Perturbation of Strongly and Polynomially Stable Riesz-Spectral Operators

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Abstract

In this paper we consider bounded and relatively bounded finite rank perturbations of a Riesz-spectral operator generating a polynomially stable semigroup of linear operators on a Hilbert space. We concentrate on a commonly encountered situation where the spectrum of the unperturbed operator is contained in the open left half-plane of the complex plane and approaches the imaginary axis asymptotically. We present conditions on the perturbing operator such that the spectrum of the perturbed operator is contained in the open left half-plane of the complex plane and additional conditions for the strong and polynomial stabilities of the perturbed semigroup. We consider two applications of the perturbation results. In the first example we apply the results to the perturbation of a polynomially stabilized one-dimensional wave equation. In the second example we consider perturbation of a closed-loop system consisting of a distributed parameter system and an observer-based feedback controller solving the robust output regulation problem related to an infinite-dimensional signal generator.

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1. Introduction

It is well-known that the robustness properties of strong stability of strongly continuous semigroups are very weak compared to the robustness properties of exponential stability. Furthermore, this is the case even if the generator of the C_0-semigroup is a well-behaving operator, for example a normal operator with a compact resolvent, and the perturbations are very simple, such as bounded and of finite rank. For example, it is well-known that for any exponentially stable C_0-semigroup $T(t)$ generated by an operator A there exists a bound $\delta > 0$ such that for any bounded perturbation B with norm $\|B\| < \delta$ the semigroup generated by the operator $A + B$ is also exponentially stable. On the other hand, it is easy to find a strongly stable C_0-semigroup $T(t)$ generated by an operator A which is normal and has compact resolvent such that there exists a bounded rank one perturbation B of arbitrarily small norm such that the perturbed semigroup generated by $A + B$ is unstable.

If X is a Hilbert space, then a concrete example of this kind of situation can be constructed by defining an operator $A : \mathcal{D}(A) \subset X \to X$ such that

$$Ax = \sum_{k=1}^{\infty} \left(-\frac{1}{k} + ik \right) \langle x, \phi_k \rangle \phi_k,$$

$$\mathcal{D}(A) = \left\{ x \in X \mid \sum_{k=1}^{\infty} k^2 |\langle x, \phi_k \rangle|^2 < \infty \right\}$$

where $\{\phi_k\}_k \subset X$ is an orthonormal basis of X. The operator A is normal, has compact resolvent and generates a strongly stable C_0-semigroup on X. For any $\varepsilon > 0$ we can choose a perturbing operator $B_\varepsilon = \frac{1}{n} \langle \cdot, \phi_n \rangle \phi_n$ with $n > \frac{1}{\varepsilon}$. Then $in \in i\mathbb{R}$ is an eigenvalue of the perturbed operator $A + B_\varepsilon$ and thus the perturbed semigroup is unstable, but on the other hand we clearly have $\|B_\varepsilon\| = \frac{1}{n} < \varepsilon$.

Even so, results on the robustness properties of strongly stable semigroups are in great demand. This is the case in particular in mathematical control theory and in robust output regulation. In the treatment of this problem it is often assumed that the used stability type is exponential stability and thus the well-known robustness properties of exponential stability can be used to guarantee the existence of an easily identifiable class of perturbations preserving the stability of the systems. However, if we want to consider reference signals generated by a signal generator having an infinite number of eigenvalues on the imaginary axis [1, 2] it is possible that exponential stability is unachievable [3, Cor 3.58]. In this case the robustness properties of strong stability are required to determine classes of perturbations preserving the stability of the systems and the output regulation property.

