Robustness of Strongly and Polynomially Stable Semigroups

L. Paunonen

Department of Mathematics, Tampere University of Technology. PO. Box 553, 33101 Tampere, Finland.

Abstract

In this paper we study the robustness properties of strong and polynomial stability of semigroups of operators. We show that polynomial stability of a semigroup is robust with respect to a large and easily identifiable class of perturbations to its infinitesimal generator. The presented results apply to general polynomially stable semigroups and bounded perturbations. The conditions on the perturbations generalize well-known criteria for the preservation of exponential stablity of semigroups. We also show that the general results can be improved if the perturbation is of finite rank or if the semigroup is generated by a Riesz-spectral operator. The theory is applied to deriving concrete conditions for the preservation of stability of a strongly stabilized one-dimensional wave equation.

Keywords: Strongly continuous semigroup, perturbation, strong stability, polynomial stability, robustness
2010 MSC: 47A55, 47D06, 93D09

1. Introduction

Characterizing classes of perturbations preserving the strong stability of a strongly continuous semigroup has for a long time been a well-known open problem. To this day, very few results are available even for semigroups generated by special classes of operators. Nevertheless, results concerning preservation of strong stability are sought after in many areas of mathematics where infinite-dimensional linear differential equations are studied. In this

Email address: `lassi.paunonen@tut.fi` (L. Paunonen)

paper we solve this problem for an important subclass of strongly stable semigroups, the so-called *polynomially stable* semigroups [1, 2, 3].

A strongly continuous semigroup $T_A(t)$ generated by a linear operator A on a Hilbert space X is said to be polynomially stable if it is uniformly bounded, if $i\mathbb{R} \subset \rho(A)$, and if

$$\|T_A(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0 \tag{1}$$

for some $M > 0$ and $\alpha > 0$. A distinguishing feature of such semigroups is that the spectrum of the operator A is fully contained in the open left half-plane $\mathbb{C}^-$ and may only approach the imaginary axis asymptotically. Polynomial stability is encountered most notably when using pole placement [4] to stabilize a bounded group with an infinite number of evenly spaced eigenvalues λ_k on the imaginary axis. It is well-known that in such a situation it is not possible to shift the eigenvalues of the generator uniformly away from the imaginary axis [5]. As a consequence, the eigenvalues μ_k of the stabilized operator necessarily approach the imaginary axis as $|\operatorname{Im} \mu_k|$ becomes large. In particular this means that the stabilized semigroup can never be exponentially stable. However, it turns out that it is indeed polynomially stable.

Studying the robustness of the stability of the semigroup $T_A(t)$ consists of introducing conditions under which the semigroup generated by the perturbed operator

$$A + \Delta$$

is strongly or polynomially stable. In this paper we are in particular interested in characterizing classes of perturbations $\Delta \in \mathcal{L}(X)$ for which the perturbed operator has the following three properties.

1. The spectrum $\sigma(A + \Delta)$ is contained in the open left half-plane of $\mathbb{C}$.
2. The semigroup generated by $A + \Delta$ is strongly stable.
3. The semigroup generated by $A + \Delta$ is polynomially stable.

This kind of subdivision of our main perturbation problem is based on two observations. First of all, if the perturbed operator has the first one of the above properties, the well-known Arendt-Batty-Lyubich-Vũ Theorem [6, 7, 8] then states that the perturbed semigroup is strongly stable provided that it is uniformly bounded. Furthermore, if the perturbed operator has the first two properties, then the polynomial stability can be determined based on the behavior of the resolvent operator of $A+\Delta$ on the imaginary axis [3].

It is well-known that if $T_A(t)$ is not exponentially stable, then for any $\varepsilon>0$ there may exist a perturbation $\Delta \in \mathcal{L}(X)$ satisfying $\|\Delta\|<\varepsilon$ such that the semigroup generated by $A+\Delta$ is unstable. The main contribution of this paper is to show that it is possible — and very easy — to introduce conditions for the preservation of the stability of the semigroup if we instead employ *graph norms* of the operators $(-A)^\beta$ and $(-A^*)^\gamma$ with suitable exponents $\beta \geq 0$ and $\gamma \geq 0$. In particular we consider structured perturbations of the form

$$A+BC$$

where $B \in \mathcal{L}(Y,X)$ and $C \in \mathcal{L}(X,Y)$ for some Hilbert space Y. We show that whenever A generates a polynomially stable semigroup on X, there exist exponents $\beta, \gamma \geq 0$ depending only on $\alpha>0$ in (1) and a constant $\delta>0$ such that the perturbed semigroup is strongly and polynomially stable whenever

$$(-A)^\beta B \in \mathcal{L}(Y,X), \qquad (-A^*)^\gamma C^* \in \mathcal{L}(Y,X), \tag{2}$$

and whenever the associated operator norms satisfy $\|(-A)^\beta B\|<\delta$ and $\|(-A^*)^\gamma C^*\|<\delta$.

Such classes of perturbations are given a natural interpretation if we consider B and C^* as operators between slighly different spaces. Indeed, condition (2) is in fact equivalent to the operators B and C^* being bounded as linear operators between the spaces

$$B: Y \to \big(\mathcal{D}((-A)^\beta), \|(-A)^\beta \cdot\|\big), \qquad C^*: Y \to \big(\mathcal{D}((-A^*)^\gamma), \|(-A^*)^\gamma \cdot\|\big).$$

Moreover, the bounds on the sizes of the perturbations can immediately be expressed using the the associated operator norms, because

$$\|B\|_{\mathcal{L}(Y,\mathcal{D}((-A)^\beta))} = \|(-A)^\beta B\|, \quad \text{and} \quad \|C^*\|_{\mathcal{L}(Y,\mathcal{D}((-A^*)^\gamma))} = \|(-A^*)^\gamma C^*\|.$$

In particular, the results presented in this paper generalize the conditions for the preservation of exponential stability of semigroups. Indeed, we will see that in the case of an exponentially stable semigroup we can in fact choose $\beta=\gamma=0$. The above conditions for the preservation of stability then simply require that the ordinary operator norms of B and C are small enough.

Considering structured perturbations of the form $A + BC$ enables us to easily study perturbations of finite rank by simply choosing $Y = \mathbb{C}^m$. In this case the perturbing operator BC can be written as

$$BC = \sum_{j=1}^{m} \langle \cdot, c_j \rangle b_j$$

for some $\{b_j\}_{j=1}^m \subset X$ and $\{c_j\}_{j=1}^m \subset X$. For finite rank perturbations the conditions for the preservation of the stability of the semigroup become very concrete. Indeed, for such operators condition (2) is in fact equivalent to

$$\{b_j\}_{j=1}^m \subset \mathcal{D}((-A)^\beta), \qquad \text{and} \qquad \{c_j\}_{j=1}^m \subset \mathcal{D}((-A^*)^\gamma),$$

and the conditions on the operator norms are satisfied if $\|(-A)^\beta b_j\|$ and $\|(-A^*)^\gamma c_j\|$ are small enough for all $j \in \{1, \ldots, m\}$. It also turns out that for finite rank perturbations we can obtain stronger results than for more general perturbations.

It should be noted that the classes of perturbations introduced in this paper are in particular very large. Indeed, since A generates a semigroup, for all $\beta \geq 0$ and $\gamma \geq 0$ both of the domains $\mathcal{D}((-A)^\beta)$ and $\mathcal{D}((-A^*)^\gamma)$ are dense in X. The sizes of the perturbation classes are most obvious for perturbations of rank one, i.e., for $BC = \langle \cdot, c \rangle b$ with $b, c \in X$. In such a case the denseness of the domains implies that the considered classes contain perturbations in directions that form a dense set in X.

The division of the main perturbation problem into parts reveals a fundamental difference between the results concerning the different subproblems. The conditions for the preservation of strong and polynomial stablity of the semigroup require $\beta \geq \alpha$ and $\gamma \geq \alpha$ for general operators B and C, and in the case of finite rank perturbations either of the two conditions is sufficient. This is in contrast with the conditions on the perturbation of the spectrum where it is sufficient to assume $\beta + \gamma \geq \alpha$. In essence this means that when considering only the preservation of the property $\sigma(A) \subset \mathbb{C}^-$, the requirement on the magnitudes of the exponents can be distributed between the two components B and C of the perturbing operator.

As was already mentioned, there are essentially no previous results on robustness of strong or polynomial stability for general semigroups. The theory presented in this paper generalizes the approach used in [9], where the problem was studied for semigroups generated by Riesz-spectral operators and finite rank perturbations.

Robustness of strong stability has also been studied in [10], where the unperturbed operator was assumed to be a skew-adjoint diagonal operator that had been stabilized strongly using a bounded feedback of rank one. The authors showed that such a feedback is robust with respect to small perturbations in a family of three dimensional half-planes. Unfortunately, the characterizations of these half-planes make use of the solutions of certain Lyapunov equations, and for this reason the conditions on the perturbations become very complicated and in particular impossible to verify in practice.

We extend the results presented in [9] in several important ways. First and foremost, the generator of the unperturbed semigroup is not required to have any special structrure, whereas in [9] it was assumed to be generated by a Riesz-spectral operator. The second most important generalization is that the perturbing operators need not be of finite rank. Instead, we will see that the conditions on the operator norms $\|(-A)^{\beta}B\|$ and $\|(-A^*)^{\gamma}C\|$ naturally generalize the ones imposed on the finite rank perturbations in [9]. We also improve the results in [9] by showing that if the perturbation satisfies the conditions for the property $\sigma(A+BC) \subset \mathbb{C}^-$, then the stability of the perturbed semigroup does not require any additional conditions on the sizes of the perturbation.

In addition to extending the results in [9] for general semigroups, we also improve them in the case where A is a Riesz-spectral operator. The most serious drawback concerning the applicability of the results presented earlier is that the conditions for the strong stablity of the perturbed semigroup require that the perturbation satisfies either $\beta \geq \alpha$ or $\gamma \geq \alpha$. In this paper we show that for Riesz-spectral operators and finite rank perturbations the condition for the exponents can in fact be distributed between the operators B and C also in the results concerning the preservation of stability. In particular, we show that for such operators the strong and polynomial stabilities of the semigroup are preserved provided that the exponents satisfy $\beta, \gamma \geq \alpha/2$, and the associated norms of the perturbing operators are small enough.

We apply the theoretic perturbation results to studying the preservation of stability of a strongly stabilized wave equation. To this end, we continue the example studied in [9]. In the previous reference it was shown that it was possible compute actual perturbation bounds for the spectrum of the perturbed equation to be contained in the open left half-plane of $\mathbb{C}$. However, the presented theory could not be used to study the preservation of stability of the equation, and the preservation of uniform boundedness had to be concluded using indirect methods. In this paper we use our improved results concerning semigroups generated by Riesz-spectral operators and complete the study of the robustness properties of this equation. In particular we use our perturbation classes to compute concrete bounds on the perturbing functions to guarantee the preservation of strong and polynomial stabilities of the wave equation.

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, and $\mathcal{R}(A)$ the domain, and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : \mathcal{D}(A) \subset X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$.

For a function $f : \mathbb{R} \to \mathbb{R}$ and for $\alpha \geq 0$ we use the notation

$$f(\omega) = \mathcal{O}\left(|\omega|^\alpha\right)$$

if there exist constants $M > 0$ and $\omega_0 \geq 0$ such that $|f(\omega)| \leq M|\omega|^\alpha$ for all $\omega \in \mathbb{R}$ with $|\omega| \geq \omega_0$.

2. Mathematical Preliminaries and the Perturbation Problem

In this section we formulate our perturbation problem mathematically. In particular this includes stating the detailed assumptions on the unperturbed semigroup $T_A(t)$ and on the perturbing operators. We conclude the section with some helpful lemmata concerning finite rank perturbations and the fundamental properties of Riesz-spectral operators.

