma_1,\gamma_2 \ge 0$ satisfy at least one of the conditions (i)-(iv). The claim is that there exists $\delta > 0$ such that if

$$\|(-A_1)^{\beta_1}B_1\| \|(-A_1^*)^{\gamma_1}C_1^*\| \|(-A_2)^{\beta_2}B_2\| \|(-A_2^*)^{\gamma_2}C_2^*\| < \delta, \tag{11}$$

then the semigroup $T(t)$ is polynomially stable with power $\alpha = \max\{\alpha_1,\alpha_2\}$.

By Lemma 4 we can choose $M_1, M_2 \ge 1$ such that

$$\|C_1R(\lambda,A_1)B_1\| \le M_1 \|(-A_1)^{\beta_1}B_1\| \|(-A_1^*)^{\gamma_1}C_1^*\| \tag{12a}$$
$$\|C_2R(\lambda,A_2)B_2\| \le M_2 \|(-A_2)^{\beta_2}B_2\| \|(-A_2^*)^{\gamma_2}C_2^*\| \tag{12b}$$

for all $\lambda \in \overline{\mathbb{C}^+}$. If we choose $0 < \delta < 1/(M_1M_2)$ and if (11) is satisfied, then $\|C_1R(\lambda,A_1)B_1C_2R(\lambda,A_2)B_2\| \le \delta M_1M_2 < 1$. In particular, this implies that $1 \in \rho(C_1R(\lambda,A_1)B_1C_2R(\lambda,A_2)B_2)$ for all $\lambda \in \overline{\mathbb{C}^+}$. Lemma 16 now shows that $\sigma(A) \subset \mathbb{C}^-$ and gives a formula for the resolvent $R(\lambda,A)$ for $\lambda \in \overline{\mathbb{C}^+}$.

To prove uniform boundedness of $T(t)$ using Lemma 6, we need to estimate the norms $\|R(\lambda,A)x\|$ and $\|R(\lambda,A)^*x\|$ for $x = (x_1,x_2)^T \in X$ and $\lambda \in \overline{\mathbb{C}^+}$. Let $\lambda \in \overline{\mathbb{C}^+}$ and denote $R_1 = R(\lambda,A_1)$, $R_2 = R(\lambda,A_2)$, and $D_\lambda = I - C_1R(\lambda,A_1)B_1C_2R(\lambda,A_2)B_2$. If $M_D = 1/(1-\delta M_1M_2)$, then a standard Neumann series argument shows that $\|D_\lambda^{-1}\| \le M_D$. If we choose $\tilde{M}_1 = M_1\|(-A_1)^{\beta_1}B_1\| \|(-A_1^*)^{\gamma_1}C_1^*\|$ and $\tilde{M}_2 = M_2\|(-A_2)^{\beta_2}B_2\| \|(-A_2^*)^{\gamma_2}C_2^*\|$, then the equations (12) imply

$$\|C_1R(\lambda,A_1)B_1\| \le \tilde{M}_1 \qquad \text{and} \qquad \|C_2R(\lambda,A_2)B_2\| \le \tilde{M}_2.$$

In the estimates we use the scalar inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ for $a,b,c \ge 0$. We have

$$\begin{aligned}
\|R(\lambda,A)x\|^2 &= \|R_1x_1 + R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1x_1 + R_1B_1C_2S_1(\lambda)^{-1}x_2\|^2 \\
&\quad + \|S_1(\lambda)^{-1}B_2C_1R_1x_1 + S_1(\lambda)^{-1}x_2\|^2 \\
&\le 3\|R_1x_1\|^2 + 3\|R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1x_1\|^2 + 3\|R_1B_1C_2S_1(\lambda)^{-1}x_2\|^2 \\
&\quad + 2\|S_1(\lambda)^{-1}B_2C_1R_1x_1\|^2 + 2\|S_1(\lambda)^{-1}x_2\|^2.
\end{aligned}$$

Using $S_1(\lambda)^{-1} = R_2 + R_2B_2C_1R_1B_1D_\lambda^{-1}C_2R_2$, the terms on the right-hand side can be further estimated by

$$\begin{aligned}
&\|R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1x_1\| \\
&\le \|R_1B_1\|\big(\|C_2R_2B_2\| + \|C_2R_2B_2\|\|C_1R_1B_1\|\|D_\lambda^{-1}\|\|C_2R_2B_2\|\big)\|C_1R_1x_1\| \\
&\le (\tilde{M}_2 + \tilde{M}_1\tilde{M}_2^2M_D)\|R_1B_1\|\|C_1R_1x_1\|,
\end{aligned}$$

and similarly we get

$$\begin{aligned}
\|R_1B_1C_2S_1(\lambda)^{-1}x_2\| &\le (1+\tilde{M}_1\tilde{M}_2M_D)\|R_1B_1\|\|C_2R_2x_2\| \\
\|S_1(\lambda)^{-1}B_2C_1R_1x_1\| &\le (1+\tilde{M}_1\tilde{M}_2M_D)\|R_2B_2\|\|C_1R_1x_1\| \\
\|S_1(\lambda)^{-1}x_2\| &\le \|R_2x_2\| + \tilde{M}_1M_D\|R_2B_2\|\|C_2R_2x_2\|.
\end{aligned}$$

