# Robustness of Polynomial Stability with Respect to Unbounded Perturbations

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**Abstract**

In this paper we present conditions for the preservation of strong and polynomial stability of a strongly continuous semigroup under unbounded finite rank perturbations of its infinitesimal generator. In addition, we also improve recent perturbation results for bounded finite rank perturbations. The results are illustrated with two examples. In the first one we consider the preservation of stability of a one-dimensional wave equation that has been stabilized polynomially with boundary feedback. In the second example we find conditions for the preservation of polynomial stability of a multiplication semigroup under unbounded rank one perturbations.

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## 1. Introduction

In this paper we study robustness properties of nonexponential stability of strongly continuous semigroups. The extreme sensitivity to perturbations is an acknowledged shortcoming of strong and weak stability types of a semigroup. Nevertheless, exponential stability is often unachievable in a variety of control problems, in particular those involving wave or beam equations [1]. Therefore, results on preservation of nonexponential stability types are essential for robust control of various classes of linear systems. Recently in [2, 3] it was shown that an important subclass of strongly stable semigroups, the so-called *polynomially stable* semigroups, do possess robustness properties. One of the key observations was that for polynomially stable semigroups, the size of the perturbations should be measured using certain graph norms instead of the usual operator norm of the underlying space. The importance of these results arise from the fact that polynomial stability is achievable in many control applications where the systems have an infinite number of eigenvalues on the imaginary axis [1, 4, 5].

In this paper we present conditions for the preservation of polynomial stability under unbounded finite rank perturbations. Moreover, we also improve the robustness results presented in [3] for bounded perturbations. Until recently, results on strong and polynomial stability of semigroups have been scarce. For strong stability, some results can be found in the literature [6, 7], but in these references either the initial assumptions or the the conditions for preservation of stability severely limit the applicability of the results. So far, the only results on preservation of polynomial stability can be found in [2, 3].

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Throughout this paper we consider a polynomially stable semigroup $T_A(t)$ generated by a linear operator $A$ on a Hilbert space $X$ [8, 9, 10]. In particular, we assume that $T_A(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$, and

$$\|T_A(t)A^{-1}\| \le \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0 \tag{1}$$

for some $M > 0$ and $\alpha > 0$. It should be noted that in some references a polynomially stable semigroup is not required to be uniformly bounded [8]. The assumption of uniform boundedness implies that the polynomially stable semigroups considered in this paper are also strongly stable.

The main results of this paper concern the preservation of the polynomial stability of $A$ under bounded and relatively bounded finite rank perturbations. On a Hilbert space, such perturbations can be written in the form

$$A + BC \qquad \text{and} \qquad A + BCA$$

where $B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^m)$. We show that there exist exponents $\beta, \gamma \ge 0$ depending only on $\alpha > 0$ in (1) and a constant $\delta > 0$ such that the perturbed operator generates a strongly and polynomially stable semigroup whenever

$$(-A)^\beta B \in \mathcal{L}(\mathbb{C}^m, X), \qquad (-A^*)^\gamma C^* \in \mathcal{L}(\mathbb{C}^m, X), \tag{2}$$

and whenever the associated operator norms satisfy $\|(-A)^\beta B\| < \delta$ and $\|(-A^*)^\gamma C^*\| < \delta$. The fractional powers of the sectorial operators $-A$ and $-A^*$ are defined as in [11].

The robustness of polynomial stability with respect to unbounded perturbations was considered in [2] for semigroups generated by Riesz-spectral operators. However, no results were presented concerning the preservation of uniform boundedness of the semigroup. In this paper we use a different approach and successfully derive conditions for the preservation of uniform boundedness and stability for a general polynomially stable semigroup. Since the perturbing operator is unbounded, this includes showing that the perturbed operator $A+BCA$ generates a strongly continuous semigroup on $X$.

In the case of bounded perturbations, we show that the polynomial stability of a semigroup is robust with respect to a considerably larger class of perturbations than the one introduced in [3]. We do this by significantly relaxing the conditions on the exponents $\beta$ and $\gamma$ in (2). In [3] the preservation of stability required that the exponents satisfy either $\beta \geq \alpha$ or $\gamma \geq \alpha$. The main result of this paper states that if one of the exponents $\beta$ and $\gamma$ is chosen to be an integer (such a choice is always possible), then the polynomial stability of the semigroup is preserved if

$$\beta + \gamma \geq \alpha \tag{3}$$

and if the corresponding graph norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$ are small enough. The result means that the condition on the size of the exponents $\beta$ and $\gamma$ can be distributed between the two components $B$ and $C$ of the perturbing operator. This improvement greatly increases the applicability of the theoretical results. In particular, for most of the polynomially stable semigroups encountered in practical applications, the fractional powers $(-A)^\beta$ and $(-A^*)^\gamma$ are difficult or even impossible to compute for $\beta \neq 1$ and $\gamma \neq 1$. We will see an example of this in Section 5, where the unperturbed operator is a skew-adjoint operator that has been polynomially stabilized with feedback. However, the condition (3) shows that in situations with $\alpha \leq 2$ we can consider perturbations with exponents $\beta = \gamma = 1$, and the fractional powers in the conditions (2) will be reduced to the unperturbed operators $A$ and $A^*$. As was illustrated in Theorem 8 in [3], the condition (3) is an optimal condition for the exponents.

The possibilities of applying the theoretic results are illustrated with two examples. In the first one we study a one-dimensional wave equation that has been stabilized polynomially with boundary feedback. We consider the preservation of the stability of the equation under rank one perturbations to the term with the first order time derivative. In the second example we consider unbounded perturbations of a polynomially stable multiplication semigroup.

If $X$ and $Y$ are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, and $\mathcal{R}(A)$ the domain, and range of $A$, respectively. The space of bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X, Y)$. If $A : \mathcal{D}(A) \subset X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of $A$, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$.

For a function $f : \mathbb{R} \to \mathbb{R}$ and for $\alpha \geq 0$ we use the notation

$$f(\omega) = \mathcal{O}(|\omega|^\alpha)$$

if there exist constants $M > 0$ and $\omega_0 \geq 0$ such that $|f(\omega)| \leq M|\omega|^\alpha$ for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$.