In this paper we concentrate on the perturbation of a strongly stable C_0-semigroup generated by a Riesz-spectral operator $A : \mathcal{D}(A) \subset X \to X$ on a Hilbert space X. We consider a commonly encountered situation where the spectrum of the generator is contained in the open left half-plane of $\mathbb{C}$ and approaches the unstable half-plane of the complex plane asymptotically, as was the case in the

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above example. More precisely, we assume that there exists a curve which from some point on limits the decay of the real parts of the eigenvalues as a function of the imaginary parts. We also assume that this curve approaches the imaginary axis at a known polynomial rate. These assumptions imply that in addition to its strong stability the semigroup generated by A is also polynomially stable [4, 5].

These kind of spectra most commonly arise from the stabilization of an operator with an infinite number of eigenvalues on the imaginary axis [6]. Nonetheless, we do not limit our attention to operators obtained this way but consider general operators whose spectra have this property. An example of a situation where we need this generality is the perturbation of a composite operator consisting of a part stabilized strongly by feedback and parts having spectrum in the open left half-plane of $\mathbb{C}$.

The perturbations we consider in this paper are bounded and relatively bounded finite rank operators. In the case of a bounded perturbation the perturbing operator can be represented as

$$B = \sum_{j=1}^{m} \langle \cdot, g_j \rangle b_j, \tag{1}$$

where $\{g_j\}_j \subset X$ and $\{b_j\}_j \subset X$.

We consider two different perturbation problems: The change of the spectrum of the generator and the preservation of the stability — strong and polynomial — of the C_0-semigroup. We solve the first problem by presenting conditions under which the spectrum of the perturbed operator is contained in the open left half-plane of $\mathbb{C}$. These conditions are given as requirements $b_j \in \mathcal{D}((-A)^\beta)$ and $g_j \in \mathcal{D}((-A^*)^\gamma)$ for some $\beta, \gamma \geq 0$ and as bounds for the corresponding graph norms. In these conditions the appropriate constants β and γ

After we have solved the problem of perturbation of the spectrum the ABLV Theorem [7, 8, 9] implies that in order to solve the problem concerning the strong stability it is sufficient to find additional conditions for the uniform boundedness of the perturbed C_0-semigroup. Using this knowledge we present two sets of conditions for the preservation of the strong stability of the C_0-semigroup. These conditions are of the same type as the ones presented for the change of the spectrum. Finally, we present separate conditions for the preservation of the polynomial stability of the semigroup. This result uses the result on the perturbation of the spectrum but is independent of the result on the preservation of uniform boundedness of the C_0-semigroup.

We consider the perturbation problems separately since the results on the perturbation of the spectrum are also useful by themselves. This is because one is not necessarily interested in the preservation of strong stability of the semigroup but only the perturbation of the spectrum. It is also possible that in some cases the uniform boundedness of the perturbed semigroup can be determined by some other means.

The robustness properties of strong stability have been studied earlier by Sklyar and Rezounenko [10] who have considered the robustness of the strongly stabilizing feedback $A + \langle \cdot, h \rangle d$ when the elements $h \in X$ and $d \in X$ are perturbed. Their approach is limited to the case where the strongly stable semigroup is obtained by stabilizing a skew-adjoint operator with compact resolvent by rank one state feedback. Another restriction on the applicability of their results is that the spectra of the operators are assumed to have a uniform gap, i.e.

$$\inf_{k \neq l} \ |\lambda_k - \lambda_l| > 0$$

for the set of eigenvalues $\{\lambda_k\}_k$. This condition rules out eigenvalues with multiplicity larger than one, the existence of continuous spectrum and the types of spectra where the eigenvalues approach infinity slowly. All of these limitations are restrictive when considering applications.

The robustness properties of strong stability have also been studied by Caraman [11] who has presented a result on the preservation of strong stability of compact semigroups. The requirement that the C_0-semigroup is compact is a very strict limitation [12, Sec. 2.3] and in particular it rules out the case where the spectrum of the generator approaches the imaginary axis asymptotically. Furthermore, the result by Caraman only concerns perturbations B which commute with the unperturbed operator A.

To our knowledge the preservation of the polynomial stability of the semigroup under perturbations of its generator has not been studied previously in the literature.