Denote $M_{\mathrm{tot}} = \max\{\tilde{M}_1M_D, 1+\tilde{M}_1\tilde{M}_2M_D, \tilde{M}_2 + \tilde{M}_1\tilde{M}_2^2M_D\}$. Combining the above estimates yields

$$\begin{aligned}
\|R(\lambda,A)x\|^2 &\le 3\|R_1x_1\|^2 + 3\|R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1x_1\|^2 \\
&\quad + 3\|R_1B_1C_2S_1(\lambda)^{-1}x_2\|^2 + 2\|S_1(\lambda)^{-1}B_2C_1R_1x_1\|^2 + 2\|S_1(\lambda)^{-1}x_2\|^2 \\
&\le 3\|R_1x_1\|^2 + 3M_{\mathrm{tot}}^2\|R_1B_1\|^2\|C_1R_1x_1\|^2 + 3M_{\mathrm{tot}}^2\|R_1B_1\|^2\|C_2R_2x_2\|^2 \\
&\quad + 2M_{\mathrm{tot}}^2\|R_2B_2\|^2\|C_1R_1x_1\|^2 + 4\|R_2x_2\|^2 + 4M_{\mathrm{tot}}^2\|R_2B_2\|^2\|C_2R_2x_2\|^2 \\
&\le 3\|R_1x_1\|^2 + 4\|R_2x_2\|^2 + 4M_{\mathrm{tot}}^2\sum_{k,l=1,2}\|R_kB_k\|^2\|C_lR_lx_l\|^2.
\end{aligned}$$

Lemmas 6 and 17 thus imply

$$\begin{aligned}
&\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)x\|^2d\eta \le 3\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A_1)x_1\|^2d\eta \\
&\quad + 4\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A_2)x_2\|^2d\eta \\
&\quad + 4M_{\mathrm{tot}}^2\sum_{k,l=1,2}\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A_k)B_k\|^2\|C_lR(\xi+i\eta,A_l)x_l\|^2d\eta < \infty.
\end{aligned}$$

Furthermore, the same estimates also show that

$$\begin{aligned}
\|R(i\omega,A)\|^2 &\le 3\|R(i\omega,A_1)\|^2 + 4\|R(i\omega,A_2)\|^2 \\
&\quad + 4M_{\mathrm{tot}}^2\sum_{k,l=1,2}\|R(i\omega,A_k)B_k\|^2\|C_lR(i\omega,A_l)\|^2.
\end{aligned}$$

This immediately implies $\|R(i\omega,A)\| = \mathcal{O}(|\omega|^\alpha)$ with $\alpha = \max\{\alpha_1,\alpha_2\}$, since we have $\|R(i\omega,A_k)B_k\|\|C_l R(i\omega,A_l)\| = \mathcal{O}(|\omega|^\alpha)$ by Lemma 17.

Because

$$R(\lambda,A)^* = \begin{pmatrix} R_1^* + \left(R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1\right)^* & \left(S_1(\lambda)^{-1}B_2C_1R_1\right)^* \\ \left(R_1B_1C_2S_1(\lambda)^{-1}\right)^* & \left(S_1(\lambda)^{-1}\right)^* \end{pmatrix}$$

we can similarly estimate the norm of $R(\lambda,A)^*x$ by

$$\begin{aligned} \|R(\lambda,A)^*x\|^2 &\le 3\|R_1^*x_1\|^2 + 3\|(R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1)^*x_1\|^2 \\ &\quad + 3\|(S_1(\lambda)^{-1}B_2C_1R_1)^*x_2\|^2 + 2\|(R_1B_1C_2S_1(\lambda)^{-1})^*x_1\|^2 \\ &\quad + 2\|(S_1(\lambda)^{-1})^*x_2\|^2. \end{aligned}$$

Since $(S(\lambda)^{-1})^* = R_2^* + R_2^*C_2^*(C_1R_1B_1D_\lambda^{-1})^*B_2^*R_2^*$ and $\|B_2^*R_2^*C_2^*\| = \|(C_2R_2B_2)^*\| = \|C_2R_2B_2\|$, we get

$$\begin{aligned} &\|(R_1B_1C_2S_1(\lambda)^{-1}B_2C_1R_1)^*x_1\| = \|R_1^*C_1^*B_2^*(S_1(\lambda)^{-1})^*C_2^*B_1^*R_1^*x_1\| \\ &\quad \le \|R_1^*C_1^*\|\left(\|C_2R_2B_2\| + \|C_2R_2B_2\|\|C_1R_1B_1\|\|D_\lambda^{-1}\|\|C_2R_2B_2\|\right)\|B_1^*R_1^*x_1\| \\ &\quad \le (\tilde{M}_2 + \tilde{M}_1\tilde{M}_2M_D)\|R_1^*C_1^*\|\|B_1^*R_1^*x_1\|, \end{aligned}$$