## 2. Robustness of Strong and Polynomial Stability of Semigroups

In this section we present our main results. We begin by stating the standing assumptions on the unperturbed operator $A$ and on the components $B$ and $C$ of the perturbation.

**Assumption 1.** *Let $X$ be a Hilbert space. Assume that the operators $A : \mathcal{D}(A) \subset X \to X$, $B \in \mathcal{L}(\mathbb{C}^m, X)$, and $C \in \mathcal{L}(X, \mathbb{C}^m)$ satisfy the following for some $\alpha > 0$, and $\beta, \gamma \geq 0$.*

1. *The operator $A$ generates a uniformly bounded semigroup, $i\mathbb{R} \subset \rho(A)$ and there exists $\alpha > 0$ such that*

$$\|T_A(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0 \tag{4}$$

2. *We have $\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta)$ and $\mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma)$*

Since $BC$ is a finite rank operator on a Hilbert space, the Riesz representation theorem implies that there exist $\{b_j\}_{j=1}^m, \{c_j\}_{j=1}^m \subset X$ such that

$$BC = \sum_{j=1}^m \langle \cdot, c_j \rangle b_j.$$

The conditions in Assumption 1 are equivalent to

$$\{b_j\}_{j=1}^m \subset \mathcal{D}((-A)^\beta), \quad \text{and} \quad \{c_j\}_{j=1}^m \subset \mathcal{D}((-A^*)^\gamma),$$

as well as to the conditions in (2).

The following theorem concerning the preservation of polynomial stability under bounded perturbations is the first main result of this paper.

**Theorem 2.** *Let $A$, $B$, and, $C$ satisfy Assumption 1 for some $\alpha > 0$, $\beta \geq 0$, and $\gamma \in \mathbb{N}_0$. If $\beta + \gamma \geq \alpha$, then there exists $\delta > 0$ such that for all $B$ and $C$ satisfying $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$ we have $\sigma(A + BC) \subset \mathbb{C}^-$, the semigroup $T_{A+BC}(t)$ generated by $A+BC$ is uniformly bounded, and there exists $M > 0$ such that*

$$\|T_{A+BC}(t)(A + BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

*In particular, the perturbed semigroup is strongly and polynomially stable.*

On a Hilbert space a semigroup $T_A(t)$ is polynomially stable if and only if the same is true for the adjoint semigroup $T_A(t)^*$ generated by $A^*$. Therefore, the conclusion of Theorem 2 is also valid in the situation where $\beta \in \mathbb{N}_0$ and $\gamma \geq 0$ satisfy $\beta + \gamma \geq \alpha$. This is a direct consequence of Theorem 2 applied to the perturbation $A^* + C^*B^*$ of the adjoint semigroup.

The second main result of the paper concerns the preservation of stability under relatively bounded perturbations $A + BCA$. If $\gamma \geq 1$ in Assumption 1, then the operator $BCA$ can be extended to a bounded operator on $X$. Therefore, the perturbation is genuinely unbounded only if $0 \leq \gamma < 1$.

**Theorem 3.** *Let $A$, $B$, and, $C$ satisfy Assumption 1 for some $\alpha > 0$, and $\beta \geq 0$, and $0 \leq \gamma < 1$. If $\beta \geq \alpha + 1$, then there exists $\delta > 0$ such that for all $B$ and $C$ satisfying $\|(-A)^\beta B\| \cdot \|C\| < \delta$ we have $\sigma(A+BCA) \subset \mathbb{C}^-$, the operator $A + BCA$ generates a uniformly bounded semigroup, and there exists $M > 0$ such that*

$$\|T_{A+BCA}(t)(A + BCA)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

*In particular, the perturbed semigroup is strongly and polynomially stable.*

In the next section we prove Theorem 2 concerning the preservation of polynomial stability under bounded perturbations. Theorem 3 dealing with unbounded perturbations is proved separately in Section 4.

## 3. Robustness with Respect to Bounded Perturbations

We use the following characterizations for the polynomial stability of a semigroup on a Hilbert space [10, Lem. 2.3, Thm. 2.4], [12, Lem. 3.2].

**Lemma 4.** *Assume $A$ generates a uniformly bounded semigroup on a Hilbert space $X$, and $i\mathbb{R} \subset \rho(A)$. For a fixed $\alpha > 0$ the following are equivalent.*

(a) $$\|T_A(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0$$

(b) $$\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$$

(c) $$\sup_{\operatorname{Re}\lambda \geq 0} \|R(\lambda, A)(-A)^{-\alpha}\| < \infty. \tag{5}$$

Our main tool in analyzing the preservation of the stability of the semigroup is the Sherman-Morrison-Woodbury formula presented below for an unbounded perturbation $A + BCA$. The lemma can be verified with a straightforward computation.

**Lemma 5.** *Let $\lambda \in \rho(A)$, $B \in \mathcal{L}(\mathbb{C}^m, X)$, $C \in \mathcal{L}(X, \mathbb{C}^m)$. If $1 \in \rho(CAR(\lambda, A)B)$, then $\lambda \in \rho(A + BCA)$ and*

$$R(\lambda, A + BCA) = R(\lambda, A) \tag{6a}$$
$$+ R(\lambda, A)B\left(I - CAR(\lambda, A)B\right)^{-1} CAR(\lambda, A). \tag{6b}$$

The first step in proving Theorems 2 and 3 consists of showing that for a small enough $\delta > 0$, the spectrum of the perturbed operator remains in the open right half-plane whenever the appropriate graph norms of the perturbations are small enough. The lemma is presented for an unbounded perturbation $A + BCA$, but the case of a bounded perturbation can be considered analogously, see Corollary 7.

