The Role of Exosystems in Output Regulation

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In this paper we study the role of the exosystem in the theory of output regulation for linear infinite-dimensional systems. The main result of this paper shows that a stabilizing autonomous controller that achieves output tracking of an almost periodic reference signal is also capable of tracking any signal generated by a full exosystem. Moreover, given an almost periodic reference signal, we present a method for constructing a minimal exosystem generating this signal.

***Index Terms*—Output regulation, distributed parameter system, feedback control.**

I. Introduction

The problem of *output regulation* for a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + w_d(t), \qquad x(0) = x_0 \in X \quad \text{(1a)}$$
$$y(t) = Cx(t) + Du(t) \quad \text{(1b)}$$

requires designing a controller in such a way that the output $y(t)$ asymptotically approaches a given reference signal $y_{ref}(t)$ despite an external disturbance signal $w_d(t)$ affecting the state of the plant. In applications the output regulation problem is encountered often in, for example, robotics, navigation and process control.

In the theory of output regulation for mathematical systems [1]–[4] the most common approach is to consider a *class* of reference and disturbance signals, instead of designing a controller to achieve tracking of a single reference signal $y_{ref}(t)$. More precisely, the controller is chosen in such a way that the controlled system achieves asymptotic tracking for any reference and disturbance signals generated by *an exosystem* of the form

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \quad \text{(2a)}$$
$$w_d(t) = Ev(t) \quad \text{(2b)}$$
$$y_{ref}(t) = Fv(t). \quad \text{(2c)}$$

The class of reference and disturbance signals generated with the different initial states v_0 of the exosystem depends on the space W and on the chosen operators S, E, and F. Recently, there has been particular interest in output regulation with reference signals generated by exosystems on infinite-dimensional spaces W [5]–[8]. Such exosystems are necessary in studying the output tracking of nonsmooth periodic reference signals or *almost periodic* functions.

Considering the output regulation problem for the class of signals generated by an exosystem (2) has several advantages. In particular, the solvability of the control problem and the controllers achieving output tracking can be characterized using the so-called *regulator equations* [1], [3], [8], which are independent of time. Moreover, there are well-known and effective techniques for controller design for output regulation with exosystems of the form (2).

However, in the case of an infinite-dimensional exosystem, the class of signals generated by (2) may be very large [9, Sec. 3]. In some applications, such as in process control or robotics, the controlled system may only be required to track a single reference signal y^*_{ref}. Nevertheless, the solution of the output regulation problem provides a controller that achieves output tracking for all reference signals generated by some exosystem capable of producing y^*_{ref} as its output. In such a situation, it is natural to ask whether the controller could be simplified by instead only considering the control problem for the single signal y^*_{ref}. In this paper we show that such an approach would not lead to a simplified controller. In particular, as our main result we prove that *a linear time-invariant controller solving the output regulation problem for some specific signals w^*_d and y^*_{ref} necessarily solves the same problem for a full exosystem generating the signals w^*_d and y^*_{ref}.* First and foremost, this result illustrates a fundamental property of linear autonomous controllers.

Throughout the paper we consider an infinite-dimensional linear system (1) with possibly unbounded input and output operators B and C, respectively. The reference and disturbance signals w^*_d and y^*_{ref} are assumed to be nonsmooth almost periodic functions. As our first main result, we present a method for constructing a minimal exosystem generating the signals w^*_d and y^*_{ref} as its output. This construction generalizes results in [9], [10]. In particular, our approach yields easily verifiable conditions on the signals w^*_d and y^*_{ref} such that the solvability of the Sylvester equation in the regulator equations is guaranteed.

The paper is organized as follows. In Section II we state the main assumptions on the plant and the controller. In Section III we construct the minimal exosystem generating the given reference and disturbance signals. The output regulation problems for both the individual signals w^*_d and y^*_{ref}, and for a full exosystem (2) are defined in Section IV. In Section V we show that a stabilizing controller solves one of these problems if and only if it solves both of them. Section VI contains concluding remarks.