and

$$\begin{aligned} \|(S_1(\lambda)^{-1}B_2C_1R_1)^*x_2\| &\le (1+\tilde{M}_1\tilde{M}_2M_D)\|R_1^*C_1^*\|\|B_2^*R_2^*x_2\| \\ \|(R_1B_1C_2S_1(\lambda)^{-1})^*x_1\| &\le (1+\tilde{M}_1\tilde{M}_2M_D)\|R_2^*C_2^*\|\|B_1^*R_1^*x_1\| \\ \|(S_1(\lambda)^{-1})^*x_2\| &\le \|R_2^*x_2\| + \tilde{M}_1M_D\|R_2^*C_2^*\|\|B_2^*R_2^*x_2\|. \end{aligned}$$

Similarly as in the case of $\|R(\lambda,A)x\|$, the above estimates further imply

$$\|R(\lambda,A)^*x\|^2 \le 3\|R_1^*x_1\|^2 + 4\|R_2^*x_2\|^2 + 4M_{\mathrm{tot}}^2 \sum_{k,l=1,2} \|R_k^*C_k^*\|^2\|B_l^*R_l^*x_l\|^2.$$

Lemmas 6 and 17 now show that

$$\begin{aligned} \sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)^*x\|^2d\eta &\le 3\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A_1)^*x_1\|^2d\eta \\ &\quad + 4\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A_2)^*x_2\|^2d\eta \\ &\quad + 4M_{\mathrm{tot}}^2\sum_{k,l=1,2}\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A_k)^*C_k^*\|^2\|B_l^*R(\xi+i\eta,A_l)^*x_l\|^2d\eta \\ &\quad < \infty. \end{aligned}$$

By Lemma 6 we finally have that the semigroup $T(t)$ is uniformly bounded. This implies that $T(t)$ is polynomially stable with power $\alpha = \max\{\alpha_1,\alpha_2\}$. □

Proof of Theorem 11. The claim of the theorem is that if $T_1(t)$ is exponentially stable and $\beta_2,\gamma_2 \ge \alpha_2$, then there exists $\delta > 0$ such that the semigroup $T(t)$ is polynomially stable with power $\alpha = \alpha_2$ provided that

$$\|B_1\|\|C_1\|\|(-A_2)^{\beta_2}B_2\|\|(-A_2^*)^{\gamma_2}C_2^*\| < \delta, \tag{13}$$

Moreover, $\dim Y_2 < \infty$, it is sufficient that $\beta_2 + \gamma_2 \ge \alpha_2$.

Since $T_1(t)$ is exponentially stable, we have $\sup_{\lambda \in \overline{\mathbb{C}^+}} \|R(\lambda, A_1)\| < \infty$. Because of this and by Lemma 4 we can choose $M_1, M_2 \ge 1$ such that

$$\begin{aligned} \|C_1 R(\lambda, A_1) B_1\| &\le M_1 \|B_1\| \|C_1\| \\ \|C_2 R(\lambda, A_2) B_2\| &\le M_2 \|(-A_2)^{\beta_2} B_2\| \|(-A_2^*)^{\gamma_2} C_2^*\| \end{aligned}$$

for all $\lambda \in \overline{\mathbb{C}^+}$. If we choose $0 < \delta < 1/(M_1 M_2)$ and if (13) is satisfied, then $\|C_1 R(\lambda, A_1) B_1 C_2 R(\lambda, A_2) B_2\| \le \delta M_1 M_2 < 1$. As in the proof of Theorem 10 we can now see that $\sigma(A) \subset \mathbb{C}^-$, and $\|D_\lambda^{-1}\|$ is uniformly bounded for $\lambda \in \overline{\mathbb{C}^+}$.

If we can verify that under the assumptions of the theorem the conditions (10) are satisfied for every $k, l \in \{1, 2\}$, then the uniform boundedness of $T(t)$ follows directly from the estimates made in the proof of Theorem 10.

If $k = 1$ and $l = 1, 2$, we have that

$$\begin{aligned} &\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_1) B_1\|^2 \|C_l R(\xi + i\eta, A_l) x_l\|^2 d\eta \\ &\quad \le \|B_1\|^2 \|C_l\|^2 \Big(\sup_{\lambda \in \overline{\mathbb{C}^+}} \|R(\lambda, A)\| \Big)^2 \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_l) x_l\|^2 d\eta < \infty \end{aligned}$$

by Lemma 6. On the other hand, if $k = l = 2$, then (10a) follows directly from Lemma 15 and our assumptions. Finally, we need to consider the case where $k = 2$ and $l = 1$. If $\beta_2 \ge \alpha_2$, then