**Theorem 6.** *Assume $\beta + \gamma \geq \alpha + 1$ in Assumption 1. There exists a constant $\delta > 0$ such that if $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$, then $\sigma(A + BCA) \subset \mathbb{C}^-$. In particular, for $\lambda \in \overline{\mathbb{C}^+}$ the operator $I - CAR(\lambda, A)B$ is invertible, and*

$$\sup_{\lambda \in \overline{\mathbb{C}^+}} \|(I - CAR(\lambda, A)B)^{-1}\| < \infty. \tag{7}$$

*Proof.* By Lemma 4 we can let $M \geq 1$ be such that $\|R(\lambda, A)(-A)^{-\alpha}\| \leq M$ for all $\lambda \in \overline{\mathbb{C}^+}$, and choose

$$\delta = \frac{1}{M\|(-A)^{\alpha+1-\beta-\gamma}\|} > 0.$$

Assume $B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^m)$ are such that $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$ and let $\lambda \in \overline{\mathbb{C}^+}$. Then for any $x \in X$ with $\|x\| = 1$ we can use the properties of the fractional powers of sectorial operators [11, Ch. 3] to estimate

$$\begin{aligned}
\|CAR(\lambda, A)Bx\| &= \sup_{\|y\|=1} |\langle C(-A)R(\lambda, A)Bx, y\rangle| \\
&= \sup_{\|y\|=1} |\langle C(-A)^\gamma R(\lambda, A)(-A)^{-\beta-\gamma+1}(-A)^\beta Bx, y\rangle| \\
&= \sup_{\|y\|=1} |\langle R(\lambda, A)(-A)^{-\beta-\gamma+1}(-A)^\beta Bx, (-A^*)^\gamma C^* y\rangle| \\
&= \sup_{\|y\|=1} \big|\big\langle R(\lambda, A)(-A)^{-\alpha}(-A)^{\alpha-\beta-\gamma+1}(-A)^\beta Bx, \\
&\qquad (-A^*)^\gamma C^* y\big\rangle\big| \\
&\leq \|R(\lambda, A)(-A)^{-\alpha}\|\|(-A)^{\alpha+1-\beta-\gamma}\| \\
&\quad \times \|(-A)^\beta B\|\|(-A^*)^\gamma C^*\| \\
&\leq M\|(-A)^{\alpha+1-\beta-\gamma}\|\|(-A)^\beta B\|\|(-A^*)^\gamma C^*\| \\
&< M\|(-A)^{\alpha+1-\beta-\gamma}\| \cdot \delta = 1.
\end{aligned}$$

and thus $\|CAR(\lambda, A)B\| < 1$. In particular we have $1 \in \rho(CAR(\lambda, A)B)$, and the Sherman-Morrison-Woodbury formula in Lemma 5 implies $\lambda \in \rho(A + BCA)$. Since $\lambda \in \overline{\mathbb{C}^+}$ was arbitrary, this concludes that $\sigma(A+BCA) \subset \mathbb{C}^-$.

Let $\lambda \in \overline{\mathbb{C}^+}$. The above estimate shows that

$$\begin{aligned}
&\|CAR(\lambda, A)B\| \\
&\quad \leq M\|(-A)^{\alpha+1-\beta-\gamma}\| \cdot \|(-A)^\beta B\|\|(-A^*)^\gamma C^*\| < 1,
\end{aligned}$$

which further implies

$$\begin{aligned}\|(I - CAR(\lambda, A)B)^{-1}\| &= \left\| \sum_{n=0}^{\infty} (CAR(\lambda, A)B)^n \right\| \\ &\leq \sum_{n=0}^{\infty} \|CAR(\lambda, A)B\|^n = \frac{1}{1 - \|CAR(\lambda, A)B\|} \\ &\leq \frac{1}{1 - M\|(-A)^{\alpha+1-\beta-\gamma}\| \|(-A)^{\beta} B\| \|(-A^*)^{\gamma} C^*\|} < \infty.\end{aligned}$$

Since the bound is independent of $\lambda \in \overline{\mathbb{C}^+}$, the proof is complete. □

With slight modifications, the proof of Theorem 6 can also be adapted to the case of a bounded perturbation $A + BC$. The conclusion is the following.

**Corollary 7.** *Assume $\beta + \gamma \geq \alpha$. There exists a constant $\delta > 0$ such that if $\|(-A)^{\beta} B\| \cdot \|(-A^*)^{\gamma} C^*\| < \delta$, then $\sigma(A + BC) \subset \mathbb{C}^-$. In particular, the operator $I - CR(\lambda, A)B$ is invertible, and*

$$\sup_{\lambda \in \overline{\mathbb{C}^+}} \|(I - CR(\lambda, A)B)^{-1}\| < \infty. \tag{8}$$

In studying the uniform boundedness of the perturbed semigroup we use the following resolvent conditions. The proof of the theorem can be found in [13, Thm. 2].

**Theorem 8.** *If $A$ generates a semigroup $T_A(t)$ on a Hilbert space $X$ and if $\sigma(A) \subset \overline{\mathbb{C}^-}$, then the following are equivalent.*

1. *The semigroup $T_A(t)$ is uniformly bounded.*
2. *For all $x, y \in X$*

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} |\langle R(\xi + i\eta, A)^2 x, y \rangle| d\eta < \infty. \tag{9}$$

3. *For all $x, y \in X$*

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)x\|^2 d\eta < \infty \tag{10a}$$

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* y\|^2 d\eta < \infty. \tag{10b}$$

We also need the following two auxiliary results. The first one concerning norms of finite rank operators can be verified with straightforward estimates. The proof can also be found in [3, Lem. 3].

**Lemma 9.** *If $R \in \mathcal{L}(X)$, then*

$$\|RB\|^2 \leq \sum_{j=1}^{m} \|Rb_j\|^2, \qquad \|CR\|^2 \leq \sum_{j=1}^{m} \|R^* c_j\|^2. \tag{11}$$

The next theorem is of key importance in showing that the condition $\beta + \gamma \geq \alpha$ on the exponents is sufficient for the preservation of uniform boundedness of the semigroup.