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A,

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respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The dual pairing on a Banach space and the inner product on a Hilbert space are both denoted by $\langle \cdot, \cdot \rangle$.

II. Assumptions on the Plant and the Controller

We assume the operators of the plant (1) on a Banach space X are such that $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$ on X on X. The input and output operators may be unbounded in such a way that $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, where X_{-1} and X_1 are scale spaces related to the operator A (see [11] for details), U is a Banach space and $Y = \mathbb{C}^p$. We denote by $A_{-1} : X \subset X_{-1} \to X_{-1}$ the extension of the operator A to the space X_{-1}. Finally, the feedthrough operator satisfies $D \in \mathcal{L}(U, Y)$.

We assume the output $y(t)$ of the plant is available for measurement, and define the regulation error as $e(t) = y(t) - y_{ref}(t)$. A dynamic error feedback controller on a Banach space Z is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z \tag{3a}$$
$$u(t) = K z(t) \tag{3b}$$

where the system operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, and $K \in \mathcal{L}(Z_1, U)$.

If the reference and disturbance signals y_{ref} and w_d, respectively, are outputs of the exosystem (2), then the closed-loop system consisting of the plant and the controller can be written on the space $X_e = X \times Z$ (with norm $\|(x,z)^T\|^2 = \|x\|^2 + \|z\|^2$) as

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0} = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \tag{4a}$$
$$e(t) = C_e x_e(t) + D_e v(t), \tag{4b}$$

where $C_e = (C \quad DK)$, $D_e = -F$,

$$A_e = \begin{pmatrix} A_{-1} & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} \quad \text{and} \quad B_e = \begin{pmatrix} E \\ -\mathcal{G}_2 F \end{pmatrix}.$$

The main standing assumption on the unboundedness of the operators in the plant and the controller is that the closed-loop system operator A_e with maximal domain generates a strongly continuous semigroup on X_e. This condition guarantees that the closed-loop system has a well-defined and continuous state $x_e(t)$.

Assumption 1. Throughout the paper (A, B, C, D) and $(\mathcal{G}_1, \mathcal{G}_2, K)$ are such that A_e with domain

$$\mathcal{D}(A_e) = \left\{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathcal{D}(C) \times \mathcal{D}(K) \;\middle|\; A_{-1}x + BKz \in X \right\}.$$

generates a strongly continuous semigroup $T_e(t)$ on X_e, and C_e is relatively bounded with respect to A_e.

In the statements of the output regulation problems for both the individual signals w_d^* and y_{ref}^* and for the full exosystem (S, E, F) in Section IV it is required that the controller stabilizes the closed-loop system strongly. Therefore, throughout the paper we assume that the controller is such that the closed-loop system semigroup $T_e(t)$ is strongly stable and $\sigma(A_e) \cap \sigma(S) = \varnothing$.

The results in this paper only use the properties of the closed-loop system (4), and therefore they do not depend on the type of the controller. For example, if $B \in \mathcal{L}(X, U)$, we can instead consider a *static state feedback control law*

$$u(t) = Kx(t) + Lv(t)$$

where $v(t)$ is the state of the exosystem (2), $K \in \mathcal{L}(X_1, U)$, and $L \in \mathcal{L}(W, U)$. For such a controller the closed-loop system can be written in the form (4) on the space $X_e = X$ with operators

$$A_e = A_{-1} + BK, \qquad B_e = BL + E$$
$$C_e = C + DK, \qquad D_e = DL - F.$$

In Assumption 1 we then again assume that the operator $A_e = A_{-1} + BK$ with the domain $\mathcal{D}(A_e) = \{ x \in \mathcal{D}(K) \mid A_{-1}x + BKx \in X \}$ generates a strongly continuous semigroup on $X_e = X$ and that $C_e = C + DK$ is relatively bounded with respect to A_e. With these modifications, the theory developed in [11] as well as all the results presented in this paper remain valid.