The problem of perturbation of a spectrum approaching the imaginary axis asymptotically also has a connection to the problem of pole placement of an infinite number of eigenvalues [6, 13, 14]. In connection to this problem it is well-known that if $\{\psi_k\}_{k=1}^\infty$ is the Riesz basis of eigenvectors of A^*, then the rate of $|\langle d, \psi_k \rangle| \to 0$ as $k \to \infty$ determines how much the eigenvalues of A can be moved asymptotically using a bounded feedback $A + \langle \cdot, h \rangle d$.

After presenting the theoretical results we present three examples of their application. In the first example we consider an undamped wave equation on an interval $(0, 1)$ resulting in a system with an infinite number of eigenvalues on the imaginary axis. We first stabilize this equation polynomially using pole placement of an infinite spectrum with state feedback [6]. Subsequently, we consider perturbations of this stabilized equation and compute explicit bounds for the pertubations such that the spectrum of the perturbed equation remains in the open left half-plane of $\mathbb{C}$ and such that the perturbed semigroup is strongly and polynomially stable. These conditions on the perturbations are ultimately given as bounds for easily computable $L^2(0, 1)$-norms of the perturbing functions and their first derivatives. The presentation of this example requires estimation of the bounding constants related to the Riesz basis of the strongly stabilized wave operator. In particular this requires estimation of the norm of an inverse of a certain infinite matrix. We show that this can be done by finding a uniform lower bound for the singular values of finite truncations of this infinite matrix. This uniform lower bound is then obtained using a generalization of the Gerschgorin Circle Theorem for block diagonally dominant matrices [15].

In the second example we apply the theory to the perturbation of an observer-based feedback controller solving a robust output regulation problem related to an infinite-dimensional signal generator [1]. In this case we are interested in the preservation of the strong stability of the closed-loop system under finite rank perturbations of the controller. The example also demonstrates how the methods presented in this paper can be applied to the study of perturbations of composite systems. Since the unperturbed operator considered in this example is not obtained by stabilizing a skew-adjoint operator with a rank one feedback, the results of Sklyar and Rezounenko are not applicable.

As the third example we briefly demonstrate why the assumption of the eigenvalues of the unperturbed operator having a uniform gap can be a limitation in applications. To this end we consider perturbation of coupled systems.

The paper is organized as follows. In Section 2 we introduce the notation and formulate the main problem of the paper. In Section 3 we present the main results on the perturbation of the spectrum. These results are proved in Section 3.3. The results on the preservation of strong and polynomial stabilities are presented in Section 4. The examples on the perturbation of the polynomially stabilized wave equation, the perturbation of the robust controller and the perturbation of coupled systems are presented in Sections 5, 6 and 7, respectively. Section 8 contains concluding remarks.

2. Main Problem

Throughout this paper we denote by X a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. The open and closed left half-planes of $\mathbb{C}$ are denoted by $\mathbb{C}^-$ and $\overline{\mathbb{C}^-}$, respectively, and similarly the open and closed right half-planes of $\mathbb{C}$ are denoted by $\mathbb{C}^+$ and $\overline{\mathbb{C}^+}$, respectively. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. The domain of a linear operator A is denoted by $\mathcal{D}(A)$. The spectrum and the resolvent set of a linear operator A are denoted by $\sigma(A)$ and $\rho(A)$, respectively.

Let X be a Hilbert space and consider an operator $A : \mathcal{D}(A) \subset X \to X$ defined by

$$Ax = \sum_{k\in\mathbb{Z}} \lambda_k \langle x, \psi_k \rangle \phi_k, \tag{2a}$$

$$\mathcal{D}(A) = \Big\{ x \in X \;\Big|\; \sum_{k\in\mathbb{Z}} |\lambda_k|^2 |\langle x, \psi_k \rangle|^2 < \infty \Big\} \tag{2b}$$