**Theorem 10.** *Let Assumption 1 be satisfied for some $\beta \geq 0$ and $\gamma \in \mathbb{N}$. Then there exist $\tilde{B} \in \mathcal{L}(\mathbb{C}^m, X)$ and $\tilde{C} \in \mathcal{L}(X, \mathbb{C}^m)$ such that $\mathcal{R}(\tilde{B}) \subset \mathcal{D}((-A)^{\beta+\gamma})$ and*

$$\|R(\lambda, A)B\| \|CR(\lambda, A)\| \leq f(\lambda) + \|R(\lambda, A)\tilde{B}\| \|\tilde{C}R(\lambda, A)\|$$

*for all $\lambda \in \overline{\mathbb{C}^+}$, where the function $f : \overline{\mathbb{C}^+} \to \mathbb{R}^+$ satisfies $f(i\omega) = \mathcal{O}(|\omega|^{\alpha})$ and*

$$\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} f(\xi + i\eta)^2 d\eta < \infty.$$

*Proof.* Denote $B_1 = A^{-1}B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C_1 = \overline{CA} \in \mathcal{L}(X, \mathbb{C}^m)$ (the unique bounded extension of $CA$ to $X$). Then $\mathcal{R}(B_1) \subset \mathcal{D}((-A)^{\beta+1})$ and $\mathcal{R}(C_1^*) \subset \mathcal{D}((-A^*)^{\gamma-1})$. For brevity, denote $R_\lambda = R(\lambda, A)$. For all $\lambda \in \overline{\mathbb{C}^+}$ we have

$$\begin{aligned}\|R_\lambda B\| \|CR_\lambda\| &= \|R_\lambda (A - \lambda + \lambda) A^{-1} B\| \|CR_\lambda\| \\ &= \|(-I + \lambda R_\lambda) B_1\| \|CR_\lambda\| \\ &\leq \|B_1\| \|CR_\lambda\| + |\lambda| \|R_\lambda B_1\| \|CR_\lambda\| \\ &= \|B_1\| \|CR_\lambda\| + \|R_\lambda B_1\| \|C(\lambda - A + A)R_\lambda\| \\ &= \|B_1\| \|CR_\lambda\| + \|R_\lambda B_1\| \|C(I + AR_\lambda)\| \\ &\leq \|B_1\| \|CR_\lambda\| + \|R_\lambda B_1\| \|C\| + \|R_\lambda B_1\| \|C_1 R_\lambda\| \\ &= f_1(\lambda) + \|R(\lambda, A)B_1\| \|C_1 R(\lambda, A)\|.\end{aligned}$$

The function $f_1(\lambda) = \|B_1\| \|CR(\lambda, A)\| + \|R(\lambda, A)B_1\| \|C\|$ satisfies (using the scalar inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and Lemma 9)

$$\begin{aligned}&\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} f_1(\xi + i\eta)^2 d\eta \\ &\leq 2\|B_1\|^2 \cdot \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|CR(\xi + i\eta, A)\|^2 d\eta \\ &\quad + 2\|C\|^2 \cdot \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)B_1\|^2 d\eta \\ &\leq 2\|B_1\|^2 \cdot \sum_{j=1}^{m} \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* c_j\|^2 d\eta \\ &\quad + 2\|C\|^2 \cdot \sum_{j=1}^{m} \sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)A^{-1} b_j\|^2 d\eta < \infty\end{aligned}$$

due to Theorem 8 and the uniform boundedness of the semigroup generated by $A$. Furthermore, the polynomial stability of the unperturbed semigroup together with Lemma 4 implies that for $\omega \in \mathbb{R}$ we have

$$\begin{aligned}f_1(i\omega) &\leq \|A^{-1}B\| \|C\| \|R(i\omega, A)\| \\ &\quad + \|R(i\omega, A)\| \|A^{-1}B\| \|C\| = \mathcal{O}(|\omega|^{\alpha}).\end{aligned}$$

If $\gamma \geq 2$, then the previous steps can be repeated for the term $\|R(\lambda, A)B_1\| \|C_1 R(\lambda, A)\|$. Continuing this we arrive at an estimate

$$\begin{aligned}\|R(\lambda, A)B\| \|CR(\lambda, A)\| &\leq f_1(\lambda) + f_2(\lambda) + \cdots + f_\gamma(\lambda) \\ &\quad + \|R(\lambda, A)B_\gamma\| \|C_\gamma R(\lambda, A)\|,\end{aligned}$$

and the choices $f(\lambda) = f_1(\lambda) + \cdots + f_\gamma(\lambda)$, $\tilde{B} = B_\gamma$, and $\tilde{C} = C_\gamma$ satisfy the conditions of the lemma. □

From the proof we can see that the operators $\tilde{B}$ and $\tilde{C}$ in Theorem 10 are $\tilde{B} = A^{-\gamma}B$ and $\tilde{C} = \overline{CA^\gamma}$ (the unique bounded extension of $CA^\gamma$ to $X$).

We are now in a position to prove Theorem 2.

*Proof of Theorem* 2. Choose $\delta > 0$ as in Corollary 7 and assume $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$. We then have $\sigma(A + BC) \subset \mathbb{C}^-$. Lemma 4 implies that we can prove the polynomial stability of the perturbed semigroup by showing the perturbed semigroup is uniformly bounded and that $\|R(i\omega, A + BC)\| = \mathcal{O}(|\omega|^\alpha)$.

Let $x, y \in X$. Our aim is to show that the integral condition (9) in Theorem 8 is satisfied. With suitable scaling it is clear that this is true for all elements $x$ and $y$ if and only if it is true for all elements with norm equal to one. We can therefore without loss of generality assume that $\|x\| = \|y\| = 1$. For brevity denote $R(\lambda, A) = R_\lambda$ and $D_\lambda = I - CR(\lambda, A)B$. By Corollary 7 there exists $M_D \geq 1$ such that $\|D_\lambda^{-1}\| \leq M_D$ for all $\lambda \in \mathbb{C}^+$, and using the scalar inequality $2ab \leq a^2 + b^2$ we get an estimate

$$\begin{aligned}
|\langle R(\lambda, A + BC)^2 x, y\rangle| &= \big|\langle R_\lambda^2 x, y\rangle + \langle R_\lambda^2 B D_\lambda^{-1} C R_\lambda x, y\rangle \\
&\quad + \langle R_\lambda B D_\lambda^{-1} C R_\lambda^2 x, y\rangle + \langle R_\lambda B D_\lambda^{-1} C R_\lambda^2 B D_\lambda^{-1} C R_\lambda x, y\rangle\big| \\
&\leq |\langle R_\lambda^2 x, y\rangle| + \|R_\lambda B\|\|D_\lambda^{-1}\|\|CR_\lambda\|\|x\|\|R_\lambda^* y\| \\
&\quad + \|R_\lambda B\|\|D_\lambda^{-1}\|\|CR_\lambda\|\|R_\lambda x\|\|y\| \\
&\quad + \|R_\lambda B\|\|D_\lambda^{-1}\|\|CR_\lambda\|\|R_\lambda B\|\|D_\lambda^{-1}\|\|CR_\lambda\|\|x\|\|y\| \\
&\leq |\langle R_\lambda^2 x, y\rangle| + \frac{M_D}{2}\left(\|R_\lambda B\|^2\|CR_\lambda\|^2 + \|R_\lambda^* y\|^2\right) \qquad \text{(12a)} \\
&\quad + \frac{M_D}{2}\left(\|R_\lambda B\|^2\|CR_\lambda\|^2 + \|R_\lambda x\|^2\right) \qquad \text{(12b)} \\
&\quad + M_D^2\|R_\lambda B\|^2\|CR_\lambda\|^2. \qquad \text{(12c)}
\end{aligned}$$