$$\|R(\lambda, A_2) B_2\| \le \|R(\lambda, A_2)(-A_2)^{-\alpha_2}\| \|(-A_2)^{\alpha_2 - \beta_2}\| \|(-A_2)^{\beta_2} B_2\|$$

for all $\lambda \in \overline{\mathbb{C}^+}$ and

$$\begin{aligned} &\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_2) B_2\|^2 \|C_1 R(\xi + i\eta, A_1) x_1\|^2 d\eta \\ &\quad \le \|(-A_2)^{\beta_2} B_2\|^2 \|(-A_2)^{\alpha_2 - \beta_2}\|^2 \|C_1\|^2 \Big(\sup_{\lambda \in \overline{\mathbb{C}^+}} \|R(\lambda, A_2)(-A_2)^{-\alpha_2}\| \Big)^2 \\ &\qquad \times \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_1) x_1\|^2 d\eta < \infty \end{aligned}$$

by Lemma 6. On the other hand, if $\dim Y_2 < \infty$, then

$$\begin{aligned} &\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_2) B_2\|^2 \|C_1 R(\xi + i\eta, A_1) x_1\|^2 d\eta \\ &\quad \le \|C_1\|^2 \|x_1\|^2 \Big(\sup_{\lambda \in \overline{\mathbb{C}^+}} \|R(\lambda, A_1)\| \Big)^2 \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A_2) B_2\|^2 d\eta < \infty \end{aligned}$$

again by Lemma 6.

It remains to show that $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^{\alpha_2})$. The estimates made in the proof of Theorem 10 show that

$$\begin{aligned} \|R(i\omega, A)\|^2 &\le 3\|R(i\omega, A_1)\|^2 + 4\|R(i\omega, A_2)\|^2 \\ &\qquad + 4M_{\mathrm{tot}}^2 \sum_{k=1,2} \|R(i\omega, A_k) B_k\|^2 \|C_l R(i\omega, A_l)\|^2. \end{aligned}$$

This implies $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^{\alpha_2})$, since in the case where $k = 1$ or $l = 1$, one of the resolvents is uniformly bounded, and we clearly have $\|R(i\omega, A_k)B_k\|\|C_l R(i\omega, A_l)\| = \mathcal{O}(|\omega|^{\alpha_2})$. In the case $k = l = 2$ the same conclusion follows from Lemma 15. □

6. Examples Concerning Optimality of the Results

In this section we present two examples to illustrate the optimality of the conditions in the results presented in Section 3. In the first example we show that the condition $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$ is in general an optimal condition for the polynomial stability of a semigroup generated by a triangular block operator matrix.

Example 18. Let $A_1 : \mathcal{D}(A_1) \subset X_1 \to X_1$ generate a semigroup $T_1(t)$ such that $T_1(t)$ is polynomially stable with power $\alpha > 0$, but

$$\sup_{t>0} t\,\|T_1(t)(-A_1)^{-\tilde{\alpha}}\| = \infty \qquad \text{for every} \quad 0 \leq \tilde{\alpha} < \alpha. \tag{14}$$

Choose $X_2 = X_1$, $A_2 = A_1$, $Y = X_1$, $B = (-A_1)^{-\beta} \in \mathcal{L}(X_1)$ and $C = (-A_1)^{-\gamma} \in \mathcal{L}(X_1)$. Consider the triangular block operator matrix

$$A = \begin{pmatrix} A_1 & BC \\ 0 & A_1 \end{pmatrix} = \begin{pmatrix} A_1 & (-A_1)^{-(\beta+\gamma)} \\ 0 & A_1 \end{pmatrix}.$$

A direct computation shows that $T(t)$ generated by A is of the form

$$T(t) = \begin{pmatrix} T_1(t) & tT_1(t)(-A_1)^{-(\beta+\gamma)} \\ 0 & T_1(t) \end{pmatrix}.$$

Since $T_1(t)$ is uniformly bounded, the semigroup $T(t)$ is uniformly bounded if and only if $\sup_{t>0}\, t\,\|T_1(t)(-A_1)^{-(\beta+\gamma)}\| < \infty$. Our assumption (14) shows that if $\beta/\alpha_1 + \gamma/\alpha_2 < 1$, or equivalently $\beta + \gamma < \alpha$, the semigroup $T(t)$ is not uniformly bounded and it is therefore unstable. This shows that the condition $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$ is in general an optimal condition for the exponents.

On the other hand, it is straightforward to verify that in this example $\beta/\alpha_1 + \gamma/\alpha_2 \geq 1$ is sufficient for polynomial stability of $T(t)$ even though Y is infinite-dimensional.

The second example shows that $\beta_2 + \gamma_2 \geq \alpha_2$ in Theorem 11 is in general an optimal condition for the exponents.