III. Exogeneous Signals and the Exosystem

In this section we start with given almost periodic reference and disturbance signals, and construct a minimal exosystem generating these signals. The signals w_d^* and y_{ref}^* are assumed to be nonsmooth almost periodic functions of the form

$$w_d^*(t) = \sum_{k \in \mathbb{Z}} a_k e^{i\omega_k t}, \qquad y_{ref}^*(t) = \sum_{k \in \mathbb{Z}} b_k e^{i\omega_k t}, \tag{5}$$

where $(\omega_k)_{k \in \mathbb{Z}}$ is a sequence of distinct frequencies, $(a_k)_{k \in \mathbb{Z}} \in \ell^1(X)$, and $(b_k)_{k \in \mathbb{Z}} \in \ell^1(Y)$. If for each $k \in \mathbb{Z}$ at least one of the coefficients a_k and b_k is nonzero, then none of the frequencies $(\omega_k)_{k \in \mathbb{Z}}$ is redundant.

Our aim is to choose a Hilbert space W and the operators $S : \mathcal{D}(S) \subset W \to W$, $E \in \mathcal{L}(W, X)$, and $F \in \mathcal{L}(W, Y)$ in such a way that the signals y_{ref}^* and w_d^* are produced as outputs of an exosystem of the form (2) for some initial state $v_0^* \in \mathcal{D}(S)$. We make the following assumption on the behaviour of the coefficients of the signals w_d^* and y_{ref}^*.

Assumption 2. Assume there exists $(\alpha_k)_{k \in \mathbb{Z}} \subset (0, 1]$ for which $((1+|\omega_k|)\alpha_k)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{R})$, and the coefficients $(a_k)_{k \in \mathbb{Z}}$ and $(b_k)_{k \in \mathbb{Z}}$ satisfy

$$\sup_{k \in \mathbb{Z}} \frac{\|a_k\| + \|b_k\|}{\alpha_k^2} < \infty$$
$$\text{and} \qquad \sup_{k \in \mathbb{Z}} \|R(i\omega_k, A_e)\|^2 \frac{\|a_k\| + \|b_k\|}{\alpha_k^2} < \infty.$$

If the closed-loop system is exponentially stable, then the resolvent $R(\lambda, A_e)$ is uniformly bounded on the imaginary axis, and the second condition becomes redundant. However, in the case of exogeneous signals with an infinite number of frequencies it is in general impossible to stabilize the closed-loop system exponentially. In such a situation the norms $\|R(i\omega_k, A_e)\|$ may grow as $|k| \to \infty$. In particular, this is the case if X_e is a Hilbert space and if the closed-loop system is stabilized polynomially as in [12]. The condition in Assumption 2 then requires that the norms $\|a_k\|$ and $\|b_k\|$ of the coefficients decay sufficiently fast compared to the growth of the resolvent norms. Finally, if the signals w_d^* and y_{ref}^* have finite numbers of frequency components, the conditions in Assumption 2 are automatically satisfied (see also the comments in Remark 5).

The following theorem gives appropriate choices for an exosystem generating the reference and disturbance signals. Similar results were presented in [10] for periodic reference signals, and in [9, Lem. 3] for polynomially growing reference signals and corresponding block diagonal exosystems. The following theorem generalizes the corresponding results in [9], [10] by introducing the sequence $(\alpha_k)_{k\in\mathbb{Z}}$ as a design parameter. As we will see in Theorem 4, the sequence $(\alpha_k)_{k\in\mathbb{Z}}$ ensures the solvability of the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ in the regulator equations. Finally, if the signals w_d^* and y_{ref}^* in (5) are such that for every $k \in \mathbb{Z}$ either a_k or b_k is nonzero, then the constructed exosystem is minimal in the sense that S contains only those frequencies ω_k that are necessary for generating the signals w_d^* and y_{ref}^*.