In order to show uniform boundedness of the semigroup generated by $A + BC$ it is now sufficient to show that for those terms on the right-hand side of (12) that depend on $\lambda = \xi + i\eta$, the integrals in Theorem 8 are uniformly bounded with respect to $\xi > 0$. This is immediately true for the integrals over the terms $|\langle R_\lambda^2 x, y\rangle|$, $\|R_\lambda^* y\|^2$, and $\|R_\lambda x\|^2$ by conditions (9) and (10) in Theorem 8.

It remains to show that the integrals over the terms $\|R_\lambda B\|^2\|CR_\lambda\|^2$ are finite. For this we can use Theorem 10. Let $\tilde{B}$, $\tilde{C}$, and $f(\cdot)$ be as in Theorem 10. Then $\mathcal{R}(\tilde{B}) \subset \mathcal{D}((-A)^{\beta+\gamma}) \subset \mathcal{D}((-A)^\alpha)$, and due to Lemma 4 there exists $M_B \geq 0$ such that

$$\|R_\lambda \tilde{B}\| \leq \|R_\lambda(-A)^{-\alpha}\|\|(-A)^\alpha \tilde{B}\| \leq M_B$$

for all $\lambda \in \mathbb{C}^+$. Let $\{\tilde{c}_j\}_{j=1}^m \subset X$ be such that

$$\tilde{C} = \left(\langle \cdot, \tilde{c}_1\rangle, \ldots, \langle \cdot, \tilde{c}_m\rangle\right)^T.$$

Using Lemma 9 we can see that the integrals over the terms $\|R_\lambda B\|^2\|CR_\lambda\|^2$ on the right-hand side of (12) satisfy

$$\begin{aligned}
&\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)B\|^2\|CR(\xi + i\eta, A)\|^2 d\eta \\
&\leq 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} f(\xi + i\eta)^2 d\eta \\
&\quad + 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)\tilde{B}\|^2\|\tilde{C}R(\xi + i\eta, A)\|^2 d\eta \\
&\leq 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} f(\xi + i\eta)^2 d\eta \\
&\quad + 2M_B^2 \cdot \sup_{\xi>0} \xi \cdot \sum_{j=1}^m \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* \tilde{c}_j\|^2 d\eta \\
&\leq 2\sup_{\xi>0} \xi \int_{-\infty}^{\infty} f(\xi + i\eta)^2 d\eta \\
&\quad + 2M_B^2 \cdot \sum_{j=1}^m \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* \tilde{c}_j\|^2 d\eta < \infty.
\end{aligned}$$

Since $x, y \in X$ were arbitrary, the above estimates conclude that the operator $A + BC$ satisfies (9), and by Theorem 8 the perturbed semigroup $T_{A+BC}(t)$ is uniformly bounded.

We complete the proof by showing that the perturbed semigroup $T_{A+BC}(t)$ is polynomially stable, and that there exists $M > 0$ such that

$$\|T_{A+BC}(t)(A + BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad t > 0.$$

Lemma 4 states that this can be done by showing that the resolvent operator of the perturbed operator satisfies $\|R(i\omega, A + BC)\| = \mathcal{O}(|\omega|^\alpha)$. For all $\omega \in \mathbb{R}$ we have $i\omega \in \mathbb{C}^+$, and thus as above we can use the Sherman-Morrison-Woodbury formula and Theorem 10 to estimate

$$\begin{aligned}
&\|R(i\omega, A + BC)\| = \big\|R(i\omega, A) \\
&\quad + R(i\omega, A)B(I - CR(i\omega, A)B)^{-1}CR(i\omega, A)\big\| \\
&\leq \|R(i\omega, A)\| \\
&\quad + \|R(i\omega, A)B\|\|(I - CR(i\omega, A)B)^{-1}\|\|CR(i\omega, A)\| \\
&\leq \|R(i\omega, A)\| + M_D\|R(i\omega, A)B\|\|CR(i\omega, A)\| \\
&\leq \|R(i\omega, A)\| + M_D f(i\omega) + M_D\|R(i\omega, A)\tilde{B}\|\|\tilde{C}R(i\omega, A)\| \\
&\leq \|R(i\omega, A)\| + M_D f(i\omega) + M_D M_B\|\tilde{C}\|\|R(i\omega, A)\| \\
&= \mathcal{O}(|\omega|^\alpha)
\end{aligned}$$

because all of the terms on the right-hand side are of order $\mathcal{O}(|\omega|^\alpha)$. This concludes that under our assumptions the perturbed semigroup is polynomially stable. ☐

## 4. Robustness with Respect to Unbounded Perturbations

In this section we prove the results for the preservation of polynomial stability under unbounded perturbations $A + BCA$.

*Proof of Theorem* 3. First of all, since $\beta \geq \alpha+1$, we have $\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta) \subset \mathcal{D}(A)$. We thus have from [14, Exer. III.3.23(2)] that the operator $A+BCA$ generates a strongly continuous semigroup $T_{A+BCA}(t)$ on $X$.

Choose $\delta > 0$ as in Theorem 6 for $\gamma = 0$, and assume $\|(-A)^\beta B\| \cdot \|C\| = \|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$. Then $\sigma(A+BCA) \subset \mathbb{C}^-$. It remains to show that $T_{A+BCA}(t)$ is uniformly bounded and that $\|R(i\omega, A+BCA)\| = \mathcal{O}(|\omega|^\alpha)$.