Example 19. Let $X_1 = X_2 = \ell^2(\mathbb{C})$, and consider

$$A_1 = \sum_{k=1}^{\infty} (-\sigma + ik)\,\langle \cdot, \phi_k \rangle \phi_k, \qquad \text{and} \qquad A_2 = \sum_{k=1}^{\infty} \left(-\frac{1}{k^{\alpha_2}} + ik \right) \langle \cdot, \phi_k \rangle \phi_k$$

where $\alpha_2 > 0$, $\sigma > 0$ and $\{\phi_k\}_{k=1}^{\infty}$ is the Euclidean basis of X_1. The operators A_1 and A_2 generate semigroups $T_1(t)$ and $T_2(t)$, respectively, such that $T_1(t)$ is exponentially stable and $T_2(t)$ is polynomially stable with power α_2. Choose $Y_1 = Y_2 = \mathbb{C}$ and for some fixed $n \in \mathbb{N}$ let $B_1 = \sigma\phi_n$, $B_2 = n^{-\alpha_2/2}\phi_n$, $C_1 =$

$\langle \cdot, \phi_n \rangle$, and $C_2 = n^{-\alpha_2/2}\langle \cdot, \phi_n \rangle$. The block operator matrix A on $X = X_1 \times X_2$ is then given by

$$A = \begin{pmatrix} A_1 & \frac{\sigma}{n^{\alpha_2/2}}\langle \cdot, \phi_n \rangle \phi_n \\ \frac{1}{n^{\alpha_2/2}}\langle \cdot, \phi_n \rangle \phi_n & A_2 \end{pmatrix}.$$

We have $\mathcal{R}(B_2) = \mathcal{R}(C_2^*) = \operatorname{span}\{\phi_n\} \subset \mathcal{D}((-A_2)^\infty)$, and therefore the range conditions on B_2 and C_2 are satisfied for any choices of the exponents $\beta_2, \gamma_2 \geq 0$. We will show that if $\beta_2 + \gamma_2 < \alpha_2$, then for any $\delta > 0$ we can choose $n \in \mathbb{N}$ in such a way that the semigroup $T(t)$ is unstable even though

$$\|B_1\| \cdot \|C_1\| \cdot \|(-A_2)^{\beta_2} B_2\| \cdot \|(-A_2^*)^{\gamma_2} C_2^*\| < \delta.$$

To this end, let $\delta > 0$ be arbitrary. A direct computation yields $\|B_1\| = \sigma$, $\|C_1\| = 1$, $\|(-A_2)^{\beta_2} B_2\| = n^{\beta_2 - \alpha_2/2}$ and $\|(-A_2^*)^{\gamma_2} C_2^*\| = n^{\gamma_2 - \alpha_2/2}$. Since $\beta_2 + \gamma_2 < \alpha_2$ by assumption, the product

$$\|B_1\| \cdot \|C_1\| \cdot \|(-A_2)^{\beta_2} B_2\| \cdot \|(-A_2^*)^{\gamma_2} C_2^*\| = \sigma n^{\beta_2 + \gamma_2 - \alpha_2}$$

can be made smaller than $\delta > 0$ by choosing a sufficiently large $n \in \mathbb{N}$. It should be noted that also both of the operator norms $\|B_1 C_2\| = \sigma n^{-\alpha_2/2}$ and $\|B_2 C_1\| = n^{-\alpha_2/2}$ can be made arbitrarily small by choosing a large enough $n \in \mathbb{N}$.

To show that $T(t)$ is unstable, consider the operator

$$\lambda - A = \begin{pmatrix} \lambda - A_1 & -\frac{\sigma}{n^{\alpha_2/2}}\langle \cdot, \phi_n \rangle \phi_n \\ -\frac{1}{n^{\alpha_2/2}}\langle \cdot, \phi_n \rangle \phi_n & \lambda - A_2 \end{pmatrix}$$

for $\lambda \in \overline{\mathbb{C}^+}$. The Schur complement $S_1(\lambda)$ of $\lambda - A_1$ in $\lambda - A$ is

$$S_1(\lambda) = \lambda - A_2 - \frac{\sigma}{n^{\alpha_2}} \langle R(\lambda, A_1)\phi_n, \phi_n \rangle \langle \cdot, \phi_n \rangle \phi_n.$$

Since $\sigma(A_1) \subset \mathbb{C}^-$, we have that $i\omega \in i\mathbb{R}$ is an eigenvalue of A if and only if 0 is an eigenvalue of $S_1(i\omega)$. But a direct computation shows that

$$\begin{aligned} S_1(in)\phi_n &= in\phi_n - \big(-\sigma + in\big)\phi_n - \frac{\sigma}{n^{\alpha_2}} \cdot \frac{1}{in + 1/n^{\alpha_2} - in}\phi_n \\ &= \sigma\phi_n - \frac{\sigma}{n^{\alpha_2}} n^{\alpha_2}\phi_n = 0. \end{aligned}$$

This implies that $\lambda = in \in i\mathbb{R}$ is an eigenvalue of A, and thus the semigroup generated by A is unstable for all choices of $n \in \mathbb{N}$.