where $\{\phi_k\}_k$ is a Riesz basis of X and $\{\psi_k\}_k$ is the corresponding biorthonormal sequence. Throughout this paper we denote by m_σ and M_σ the positive constants such that for all $x \in X$

$$m_\sigma \sum_{k\in\mathbb{Z}} |\langle x, \psi_k \rangle|^2 \le \|x\|^2 \le M_\sigma \sum_{k\in\mathbb{Z}} |\langle x, \psi_k \rangle|^2$$

$$\frac{1}{M_\sigma} \sum_{k\in\mathbb{Z}} |\langle x, \phi_k \rangle|^2 \le \|x\|^2 \le \frac{1}{m_\sigma} \sum_{k\in\mathbb{Z}} |\langle x, \phi_k \rangle|^2$$

We assume that the set $\{\lambda_k\}_k$ is contained in $\mathbb{C}^-$ and that it has no accumulation points on $i\mathbb{R}$. As in [16, Sec 2.3] it can be shown that A generates a uniformly bounded C_0-semigroup $T(t)$ on X. Due to the ABLV Theorem [7, 8, 9] the assumptions on the eigenvalues $\{\lambda_k\}_k$ of A further imply that this semigroup is strongly stable.

We make the following assumption on the asymptotic behaviour of the eigenvalues $\{\lambda_k\}_k$ of the operator A.

Assumption 1 (Geometric assumption on $\sigma(A)$)**.** *The set $\{\lambda_k\}_k$ is contained in $\mathbb{C}^-$, it has no accumulation points on $i\mathbb{R}$ and there exist constants $\alpha, c_\sigma > 0$ and $y_\sigma > 0$ such that*

$$\operatorname{Re}\lambda_k \le -\frac{c_\sigma}{|\operatorname{Im}\lambda_k|^\alpha} \qquad \text{if} \quad |\operatorname{Im}\lambda_k| \ge y_\sigma.$$

Because A is similar to a normal operator Assumption 1 also implies that the semigroup generated by A is polynomially stable. More precisely, if A satisfies Assumption 1 for some $\alpha > 0$, there exists a constant $C_\sigma > 0$ such that

$$\|T(t)A^{-1}\| \le \frac{C_\sigma}{t^{1/\alpha}}$$

for all $t > 0$ [4, Prop. 4.1]. In general it is possible that a polynomially stable semigroup is not uniformly bounded, and because of this, not strongly stable. However, in the case of semigroups generated by operators of form (2) a polynomially stable semigroup is also always strongly stable.

Since $\{\lambda_k\}_k \subset \mathbb{C}^-$ and it has no finite accumulation points on $i\mathbb{R}$ the operator $-A$ is an invertible sectorial operator in the sense of [17]. We can therefore make the following definitions using the fractional domains of the operators $-A$ and $-A^*$. For $\beta \ge 0$ define

$$\mathcal{D}_\beta = \mathcal{D}((-A)^\beta) = \Big\{ x \in X \;\Big|\; \sum_{k\in\mathbb{Z}} |\lambda_k|^{2\beta} |\langle x, \psi_k \rangle|^2 < \infty \Big\}$$

$$\mathcal{D}_\beta^* = \mathcal{D}((-A^*)^\beta) = \Big\{ x \in X \;\Big|\; \sum_{k\in\mathbb{Z}} |\lambda_k|^{2\beta} |\langle x, \phi_k \rangle|^2 < \infty \Big\}.$$