Let $x, y \in X$. In verifying condition (12) in Theorem 8 we can again without loss of generality assume $\|x\| = \|y\| = 1$. For brevity denote $R(\lambda, A) = R_\lambda$ and $D_\lambda = I - CAR(\lambda, A)B$. By Theorem 6 there exists $M_D \geq 1$ such that $\|D_\lambda^{-1}\| \leq M_D$ for all $\lambda \in \mathbb{C}^+$. Let $\lambda \in \overline{\mathbb{C}^+}$. Since $CAR(\lambda, A)$ is a bounded operator, we can estimate as in the proof of Theorem 2

$$
\begin{aligned}
&|\langle R(\lambda, A+BCA)^2 x, y\rangle| \\
&= \big|\langle R_\lambda^2 x, y\rangle + \langle R_\lambda^2 B D_\lambda^{-1} CAR_\lambda x, y\rangle \\
&\qquad + \langle R_\lambda B D_\lambda^{-1} CAR_\lambda^2 x, y\rangle \\
&\qquad + \langle R_\lambda B D_\lambda^{-1} CAR_\lambda^2 B D_\lambda^{-1} CAR_\lambda x, y\rangle\big| \\
&\leq |\langle R_\lambda^2 x, y\rangle| + \frac{M_D}{2}\left(\|R_\lambda B\|^2 \|CAR_\lambda\|^2 + \|R_\lambda^* y\|^2\right) \\
&\quad + \frac{M_D}{2}\left(\|R_\lambda B\|^2 \|CAR_\lambda\|^2 + \|R_\lambda x\|^2\right) \\
&\quad + M_D^2 \|R_\lambda B\|^2 \|CAR_\lambda\|^2.
\end{aligned}
$$

To show uniform boundedness of the semigroup generated by $A+BCA$ it is again sufficient to show that for those terms on the right-hand side that depend on $\lambda = \xi + i\eta$, the integrals in Theorem 8 are uniformly bounded with respect to $\xi > 0$. This is immediately true for the integrals over the terms $|\langle R_\lambda^2 x, y\rangle|$, $\|R_\lambda^* y\|^2$, and $\|R_\lambda x\|^2$ by conditions (9) and (10) in Theorem 8.

It remains to show that the integrals over the terms $\|R_\lambda B\|^2 \|CAR_\lambda\|^2$ are finite. This can be done using similar techniques as in Theorem 10. Let $M_B \geq 0$ be such that $\|R(\lambda, A)(-A)^{-\alpha}\| \leq M_B$ for all $\lambda \in \overline{\mathbb{C}^+}$. Then (denoting $R_\lambda = R(\lambda, A)$ for brevity)

$$
\begin{aligned}
&\|R_\lambda B\| \|CAR_\lambda\| = \|R_\lambda B\| \|C(A - \lambda + \lambda)R_\lambda\| \\
&\quad \leq \|R_\lambda B\| \|C\| + |\lambda| \|R_\lambda B\| \|CR_\lambda\| \\
&\quad = \|R_\lambda B\| \|C\| + \|R_\lambda(\lambda - A + A)B\| \|CR_\lambda\| \\
&\quad = \|R_\lambda B\| \|C\| + \|(I + R_\lambda A)B\| \|CR_\lambda\| \\
&\quad \leq \|R_\lambda B\| \|C\| + \|B\| \|CR_\lambda\| + \|R_\lambda AB\| \|CR_\lambda\| \\
&\quad \leq \|R_\lambda B\| \|C\| + \|B\| \|CR_\lambda\| \\
&\qquad + \|R_\lambda(-A)^{-\alpha}\| \|(-A)^\alpha AB\| \|CR_\lambda\| \\
&\quad \leq \|R_\lambda B\| \|C\| + \|B\| \|CR_\lambda\| + M_B \|(-A)^\alpha AB\| \|CR_\lambda\|
\end{aligned}
$$

since $\mathcal{R}(AB) \subset \mathcal{D}((-A)^\alpha)$. From here it is straightforward to verify, as in the proof of Theorem 2 that

$$
\sup_{\xi > 0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)B\|^2 \|CAR(\xi + i\eta, A)\|^2 d\eta < \infty.
$$

Together with the earlier estimates this concludes that the operator $A+BCA$ satisfies condition (9), and thus the perturbed semigroup is uniformly bounded.

Finally, as in the proof of Theorem 2, the Sherman-Morrison-Woodbury formula implies that

$$
\begin{aligned}
&\|R(i\omega, A+BCA)\| \leq \|R(i\omega, A)\| \\
&\quad + \|R(i\omega, A)B\| \|(I - CAR(i\omega, A)B)^{-1}\| \|CAR(i\omega, A)\| \\
&\leq \|R(i\omega, A)\| + M_D \|R(i\omega, A)B\| \|CAR(i\omega, A)\| \\
&\leq \|R(i\omega, A)\| + M_D \|R(i\omega, A)B\| \|C\| \\
&\quad + M_D \|B\| \|CR(i\omega, A)\| + M_D M_B \|(-A)^\alpha AB\| \|CR(i\omega, A)\| \\
&\leq \big(1 + 2M_D \|B\| \|C\| + M_D M_B \|(-A)^\alpha AB\| \|C\|\big) \|R(i\omega, A)\| \\
&\quad = \mathcal{O}(|\omega|^\alpha)
\end{aligned}
$$

since $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$. Together with Lemma 4 this concludes that the semigroup generated by $A+BCA$ is polynomially stable. □

## 5. Robustness of a Polynomially Stable Wave Equation

In this section we consider perturbation of a wave equation that has been stabilized polynomially using boundary feedback and pole placement. As in [4, Ex. 1], the equation

$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(z,t) &= \frac{\partial^2 w}{\partial z^2}(z,t) \\
w(0,t) = 0, &\quad \frac{\partial w}{\partial z}(1,t) = u(t)
\end{aligned}
$$

can be written formally as

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} w(t) \\ \frac{dw}{dt}(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}\begin{bmatrix} w(t) \\ \frac{dw}{dt}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g_0(z) \end{bmatrix} u(t) \\
&= A\begin{bmatrix} w(t) \\ \frac{dw}{dt}(t) \end{bmatrix} + Gu(t),
\end{aligned}
$$

where $g_0(z) = \delta(1-z)$ (the Dirac delta function), $G = g(\cdot) = [0, g_0(\cdot)]^T$, and $A_0 = -\frac{d^2}{dz^2}$ with domain

$$
\mathcal{D}(A_0) = \big\{ f \in L^2(0,1) \;\big|\; f, f' \text{ abs. cont.}, \quad f'' \in L^2(0,1),\; f(0) = f'(1) = 0 \big\}.
$$