7. Coupled Wave Equations

In this section we use the results in Section 3 to study the stability properties of a system consisting of two coupled wave equations,

$$v_{tt}(z,t) - \Delta v(z,t) + a(z)v_t(z,t) = B_0 C_0 w(\cdot, t) \tag{15a}$$

$$w_{tt}(r,t) - w_{rr}(r,t) + (1-r)u(t) = 0 \tag{15b}$$

on $z \in \Omega = (0,\pi)\times(0,\pi) \subset \mathbb{R}^2$ and $r \in (0,1)$, respectively, and with boundary and initial conditions

$$\begin{aligned} v(z,t) &= 0 \qquad z \in \partial\Omega \\ v(z,0) &= v_0(z), \quad v_t(z,0) = v_1(z) \\ w(0,t) &= w(1,t) = 0 \\ w(r,0) &= w_0(r), \quad w_t(r,0) = w_1(r). \end{aligned}$$

The equation (15a) is a two-dimensional wave equation with local viscous damping term $a(z)v_t(z,t)$ [12, Sec 3, Ex. 3]. The function $a(z)$ is chosen as

$$a(z) = \begin{cases} 1 & 0 \le z_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

for $z = (z_1, z_2) \in \Omega$. The function $u(t)$ in (15b) is chosen in such a way that the one-dimensional subsystem is polynomially stable. This is done in Section 7.2. Our main aim in this example is to derive conditions for the operators B_0 and C_0 in the coupling between the equations so that the coupled system (15) is polynomially stable. To accomplish this, we will write (15) as a triangular system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 & BC \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{17}$$

on a suitable space $X = X_1 \times X_2$, and subsequently use Theorem 7 to study the stability of the semigroup generated by its system operator.

7.1. The Two-Dimensional System

The equation (15a) with the boundary conditions in (16) can be written as a first order linear system on a Hilbert space $X = H_0^1(\Omega) \times L^2(\Omega)$ with inner product $\langle x, y\rangle_X = \langle \nabla x_1, \nabla y_1\rangle_{L^2(\Omega)^2} + \langle x_2, y_2\rangle_{L^2(\Omega)}$ by choosing (see [12, Sec. 3, Ex. 3]) $x = (v, v_t)$, and

$$A = \begin{pmatrix} 0 & I \\ \Delta & -a(z) \end{pmatrix}, \quad \mathcal{D}(A) = \left\{ (x_1, x_2) \;\middle|\; x_2 \in H_0^1(\Omega),\ \Delta x_1 \in L^2(\Omega) \right\}.$$

With these choices (15a) without the term $B_0 C_0 w(\cdot,t)$ on the right-hand side becomes

$$\dot{x} = Ax, \qquad x(0) = x_0,$$

where $x_0 = (v_0, v_1)^T$. Since $a(z)$ is strictly positive on a vertical strip of Ω, we have from [12, Sec. 3, Ex. 3] that A generates a strongly stable semigroup and $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^2)$, and thus by Theorem 3 the semigroup generated by A is polynomially stable with power $\alpha = 2$.

In the composite system (17) we choose the first subsystem as $X_1 = H_0^1(\Omega) \times L^2(\Omega)$ and $A_1 = A$. The semigroup $T_1(t)$ generated by A_1 is then polynomially stable with power $\alpha_1 = 2$.

7.2. The One-Dimensional System

Now we turn our attention to the one-dimensional equation (15b) with the boundary and initial conditions in (16). We define $g_0(r) = 1 - r$ and $A_0 : \mathcal{D}(A_0) \subset L^2(0,1) \to L^2(0,1)$ as $A_0 = -\frac{d^2}{dr^2}$ with the domain

$$\mathcal{D}(A_0) = \left\{ x \in L^2(0,1) \mid x, x' \text{ abs. cont., } x'' \in L^2(0,1),\ x(0) = x(1) = 0 \right\},$$

(i.e., $\mathcal{D}(A_0) = H_0^1(0,1) \cap H^2(0,1)$). The operator A_0 has a positive self-adjoint square root

$$A_0^{1/2} x = \sum_{k=1}^{\infty} k\pi \langle x(\cdot), \sqrt{2}\sin(k\pi\cdot)\rangle_{L^2} \sqrt{2}\sin(k\pi\cdot)$$

and the space $X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$ is a Hilbert space with the inner product $\langle x, y\rangle_X = \langle A_0^{1/2} x_1, A_0^{1/2} y_1\rangle_{L^2} + \langle x_2, y_2\rangle_{L^2}$ for $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$. Choosing

$$x = \begin{pmatrix} w \\ w_r \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}),$$

$$Gu = gu = \begin{pmatrix} 0 \\ g_0 \end{pmatrix} u, \quad x_0 = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix},$$

the wave equation (15b) can be written as

$$\dot{x} = Ax + Gu, \qquad x(0) = x_0. \tag{18}$$

We will now show that we can choose $K = \langle \cdot, h\rangle \in \mathcal{L}(X, \mathbb{C})$ in such a way that with feedback input $u(t) = Kx(t)$ the system (18) is polynomially stable with power $\alpha = 5/3$. The eigenvalues of the operator A are $\lambda_k = ik\pi$ for $k \in \mathbb{Z}\backslash\{0\}$, and the corresponding eigenvectors

$$\varphi_k(z) = \frac{1}{\lambda_k} \begin{pmatrix} \sin(k\pi z) \\ \lambda_k \sin(k\pi z) \end{pmatrix}$$

form an orthonormal basis of X and

$$\langle g, \varphi_k\rangle_X = \langle g_0, \sin(k\pi\cdot)\rangle_{L^2} = \frac{1}{k\pi}.$$