Theorem 3. *Let $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$ be as in Assumption 2. Choose $W = \ell^2(\mathbb{C})$ with an orthonormal basis $\{\phi_k\}_{k\in\mathbb{Z}}$, and choose the system operator of the exosystem (2) to be*

$$Sv = \sum_{k\in\mathbb{Z}} i\omega_k \langle v, \phi_k\rangle \phi_k,$$
$$\mathcal{D}(S) = \Big\{ v \in W \;\Big|\; \sum_{k\in\mathbb{Z}} \omega_k^2 |\langle v, \phi_k\rangle|^2 < \infty \Big\}.$$

The output operators are chosen as

$$Ev = \sum_{k\in\mathbb{Z}} \frac{1}{\alpha_k}\langle v, \phi_k\rangle a_k, \qquad Fv = \sum_{k\in\mathbb{Z}} \frac{1}{\alpha_k}\langle v, \phi_k\rangle b_k.$$

The signals w_d^* and y_{ref}^* are then generated with the choice

$$v_0^* = \sum_{k\in\mathbb{Z}} \alpha_k \phi_k \in \mathcal{D}(S)$$

of the initial state of the exosystem.

Proof. The chosen operator S generates an isometric group $T_S(t)$ on W, and its spectrum satisfies $\sigma(S) = \overline{\sigma_p(S)} = \overline{\{i\omega_k\}_k}$.

For every $v \in W$ we have

$$\|Ev\| \le \sum_{k\in\mathbb{Z}} \frac{\|a_k\|}{\alpha_k} |\langle v, \phi_k\rangle| \le \left[\sum_{k\in\mathbb{Z}} \frac{\|a_k\|^2}{\alpha_k^2}\right]^{\frac{1}{2}} \left[\sum_{k\in\mathbb{Z}} |\langle v, \phi_k\rangle|^2\right]^{\frac{1}{2}}$$
$$\le \|v\| \left(\sup_{k\in\mathbb{Z}} \frac{\|a_k\|}{\alpha_k^2} \sum_{k\in\mathbb{Z}} \|a_k\| \right)^{\frac{1}{2}},$$

where $\sup_{k\in\mathbb{Z}} \frac{\|a_k\|}{\alpha_k^2} \sum_{k\in\mathbb{Z}} \|a_k\| < \infty$ due to Assumption 2 and $(a_k)_{k\in\mathbb{Z}} \in \ell^1(X)$. We therefore have $E \in \mathcal{L}(W, Y)$. Analogously it can be shown that $F \in \mathcal{L}(W, Y)$.

The assumption $((1+|\omega_k|)\alpha_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{R})$ and $\langle v_0^*, \phi_k\rangle = \alpha_k$ immediately imply that the initial state v_0^* satisfies $v_0^* \in W$. Moreover, we have

$$\sum_{k\in\mathbb{Z}} \omega_k^2 |\langle v_0^*, \phi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} \omega_k^2 \alpha_k^2 \le \sum_{k\in\mathbb{Z}} (1+|\omega_k|)^2 \alpha_k^2 < \infty,$$

and thus $v_0^* \in \mathcal{D}(S)$.

Finally, the reference and disturbance signals generated by the exosystem with the initial state v_0^* are given by

$$\begin{aligned} w_d(t) &= ET_S(t)v_0^* = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle E\phi_k \\ &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\alpha_k \frac{a_k}{\alpha_k} = \sum_{k\in\mathbb{Z}} a_k e^{i\omega_k t} = w_d^*(t) \\ y_{ref}(t) &= FT_S(t)v_0^* = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle F\phi_k \\ &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\alpha_k \frac{b_k}{\alpha_k} = \sum_{k\in\mathbb{Z}} b_k e^{i\omega_k t} = y_{ref}^*(t). \end{aligned}$$

□

We chose the exosystem in such a way that the initial state v_0^* generating the reference and disturbance signals w_d^* and y_{ref}^* is in the domain $\mathcal{D}(S)$ of the system operator. This is necessary since we allow the operators C and K to be unbounded, which in turn requires that the initial states of the exosystem in the output regulation problem satisfy $v_0 \in \mathcal{D}(S)$ [11]. If these two operators are bounded, there is no difficulty in considering the output regulation problem for initial states $v_0 \in W$ of exosystem, see [8] for details. In this case, the conditions of Assumption 2 can be relaxed by replacing the condition $((1+|\omega_k|)\alpha_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{R})$ with $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{R})$ (which is a weaker requirement whenever the sequence $(\omega_k)_{k\in\mathbb{Z}}$ is unbounded).