We can consider the system on the space $X = H \times L^2(0,1)$, where $H = \big\{ f \;\big|\; f, f' \in L^2(0,1),\; f(0) = 0 \big\}$. The space $X$ is a Hilbert space with the inner product

$$
\left\langle \begin{bmatrix} f_1 \\ f_1 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\rangle = \int_0^1 \left[ f_1'(z)\overline{g_1'(z)} + f_2(z)\overline{g_2(z)} \right] dz.
$$

The operator $A$ with domain $\mathcal{D}(A) = \mathcal{D}(A_0) \times H$ is skew-adjoint. It was shown in [4, Ex. 2] that there exists $K = \langle \cdot, h\rangle$ with $h \in X$ such that the operator

$$
(A + GK)^* = -A + h\langle \cdot, g\rangle
$$

with domain $\mathcal{D}((A+GK)^*) = \mathcal{D}(A)$ generates a polynomially stable semigroup on $X$ with $\alpha = 1$.

We consider conditions for polynomial stability of the wave equation after addition of a perturbing term

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)\left\langle \frac{\partial w}{\partial t}(\cdot,t), c_0 \right\rangle_{L^2}$$
$$w(0,t) = 0, \quad \frac{\partial w}{\partial z}(1,t) = u(t),$$

where $b_0, c_0 \in L^2(0,1)$. The perturbed system operator can then be formally written as

$$\begin{aligned} A + GK + BC &= \begin{bmatrix} 0 & I \\ A_0 & B_0 C_0 \end{bmatrix} + \begin{bmatrix} 0 \\ g_0 \end{bmatrix} K \\ &= A + GK + b\langle \cdot, c\rangle \end{aligned}$$

where $b = [0,\ b_0(\cdot)]^T$, $c = [0,\ c_0(\cdot)]^T$, $B = b \in \mathcal{L}(\mathbb{C}, X)$, and $C = \langle \cdot, c\rangle \in \mathcal{L}(X, \mathbb{C})$. As the perturbed equation does not have an exact representation as a system on $X$, we instead choose to consider the stability of the adjoint system.

$$\begin{aligned} (A + GK + BC)^* &= (A+GK)^* + C^* B^* \\ &= -A + h\langle \cdot, g\rangle + c\langle \cdot, b\rangle. \end{aligned}$$

The unperturbed operator is $(A+GK)^* = -A + h\langle \cdot, g\rangle$. Since $\alpha = 1$, Theorem 2 tells us that the perturbed operator $(A+GK)^* + C^*B^*$ generates a polynomially stable semigroup if $c \in \mathcal{D}((A+GK)^*) = \mathcal{D}(A) = \mathcal{D}(A_0) \times H$ and $b \in X$, and if the norms $\|(A+GK)^* c\|$ and $\|b\|$ are small enough. Since $c = [0,\ c_0(\cdot)]^T$, the condition $c \in \mathcal{D}(A)$ is equivalent to $c_0(\cdot) \in H$, i.e., $c_0(\cdot)$ is required to be absolutely continuous and satisfy $c_0(0) = 0$. We have $\|B\| = \|b\| = \|b_0\|_{L^2}$ and

$$\begin{aligned} \|(A+GK)^* C^*\| &= \|-Ac + h\langle c, g\rangle\| \\ &= \left\| -\begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ c_0 \end{bmatrix} + h\left\langle \begin{bmatrix} 0 \\ c_0 \end{bmatrix}, \begin{bmatrix} 0 \\ g_0 \end{bmatrix} \right\rangle \right\| \\ &= \left\| -\begin{bmatrix} c_0 \\ 0 \end{bmatrix} + h\int_0^1 c_0(z)\delta(1-z)dz \right\| \\ &= \left\| -\begin{bmatrix} c_0 \\ 0 \end{bmatrix} + hc_0(1) \right\| \le \|c_0'\|_{L^2} + |c_0(1)|\|h\|. \end{aligned}$$

This concludes that the perturbed wave equation is polynomially stable for all $b_0 \in L^2(0,1)$, $c_0 \in H$ whenever the functions are such that

$$\|b_0\|_{L^2}, \quad \|c_0'\|_{L^2}, \quad \text{and} \quad |c_0(1)|$$

are small enough.

## 6. Robustness of a Polynomially Stable Multiplication Semigroup

In this section we consider preservation of polynomial stability of a multiplication semigroup under unbounded perturbations. To this end, we consider a multiplication semigroup [14, Par. II.2.9]

$$(T_A(t)f)(\mu) = e^{t\cdot\mu} f(\mu)$$

on $X = L^2(\Omega)$, where $\Omega \subset \mathbb{C}^-$ is polynomially bounded away from the imaginary axis. More precisely, in this example we assume $\Omega$ is such that every $\mu \in \Omega$ satisfies

$$\begin{cases} \operatorname{Re}\mu \le -1 & \text{if } |\operatorname{Im}\mu| \le 1 \\ \operatorname{Re}\mu \le -|\operatorname{Im}\mu|^{-1} & \text{if } |\operatorname{Im}\mu| > 1. \end{cases}$$

The infinitesimal generator $A$ of the semigroup $T_A(t)$ is a multiplication operator $(Af)(\mu) = \mu f(\mu)$ with domain

$$\mathcal{D}(A) = \left\{ f \in L^2(\Omega) \;\middle|\; \int_\Omega |\mu|^2 |f(\mu)|^2 d\mu < \infty \right\}.$$

The spectrum of $A$ is given by $\sigma(A) = \Omega$, and the semigroup is uniformly bounded. Furthermore, for all $f \in X$ and for $\omega \in \mathbb{R}$ with large $|\omega|$ we have

$$\begin{aligned} \|R(i\omega, A)f\|^2 &= \int_\Omega \frac{|f(\mu)|^2}{|i\omega - \mu|^2} d\mu \\ &\le \frac{1}{\operatorname{dist}(i\omega, \Omega)^2} \int_\Omega |f(\mu)|^2 d\mu = \frac{\|f\|^2}{\operatorname{dist}(i\omega,\Omega)^2} = \mathcal{O}(|\omega|^2), \end{aligned}$$

and thus $\|R(i\omega, A)\| = \mathcal{O}(|\omega|)$. Together these properties conclude that $T_A(t)$ is polynomially stable with $\alpha = 1$.