For $k \neq 0$ denote $\mu_k = -\frac{1}{|k|^{5/3}} + ik\pi$. Then for any $\lambda \in \mathbb{C}$ with $\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{\pi}{3} = \frac{1}{3} \inf_{k\neq l} |\lambda_k - \lambda_l|$ and for any $l \neq 0$ we have

$$\sum_{k\neq 0} \frac{|\langle g, \varphi_k\rangle|^2}{|\lambda - \lambda_k|^2} \leq \frac{1}{\pi^2 \operatorname{dist}(\lambda, \sigma(A))^2} \sum_{k\neq 0} \frac{1}{k^2} < \infty \tag{19a}$$

$$\sum_{\substack{k\neq 0\\ k\neq l}} \frac{|\langle g, \varphi_k\rangle|^2}{|\lambda_k - \lambda_l|^2} \leq \frac{1}{\pi^2} \sum_{\substack{k\neq 0\\ k\neq l}} \frac{1}{k^2\pi^2} < \infty \tag{19b}$$

$$\sum_{k\neq 0} \frac{|\mu_k - \lambda_k|^2}{|\langle g, \varphi_k\rangle|^2} = \sum_{k\neq 0} \frac{\frac{1}{|k|^{10/3}}}{\frac{1}{\pi^2 k^2}} = \pi^2 \sum_{k\neq 0} \frac{1}{|k|^{4/3}} < \infty. \tag{19c}$$

We now have from [19, Thm. 1] that there exists $h \in X$ such that $A+GK$ with $K = \langle \cdot, h\rangle_X$ is a Riesz-spectral operator with eigenvalues $\{\mu_k\}_{k\neq 0}$ and $A+GK$ has at most finite number of nonsimple eigenvalues. This immediately implies that for some constant $M \geq 1$ and for $\omega \in \mathbb{R}$ we have

$$\|R(i\omega, A+GK)\| \leq \frac{M}{\inf_k \operatorname{dist}(i\omega, \mu_k)} = \mathcal{O}(|\omega|^{5/3}),$$

and thus the semigroup generated by $A+GK$ is polynomially stable with power $\alpha = 5/3$.

In the composite system (17) we choose $X_2 = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$ and $A_2 = A+GK$. We then have that the semigroup $T_2(t)$ generated by A_2 is polynomially stable with power $\alpha_2 = 5/3$.

7.3. The Composite System

If the space X_1 and X_2 and the operators A_1 and A_2 are chosen as in Sections 7.1 and 7.2, then the coupled wave equations (15) can be written as a triangular system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 & BC \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where the operators $B \in \mathcal{L}(Y, X_1)$ and $C \in \mathcal{L}(X_2, Y)$ are such that $By = (0, B_0 y)^T \in X_1$ for $y \in Y$ and $C(x_2^1, x_2^2) = C_0 x_2^1$ for $x_2 = (x_2^1, x_2^2)^T \in X_2$.

We can now use Theorem 7 to impose conditions on B and C so that the triangular block operator matrix generates a polynomially stable semigroup. Indeed, if these operators are such that $\mathcal{R}(B) \subset \mathcal{D}(A_1)$ and $C(-A_2) : \mathcal{D}(A_2) \to X_2$ extends to a bounded operator on X_2 (i.e., if $\beta = \gamma = 1$), then $\beta/\alpha_1 + \gamma/\alpha_2 = 1/2 + 3/5 = 11/10 > 1$, and Theorem 7 shows that the system (15) is polynomially stable with power $\alpha = \max\{\alpha_1, \alpha_2\} = 2$. In particular, the space Y does not have to be finite-dimensional. As an example, we can consider an interconnection of the form

$$(B_0 C_0 w(\cdot, t))(z) = \sum_{k\neq 0} \frac{1}{k^2}\langle w(\cdot,t), \sin(k\pi \cdot)\rangle_{L^2} \sin(kz_1)\sin(kz_2) \tag{20}$$

for $z = (z_1, z_2) \in \Omega$. Here we can choose $Y = \ell^2(\mathbb{C})$ with the Euclidean basis vectors $\{e_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, and define $B_0 \in \mathcal{L}(Y, L^2(\Omega))$ and $C_0 \in \mathcal{L}(\mathcal{D}(A_0^{1/2}), Y)$ such that