The spaces $(\mathcal{D}_\beta, \|\cdot\|_\beta)$ and $(\mathcal{D}_\beta^*, \|\cdot\|_{*,\beta})$ are Hilbert spaces with norms defined by

$$\|x\|_\beta^2 = \sum_{k\in\mathbb{Z}} |\lambda_k|^{2\beta} |\langle x, \psi_k \rangle|^2, \quad \|y\|_{*,\beta}^2 = \sum_{k\in\mathbb{Z}} |\lambda_k|^{2\beta} |\langle y, \phi_k \rangle|^2$$

for every $x \in \mathcal{D}_\beta$ and $y \in \mathcal{D}_\beta^*$. It is also straight-forward to verify that under our assumptions for any $\gamma \ge \beta \ge 0$ we have $\mathcal{D}_\gamma \subset \mathcal{D}_\beta$ and $\mathcal{D}_\gamma^* \subset \mathcal{D}_\beta^*$ and that there exist constants $c_{\beta,\gamma} > 0$ and $c_{\beta,\gamma}^* > 0$ such that

$$\|x\|_\beta \leq c_{\beta,\gamma}\|x\|_\gamma \qquad \text{and} \qquad \|y\|_{*,\beta} \leq c_{\beta,\gamma}^*\|y\|_{*,\gamma}$$

for all $x \in \mathcal{D}_\gamma$ and $y \in \mathcal{D}_\gamma^*$.

The main problems of this paper are the following.

Problem 1. *Let A satisfy the geometric assumption on $\sigma(A)$ for some $\alpha > 0$. Find conditions on the operator $B : \mathcal{D}(B) \subset X \to X$ with $\mathcal{D}(B) \supset \mathcal{D}(A)$ such that for the perturbed operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$*

(a) $\sigma(A + B) \subset \mathbb{C}^-$;
(b) *$A+B$ generates a strongly stable C_0-semigroup on X.*
(c) *$A + B$ generates a polynomially stable C_0-semigroup on X.*

In this paper we consider perturbations B which are relatively bounded with respect to the operator A (or A-bounded) and of finite rank, i.e. $\mathcal{R}(B)$ is finite-dimensional. In this case $BA^{-1} \in \mathcal{L}(X)$ and there exist linearly independent sets $\{b_j\}_{j=1}^m \subset X$ and $\{g_j\}_{j=1}^m \subset X$ such that

$$BA^{-1} = \sum_{j=1}^m \langle \cdot, g_j\rangle b_j. \tag{3}$$

If B is a bounded operator we can also write

$$B = \sum_{j=1}^m \langle \cdot, \tilde{g}_j\rangle \tilde{b}_j, \tag{4}$$

where the sets $\{\tilde{b}_j\}_{j=1}^m \subset X$ and $\{\tilde{g}_j\}_{j=1}^m \subset X$ are linearly independent.

The conditions for the perturbation of the spectrum and the preservation of the strong stability of the C_0-semigroup presented in this paper are given in terms of the elements $\{b_j\}_j$ of the perturbation B belonging to a space $\mathcal{D}_\beta$ for some $\beta \geq 0$ and the norms $\|b_j\|_\beta$ being less than a certain bound $\delta > 0$. Similar conditions are also imposed on the elements $\{g_j\}_j$ or alternatively — in the case of a bounded perturbation B — on the elements $\{\tilde{b}_j\}_j$ and $\{\tilde{g}_j\}_j$.

3. Results on The Perturbation of The Spectrum

In this Section we consider the perturbation of the spectrum of the operator A. Since our main concern in this paper are the conditions for the preservation of strong stability of the semigroup, we are only interested in the property that the spectrum of the perturbed operator is contained in the open left half-plane of $\mathbb{C}$. The following Theorem is the main result concerning the perturbation of the spectrum.

Theorem 1. *Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$. For every $m \in \mathbb{N}$ and $\alpha_{max} \geq \alpha + 1$ there exists a constant $\delta > 0$ such that if $\beta, \gamma \geq 0$ are such that $\alpha + 1 \leq \beta + \gamma \leq \alpha_{max}$ and if B is A-bounded and of rank m with $\{b_j\}_j \subset \mathcal{D}_\beta$ and $\{g_j\}_j \subset \mathcal{D}_\gamma^*$, then*

$$\sigma(A + B) \subset \mathbb{C}^-$$

whenever $\|b_j\|_\beta \cdot \|g_k\|_{*,\gamma} < \delta$ for all $j, k \in \{1, \ldots, m\}$.