We consider relatively bounded rank one perturbations, $A + BCA$, where $Cf = \langle f, c\rangle_{L^2}$ for a function $c \in L^2(\Omega)$ and $B = b(\cdot) \in L^2(\Omega)$. For all $f \in \mathcal{D}(A)$ we have

$$(BCAf)(\cdot) = b(\cdot) \int_\Omega \mu f(\mu) c(\mu) d\mu.$$

For $\beta \ge 0$ domains of the operators $(-A)^\beta$ and $(-A^*)^\beta$ are given by

$$\mathcal{D}((-A)^\beta) = \mathcal{D}((-A^*)^\beta) = \left\{ f \;\middle|\; \int_\Omega |\mu|^{2\beta} |f(\mu)|^2 d\mu < \infty \right\}.$$

In this example we choose $c \in X$ such that $c(\mu) = \frac{1}{|\mu|^{5/4}}$ for all $\mu \in \Omega$. Then

$$\int_\Omega |\mu|^{2\gamma} |c(\mu)|^2 d\mu = \int_\Omega \frac{|\mu|^{2\gamma}}{|\mu|^{5/2}} d\mu = \int_\Omega |\mu|^{2\gamma - 5/2} d\mu$$

and thus $c \in \mathcal{D}((-A^*)^\gamma)$ whenever $2\gamma - 5/2 < -1$, or $\gamma < 3/4$. In particular, since $c \notin \mathcal{D}((-A^*))$, we can see that $CA : \mathcal{D}(A) \to X$ does not extend to a bounded operator on $X$, and the perturbation is therefore genuinely unbounded.

Theorem 3 states that the perturbed operator $A + BCA$ generates a polynomially stable semigroup on $X$ whenever $B = b(\cdot)$ is such that $b \in \mathcal{D}((-A)^{\alpha+1}) = \mathcal{D}((-A)^2)$, i.e., whenever $b(\cdot) \in L^2(\Omega)$ satisfies

$$\int_\Omega |\mu|^4 |b(\mu)|^2 d\mu < \infty,$$

and when the norms

$$\|C\|^2 = \int_\Omega |c(\mu)|^2 d\mu \quad \text{and} \quad \|(-A)^2 B\|^2 = \int_\Omega |\mu|^4 |b(\mu)|^2 d\mu$$

are small enough.

In this example it is also easy to compute the actual bound $\delta > 0$ for the norms such that the polynomial stability of $T_A(t)$ is preserved. To do this, we need a uniform upper bound $M \geq 1$ for $\|R(\lambda, A)(-A)^{-1}\|$ for $\lambda \in \mathbb{C}^+$. For all $f \in X$ with $\|f\| = 1$ and for all $\lambda \in \overline{\mathbb{C}^+}$ we have

$$\begin{aligned}\|R(\lambda, A)(-A)^{-1} f\|^2 &= \int_\Omega \frac{|f(\mu)|^2}{|\lambda - \mu|^2 |-\mu|^2} d\mu \\ &\leq \sup_{\mu \in \Omega} \frac{1}{|\mu|^2 |\lambda - \mu|^2} \int_\Omega |f(\mu)|^2 d\mu = \sup_{\mu \in \Omega} \frac{1}{|\mu|^2 |\lambda - \mu|^2}.\end{aligned}$$

Denote $\lambda = \xi + i\eta$ and $\mu = a + ib$. The properties of the domain $\Omega \subset \mathbb{C}^-$ imply that whenever $|b| \geq 1$, we have $|a| \geq 1/|b|$ and

$$\begin{aligned}\frac{1}{|\mu|^2 |\lambda - \mu|^2} &= \frac{1}{(a^2 + b^2)((\xi - a)^2 + (\eta - b)^2)} \\ &\leq \frac{1}{(a^2 + b^2) a^2} \leq \frac{1}{b^2 (1/b)^2} = 1.\end{aligned}$$

On the other hand, if $|b| < 1$, then $|a| \geq 1$ and a similar estimate yields

$$\frac{1}{|\mu|^2 |\lambda - \mu|^2} \leq \frac{1}{(a^2 + b^2) a^2} \leq \frac{1}{a^2 a^2} \leq 1.$$

This concludes that $\|R(\lambda, A)(-A)^{-1}\| \leq 1$ for all $\lambda \in \overline{\mathbb{C}^+}$, and thus we can choose $M = 1$. The proof of Theorem 6 shows that if we choose $\delta = 1/M = 1$, then the perturbed semigroup is polynomially stable whenever $b$ and $c$ are such that

$$\|C\|^2 < 1 \quad \text{and} \quad \|(-A)^2 B\|^2 = \int_\Omega |\mu|^4 |b(\mu)|^2 d\mu < 1.$$

This is particularly true for any functions $b, c \in L^2(\Omega)$ that have compact supports and small enough norms.

## 7. Conclusions

In this paper we have considered the preservation of polynomial stability of a semigroup with respect to bounded and unbounded finite rank perturbations of its generator. The main results extend and improve recent results in [2, 3].

The main topic for future research is the possibility to further relax the conditions for the exponents $\beta$ and $\gamma$ related to the perturbation. In this paper we showed that in the case where either $\beta$ or $\gamma$ is an integer, then the polynomial stability is preserved if $\beta + \gamma \geq \alpha$ and if the associated graph norms are small enough. If the requirement for one of $\beta$ and $\gamma$ being an integer could be removed, then the perturbation results would be optimal. More precisely, it was shown in [3, Thm. 8] that it is easy to construct a polynomially stable semigroup generated by a diagonal operator in such a way that any perturbation failing to satisfy $\beta + \gamma \geq \alpha$ will destroy the stability of the semigroup regardless of the size of the norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$.

The results presented in this paper are only valid for semigroups on Hilbert spaces. Deriving conditions for the preservation of polynomial stability on Banach spaces is an important topic for further research.


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