Throughout the paper we consider a strongly continuous semigroup $T_A(t)$ generated by $A : \mathcal{D}(A) \subset X \to X$ on a Hilbert space X. We assume $T_A(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$, and for some $\alpha > 0$ and $M > 0$ the semigroup satisfies

$$\|T_A(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad t > 0. \tag{3}$$

A semigroup satisfying these conditions is called polynomially stable [1, 2, 3]. In some references the polynomially stable semigroups are not necessarily strongly stable. However, our assumption of uniform boundedness of $T_A(t)$ together with the estimate (3) implies that $T_A(t)$ also satisfies $T_A(t)x \to 0$ as $t \to \infty$ for all $x \in X$.

It is shown in [3] that the polynomial decay (3) of a uniformly bounded semigroup is completely characterized by the behavior of the resolvent operator of A on the imaginary axis. More precisely, whenever $T_A(t)$ is uniformly bounded and $i\mathbb{R} \subset \rho(A)$, for a fixed $\alpha > 0$ the semigroup satisfies (3) for some $M > 0$ if and only if [3, Thm. 2.4].

$$\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha).$$

Since $\sigma(A) \subset \mathbb{C}^-$, the operators $-A$ and $-A^*$ are sectorial in the sense of [11]. For $\beta \geq 0$ we can therefore consider the fractional powers $(-A)^\beta$ and $(-A^*)^\beta$ as defined in [11, Ch. 3]. Since the operators are boundedly invertible, the mappings

$$x \mapsto \|(-A)^\beta x\|, \qquad x \in \mathcal{D}((-A)^\beta)$$
$$y \mapsto \|(-A^*)^\beta y\|, \qquad y \in \mathcal{D}((-A^*)^\beta)$$

define norms that are equivalent to the graph norms of the operators $(-A)^\beta$ and $(-A^*)^\beta$, respectively. In particular, since the operators are closed, the spaces

$$\big(\mathcal{D}((-A)^\beta), \|(-A)^\beta \cdot\|\big), \qquad \text{and} \qquad \big(\mathcal{D}((-A^*)^\beta), \|(-A^*)^\beta \cdot\|\big)$$

are Banach spaces for all $\beta \geq 0$.

In the following we list the standing assumptions on our unperturbed operator A and on the components B and C of the perturbation.

Assumption 1. *Let X and Y be Hilbert spaces. Assume that the operators $A : \mathcal{D}(A) \subset X \to X$, $B \in \mathcal{L}(Y, X)$, and $C \in \mathcal{L}(X, Y)$ satisfy the following for some $\alpha > 0$, and $\beta, \gamma \geq 0$.*

1. *The operator A generates a uniformly bounded semigroup, $i\mathbb{R} \subset \rho(A)$ and there exists $\alpha > 0$ such that*

$$\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha). \tag{4}$$

2. *We have $\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta)$ and $(-A)^\beta B \in \mathcal{L}(Y, X)$*

3. *We have $\mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma)$ and $(-A^*)^\gamma C^* \in \mathcal{L}(Y, X)$*

In the case of a single perturbing operator, i.e., if we want to study $A+\Delta$ for some $\Delta \in \mathcal{L}(X)$, we can choose $Y = X$ in Assumption 1. The results can then be applied to a structured perturbation BC with either

1. $B = \Delta$, $\gamma = 0$, and $C = I$, or
2. $C = \Delta$, $\beta = 0$, and $B = I$.

As was discussed in the introduction, we subdivide the main perturbation problem concerning the preservation of stability into three parts. The first one of these parts concerns the change of the spectrum of the operator, and latter two consist of finding additional conditions for the strong and polynomial stability of the perturbed semigroup.

Problem 2. *Under the conditions of Assumption* 1*, characterize classes of operators B and C with the following properties.*

1. *The spectrum of the perturbed operator satisfies $\sigma(A+BC) \subset \mathbb{C}^-$.*
2. *The semigroup generated by the perturbed operator $A+BC$ is strongly stable.*
3. *The semigroup generated by the perturbed operator $A+BC$ is polynomially stable.*

We know from the theory of strong stability of semigroups that if the perturbed operator satisfies $\sigma(A+BC) \subset \mathbb{C}^-$, then for preservation of strong stability it is sufficient to find conditions under which the perturbed semigroup is uniformly bounded [6, 7, 8]. To further show that the perturbed semigroup is also polynomially stable, we need to show that the resolvent operator of the perturbed operator $A+BC$ is polynomially bounded on the imaginary axis [3].

2.1. Perturbations of Finite Rank

In this section we make a few quick remarks concerning perturbations of finite rank. On a Hilbert space X, any finite rank perturbation BC can be written in the form

$$BC = \sum_{j=1}^{m} \langle \cdot, c_j \rangle b_j \in \mathcal{L}(X),$$

with $\{b_j\}_{j=1}^m \subset X$ and $\{c_j\}_{j=1}^m \subset X$. In Assumption 1 we can therefore take $Y = \mathbb{C}^m$, and choose the operators $B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^m)$ as

$$Cx = \begin{pmatrix} \langle x, c_1 \rangle \\ \vdots \\ \langle x, c_m \rangle \end{pmatrix}, \qquad B \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \sum_{j=1}^m y_j b_j.$$

The conditions on B and C in Assumption 1 are then equivalent to requiring that for some constants $\beta \geq 0$ and $\gamma \geq 0$ we have

$$b_j \in \mathcal{D}((-A)^\beta), \qquad \text{and} \qquad c_j \in \mathcal{D}((-A^*)^\gamma)$$

for all $j \in \{1, \dots, m\}$. Furthermore, it is straightforward to verify

$$\max_j \|(-A)^\beta b_j\| \leq \|(-A)^\beta B\| \leq m \cdot \max_j \|(-A)^\beta b_j\|$$
$$\max_j \|(-A^*)^\gamma c_j\| \leq \|(-A^*)^\beta C^*\| \leq m \cdot \max_j \|(-A^*)^\gamma c_j\|.$$

This immediately implies that any conditions of the form $\|(-A)^\beta B\| < \delta$ and $\|(-A^*)^\gamma C^*\| < \delta$ can be replaced with a requirement that the norms

$$\|(-A)^\beta b_j\| \qquad \text{and} \qquad \|(-A^*)^\gamma c_j\|,$$

respectively, are small enough for all $j \in \{1, \dots, m\}$.

The following lemma will be useful when considering finite rank perturbations.

Lemma 3. *Let $Y = \mathbb{C}^m$. If $R \in \mathcal{L}(X)$, then*

$$\|RB\|^2 \leq \sum_{j=1}^m \|Rb_j\|^2, \qquad \|CR\|^2 \leq \sum_{j=1}^m \|R^* c_j\|^2. \tag{5}$$

Proof. The estimates follow directly from

$$\|RBy\|^2 = \left\| \sum_{j=1}^m y_j R b_j \right\|^2 \leq \left(\sum_{j=1}^m |y_j| \|Rb_j\| \right)^2 \leq \left(\sum_{j=1}^m |y_j|^2 \right) \left(\sum_{j=1}^m \|Rb_j\|^2 \right).$$

and

$$\|CRx\|^2 = \sum_{j=1}^m |\langle Rx, c_j \rangle|^2 = \sum_{j=1}^m |\langle x, R^* c_j \rangle|^2 \leq \|x\|^2 \sum_{j=1}^m \|R^* c_j\|^2.$$

□

2.2. Fundamental Properties of Riesz-Spectral Operators

In this section we introduce the notation and briefly state the most relevant properties of Riesz-spectral operators [12, Sec. 2.3]. If A is a Riesz-spectral operator, it can be written in the form

$$Ax = \sum_{k=0}^{\infty} \lambda_k \langle x, \psi_k \rangle \phi_k, \quad x \in \mathcal{D}(A) = \left\{ x \in X \;\middle|\; \sum_{k=0}^{\infty} |\lambda_k|^2 |\langle x, \psi_k \rangle|^2 < \infty \right\},$$

where the sequences $(\phi_k)_{k=0}^{\infty}$ and $(\psi_k)_{k=0}^{\infty}$ are biorthonormal, and both of them are bases of X. The eigenvalues of A are $\sigma_p(A) = \{\lambda_k\}_{k=0}^{\infty}$, and the full spectrum of A is the closure of its point spectrum, i.e., $\sigma(A) = \overline{\sigma_p(A)}$.

There exist constants $M_\sigma, m_\sigma > 0$ such that all $x \in X$ satisfy

$$m_\sigma \sum_{k=0}^{\infty} |\langle x, \psi_k \rangle|^2 \le \|x\|^2 \le M_\sigma \sum_{k=0}^{\infty} |\langle x, \psi_k \rangle|^2$$

$$\frac{1}{M_\sigma} \sum_{k=0}^{\infty} |\langle x, \phi_k \rangle|^2 \le \|x\|^2 \le \frac{1}{m_\sigma} \sum_{k=0}^{\infty} |\langle x, \phi_k \rangle|^2.$$

For a Riesz-spectral operator the fractional domains $\mathcal{D}((-A)^\beta)$ and $\mathcal{D}((-A^*)^\gamma)$ have particularly simple representations as

$$\mathcal{D}((-A)^\beta) = \left\{ x \in X \;\middle|\; \sum_{k=0}^{\infty} |\lambda_k|^{2\beta} |\langle x, \psi_k \rangle|^2 < \infty \right\}$$

$$\mathcal{D}((-A^*)^\beta) = \left\{ y \in X \;\middle|\; \sum_{k=0}^{\infty} |\lambda_k|^{2\beta} |\langle y, \phi_k \rangle|^2 < \infty \right\}.$$

3. Robustness of Strong and Polynomial Stability of Semigroups

In this section we state our main results. We begin by answering the first part of Problem 2. The following theorem characterizes classes of operators B and C for which the spectrum of the operator $A + BC$ is contained in the left half-plane of $\mathbb{C}$. The result also concludes that for all such perturbations the elements λ of the spectrum of the perturbed operator may approach the imaginary axis only as $|\mathrm{Im}\, \lambda|$ becomes large.

Theorem 4. *Let A, B, and, C satisfy Assumption 1 for some $\alpha > 0$, and $\beta, \gamma \geq 0$. If $\beta + \gamma \geq \alpha$, then there exists $\delta > 0$ such that the perturbed operator satisfies*

$$\sigma(A + BC) \subset \mathbb{C}^-$$

whenever $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$.

Moreover, there exist constants $c_A > 0$ and $r_A > 0$ such that for any such perturbation we have

$$\operatorname{Re} \lambda \leq -c_A |\operatorname{Im} \lambda|^{-\alpha},$$

for all $\lambda \in \sigma(A + BC)$ with $|\operatorname{Im} \lambda| \geq r_A$.

The second one of our main results concerns the preservation of the stability of the semigroup $T_A(t)$. The following theorem presents conditions under which the perturbed semigroup is polynomially stable. For strong stability it would have been sufficient to state additional conditions for the preservation uniform boundedness. However, it turns out that our approach gives the polynomial stability of the perturbed semigroup for free.

Theorem 5. *Let A, B, and C satisfy Assumption 1 for some $\alpha > 0$, and $\beta, \gamma \geq \alpha$, and choose $\delta > 0$ as in Theorem 4 for $\beta = \alpha$ and $\gamma = 0$. Then for all B and C satisfying $\|(-A)^\alpha B\| \cdot \|C^*\| < \delta$ we have $\sigma(A + BC) \subset \mathbb{C}^-$, the semigroup $T_{A+BC}(t)$ generated by $A + BC$ is uniformly bounded, and for some $M > 0$ we have*

$$\|T_{A+BC}(t)(A + BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

In particular, the perturbed semigroup is strongly and polynomially stable.

If $\dim Y < \infty$, the conclusions remain valid for any $\gamma \geq 0$.

On a Hilbert space a semigroup $T_A(t)$ is uniformly bounded if and only if the same is true for the adjoint semigroup $T_A(t)^*$ generated by A^*. It is therefore reasonable to expect that Theorem 5 has an analogue with the roles of the operators B and C reversed. Indeed, this follows directly from the fact that we can apply Theorem 5 to the operator A^* and the perturbation $(BC)^* = C^* B^*$.