Theorem 4. *Assume $T_e(t)$ is strongly stable and $\sigma(A_e) \cap \sigma(S) = \varnothing$. If the signals w_d^* and y_{ref}^* satisfy Assumption 2 and if the exosystem (S, E, F) is chosen as in Theorem 3, then the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$.*

Proof. If E and F are chosen as in Theorem 3, then for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} \|B_e\phi_k\| &= \left\| \begin{pmatrix} E\phi_k \\ -\mathcal{G}_2 F\phi_k \end{pmatrix} \right\| \le \max\{1, \|\mathcal{G}_2\|\} \frac{1}{\alpha_k} \left\| \begin{pmatrix} a_k \\ b_k \end{pmatrix} \right\| \\ &= \max\{1, \|\mathcal{G}_2\|\} \frac{1}{\alpha_k} \sqrt{\|a_k\|^2 + \|b_k\|^2} \\ &= \sqrt{2}\max\{1, \|\mathcal{G}_2\|\} \frac{1}{\alpha_k} (\|a_k\| + \|b_k\|). \end{aligned}$$

Let $x_e' \in X_e'$ be such that $\|x_e'\| \le 1$. Then

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} |\langle R(i\omega_k, A_e)B_e\phi_k, x_e'\rangle|^2 \le \sum_{k\in\mathbb{Z}} \|R(i\omega_k, A_e)\|^2 \|B_e\phi_k\|^2 \\ &\quad \le 2\max\{1, \|\mathcal{G}_2\|^2\} \sum_{k\in\mathbb{Z}} \|R(i\omega_k, A_e)\|^2 \frac{(\|a_k\| + \|b_k\|)^2}{\alpha_k^2} \\ &\quad \le 2\max\{1, \|\mathcal{G}_2\|^2\} \left(\sup_{k\in\mathbb{Z}} \|R(i\omega_k, A_e)\|^2 \frac{\|a_k\| + \|b_k\|}{\alpha_k^2} \right) \\ &\qquad \times \sum_{k\in\mathbb{Z}} (\|a_k\| + \|b_k\|) < \infty \end{aligned}$$

due to Assumption 2, $(a_k)_{k\in\mathbb{Z}} \in \ell^1(X)$, and $(b_k)_{k\in\mathbb{Z}} \in \ell^1(Y)$. Since the bound is independent of $x_e' \in X_e$, and since $\sigma(A_e)\cap\sigma(S) = \varnothing$ by assumption, we have from [7, Lem. 6] that the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ has a unique bounded solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$, and Σ is given by

$$\Sigma v = \sum_{k\in\mathbb{Z}} \langle v, \phi_k\rangle R(i\omega_k, A_e)B_e\phi_k, \qquad v \in W.$$

□

Remark 5. In the beginning of this section we assumed that the reference and disturbance signals are of the form (5). However, if only a finite number of the coefficients in (5) are nonzero, the signals can instead be written as

$$w_d^*(t) = \sum_{k=1}^N a_k e^{i\omega_k t} \qquad y_{\mathit{ref}}^*(t) = \sum_{k=1}^N b_k e^{i\omega_k t},$$