$$(B_0 y)(z) = \sum_{k\neq 0} \frac{1}{k^2}\langle y, e_k\rangle \sin(kz_1)\sin(kz_2)$$

$$C_0 x_2^1 = \sum_{k\neq 0} \langle x_2^1, \sin(k\pi \cdot)\rangle_{L^2} e_k$$

for $z = (z_1, z_2) \in \Omega$. We then have

$$
\begin{aligned}
\|C_0 x_2^1\|^2 &= \sum_{k \neq 0} |\langle x_2^1, \sin(k\pi \cdot)\rangle_{L^2}|^2 = 2 \sum_{k=1}^{\infty} |\langle x_2^1, \sin(k\pi \cdot)\rangle_{L^2}|^2 \\
&\leq \sum_{k=1}^{\infty} k^2 \pi^2 |\langle x_2^1, \sqrt{2} \sin(k\pi \cdot)\rangle_{L^2}|^2 = \|x_2^1\|^2_{\mathcal{D}(A_0^{1/2})}.
\end{aligned}
$$

For every $(x^1, x^2)^T \in \mathcal{D}(A_2) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ we also have that

$$
\begin{aligned}
C(-A_2) \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} &= -C \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} - CGK \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \\
&= -C_0 x^2 - CGK \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.
\end{aligned}
$$

Since $CGK \in \mathcal{L}(X_2, Y)$ and since

$$
\begin{aligned}
\|C_0 x^2\|^2 &= \sum_{k \neq 0} |\langle x^2, \sin(k\pi \cdot)\rangle_{L^2}|^2 = \sum_{k=1}^{\infty} |\langle x^2, \sqrt{2} \sin(k\pi \cdot)\rangle_{L^2}|^2 = \|x^2\|^2_{L^2(0,1)} \\
&\leq \|x^1\|^2_{\mathcal{D}(A_0^{1/2})} + \|x^2\|^2_{L^2(0,1)} = \left\| \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \right\|^2,
\end{aligned}
$$

we have that $C(-A_2)$ extends to a bounded operator on X_2, and thus we can let $\gamma = 1$.

In order to verify that B satisfies $\mathcal{R}(B) \subset \mathcal{D}((-A_1)^\beta)$ for $\beta = 1$, we need to show that $\mathcal{R}(B_0) \subset H_0^1(\Omega)$. Let $y \in Y = \ell^2(\mathbb{C})$ be arbitrary. We have $(\langle y, e_k \rangle / k)_{k \neq 0} \in \ell^1(\mathbb{C})$, and therefore

$$
\begin{aligned}
\frac{\partial}{\partial z_1} (B_0 y)(z) &= \sum_{k \neq 0} \frac{\langle y, e_k \rangle}{k^2} k \cos(k z_1) \sin(k z_2) = \sum_{k \neq 0} \frac{\langle y, e_k \rangle}{k} \cos(k z_1) \sin(k z_2) \\
\frac{\partial}{\partial z_2} (B_0 y)(z) &= \sum_{k \neq 0} \frac{\langle y, e_k \rangle}{k} \sin(k z_1) \cos(k z_2).
\end{aligned}
$$

Moreover, the property $(\langle y, e_k \rangle / k)_{k \neq 0} \in \ell^1(\mathbb{C})$ implies that $\frac{\partial}{\partial z_1}(B_0 y)(\cdot)$ and $\frac{\partial}{\partial z_2}(B_0 y)(\cdot)$ are bounded uniformly continuous functions on $\overline{\Omega} = [0, \pi] \times [0, \pi]$. Indeed, if we for instance denote $f(z) = \frac{\partial}{\partial z_1}(B_0 y)(z)$ and $y_k = \langle y, e_k \rangle / k$, and if $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $\sum_{|k| > N} |y_k| < \varepsilon / 4$. Moreover, since the function $z \mapsto f_N(z) = \sum_{0 < |k| \leq N} y_k \cos(k z_1) \sin(k z_2)$ is uniformly continuous, there exists $\delta > 0$ such that $|f_N(z) - f_N(\tilde{z})| < \varepsilon / 2$ whenever $\|z - \tilde{z}\| < \delta$. Thus if $\|z - \tilde{z}\| < \delta$, then

$$
|f(z) - f(\tilde{z})| \leq |f_N(z) - f_N(\tilde{z})| + 2 \sum_{|k| > N} |y_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

Since we also clearly have $(B_0 y)(z) = 0$ for every $z \in \partial\Omega$, this shows that $B_0 y \in H_0^1(\Omega)$. The element $y \in Y$ was arbitrary, and we have thus shown that $\mathcal{R}(B_0) \subset H_0^1(\Omega)$.

Since the conditions of Theorem 7 are satisfied, we have that the system (15) of wave equations with the coupling (20) is polynomially stable with power $\alpha = \max\{\alpha_1, \alpha_2\} = 2$.

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