Corollary 6. *Let A, B, and C satisfy Assumption 1 for some $\alpha > 0$, and $\beta, \gamma \geq \alpha$ and choose $\delta > 0$ as in Theorem 4 for $\beta = 0$ and $\gamma = \alpha$. Then for all B and C satisfying $\|B\| \cdot \|(-A^*)^\alpha C^*\| < \delta$ we have $\sigma(A + BC) \subset \mathbb{C}^-$, the semigroup $T_{A+BC}(t)$ generated by $A + BC$ is uniformly bounded, and for some $M > 0$ we have*

$$\|T_{A+BC}(t)(A+BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

In particular, the perturbed semigroup is strongly and polynomially stable.

If $\dim Y < \infty$, the conclusions remain valid for any $\beta \geq 0$.

Finally, for semigroups generated by Riesz-spectral operators we can improve the conditions in Theorem 5. In particular, in this case the conditions on the exponents β and γ can be distributed between the components B and C of the perturbing operator, as was possible when considering perturbation of the spectrum of A in Theorem 4. We will see in Section 5 that besides Riesz-spectral operators, part of the result can also be formulated for general operators A whose resolvents satisfy an additional integrability condition.

Theorem 7. *Assume $\dim Y < \infty$, let A be a Riesz-spectral operator, and let A, B, and C satisfy the conditions of Assumption 1 for $\alpha > 0$, and $\beta, \gamma \geq \alpha/2$. Choose $\delta > 0$ as in Theorem 4 for $\beta = \gamma = \alpha/2$. Then for all B and C satisfying $\|(-A)^{\frac{\alpha}{2}} B\| \cdot \|(-A^*)^{\frac{\alpha}{2}} C^*\| < \delta$ we have $\sigma(A + BC) \subset \mathbb{C}^-$, the semigroup $T_{A+BC}(t)$ generated by $A + BC$ is uniformly bounded, and for some $M > 0$ we have*

$$\|T_{A+BC}(t)(A+BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

In particular, the perturbed semigroup is strongly and polynomially stable.

The proofs of the Theorems 4, 5, and 7 are presented in Sections 4 and 5. Before moving on, however, we will take a moment to address the optimality of our conditions. The following theorem presents a simple counterexample to demonstrate that the condition $\beta + \gamma \geq \alpha$ in Theorem 4 is necessary for the property $\sigma(A + BC) \subset \mathbb{C}^-$ and the stability of the perturbed semigroup to be achieved using bounds on the norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$.

Theorem 8. *Consider the semigroup generated by the diagonal operator*

$$Ax = \sum_{k=1}^\infty \lambda_k \langle \cdot, \phi_k \rangle \phi_k, \qquad \mathcal{D}(A) = \left\{ x \in X \;\middle|\; \sum_{k=1}^\infty |\lambda_k|^2 |\langle x, \phi_k \rangle|^2 < \infty \right\},$$

where $\{\phi_k\}_{k=1}^\infty$ *is an orthonormal basis of* X*, and* $\lambda_k = -|k|^{-\alpha} + ik$ *for all* $k \in \mathbb{N}$ *and for some* $\alpha > 0$*.*

For any $\beta, \gamma \geq 0$ *such that* $\beta + \gamma < \alpha$ *and for all* $\delta > 0$ *there exist* B *and* C *for which* $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$ *and* $\sigma_p(A + BC) \cap \overline{\mathbb{C}^+} \neq \varnothing$*.*

Proof. Let $\beta, \gamma \geq 0$ be such that $\beta + \gamma < \alpha$, and let $\delta > 0$. Consider a rank one perturbation BC with $B = b$ and $C = \langle \cdot, c \rangle$, where $b = c = \sqrt{|\mathrm{Re}\, \lambda_n|}\phi_n$ for some $n \in \mathbb{N}$. Now

$$\begin{aligned} \|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| &= |\lambda_n|^\beta \sqrt{|\mathrm{Re}\, \lambda_n|} |\langle \phi_n, \phi_n \rangle| \cdot |\lambda_n|^\gamma \sqrt{|\mathrm{Re}\, \lambda_n|} |\langle \phi_n, \phi_n \rangle| \\ &= |\lambda_n|^{\beta+\gamma} |\mathrm{Re}\, \lambda_n| = \left(\sqrt{\frac{1}{n^{2\alpha}} + n^2} \right)^{\beta+\gamma} \frac{1}{n^\alpha} \\ &\leq \left(\sqrt{2n^2} \right)^{\beta+\gamma} \frac{1}{n^\alpha} = 2^{\frac{\beta+\gamma}{2}} n^{\beta+\gamma-\alpha} \to 0, \end{aligned}$$

as $n \to \infty$, since $\beta + \gamma - \alpha < 0$. If we choose $n > \left(2^{-\frac{\beta+\gamma}{2}}\delta\right)^{\alpha-\beta-\gamma}$, we then have $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$, but

$$(A + BC)\phi_n = \lambda_n \phi_n + |\mathrm{Re}\, \lambda_n| \langle \phi_n, \phi_n \rangle \phi_n = i\, \mathrm{Im}\, \lambda_n \phi_n.$$

This concludes that $\sigma_p(A + BC) \cap \overline{\mathbb{C}^+} \neq \varnothing$, and further that the semigroup generated by $A + BC$ is unstable. □

4. Perturbation of the Spectrum

In this section we present the proof of Theorem 4 describing the change of the spectrum of A under perturbations. For this we use the following lemma relating the order of growth of the resolvent operator of A on the imaginary axis to its behavior in the right half-plane of $\mathbb{C}$. The proof of this convenient result can be found in [3, Lem. 2.3], [13, Lem. 3.2].

Lemma 9. *Assume* A *generates a uniformly bounded semigroup on a Hilbert space* X*, and* $i\mathbb{R} \subset \rho(A)$*. For a fixed* $\alpha > 0$ *we have*

$$\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$$

if and only if

$$\sup_{\mathrm{Re}\, \lambda \geq 0} \|R(\lambda, A)(-A)^{-\alpha}\| < \infty. \tag{6}$$

In analyzing the change of the spectrum of A — as well as the preservation of the stability of the semigroup in the next section — we also use the Sherman-Morrison-Woodbury formula given in the next lemma. This well-known operator identity can be verified with a straightforward computation.

Lemma 10. *Let $\lambda \in \rho(A)$, $B \in \mathcal{L}(Y,X)$, $C \in \mathcal{L}(X,Y)$. If $1 \in \rho(CR(\lambda,A)B)$, then $\lambda \in \rho(A+BC)$ and*

$$R(\lambda, A+BC) = R(\lambda,A) + R(\lambda,A)B(I - CR(\lambda,A)B)^{-1}CR(\lambda,A). \tag{7}$$

We will begin the proof of Theorem 4 by showing that if A is polynomially stable, we can extend the estimate (6) to an appropriately chosen open domain $\Delta_\alpha \subset \mathbb{C}$ containing the closed right half-plane of $\mathbb{C}$. The proof of Theorem 4 is then completed by showing that under the given assumptions we can choose $\delta > 0$ in such a way that $\Delta_\alpha \subset \rho(A+BC)$ whenever the perturbing operators satisfy $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$.

We remark that the construction of the domain Δ_α is mainly required for showing that the spectrum of the perturbed operator may only approach the imaginary axis at a rate $|\mathrm{Im}\,\lambda|^{-\alpha}$. The proof of Theorem 4 can be simplified if we are only interested in showing $\sigma(A+BC) \subset \mathbb{C}^-$, see Remark 13 for details.

Theorem 11. *Let A satisfy the conditions of Assumption 1 for some $\alpha > 0$. Then there exists an open set $\Delta_\alpha \subset \mathbb{C}$ with the following properties.*

1. *We have $\overline{\mathbb{C}^+} \subset \Delta_\alpha \subset \rho(A)$ and there exist constants $c_A > 0$ and $r_A > 0$ such that any $\lambda \in \mathbb{C} \setminus \Delta_\alpha$ with $|\mathrm{Im}\,\lambda| \ge r_A$ satisfies*

$$\mathrm{Re}\,\lambda \le -c_A |\mathrm{Im}\,\lambda|^{-\alpha}.$$

2. *We have*

$$\sup_{\lambda \in \Delta_\alpha} \|R(\lambda,A)(-A)^{-\alpha}\| < \infty. \tag{8}$$

Proof. Let $0 < \kappa < 1$. Since the resolvent operator of A satisfies (4), there exists $\omega_0 \ge 1$ and $M_A > 0$ such that

$$\|R(i\omega, A)\| \le M_A |\omega|^\alpha, \qquad \text{whenever} \quad |\omega| \ge \omega_0. \tag{9}$$

Every $\lambda \in \rho(A)$ satisfies $\operatorname{dist}(\lambda, \sigma(A)) \geq \|R(\lambda, A)\|^{-1}$ [14, Cor. IV.1.14], and therefore for every $i\omega$ with $|\omega| \geq \omega_0$ we necessarily have

$$\operatorname{dist}(i\omega, \sigma(A)) \geq \frac{1}{M_A |\omega|^\alpha}.$$

In particular this implies that any $\lambda \in \sigma(A)$ with $|\operatorname{Im} \lambda| \geq \omega_0$ must satisfy $\operatorname{Re} \lambda \leq -(1/M_A)|\operatorname{Im} \lambda|^{-\alpha}$. Therefore the spectrum of A can approach the imaginary axis only as $|\operatorname{Im} \lambda|$ becomes large, and at a rate that is at most $|\operatorname{Im} \lambda|^{-\alpha}$.

Our aim now is to construct the domain Δ_α in such a way that it contains the closed right half-plane of $\mathbb{C}^-$, and its boundary lies between the imaginary axis and the spectrum of A. The final result of the construction is illustrated in Figure 1. The mathematical details are written out in the following.

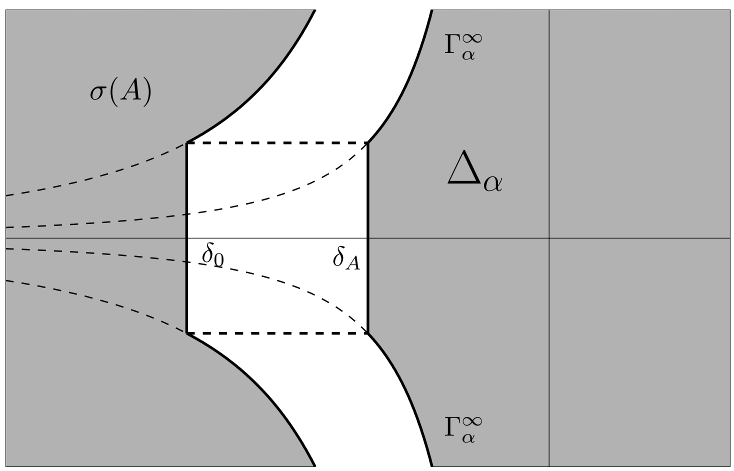

Figure 1: The domain $\Delta_\alpha \supset \overline{\mathbb{C}^+}$

Since $i\mathbb{R} \subset \rho(A)$, the points $\lambda \in \sigma(A)$ with $|\operatorname{Im} \lambda| \leq \omega_0$ are uniformly bounded away from the imaginary axis, i.e., we have

$$\delta_0 = \sup \left\{ \operatorname{Re} \lambda \;\middle|\; \lambda \in \sigma(A), \text{ and } |\operatorname{Im} \lambda| \leq \omega_0 \right\} < 0.$$

We can therefore define $\delta_A < 0$ by

$$\delta_A = \max \left\{ \kappa \delta_0, -\frac{\kappa}{M_A |\omega_0|^\alpha} \right\} < 0.$$

The domain $\Delta_\alpha \subset \mathbb{C}$ is defined in such a way that the real part of $\lambda \in \Delta_\alpha$ is bounded from below by both the vertical line $\Gamma_\alpha^0 = \{ \lambda \in \mathbb{C}^- \mid \operatorname{Re} \lambda = \delta_A \}$ and the curve

$$\Gamma_\alpha^\infty = \Big\{ \lambda \in \mathbb{C}^- \;\Big|\; \lambda = -\frac{\kappa}{M_A |s|^\alpha} + is, \;\; s \neq 0 \Big\}.$$

The paths Γ_α^0 and Γ_α^∞ intersect each other at two points $\lambda_0^\pm \in \mathbb{C}^-$. If we denote by $r_A > 0$ the modulus of the imaginary part of the intersections, i.e., $\lambda_0^\pm = \operatorname{Re} \lambda_0 \pm i r_A$, then r_A is determined by

$$\delta_A = \operatorname{Re} \lambda_0 = -\frac{\kappa}{M_A |r_A|^\alpha} \qquad \Leftrightarrow \qquad r_A = \left(\frac{\kappa}{|\delta_A| M_A} \right)^{1/\alpha}.$$

The fact that $|\delta_A| \leq (\kappa / M_A) |\omega_0|^{-\alpha}$ finally ensures $r_A \geq \omega_0$. By definition we clearly have $\overline{\mathbb{C}^+} \subset \Delta_\alpha \subset \rho(A)$, and the domain Δ_α satisfies the properties in the theorem.