for some $N \in \mathbb{N}$, a sequence of distinct frequencies $(\omega_k)_{k=1}^N \subset \mathbb{R}$, and coefficients $(a_k)_{k=1}^N \subset X$ and $(b_k)_{k=1}^N \subset Y$. In this situation it is possible to simplify many of the proofs presented in this paper. Indeed, if the set $\mathbb{Z}$ of indices is replaced with the finite index set $\{1,\dots,N\}$, the conditions in Assumption 2 are automatically satisfied for any sequence $(\alpha_k)_{k=1}^N \subset (0,1]$, the unique solvability of the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ is guaranteed by the assumption $\sigma(A_e)\cap\sigma(S) = \varnothing$, and many of the series appearing in the proofs are reduced to finite sums (and are thus automatically convergent). Most importantly, the exosystem constructed in Theorem 3 can be chosen to be a finite-dimensional linear system on the space $W = \mathrm{span}\{\phi_1,\dots,\phi_N\}$ with a system operator

$$S = \sum_{k=1}^N i\omega_k\langle\cdot,\phi_k\rangle\phi_k \in \mathcal{L}(W).$$

IV. The Two Output Regulation Problems

In this section we formulate the output regulation problem for individual reference and disturbance signals, as well as for the class of signals generated by an exosystem of the form (2). We begin with the problem for individual signals w_d^* and y_{ref}^*. In this version, it is also only required that the regulation error decays asymptotically for some initial state $x_{e0} \in \mathcal{D}(A_e)$ of the closed-loop system.

The Output Regulation Problem for $(y_{\mathit{ref}}^*, w_d^*)$**.** Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:

(1) The semigroup $T_e(t)$ generated by the closed-loop system operator A_e is strongly stable.
(2) For the exogeneous signals w_d^* and y_{ref}^* and for some initial state $x_{e0} \in \mathcal{D}(A_e)$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.

On the other hand, the output regulation problem for an exosystem with operators (S, E, F) is defined as follows [7], [8], [11].

The Output Regulation Problem for (S, E, F)**.** Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:

(1) The semigroup $T_e(t)$ generated by the closed-loop system operator A_e is strongly stable.
(2) For all initial states $v_0 \in \mathcal{D}(S)$ and $x_{e0} \in \mathcal{D}(A_e)$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.

The following theorem gives conditions for the solvability of the output regulation problem for the exosystem (S, E, F) using the regulator equations. This well-known result has appeared in many different settings from classical finite-dimensional control [1] to control of infinite-dimensional system with various classes of exosystems [3], [6]–[8]. In [11] it was presented for systems with unbounded control and observation operators and infinite-dimensional exosystems.

Theorem 6. *Let Assumption* 2 *be satisfied. If the controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is such that* A_e *generates a strongly stable semigroup and* $\sigma(A_e)\cap\sigma(S) = \varnothing$*, then it solves the output regulation problem for the exosystem* (S, E, F) *if and only if the regulator equations*

$$\Sigma S = A_e\Sigma + B_e \tag{6a}$$
$$0 = C_e\Sigma + D_e \tag{6b}$$

on $\mathcal{D}(S)$ *have a solution* $\Sigma \in \mathcal{L}(W, X_e)$ *satisfying* $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$.

Proof. We have from Theorem 4 that the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ has a solution $\Sigma \in \mathcal{L}(W, X_e)$. The result now follows from [11, Thm. 3.1]. □

V. Equivalence of the Output Regulation Problems

In this section we show that any error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solving the output regulation problem for the given signals w_d^* and y_{ref}^* solves the problem for all signals generated by the corresponding exosystem given in Theorem 3. The following theorem is the main result of this paper.

Theorem 7. *Let Assumption* 2 *be satisfied and let the exosystem be as in Theorem* 3*.*

A controller which stabilizes the closed-loop system strongly in such a way that $\sigma(A_e)\cap\sigma(S) = \varnothing$ *solves the output regulation problem for the signals* w_d^* *and* y_{ref}^* *if and only if it solves the output regulation problem for the exosystem* (S, E, F)*.*

Proof. The "if" part of the theorem follows directly from the fact that the signals w_d^* and y_{ref}^* belong to the class of signals generated by the exosystem (S, E, F).