Our second result on the perturbation of the spectrum concerns the case where B is a bounded operator. It makes use of the alternative representation (4).

Theorem 2. *Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$. For every $m \in \mathbb{N}$ and $\alpha_{max} \geq \alpha$ there exists a constant $\delta > 0$ such that if $\beta, \gamma \geq 0$ are such that $\alpha \leq \beta + \gamma \leq \alpha_{max}$ and if $B \in \mathcal{L}(X)$ is of rank m with $\{\tilde{b}_j\}_j \subset \mathcal{D}_\beta$ and $\{\tilde{g}_j\}_j \subset \mathcal{D}_\gamma^*$, then*

$$\sigma(A + B) \subset \mathbb{C}^-$$

whenever $\|\tilde{b}_j\|_\beta \cdot \|\tilde{g}_k\|_{*,\gamma} < \delta$ for all $j, k \in \{1, \ldots, m\}$.

Before proving these results we make a quick remark concerning the bounds for the norms in the previous Theorems. If $\beta, \gamma \geq 0$, then due to the properties of the Riesz basis we have that for any $x \in \mathcal{D}_\beta$ and $y \in \mathcal{D}_\gamma^*$

$$\begin{aligned}
\|x\|_\beta^2 &= \sum_{k\in\mathbb{Z}} |\lambda_k|^{2\beta} |\langle x, \psi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle (-A)^\beta x, \psi_k\rangle|^2 \\
&\leq \frac{1}{m_\sigma} \|(-A)^\beta x\|_X^2 \\
\|y\|_{*,\gamma}^2 &= \sum_{k\in\mathbb{Z}} |\lambda_k|^{2\gamma} |\langle y, \phi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle (-A^*)^\gamma y, \phi_k\rangle|^2 \\
&\leq M_\sigma \|(-A^*)^\gamma y\|_X^2.
\end{aligned}$$

This implies that bounds in Theorems 1 and 2 can alternatively be given for the norms $\|(-A)^\beta x\|_X$ and $\|(-A^*)^\gamma y\|_X$. This is particularly useful when the constants β and γ can be chosen to be natural numbers. We make use of this fact in the example presented in Section 5.

3.1. The Weinstein-Aronszajn Determinant

In order to prove Theorems 1 and 2 we recall the definition of the Weinstein-Aronszajn determinant [18]. This is a complex-valued function describing the change of the spectrum of an operator under finite rank perturbations.

If B is A-bounded and of finite rank, then $BA^{-1} \in \mathcal{L}(X)$ is of form (3) and for all $\lambda \in \rho(A)$ and $x \in X$ we have

$$\begin{aligned}
BR(\lambda, A)x = BA^{-1}AR(\lambda, A)x &= \sum_{j=1}^m \langle AR(\lambda, A)x, g_j\rangle b_j \\
&= \sum_{j=1}^m \langle x, (AR(\lambda, A))^* g_j\rangle b_j.
\end{aligned}$$

Now $Z = \mathcal{R}(BR(\lambda, A)) = \operatorname{span}\{b_j\}_{j=1}^m$ is a finite-dimensional invariant subspace of the operator $BR(\lambda, A)$. The

Weinstein-Aronszajn determinant of the first kind [18, Sec IV.6] is given by

$$\begin{aligned}\omega(\lambda; A, B) &= \det(I_Z - BR(\lambda, A)|_Z)\\ &= \det\left(\delta_{jk} - \langle b_j, (AR(\lambda, A))^* g_k\rangle\right)_{jk}\\ &= \det\left(\delta_{jk} - \langle AR(\lambda, A)b_j, g_k\rangle\right)_{jk}.\end{aligned}$$

The following Lemma, which is based on the First Weinstein-Aronszajn Formula [18, Thm IV.6.2], is essential in the proof of Theorems 1 and 2. The First Weinstein-Aronszajn Formula states that the Weinstein-Aronszajn determinant can be used to determine the change of the spectrum of a closed operator under a relatively bounded finite rank perturbation. The following less general formulation of the formula is sufficient for our purposes.