It remains to show that (8) is satisfied for this choice of Δ_α. By Lemma 9 it suffices to show that $\|R(\lambda, A)(-A)^{-\alpha}\|$ is uniformly bounded with respect to $\lambda \in \Delta_\alpha \cap \mathbb{C}^-$. Furthermore, since the compact set

$$\Big\{ \lambda \in \overline{\mathbb{C}^-} \;\Big|\; \delta_A \leq \operatorname{Re} \lambda \leq 0, \;\; |\operatorname{Im} \lambda| \leq r_A \Big\} \subset \overline{\mathbb{C}^-}$$

is contained in $\rho(A)$, the mapping $\lambda \mapsto \|R(\lambda, A)(-A)^{-\alpha}\|$ is continuous, and thus uniformly bounded, on this set. It therefore remains to show that $\|R(\lambda, A)(-A)^{-\alpha}\|$ is uniformly bounded for $\lambda \in \Delta_\alpha \cap \mathbb{C}^-$ with $|\operatorname{Im} \lambda| \geq r_A$.

By construction of Δ_α, for any $\lambda \in \Delta_\alpha$ satisfying $|\operatorname{Im} \lambda| \geq r_A \geq \omega_0$ and $\operatorname{Re} \lambda \leq 0$ we have $|\operatorname{Re} \lambda| < (\kappa / M_A) |\operatorname{Im} \lambda|^{-\alpha}$. We can use the resolvent equation to estimate

$$\begin{aligned}
&\|R(\lambda, A)(-A)^{-\alpha}\| = \|R(\operatorname{Re} \lambda + i \operatorname{Im} \lambda, A)(-A)^{-\alpha}\| \\
&\quad = \|R(i \operatorname{Im} \lambda, A)(-A)^{-\alpha} + (-\operatorname{Re} \lambda) R(i \operatorname{Im} \lambda, A) R(\operatorname{Re} \lambda + i \operatorname{Im} \lambda, A)(-A)^{-\alpha}\| \\
&\quad \leq \|R(i \operatorname{Im} \lambda, A)(-A)^{-\alpha}\| + |\operatorname{Re} \lambda| \cdot \|R(i \operatorname{Im} \lambda, A)\| \|R(\lambda, A)(-A)^{-\alpha}\|,
\end{aligned}$$

and thus

$$\big(1 - |\operatorname{Re} \lambda| \cdot \|R(i \operatorname{Im} \lambda, A)\|\big) \|R(\lambda, A)(-A)^{-\alpha}\| \leq \|R(i \operatorname{Im} \lambda, A)(-A)^{-\alpha}\|.$$

But since $|\operatorname{Re} \lambda| < (\kappa / M_A) |\operatorname{Im} \lambda|^{-\alpha}$, we have from (9) that

$$|\operatorname{Re} \lambda| \cdot \|R(i \operatorname{Im} \lambda, A)\| < \frac{\kappa}{M_A |\operatorname{Im} \lambda|^\alpha} \cdot M_A |\operatorname{Im} \lambda|^\alpha = \kappa < 1,$$

and thus

$$\|R(\lambda,A)(-A)^{-\alpha}\| \le \frac{\|R(i\operatorname{Im}\lambda,A)(-A)^{-\alpha}\|}{1-|\operatorname{Re}\lambda|\cdot\|R(i\operatorname{Im}\lambda,A)\|} \le \frac{1}{1-\kappa}\cdot\sup_{\operatorname{Re}\mu\ge 0}\|R(\mu,A)(-A)^{-\alpha}\|.$$

Since the bound on the right-hand side is finite and independent of λ, this concludes the proof. □

We can now prove Theorem 4 using the domain Δ_α and the estimate (8).

Proof of Theorem 4. Let Δ_α be as in Theorem 11 and choose $M>0$ as

$$M=\sup_{\lambda\in\Delta_\alpha}\|R(\lambda,A)(-A)^{-\alpha}\|<\infty.$$

Since $\beta+\gamma\ge\alpha$, we have $(-A)^{\alpha-\beta-\gamma}\in\mathcal{L}(X)$, and we can thus define

$$\delta=\frac{1}{M\|(-A)^{\alpha-\beta-\gamma}\|}>0.$$

Let $B\in\mathcal{L}(Y,X)$ and $C\in\mathcal{L}(X,Y)$ be such that $\|(-A)^\beta B\|\cdot\|(-A^*)^\gamma C^*\|<\delta$ and let $\lambda\in\Delta_\alpha$. Then for any $x\in X$ with $\|x\|=1$ we can use the properties of the fractional powers of sectorial operators [11, Ch. 3] to estimate

$$\begin{aligned}
\|CR(\lambda,A)Bx\| &= \sup_{\|y\|=1}|\langle C(-A)^\gamma(-A)^{-\gamma}R(\lambda,A)(-A)^{-\beta}(-A)^\beta Bx,y\rangle| \\
&= \sup_{\|y\|=1}|\langle R(\lambda,A)(-A)^{-\beta-\gamma}(-A)^\beta Bx,(-A^*)^\gamma C^*y\rangle| \\
&= \sup_{\|y\|=1}|\langle R(\lambda,A)(-A)^{-\alpha}(-A)^{\alpha-\beta-\gamma}(-A)^\beta Bx,(-A^*)^\gamma C^*y\rangle| \\
&\le \sup_{\|y\|=1}\|R(\lambda,A)(-A)^{-\alpha}\|\|(-A)^{\alpha-\beta-\gamma}\|\|(-A)^\beta B\|\|x\|\|(-A^*)^\gamma C^*\|\|y\| \\
&\le M\|(-A)^{\alpha-\beta-\gamma}\|\cdot\|(-A)^\beta B\|\cdot\|(-A^*)^\gamma C^*\| \\
&< M\|(-A)^{\alpha-\beta-\gamma}\|\cdot\delta=1,
\end{aligned}$$

and thus $\|CR(\lambda,A)B\|<1$. In particular we have $1\in\rho(CR(\lambda,A)B)$, and the Sherman-Morrison-Woodbury formula in Lemma 10 implies that $\lambda\in\rho(A+BC)$.

Since $\lambda\in\Delta_\alpha$ was arbitrary, we have $\Delta_\alpha\subset\rho(A+BC)$, which finally concludes $\sigma(A+BC)\subset\mathbb{C}\setminus\Delta_\alpha\subset\mathbb{C}^-$. The estimate for the real part of $\lambda\in\sigma(A+BC)$ follows directly from Theorem 11. □

At the end of the previous proof we saw that the norm of the operator $CR(\lambda, A)B$ could be estimated independently of $\lambda \in \Delta_\alpha$. We can further prove the following lemma which will be used repeatedly in studying the preservation of stability in the next section.

Lemma 12. *If B and C are such that $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$ in Theorem 4, then*

$$\sup_{\lambda \in \Delta_\alpha} \|(I - CR(\lambda, A)B)^{-1}\| < \infty.$$

Proof. Let $\lambda \in \Delta_\alpha$. We saw in the proof of Theorem 4 that

$$\|CR(\lambda, A)B\| \le M\|(-A)^{\alpha-\beta-\gamma}\| \cdot \|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < 1,$$

which in particular implied $1 \in \rho(CR(\lambda, A)B)$. A standard argument further shows that

$$\begin{aligned} \|(I - CR(\lambda, A)B)^{-1}\| &= \left\| \sum_{n=0}^{\infty} (CR(\lambda, A)B)^n \right\| \le \sum_{n=0}^{\infty} \|CR(\lambda, A)B\|^n \\ &= \frac{1}{1 - \|CR(\lambda, A)B\|} \le \frac{1}{1 - M\|(-A)^{\alpha-\beta-\gamma}\| \|(-A)^\beta B\| \|(-A^*)^\gamma C^*\|} < \infty. \end{aligned}$$

Since the bound is independent of $\lambda \in \Delta_\alpha$, this concludes the proof. □

Remark 13. It should be noted that if we are only interested in the property $\sigma(A + BC) \subset \mathbb{C}^-$, and not on the asymptotic behavior of the spectrum of $A + BC$ near the imaginary axis, then Theorem 4 can be proved without the construction of the domain Δ_α. Indeed, in this case we can replace $\lambda \in \Delta_\alpha$ by $\lambda \in \overline{\mathbb{C}^+}$ in the proof, and similarly arrive to a conclusion that if

$$\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \frac{1}{M\|(-A)^{\alpha-\beta-\gamma}\|}, \quad M = \sup_{\operatorname{Re}\lambda \ge 0} \|R(\lambda, A)(-A)^{-\alpha}\|,$$

we then have $\overline{\mathbb{C}^+} \subset \rho(A + BC)$. Moreover, also the conclusion of Lemma 12 remains valid. ■

5. The Preservation of Stability

In this section we study the preservation of strong and polynomial stability of the semigroup. As was discussed earlier, this can be done by posing additional conditions under which the perturbed semigroup is uniformly bounded and the resolvent operator of $A+BC$ is polynomially bounded on the imaginary axis. In particular we will see that under the assumptions of Theorem 5, the preservation of these properties does not require additional conditions on the sizes of the perturbations. Instead, it is sufficient to choose $\delta > 0$ as in Theorem 4.

We use the following resolvent conditions to study the uniform boundedness of the perturbed semigroup. The proof of the theorem can be found in [15, Thm. 2].

Theorem 14. *If A generates a semigroup $T_A(t)$ on a Hilbert space X and if $\sigma(A) \subset \overline{\mathbb{C}^-}$, then the following are equivalent.*

1. *The semigroup $T_A(t)$ is uniformly bounded.*
2. *For all $x, y \in X$*
$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} |\langle R(\xi+i\eta, A)^2 x, y\rangle| d\eta < \infty. \tag{10}$$
3. *For all $x, y \in X$*
$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta, A)x\|^2 + \|R(\xi+i\eta, A)^* y\|^2 d\eta < \infty. \tag{11}$$

We are now in a position to prove Theorem 5. We begin by showing that the perturbed semigroup is uniformly bounded for general operators B and C in the perturbation. Subsequently, we show that the resolvent operator of $A+BC$ is polynomially bounded on the imaginary axis. The proof is completed by showing that for finite rank perturbations the conclusions of the theorem remain valid even without the requirement $\gamma \geq \alpha$.