To prove the converse statement, assume that the controller solves the output regulation problem for the signals w_d^* and y_{ref}^*. Let $v_0^* \in \mathcal{D}(S)$ be the initial state of the exosystem generating these signals, i.e., $w_d^*(t) = ET_S(t)v_0^*$ and $y_{\mathit{ref}}^*(t) = FT_S(t)v_0^*$ for all $t \geq 0$. Moreover, let $x_{e0} \in \mathcal{D}(A_e)$ be the initial state of the closed-loop system for which the controller solves the output regulation problem.

By Theorem 4 the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ has a solution $\Sigma \in \mathcal{L}(W, X_e)$. Since the regulation error decays asymptotically, we have from [11, Thm. 3.3] that

$$\|(C_e\Sigma + D_e)T_S(t)v_0^*\| \tag{7a}$$
$$\leq \|(C_e\Sigma + D_e)T_S(t)v_0^* - e(t)\| + \|e(t)\| \to 0 \tag{7b}$$

as $t \to \infty$.

By [11, Lem. 3.2] we have that $(C_e\Sigma + D_e)(I - S)^{-1} \in \mathcal{L}(W, Y)$. Since the space $Y = \mathbb{C}^p$ is finite-dimensional, the Riesz Representation Theorem implies $((C_e\Sigma + D_e)(I - S)^{-1}\phi_k)_{k\in\mathbb{Z}} \in \ell^2(Y)$ (i.e., $(C_e\Sigma+D_e)(I-S)^{-1}$ is a Hilbert–Schmidt operator), and thus

$$\sum_{k\in\mathbb{Z}} \frac{\|(C_e\Sigma + D_e)\phi_k\|^2}{1+\omega_k^2} = \sum_{k\in\mathbb{Z}} \left\|(C_e\Sigma + D_e)\frac{1}{1-i\omega_k}\phi_k\right\|^2$$
$$= \sum_{k\in\mathbb{Z}} \|(C_e\Sigma + D_e)(I - S)^{-1}\phi_k\|^2 < \infty.$$

Combining this with $((1 + |\omega_k|)\langle v_0^*, \phi_k\rangle)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$ implies

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \|\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k\| \\
&\qquad = \sum_{k\in\mathbb{Z}} (1 + |\omega_k|)|\langle v_0^*, \phi_k\rangle| \frac{\|(C_e\Sigma + D_e)\phi_k\|}{1 + |\omega_k|} \\
&\qquad = \left(\sum_{k\in\mathbb{Z}} (1 + |\omega_k|)^2|\langle v_0^*, \phi_k\rangle|^2\right)^{\frac{1}{2}} \\
&\qquad\quad \times \left(\sum_{k\in\mathbb{Z}} \frac{\|(C_e\Sigma + D_e)\phi_k\|^2}{(1 + |\omega_k|)^2}\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

i.e., $(\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k)_{k\in\mathbb{Z}} \in \ell^1(Y)$, and

$$(C_e\Sigma + D_e)T_S(t)v_0^* = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k.$$

For all $N \in \mathbb{N}$ the functions

$$t \mapsto \sum_{k=-N}^{N} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k$$

are trigonometric polynomials and

$$\begin{aligned}
&\left\|(C_e\Sigma + D_e)T_S(t)v_0^* - \sum_{k=-N}^{N} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k\right\| \\
&\quad = \left\|\sum_{|k|>N} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k\right\| \\
&\quad \leq \sum_{|k|>N} \|\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k\| \longrightarrow 0
\end{aligned}$$

uniformly in $t \in \mathbb{R}$ as $N \to \infty$. This concludes that the function $t \mapsto (C_e\Sigma + D_e)T_S(t)v_0^*$ is almost periodic [13, Def. 4.5.6].

We will now show that the limit in (7) implies

$$(C_e\Sigma + D_e)T_S(t)v_0^* = \sum_{k\in\mathbb{Z}} e^{i\omega_k t}\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k = 0$$

for every $t \in \mathbb{R}$. To this end, let $\varepsilon > 0$ and $t \in \mathbb{R}$ be arbitrary. Due to (7) we can choose $t_0 > 0$ such that $\|(C_e\Sigma + D_e)T_S(s)v_0^*\| < \varepsilon$ for every $s \geq t_0$. The function $(C_e\Sigma + D_e)T_S(\cdot)v_0^*$ is almost periodic, and thus by [13, Thm. 4.5.7 & (4.23)] we have

$$\begin{aligned}
\|(C_e\Sigma + D_e)T_S(t)v_0^*\| &\leq \sup_{s\in\mathbb{R}} \|(C_e\Sigma + D_e)T_S(s)v_0^*\| \\
&= \sup_{s\geq t_0} \|(C_e\Sigma + D_e)T_S(s)v_0^*\| \leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $(C_e\Sigma + D_e)T_S(t)v_0^* = 0$. Also $t \in \mathbb{R}$ was arbitrary, and we can thus conclude that $(C_e\Sigma + D_e)T_S(\cdot)v_0^* \equiv 0$.

Because the frequencies $i\omega_k$ are distinct by assumption, $(C_e\Sigma + D_e)T_S(\cdot)v_0^* \equiv 0$ is only possible if we have $\langle v_0^*, \phi_k\rangle(C_e\Sigma + D_e)\phi_k = 0$ for all $k \in \mathbb{Z}$ [13, Cor. 4.5.9(a)]. The assumption $\langle v_0^*, \phi_k\rangle = \alpha_k \neq 0$ for every $k \in \mathbb{Z}$ further implies $(C_e\Sigma + D_e)\phi_k = 0$ for all $k \in \mathbb{Z}$. Finally, since $\{\phi_k\}_{k\in\mathbb{Z}}$ is a basis of W, we must have $C_e\Sigma + D_e = 0$. The operator Σ is the solution of the Sylvester equation (6a), and we have thus concluded that the regulator equations

$$\Sigma S = A_e\Sigma + B_e$$
$$0 = C_e\Sigma + D_e$$

have a solution. By Theorem 6 the controller solves the output regulation problem for the exosystem (S, E, F). □

VI. Conclusions

The results in this paper shed light on properties and limitations of linear autonomous controllers. In particular we saw that the tracking of a single nonsmooth signal $y_{ref}^*(\cdot) : \mathbb{R} \to \mathbb{C}$ requires the autonomous controller to have the ability to track a class of signals that consist of any combination of the frequency components in y_{ref}^*. The controller design, on the other hand, requires knowledge of the frequencies in the reference and disturbance signals. In the case of, for example, an unknown disturbance signal w_d^*, the spectral content of the signal must first be estimated and subsequently the controller must be tuned to those frequencies. However, rejection of unknown disturbance signals is a good example of a problem where a better approach would be to use a control law that depends on time [14].

Combined with the existing theory, the results presented in this paper imply that output regulation of reference and disturbance signals with an infinite number of frequency components requires an infinite-dimensional controller. However, if the reference and disturbance signals only have a finite number of frequencies, they can be generated with a finite-dimensional exosystem (see Remark 5). Output regulation for a finite-dimensional exosystem can be achieved with a finite-dimensional controller, for example, whenever the system to be controlled is exponentially stable and its transfer function belongs to the Callier–Desoer algebra [15]. Implementing and approximating infinite-dimensional controllers, and the study of their performance is an important topic for further research. The challenges of implementing controllers for output regulation of periodic signals have been discussed in detail in [16].

In this paper we have considered almost periodic reference and disturbance signals generated by diagonal infinite-dimensional exosystems. Such signals are always uniformly bounded. However, if the exosystem is allowed to be block diagonal, the output regulation problem can be studied for polynomially growing signals of the form

$$y^*_{ref}(t) = t^n y^*_n(t) + t^{n-1} y^*_{n-1}(t) + \cdots + y^*_0(t),$$

where $y^*_j(\cdot)$ are almost periodic functions of the form (5) [9, Sec. 3]. In this situation the analysis becomes a bit more involved, but all the results corresponding to the ones presented in this paper are also valid for infinite-dimensional block diagonal exosystems.

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