Proof of Theorem 5. Let Δ_α be as in Theorem 11, and let B and C be such that $(-A)^\alpha B \in \mathcal{L}(Y, X)$ and $(-A^*)^\alpha C^* \in \mathcal{L}(Y, X)$. Choose $\delta > 0$ as in Theorem 4 for $\beta = \alpha$ and $\gamma = 0$, and assume $\|(-A)^\alpha B\| \cdot \|C^*\| < \delta$. By Theorem 4 we then have $\overline{\mathbb{C}^+} \subset \Delta_\alpha \subset \rho(A+BC)$ and Lemma 12 implies

$$\sup_{\lambda\in\Delta_\alpha} \|(I - CR(\lambda, A)B)^{-1}\| < \infty. \tag{12}$$

We begin by showing that the semigroup generated by $A+BC$ is uniformly bounded. In order to do this, we will show that the perturbed operator

$A+BC$ satisfies the condition (11) in Theorem 14. The Sherman-Morrison-Woodbury formula in Lemma 10 shows us that under our assumptions the resolvent operator $R(\lambda, A+BC)$ is given by the formulas

$$R(\lambda, A+BC) = \left[I + R(\lambda, A)B(I - CR(\lambda, A)B)^{-1}C\right] R(\lambda, A) \tag{13a}$$
$$= R(\lambda, A)\left[I + B(I - CR(\lambda, A)B)^{-1}CR(\lambda, A)\right] \tag{13b}$$

for all $\lambda \in \overline{\mathbb{C}^+}$. Now by (12) and Lemma 9 there exists $M_1 \geq 1$ independent of $\lambda \in \overline{\mathbb{C}^+}$ such that

$$\begin{aligned}
&\|I + R(\lambda, A)B(I - CR(\lambda, A)B)^{-1}C\| \\
&\quad = \|I + R(\lambda, A)(-A)^{-\alpha}(-A)^{\alpha}B(I - CR(\lambda, A)B)^{-1}C\| \\
&\quad \leq 1 + \|R(\lambda, A)(-A)^{-\alpha}\|\|(-A)^{\alpha}B\|\|(I - CR(\lambda, A)B)^{-1}\|\|C\| \leq M_1.
\end{aligned}$$

For any $x \in X$ we therefore have

$$\begin{aligned}
\|R(\lambda, A+BC)x\| &= \left\|\left[I + R(\lambda, A)B(I - CR(\lambda, A)B)^{-1}C\right] R(\lambda, A)x\right\| \\
&\leq M_1\|R(\lambda, A)x\|
\end{aligned}$$

for $\lambda \in \mathbb{C}^+$, and further

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A+BC)x\|^2 d\eta \tag{14a}$$
$$\leq \; M_1^2 \cdot \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)x\|^2 d\eta < \infty \tag{14b}$$

by condition (11) in Theorem 14.

Moreover, since for all $x \in X$ with $\|x\| = 1$

$$\begin{aligned}
\|CR(\lambda, A)x\| &= \sup_{\|y\|=1} |\langle CR(\lambda, A)x, y\rangle| = \sup_{\|y\|=1} |\langle C(-A)^{\alpha}(-A)^{-\alpha}R(\lambda, A)x, y\rangle| \\
&\leq \sup_{\|y\|=1} |\langle R(\lambda, A)(-A)^{-\alpha}x, (-A^*)^{\alpha}C^*y\rangle| \leq \|R(\lambda, A)(-A)^{-\alpha}\|\|(-A^*)^{\alpha}C^*\|,
\end{aligned}$$

we have from (12) and Lemma 9 that there exists $M_2 \geq 1$ independent of $\lambda \in \overline{\mathbb{C}^+}$ such that

$$\begin{aligned}
&\left\|\left[I + B(I - CR(\lambda, A)B)^{-1}CR(\lambda, A)\right]^*\right\| \\
&\quad = \|I + B(I - CR(\lambda, A)B)^{-1}CR(\lambda, A)\| \\
&\quad = 1 + \|B\|\|(I - CR(\lambda, A)B)^{-1}\|\|R(\lambda, A)(-A)^{-\alpha}\|\|(-A^*)^{\alpha}C^*\| \leq M_2.
\end{aligned}$$

For all $y \in X$ we thus have

$$\begin{aligned}\|R(\lambda, A+BC)^* y\| &= \left\| \left[I + B(I - CR(\lambda, A)B)^{-1} CR(\lambda, A) \right]^* R(\lambda, A)^* y \right\| \\ &\leq M_2 \|R(\lambda, A)^* y\|\end{aligned}$$

for $\lambda \in \mathbb{C}^+$, and again

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A+BC)^* y\|^2 d\eta \tag{15a}$$

$$\leq \; M_2^2 \cdot \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* y\|^2 d\eta < \infty, \tag{15b}$$

by condition (11) in Theorem 14. Together the estimates (14) and (15) conclude that $A + BC$ satisfies condition (11), and therefore by Theorem 14 the semigroup $T_{A+BC}(t)$ is uniformly bounded.

We continue the proof by showing that the perturbed semigroup $T_{A+BC}(t)$ is polynomially stable, and that there exists $M > 0$ such that

$$\|T_{A+BC}(t)(A+BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad t > 0.$$

For this it is sufficient to show that the resolvent operator of the perturbed operator satisfies [3, Thm. 2.4]

$$\|R(i\omega, A+BC)\| = \mathcal{O}(|\omega|^\alpha).$$

This, on the other hand, follows immediately from the fact that for all $\omega \in \mathbb{R}$ we have $i\omega \in \overline{\mathbb{C}^+}$, and thus as above we can estimate

$$\begin{aligned}\|R(i\omega, A+BC)\| &= \left\| \left[I + R(i\omega, A)B(I - CR(i\omega, A)B)^{-1} C \right] R(i\omega, A) \right\| \\ &\leq \|I + R(i\omega, A)B(I - CR(i\omega, A)B)^{-1} C\| \|R(i\omega, A)\| \\ &\leq M_1 \|R(i\omega, A)\|.\end{aligned}$$

Therefore the property $\|R(i\omega, A+BC)\| = \mathcal{O}(|\omega|^\alpha)$ follows directly from $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$. This concludes that under our assumptions the perturbed semigroup is polynomially stable.

It now only remains to show that if $Y = \mathbb{C}^m$, then the requirement $\gamma \geq \alpha$ may be omitted, and the conclusions of the theorem remain valid for any $C \in \mathcal{L}(X, \mathbb{C}^m)$. We may notice that in proving $\|R(i\omega, A+BC)\| = \mathcal{O}(|\omega|^\alpha)$

we made no use of the assumption $\gamma \geq \alpha$. This property is indeed independent of the value of $\gamma \geq 0$, and it is therefore sufficient to show that the semigroup generated by $A + BC$ is uniformly bounded. In order to accomplish this, we this time show that the resolvent operator of $A + BC$ satisfies condition (10) in Theorem 14.

Let $x, y \in X$. The Sherman-Morrison-Woodbury formula in Lemma 10 shows that under our assumptions the resolvent operator $R(\lambda, A + BC)$ is given by the formula (7) for all $\lambda \in \mathbb{C}^+$. For brevity we denote $R(\lambda, A) = R_\lambda$ and $D_\lambda = I - CR(\lambda, A)B$. By Lemma 12 there exists $M_D \geq 1$ such that $\|D_\lambda^{-1}\| \leq M_D$ for all $\lambda \in \mathbb{C}^+$, and using the scalar inequality $2ab \leq a^2 + b^2$ we can estimate

$$
\begin{aligned}
|\langle R(\lambda, A+BC)^2 x, y\rangle| = \big|&\langle R_\lambda^2 x, y\rangle + \langle R_\lambda^2 B D_\lambda^{-1} C R_\lambda x, y\rangle \\
&+\langle R_\lambda B D_\lambda^{-1} C R_\lambda^2 x, y\rangle + \langle R_\lambda B D_\lambda^{-1} C R_\lambda^2 B D_\lambda^{-1} C R_\lambda x, y\rangle\big| \\
\leq\ & |\langle R_\lambda^2 x, y\rangle| + \|R_\lambda B\| \|D_\lambda^{-1}\| \|C R_\lambda\| \|x\| \|R_\lambda^* y\| \\
&+ \|R_\lambda B\| \|D_\lambda^{-1}\| \|C R_\lambda\| \|R_\lambda x\| \|y\| \\
&+ \|R_\lambda B\| \|D_\lambda^{-1}\| \|C R_\lambda\| \|R_\lambda B\| \|D_\lambda^{-1}\| \|C R_\lambda\| \|x\| \|y\| \\
\leq\ & |\langle R_\lambda^2 x, y\rangle| + \frac{M_D}{2} \|x\| \left(\|R_\lambda B\|^2 \|C R_\lambda\|^2 + \|R_\lambda^* y\|^2 \right) && \text{(16a)} \\
&+ \frac{M_D}{2} \|y\| \left(\|R_\lambda B\|^2 \|C R_\lambda\|^2 + \|R_\lambda x\|^2 \right) && \text{(16b)} \\
&+ M_D^2 \|x\| \|y\| \|R_\lambda B\|^2 \|C R_\lambda\|^2. && \text{(16c)}
\end{aligned}
$$

To show uniform boundedness of the semigroup generated by $A + BC$ it is now sufficient to show that for those terms on the right-hand side of (16) that depend on $\lambda = \xi + i\eta$, the integrals in Theorem 14 are uniformly bounded with respect to $\xi > 0$. This is immediately true for the integrals over the terms $|\langle R_\lambda^2 x, y\rangle|$, $\|R_\lambda^* y\|^2$, and $\|R_\lambda x\|^2$ by conditions (10) and (11) in Theorem 14.

It remains to show that the integrals over the terms $\|R_\lambda B\|^2 \|C R_\lambda\|^2$ are finite. Similarly as earlier in the proof, Lemma 9 implies that there exists $M_B \geq 1$ such that

$$
\|R_\lambda B\| \leq \|R_\lambda (-A)^{-\alpha}\| \|(-A)^\alpha B\| \leq M_B
$$

for all $\lambda \in \mathbb{C}^+$. Using Lemma 3 we can therefore see that the integrals over the terms $\|R_\lambda B\|^2\|CR_\lambda\|^2$ on the right-hand side of (16) satisfy

$$\begin{aligned}
&\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2d\eta\\
\leq\ & M_B^2\cdot\sup_{\xi>0}\xi\cdot\sum_{j=1}^{m}\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)^*c_j\|^2d\eta\\
\leq\ & M_B^2\cdot\sum_{j=1}^{m}\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)^*c_j\|^2d\eta<\infty.
\end{aligned}$$

again by condition (11) in Theorem 14. Together these estimates and the fact that $x,y\in X$ were arbitrary conclude that the operator $A+BC$ satisfies (10), and by Theorem 14 the perturbed semigroup $T_{A+BC}(t)$ is uniformly bounded. □

As was discussed earlier in the paper, the fundamental difference between the conditions for the change of the spectrum in Theorem 4 and for the preservation of strong and polynomial stability types in Theorem 5 arises from the conditions on the exponents β and γ. The following theorem shows that under certain additional assumptions on the unperturbed operator A, the requirements for the magnitudes of the exponents can be distributed between the operators B and C also when studying the preservation of stability. In Section 5.1 we will show that these additional assumptions are satisfied in particular for all Riesz-spectral operators.

Theorem 15. *Assume* $\dim Y<\infty$ *and let* A, B, *and* C *satisfy the conditions of Assumption* 1 *for some* $\alpha>0$ *and* $\beta,\gamma\geq\alpha/2$. *Assume further that*

$$\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)x\|^2\|R(\xi+i\eta,A)^*y\|^2d\eta<\infty \tag{17}$$

for all $x\in\mathcal{D}((-A)^{\frac{\alpha}{2}})$ *and* $y\in\mathcal{D}((-A^*)^{\frac{\alpha}{2}})$. *Choose* $\delta>0$ *as in Theorem* 4 *for* $\beta=\gamma=\alpha/2$. *Then for all* B *and* C *satisfying* $\|(-A)^{\frac{\alpha}{2}}B\|\cdot\|(-A^*)^{\frac{\alpha}{2}}C^*\|<\delta$ *the semigroup generated by* $A+BC$ *is strongly stable.*

Proof. Let $Y=\mathbb{C}^m$ and let B and C satisfy $\|(-A)^{\frac{\alpha}{2}}B\|\cdot\|(-A^*)^{\frac{\alpha}{2}}C^*\|<\delta$. Then by Lemma 12 there exists $M_D\geq 1$ such that

$$\sup_{\lambda\in\Delta_\alpha}\|D_\lambda^{-1}\|\leq M_D,$$

and Theorem 4 implies $\sigma(A+BC) \subset \mathbb{C}^-$. To prove the strong stability of the semigroup $T_{A+BC}(t)$ it is sufficient to show that it is uniformly bounded.

We prove the uniform boundedness of the perturbed semigroup by showing that the resolvent operator of $A+BC$ satisfies (10) in Theorem 14. To this end, we let $x, y \in X$ and use the estimate (16).

The integrals in Theorem 14 over the terms $|\langle R_\lambda^2 x, y\rangle|$, $\|R_\lambda^* y\|^2$ and $\|R_\lambda x\|^2$ on the right-hand side of (16) are again finite by (10) and (11), since A generates a uniformly bounded semigroup. Furthermore, since the perturbation satisfies $\{b_j\}_{j=1}^m \subset \mathcal{D}((-A)^{\frac{\alpha}{2}})$ and $\{c_j\}_{j=1}^m \subset \mathcal{D}((-A)^{\frac{\alpha}{2}})$, Lemma 3 implies that the integrals over the terms $\|R_\lambda B\|^2 \|C R_\lambda\|^2$ satisfy

$$\begin{aligned}
&\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)B\|^2 \|CR(\xi+i\eta,A)\|^2 d\eta \\
&\quad \leq \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \left(\sum_{j=1}^m \|R(\xi+i\eta,A)b_j\|^2\right)\left(\sum_{l=1}^m \|R(\xi+i\eta,A)^* c_l\|^2\right) d\eta \\
&\quad \leq \sum_{j=1}^m \sum_{l=1}^m \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)b_j\|^2 \|R(\xi+i\eta,A)^* c_l\|^2 d\eta < \infty,
\end{aligned}$$

due to our assumption (17). Since $x, y \in X$ were arbitrary, this concludes that the resolvent operator of $A+BC$ satisfies (10), and thus the perturbed semigroup $T_{A+BC}(t)$ is uniformly bounded. □

5.1. Preservation of Stability for Riesz-Spectral Operators

In this section we prove Theorem 7. To this end, we first show that the additional condition in Theorem 15 is satisfied if A is a Riesz-spectral operator. This allows us to use the theorem to conclude uniform boundedness of the perturbed semigroup $T_{A+BC}(t)$. The proof of Theorem 7 is then completed by studying behavior of the resolvent operator of $A+BC$ on the imaginary axis, and in this way showing that the perturbed semigroup is polynomially stable.

Lemma 16. *Assume that A is a Riesz-spectral operator satisfying the conditions of Assumption 1 for some $\alpha > 0$. Then*

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)x\|^2 \|R(\xi+i\eta,A)^* y\|^2 d\eta < \infty$$

for all $x \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$ and $y \in \mathcal{D}((-A^*)^{\frac{\alpha}{2}})$.

Proof. The scalar inequality $2ab \leq a^2 + b^2$ implies that it is sufficient to show

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)x\|^4 d\eta < \infty, \quad \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)^* y\|^4 d\eta < \infty$$

whenever $x \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$ and $y \in \mathcal{D}((-A^*)^{\frac{\alpha}{2}})$. We prove that A satisfies first condition. Since A is a Riesz-spectral operator, the remaining condition can be shown in an analogous way.

We first remark that we can without loss of generality assume there exists $c > 0$ such that for all $k \in \mathbb{N}_0$ we have $\operatorname{Im} \lambda_k \neq 0$ and $\operatorname{Re} \lambda_k \leq -c|\operatorname{Im} \lambda_k|^{-\alpha}$. Indeed, if this is not the case, then under the standing assumptions we can decompose A into an operator generating an exponentially stable semigroup and a Riesz-spectral operator whose eigenvalues satisfy the above condition. It is then fairly easy to see that A satisfies the integral condition if and only if it is satisfied for the non-exponentially stable part of the operator.

Let $x \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$ and for brevity denote $x_k = \langle x, \psi_k \rangle$, $a_k = \operatorname{Re} \lambda_k < 0$, and $b_k = \operatorname{Im} \lambda_k \neq 0$. For all $\lambda \in \rho(A)$ we have

$$\begin{aligned} \|R(\lambda, A)x\|^4 &= \left\| \sum_{k=0}^{\infty} \frac{\langle x, \psi_k \rangle}{\lambda - \lambda_k} \phi_k \right\|^4 \leq M_\sigma^2 \left(\sum_{k=0}^{\infty} \frac{|x_k|^2}{|\lambda - \lambda_k|^2} \right)^2 \\ &= M_\sigma^2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{|x_k|^2}{|\lambda - \lambda_k|^2} \cdot \frac{|x_{n-k}|^2}{|\lambda - \lambda_{n-k}|^2} \right) \end{aligned}$$

by the Cauchy product formula, since the series are absolutely convergent. Let $\xi > 0$. If $\lambda_k \neq \lambda_{n-k}$, then the integral of a single term in the series can be estimated

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{|x_k|^2}{|\xi + i\eta - \lambda_k|^2} \cdot \frac{|x_{n-k}|^2}{|\xi + i\eta - \lambda_{n-k}|^2} d\eta \\ &\quad = |x_k|^2 |x_{n-k}|^2 \int_{-\infty}^{\infty} \frac{d\eta}{[(\xi - a_k)^2 + (\eta - b_k)^2] \cdot [(\xi - a_{n-k})^2 + (\eta - b_{n-k})^2]} \\ &\quad = |x_k|^2 |x_{n-k}|^2 \frac{\pi(2\xi - a_k - a_{n-k})}{(\xi - a_k)(\xi - a_{n-k}) \left[(2\xi - a_k - a_{n-k})^2 + (b_k - b_{n-k})^2 \right]} \\ &\quad \leq \pi |x_k|^2 |x_{n-k}|^2 \frac{2\xi - a_k - a_{n-k}}{|a_k| \cdot |a_{n-k}| \cdot (2\xi - a_k - a_{n-k})^2} \\ &\quad \leq \frac{\pi}{c^2} |x_k|^2 |x_{n-k}|^2 \frac{|b_k|^\alpha \cdot |b_{n-k}|^\alpha}{2\xi - a_k - a_{n-k}} \leq \frac{\pi}{2c^2} \frac{1}{\xi} |x_k|^2 |x_{n-k}|^2 |\lambda_k|^\alpha \cdot |\lambda_{n-k}|^\alpha, \end{aligned}$$

where we have used $|a_k| \geq c|b_k|^{-\alpha}$. On the other hand, if $\lambda_{n-k} = \lambda_k$, the integral over the term in the Cauchy product satisfies

$$\int_{-\infty}^{\infty} \frac{|x_k|^2|x_{n-k}|^2}{|\xi + i\eta - \lambda_k|^4} d\eta = |x_k|^2|x_{n-k}|^2 \int_{-\infty}^{\infty} \frac{d\eta}{[(\xi - a_k)^2 + (\eta - b_k)^2]^2}$$

$$= \frac{\pi}{2}|x_k|^2|x_{n-k}|^2 \frac{1}{(\xi - a_k)(\xi - a_k)^2} \leq \frac{\pi}{2}|x_k|^2|x_{n-k}|^2 \frac{1}{\xi \cdot a_k^2}$$

$$\leq \frac{\pi}{2c^2}|x_k|^2|x_{n-k}|^2 \frac{|b_k|^{2\alpha}}{\xi} \leq \frac{\pi}{2c^2}|x_k|^2|x_{n-k}|^2 \frac{|\lambda_k|^{2\alpha}}{\xi}.$$

Combining these estimates we therefore have

$$\int_{-\infty}^{\infty} \frac{|x_k|^2}{|\xi + i\eta - \lambda_k|^2} \cdot \frac{|x_{n-k}|^2}{|\xi + i\eta - \lambda_{n-k}|^2} d\eta \leq \frac{\pi}{2c^2\xi}|x_k|^2|x_{n-k}|^2|\lambda_k|^\alpha \cdot |\lambda_{n-k}|^\alpha$$

for all $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, n\}$. Since the integrals over the terms in the series are finite, and since the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\int_{-\infty}^{\infty} \frac{|x_k|^2}{|\xi + i\eta - \lambda_k|^2} \cdot \frac{|x_{n-k}|^2}{|\xi + i\eta - \lambda_{n-k}|^2} d\eta$$

$$\leq \frac{\pi}{2c^2\xi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(|x_k|^2|x_{n-k}|^2|\lambda_k|^\alpha \cdot |\lambda_{n-k}|^\alpha\right) = \frac{\pi}{2c^2\xi}\left(\sum_{k=0}^{\infty}|\lambda_k|^\alpha \cdot |x_k|^2\right)^2$$

converges due to the fact that $x \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$, we have

$$\xi\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\frac{|x_k|^2}{|\lambda - \lambda_k|^2}\right)^2 d\eta = \xi\sum_{n=0}^{\infty}\sum_{k=0}^{n}\int_{-\infty}^{\infty}\frac{|x_k|^2}{|\xi + i\eta - \lambda_k|^2} \cdot \frac{|x_{n-k}|^2}{|\xi + i\eta - \lambda_{n-k}|^2} d\eta$$

$$\leq \frac{\pi\xi}{2c^2\xi}\left(\sum_{k=0}^{\infty}|\lambda_k|^\alpha \cdot |x_k|^2\right)^2 \leq \frac{\pi}{2c^2} \cdot \frac{1}{m_\sigma^2}\|(-A)^{\frac{\alpha}{2}}x\|^4.$$

Using this estimate we can see that

$$\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi + i\eta, A)x\|^4 d\eta \leq M_\sigma^2 \cdot \sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\frac{|\langle x, \psi_k\rangle|^2}{|\xi + i\eta - \lambda_k|^2}\right)^2 d\eta$$

$$\leq \frac{\pi}{2c^2} \cdot \frac{M_\sigma^2}{m_\sigma^2}\|(-A)^{\frac{\alpha}{2}}x\|^4 < \infty.$$

Since $x \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$ was arbitrary, this concludes the proof. $\square$

Theorem 7 can now be proved using Theorem 15 and Lemma 16.

Proof of Theorem 7. Let $Y = \mathbb{C}^m$, and assume the operators B and C satisfy $\|(-A)^{\frac{\alpha}{2}}B\| \cdot \|(-A^*)^{\frac{\alpha}{2}}C^*\| < \delta$. The strong stability of the semigroup generated by $A + BC$ follows from Theorem 15 together with Lemma 16. It therefore remains to show $\|R(i\omega, A + BC)\| = \mathcal{O}(|\omega|^\alpha)$, which will conclude that the perturbed semigroup is polynomially stable.

Lemma 12 implies that there exists $M > 0$ such that

$$\sup_{\omega\in\mathbb{R}} \|(I - CR(i\omega, A)B)^{-1}\| \leq M.$$

This together with Lemma 3 and the Sherman-Morrison-Woodbury formula in Lemma 10 allow us to estimate

$$\begin{aligned}
\|R(i\omega, A + BC)\| &= \left\|R(i\omega, A) + R(i\omega, A)B(I - CR(i\omega, A)B)^{-1}CR(i\omega, A)\right\| \\
&\leq \|R(i\omega, A)\| + M\|R(i\omega, A)B\|\|CR(i\omega, A)\| \\
&\leq \|R(i\omega, A)\| + M\left(\sum_{j=1}^{m}\|R(i\omega, A)b_j\|^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{m}\|R(i\omega, A)^*c_l\|^2\right)^{\frac{1}{2}} \\
&\leq \|R(i\omega, A)\| + M\left(\sum_{j=1}^{m}\sum_{l=1}^{m}\|R(i\omega, A)b_j\|^2\|R(i\omega, A)^*c_l\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\|R(i\omega, A)\| = \mathcal{O}(|\omega|^\alpha)$, we can now prove $\|R(i\omega, A + BC)\| = \mathcal{O}(|\omega|^\alpha)$ by showing that

$$\|R(i\omega, A)b\|^2 = \mathcal{O}(|\omega|^\alpha), \quad \text{and} \quad \|R(i\omega, A)^*c\|^2 = \mathcal{O}(|\omega|^\alpha)$$

whenever $b \in \mathcal{D}((-A)^{\frac{\alpha}{2}})$ and $c \in \mathcal{D}((-A^*)^{\frac{\alpha}{2}})$. We will show that A satisfies the first one of these conditions. Since A is a Riesz-spectral operator, the second one can be verified analogously.

If we let $r_A > 0$ be as in the proof of Theorem 4, then by construction of Δ_α there exists $c_1 > 0$ such that if $|\omega| \geq r_A$, then for any eigenvalue λ_k of A we have

$$|i\omega - \lambda_k| \geq c_1|\omega|^{-\alpha}.$$

Moreover, we can also choose $c_2 > 0$ in such a way that for all $k \in \mathbb{N}_0$

$$\operatorname{Re}\lambda_k \leq \begin{cases} -c_2 & \text{if} \quad |\operatorname{Im}\lambda_k| < r_A, \\ -c_2|\operatorname{Im}\lambda_k|^{-\alpha} & \text{if} \quad |\operatorname{Im}\lambda_k| \geq r_A. \end{cases}$$

Denote by $I \subset \mathbb{N}_0$ the set of indices k for which $|\operatorname{Im}\lambda_k| < r_A$, and let $\omega \in \mathbb{R}$ be such that $|\omega| \geq r_A$. We then have

$$
\begin{aligned}
\|R(i\omega, A)b\|^2 &\leq M_\sigma \sum_{k=0}^{\infty} \frac{|\langle b, \psi_k\rangle|^2}{|i\omega - \lambda_k|^2} \\
&\leq \frac{M_\sigma}{\inf_{k\in\mathbb{N}_0} |i\omega - \lambda_k|} \cdot \sum_{k=0}^{\infty} \frac{|\langle b, \psi_k\rangle|^2}{\sqrt{(\operatorname{Re}\lambda_k)^2 + (\omega - \operatorname{Im}\lambda_k)^2}} \\
&\leq \frac{M_\sigma}{c_1|\omega|^{-\alpha}} \cdot \sum_{k=0}^{\infty} \frac{|\langle b, \psi_k\rangle|^2}{|\operatorname{Re}\lambda_k|} \leq |\omega|^\alpha \frac{M_\sigma}{c_1} \cdot \left(\sum_{k\in I} \frac{|\langle b, \psi_k\rangle|^2}{|\operatorname{Re}\lambda_k|} + \sum_{k\notin I} \frac{|\langle b, \psi_k\rangle|^2}{|\operatorname{Re}\lambda_k|} \right) \\
&\leq |\omega|^\alpha \frac{M_\sigma}{c_1} \cdot \left(\frac{1}{c_2} \sum_{k\in I} |\langle b, \psi_k\rangle|^2 + \frac{1}{c_2} \sum_{k\notin I} |\operatorname{Im}\lambda_k|^\alpha |\langle b, \psi_k\rangle|^2 \right) \\
&\leq |\omega|^\alpha \frac{M_\sigma}{c_1 c_2} \cdot \left(\sum_{k=0}^{\infty} |\langle b, \psi_k\rangle|^2 + \sum_{k=0}^{\infty} |\lambda_k|^\alpha |\langle b, \psi_k\rangle|^2 \right) \\
&\leq |\omega|^\alpha \frac{M_\sigma}{m_\sigma c_1 c_2} \cdot \left(\|b\|^2 + \|(-A)^{\frac{\alpha}{2}} b\|^2 \right).
\end{aligned}
$$

This shows that we indeed have $\|R(i\omega, A)b\|^2 = \mathcal{O}(|\omega|^\alpha)$, and thus concludes the proof. □

6. Robustness of a Polynomially Stable Wave Equation

We conclude the paper by applying the perturbation results in Section 3 to analyzing robustness of a strongly stable partial differential equation with respect to rank one perturbations. To this end, we consider a one-dimensional wave equation with distributed control. We begin by using state feedback to stabilize the system polynomially, and then consider perturbations to this stabilized equation.

The same system was considered earlier in [9], where it was used to demonstrate a method for converting the norm conditions in the perturbation results into easily verifiable criteria involving L^2-norms of the perturbing functions and their derivatives. However, it was remarked that the available results on preservation of uniform boundedness and polynomial stability became difficult to check and led to impractical conditions on the perturbations.

In this section we complete the study of the example by improving the conclusions in [9] with the aid of the new results presented in this paper.

We show that a direct application of Theorem 7 makes it possible to derive easily verifiable conditions for the preservation of the strong and polynomial stability of the wave equation. In particular this approach allows us to avoid posing any restrictions on the stabilizing feedback. This, in turn, greatly increases the applicability of the resulting conditions on the perturbations.

We begin by considering a one-dimensional controlled wave equation

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + g_0(z)u(t) \tag{18a}$$

$$w(0,t) = w(1,t) = 0 \tag{18b}$$

$$w(z,0) = w_0(z), \quad \frac{\partial w}{\partial t}(z,0) = w_1(z), \tag{18c}$$

on $(0,1)$ with $g_0(z) = \sqrt{3}(1-z)$. It is well-known that the equation can be written as a first order linear system on a Hilbert space. To this end, define $A_0 : \mathcal{D}(A_0) \subset L^2(0,1) \to L^2(0,1)$ as $A_0 = -\frac{d^2}{dz^2}$ with the domain

$$\mathcal{D}(A_0) = \left\{ x \in L^2(0,1) \;\middle|\; x, x' \text{ abs. cont.}, \ x'' \in L^2(0,1), \ x(0) = x(1) = 0 \right\}.$$

The operator A_0 has a positive self-adjoint square root $A_0^{1/2}$, and the space $X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$ is a Hilbert space when equipped with an inner product $\langle x, y\rangle_X = \langle A_0^{1/2} x_1, A_0^{1/2} y_1\rangle_{L^2} + \langle x_2, y_2\rangle_{L^2}$. Choosing

$$x = \begin{bmatrix} w \\ \frac{dw}{dt} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}),$$

$$Gu = gu = \begin{bmatrix} 0 \\ g_0 \end{bmatrix} u, \quad x_0 = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix},$$

the wave equation (18) can be written as

$$\dot{x} = Ax + Gu, \qquad x(0) = x_0. \tag{19}$$

The eigenvalues of the operator A are $\lambda_k = ik\pi$ for $k \in \mathbb{Z} \setminus \{0\}$, and the corresponding eigenvectors

$$\varphi_k(z) = \frac{1}{\lambda_k} \begin{bmatrix} \sin(k\pi z) \\ \lambda_k \sin(k\pi z) \end{bmatrix}$$

form an orthonormal basis of X.

In the following theorem we choose a feedback law $u = Kx = \langle x, h\rangle$ in such a way that $A + GK$ is a Riesz-spectral operator generating a polynomially stable semigroup on X. The theorem also gives us the constants $m_\sigma > 0$ and $M_\sigma > 0$ related to the Riesz basis of eigenvectors of $A + GK$. These constants are essential in computing explicit perturbation bounds later in the section. The proof of the theorem can be found in [9, Thm. 13].

Theorem 17. *Choose $K = \langle \cdot, h\rangle \in \mathcal{L}(X, \mathbb{C})$ in such a way that*

$$h = -\frac{\pi^2}{10\sqrt{3}} \sum_{k\neq 0} \frac{\overline{\alpha_k}}{k} \varphi_k, \quad \textit{where} \quad \alpha_k = \prod_{l\neq 0,k} \left(1 + i\frac{1}{10l^2(l-k)}\right).$$

Then $A + GK$ is a strongly stable Riesz-spectral operator with eigenvalues $\sigma(A + GK) = \{-\frac{\pi}{10k^2} + ik\pi\}_{k\neq 0}$. The Riesz basis $\{\phi_k\}_{k\neq 0}$ of eigenfunctions of the operator $A + GK$ satisfies

$$\frac{1}{M_\sigma} \sum_{k\neq 0} |\langle x, \phi_k\rangle|^2 \leq \|x\|_X^2 \leq \frac{1}{m_\sigma} \sum_{k\neq 0} |\langle x, \phi_k\rangle|^2$$

for $m_\sigma = \frac{3}{5}$ and $M_\sigma = \frac{5}{3}$.

The spectrum of the stabilized system operator consists of simple eigenvalues and satisfies $\sigma(A + GK) \subset \mathbb{C}^-$, and thus the strong stability of the semigroup follows immediately from the fact that $A + GK$ is a Riesz-spectral operator. Moreover, for all $\omega \in \mathbb{R}$ we have

$$\|R(i\omega, A + GK)x\|^2 \leq M_\sigma \sum_{k\neq 0} \frac{|\langle x, \psi_k\rangle|^2}{|i\omega - \mu_k|^2} \leq \frac{M_\sigma}{m_\sigma} \frac{\|x\|^2}{\operatorname{dist}(i\omega, \sigma(A + GK))^2}.$$

If $\omega \geq 1$, then properties of the spectrum $\sigma(A + GK) = \{\mu_k\}_{k\neq 0}$ imply

$$\begin{aligned}\operatorname{dist}(i\omega, \sigma(A+GK)) \geq \operatorname{dist}(i\lceil\omega\rceil, \sigma(A+GK)) &= \left| i\lceil\omega\rceil - \left(-\frac{\pi}{10\lceil\omega\rceil^2} + i\lceil\omega\rceil\right)\right| \\ &= \frac{\pi}{10} \cdot \frac{1}{\lceil\omega\rceil^2} \geq \frac{\pi}{10} \cdot \frac{1}{\omega^2} \cdot \inf_{\omega\geq 1} \left(\frac{\omega}{\lceil\omega\rceil}\right)^2 = \frac{\pi}{40} \cdot \frac{1}{\omega^2},\end{aligned}$$

where $\lceil\omega\rceil = k$ if $\omega \in (k-1, k]$. Since $\sigma(A+GK)$ is symmetrical with respect to the real axis, we have $\operatorname{dist}(i\omega, \sigma(A + GK)) \geq (\pi/40)|\omega|^{-2}$ for all $\omega \in \mathbb{R}$ with $|\omega| \geq 1$. Therefore for all such $\omega \in \mathbb{R}$ we also have

$$\|R(i\omega, A + GK)x\| \leq \sqrt{\frac{M_\sigma}{m_\sigma}} \frac{40}{\pi} |\omega|^2 \|x\| = \frac{200}{3\pi} |\omega|^2 \|x\|,$$

which concludes $\|R(i\omega, A + GK)\| = \mathcal{O}(|\omega|^2)$, and $A + GK$ satisfies the conditions of Assumption 1 for $\alpha = 2$. In particular, the wave equation with state feedback $u = Kx$ is strongly and polynomially stable.

We can now use the perturbation results in Section 3 to study the robustness properties of the stabilized wave equation. The perturbations we consider are of the form

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + g_0(z)u(t) \tag{20a}$$

$$+ b_0(z)\left(\langle w(\cdot,t), c_1\rangle_{L^2} + \langle \frac{\partial w}{\partial t}(\cdot,t), c_2\rangle_{L^2}\right), \tag{20b}$$

where $b_0, c_2 \in \mathcal{D}(A_0)$ and $c_1 \in L^2(0,1)$. If we denote

$$b = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, \qquad c = \begin{bmatrix} A_0^{-1}c_1 \\ c_2 \end{bmatrix},$$

then the perturbed equation can be written as

$$\dot{x} = (A + GK)x + \langle x, c\rangle_X b, \qquad x(0) = x_0,$$

and $b, c \in \mathcal{D}(A_0) \times \mathcal{D}(A_0) \subset \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}) = \mathcal{D}(A)$. The operators $B = b \in \mathcal{L}(\mathbb{C}, X)$ and $C = \langle \cdot, c\rangle \in \mathcal{L}(X, \mathbb{C})$ therefore satisfy the conditions of Assumption 1 for $Y = \mathbb{C}$ and for $\beta = \gamma = 1$.

In the following we use Theorem 7 to determine classes of functions b_0, c_1, and c_2 for which the perturbed wave equation (20) remains strongly and polynomially stable. Since we now have $\beta = \gamma = 1 = \alpha/2$, Theorem 7 states that there exists $\delta > 0$ such that the the semigroup generated by the operator $(A + GK) + BC$ is strongly and polynomially stable whenever $\|(A+GK)B\| \cdot \|(A+GK)^*C^*\| < \delta$. In the case of the rank one perturbation $BC = \langle \cdot, c\rangle b$, it is sufficient to require

$$\|(A + GK)b\| < \sqrt{\delta} \qquad \text{and} \qquad \|(A + GK)^*c\| < \sqrt{\delta}.$$

In order to choose an appropriate $\delta > 0$, we need to estimate the behavior of $\|R(\lambda, A+GK)(-A-GK)^{-2}\|$ for $\lambda \in \overline{\mathbb{C}^+}$. Since we now have $\alpha - \beta - \gamma = 0$, by Remark 13 we can choose any $\delta > 0$ satisfying

$$\delta \le \frac{1}{\sup_{\operatorname{Re}\lambda \ge 0} \|R(\lambda, A + GK)(-A - GK)^{-2}\|}.$$

For any $\lambda \in \overline{\mathbb{C}^+}$ and any $k \in \mathbb{Z}$ we can estimate

$$
\begin{aligned}
|\mu_k|^4|\lambda-\mu_k|^2 &= \big((\operatorname{Re}\mu_k)^2+(\operatorname{Im}\mu_k)^2\big)^2\big((\operatorname{Re}\lambda-\operatorname{Re}\mu_k)^2+(\operatorname{Im}\lambda-\operatorname{Im}\mu_k)^2\big)\\
&\geq (\operatorname{Im}\mu_k)^4(\operatorname{Re}\lambda-\operatorname{Re}\mu_k)^2 \geq (\operatorname{Im}\mu_k)^4(\operatorname{Re}\mu_k)^2 = \pi^4k^4\cdot\frac{\pi^2}{100k^4} = \frac{\pi^6}{100}.
\end{aligned}
$$

Therefore, for all $\lambda \in \overline{\mathbb{C}^+}$ and for all $x \in X$ we have

$$
\begin{aligned}
\|R(\lambda, A+GK)(-A-GK)^{-2}x\|^2 &\leq M_\sigma \sum_{k\neq 0} \frac{|\langle x,\psi_k\rangle|^2}{|\mu_k|^4|\lambda-\mu_k|^2}\\
&\leq \frac{M_\sigma}{m_\sigma}\|x\|^2\cdot\sup_{k\neq 0}\frac{1}{|\mu_k|^4|\lambda-\mu_k|^2} \leq \frac{M_\sigma}{m_\sigma}\|x\|^2\frac{100}{\pi^6}.
\end{aligned}
$$

This concludes that we can choose $\delta > 0$ in Theorem 4 as

$$
\delta = \sqrt{\frac{m_\sigma}{M_\sigma}}\frac{\pi^3}{10} = \frac{3\pi^3}{50}.
$$

We clearly have $\|G\| = \|g\|_X = \|g_0\|_{L^2} = 1$ and $\|b\|_X = \|b_0\|_{L^2}$. It was further shown in [9, Sec. 5] that

$$
\begin{aligned}
\|K\| &\leq \frac{\pi}{3}, \qquad \|c\|_X^2 \leq \frac{1}{\pi^2}\|c_1\|_{L^2}^2 + \|c_2\|_{L^2}^2\\
\|Ab\|_X^2 &= \|b_0'\|_{L^2}^2, \qquad \|A^*c\|_X^2 = \|c_1\|_{L^2}^2 + \|c_2'\|_{L^2}^2.
\end{aligned}
$$

We can therefore estimate

$$
\|(A+GK)b\|_X \leq \|Ab\|_X + \|G\|\|K\|\|b\| \leq \|b_0'\|_{L^2} + \frac{\pi}{3}\|b_0\|_{L^2},
$$

and

$$
\begin{aligned}
&\|(A+GK)^*c\|_X \leq \|A^*c\|_X + \|K^*\|\|G^*\|\|c\|_X\\
\leq\ & \sqrt{\|c_1\|_{L^2}^2 + \|c_2'\|_{L^2}^2} + \frac{\pi}{3}\sqrt{\frac{1}{\pi^2}\|c_1\|_{L^2}^2 + \|c_2\|_{L^2}^2}\\
\leq\ & \sqrt{2}\,(\|c_1\|_{L^2} + \|c_2'\|_{L^2}) + \frac{\pi\sqrt{2}}{3}\left(\frac{1}{\pi}\|c_1\|_{L^2} + \|c_2\|_{L^2}\right)\\
=\ & \frac{4\sqrt{2}}{3}\|c_1\|_{L^2} + \sqrt{2}\left(\frac{\pi}{3}\|c_2\|_{L^2} + \|c_2'\|_{L^2}\right)
\end{aligned}
$$

Combining these estimates we arrive at the following conditions for the preservation of the strong and polynomial stability of the wave equation. In particular, the theorem shows that the perturbed semigroup is stable whenever the norms of the functions b_0, b_0', c_1, c_2 and c_2' are small enough.

Theorem 18. *The perturbed wave equation* (20) *with the system operator* $A + GK + BC$ *is strongly and polynomially stable whenever*

$$b_0, c_2 \in \mathcal{D}(A_0) = \left\{ x \in L^2(0,1) \;\middle|\; x, x' \text{ abs. cont.}, \right.$$
$$\left. x'' \in L^2(0,1),\ x(0) = x(1) = 0 \right\},$$

and $c_1 \in L^2(0,1)$ *are such that the* L^2*-norms of the functions* b_0*,* b_0'*,* c_1*,* c_2 *and* c_2' *satisfy*

$$\frac{\pi}{3}\|b_0\|_{L^2} + \|b_0'\|_{L^2} < \frac{3\pi^3}{50}, \quad \|c_1\|_{L^2} < \frac{9\pi^3}{400\sqrt{2}}, \quad \frac{\pi}{3}\|c_2\|_{L^2} + \|c_2'\|_{L^2} < \frac{3\pi^3}{100\sqrt{2}}.$$

For example for functions

$$b_0(z) = a_1 \sin(\pi z)\cos(\sqrt{2}\pi z),$$
$$c_1(z) = a_2(\chi_{[.25,.75]}(z) + iz),$$
$$c_2(z) = a_3 e^{-\tan(\pi z + \frac{\pi}{2})^2}$$

the conditions in Theorem 18 are satisfied whenever $a_1, a_2, a_3 \leq \frac{1}{5}$.

7. Conclusions

In this paper we have studied the preservation of strong and polynomial stability of a semigroup under perturbations to its infinitesimal generator. In particular we saw that polynomial stability of a semigroup is robust with respect to a large class of perturbations. We demonstrated that the well-known difficulties in studying the preservation of non-exponential stability can be overcome by measuring the sizes of the perturbations using the graph norms of the operators $(-A)^\beta$ and $(-A^*)^\gamma$, instead of the norm of the underlying Hilbert space. This approach produced easily characterizable classes of perturbations preserving properties of the spectrum of the generator and the stability of the semigroup.

The conditions on the perturbations are very simple, but their usefulness and the ease of verifying them for actual operators depend on several factors.

This is well illustrated by the example considered in Section 6. There the theoretic results presented in Section 3 were used to derive concrete and easily checkable conditions on the perturbing functions b_0, c_1 and c_2. Existence of such conditions was, most of all, a consequence of the possibility to choose the appropriate exponents as $\beta = \gamma = 1$. First of all, the identities

$$\mathcal{D}((-A-GK)^\beta) = \mathcal{D}(-A-GK) = \mathcal{D}(A),$$
$$\mathcal{D}(((-A-GK)^*)^\gamma) = \mathcal{D}((-A-GK)^*) = \mathcal{D}(-A^*-K^*G^*) = \mathcal{D}(A^*),$$

made the conditions

$$b \in \mathcal{D}((-A-GK)^\beta), \qquad \text{and} \qquad c \in \mathcal{D}(((-A-GK)^*)^\gamma) \tag{21}$$

very straightforward to verify. In particular, they could be expressed as simple boundary conditions and using the differentiability properties of the component functions b_0, c_1, and c_2. Furthermore, the choices $\beta = \gamma = 1$ also made it possible to derive estimates for the graph norms in the conditions by simply estimating $\|(-A-GK)b\|$ and $\|(-A-GK)^*c\|$.

However, as was already remarked in [9], the conditions in (21) become much more restrictive and much more complicated to verify as soon as we have either $\beta > 1$ or $\gamma > 1$. Indeed, even in the case $\beta = 2$ the usefulness of the results is greatly diminished by the fact that the elements in the domain

$$\mathcal{D}((-A-GK)^2) = \left\{ x \in \mathcal{D}(A) \;\middle|\; (A+GK)x \in \mathcal{D}(A) \right\}$$

are required to satisfy conditions that are more sophisticated than simply having continuous derivatives of high enough order. Difficulties like this motivate especially the search for conditions with lowest possible requirements for the exponents β and γ, as well as for conditions providing maximal freedom in choosing these parameters.

[1] A. Bátkai, K.-J. Engel, J. Prüss, R. Schnaubelt, Polynomial stability of operator semigroups, Mathematische Nachrichten 279 (13–14) (2006) 1425–1440.

[2] C. J. K. Batty, T. Duyckaerts, Non-uniform stability for bounded semigroups on banach spaces, J. Evol. Equ. 8 (2008) 765–780.

[3] A. Borichev, Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, Math. Ann. 347 (2) (2010) 455–478.

[4] C.-Z. Xu, G. Sallet, On spectrum and Riesz basis assignment of infinite-dimensional linear systems by bounded linear feedbacks, SIAM J. Control Optim. 34 (2) (1996) 521–541.

[5] S. Shun-Hua, On spectrum distribution of completely controllable linear systems, SIAM J. Control Optim. 19 (6) (1981) 730–743.

[6] W. Arendt, C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Amer. Math. Soc. 306 (1988) 837–841.

[7] Y. I. Lyubich, V. Q. Phông, Asymptotic stability of linear differential equations in Banach spaces, Studia Mathmatica 88 (1988) 37–42.

[8] Y. Tomilov, A resolvent approach to stability of operator semigroups, Journal of Operator Theory 46 (2001) 63–98.

[9] L. Paunonen, Perturbation of strongly and polynomially stable Riesz-spectral operators, Systems Control Lett. 60 (2011) 234–248.

[10] G. M. Sklyar, A. V. Rezounenko, Stability of a strongly stabilizing control for systems with a skew-adjoint operator in Hilbert space, J. Math. Anal. Appl. 254 (2001) 1–11.

[11] M. Haase, Functional Calculus for Sectorial Operators, Birkhäuser, 2006.

[12] R. F. Curtain, H. J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, New York, 1995.

[13] Y. Latushkin, R. Shvydkoy, Hyperbolicity of semigroups and Fourier multipliers, in: Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), Vol. 129 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2001, pp. 341–363.

[14] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, 2000.

[15] A. Gomilko, Conditions on the generator of a uniformly bounded C_0-semigroup, Funct. Anal. Appl. 33 (4) (1999) 294–296.