Lemma 3. *Let B be A-bounded and of finite rank and let $\omega(\lambda; A, B)$ be the associated Weinstein-Aronszajn determinant of the first kind. If $\Delta \subset \rho(A)$ is a domain of $\mathbb{C}$, then for all $\lambda \in \Delta$*

$$\lambda \in \sigma(A + B) \qquad \Leftrightarrow \qquad \omega(\lambda; A, B) = 0.$$

Proof. This is a direct consequence of [18, Thm IV.6.2]. □

We will use Lemma 3 to prove Theorems 1 and 2. To this end we will construct a domain $\Delta \subset \mathbb{C}$ such that it contains the closed right half-plane of $\mathbb{C}$ and the distance of the spectrum of A from the domain Δ satisfies certain bounds. This is done in Section 3.2. The results of this geometrical consideration are then used together with Lemma 3 to prove Theorems 1 and 2 in Section 3.3.

3.2. Construction of The Domain Δ

In this Section we construct a domain Δ of $\mathbb{C}$ satisfying $\overline{\mathbb{C}^+} \subset \Delta \subset \rho(A)$ such that there exists a constant $a_\sigma > 0$ for which $\operatorname{dist}(\lambda_k, \Delta) \geq a_\sigma |\lambda_k|^{-\alpha}$ for all $k \in \mathbb{Z}$. The main result of the Section is presented in Theorem 6.

Choose $\delta_\sigma > 0$ such that $\operatorname{Re}\lambda_k \leq -\delta_\sigma$ for all $k \in \mathbb{Z}$ for which $|\operatorname{Im}\lambda_k| < y_\sigma$. This is possible since the set $\{\lambda_k\}$ has no accumulation points on $i\mathbb{R}$. For $0 < \kappa < 1$ denote by $\mathcal{C}$ and $\mathcal{C}_\kappa$ the paths

$$\begin{aligned}\mathcal{C} &= \left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda = -c_\sigma (\operatorname{Im}\lambda)^{-\alpha},\ \operatorname{Im}\lambda > 0 \right\},\\ \mathcal{C}_\kappa &= \left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda = -\kappa c_\sigma (\operatorname{Im}\lambda)^{-\alpha},\ \operatorname{Im}\lambda > 0 \right\}.\end{aligned}$$

Also, denote by $\mathcal{C}^*$ and $\mathcal{C}_\kappa^*$ their mirror images with respect to the real axis. Define the domain $\Delta \subset \mathbb{C}$ and closed sets Ω_i for $i = 1, 2, 3$ as shown in Figure 1. The boundary of the domain Δ consists of the curves $\mathcal{C}_\kappa$, $\mathcal{C}_\kappa^*$ and the line segment $\left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda = -\kappa\delta_\sigma,\ |\operatorname{Im}\lambda| \leq \delta_\sigma^\alpha c_\sigma^{-\alpha} \right\}$.

Denote $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Our aim is to show that there exists a constant $a_\sigma > 0$ such that

$$\operatorname{dist}(\lambda, \Delta) \geq a_\sigma |\lambda|^{-\alpha}, \qquad \forall \lambda \in \Omega.$$

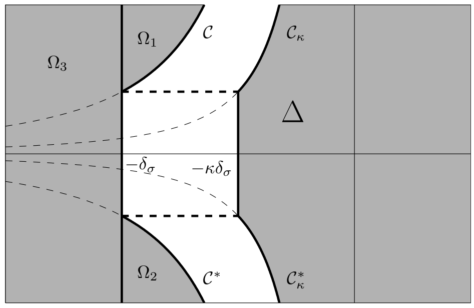

Figure 1: The domain Δ and the closed sets Ω_i.

Since clearly $\{\lambda_k\}_{k \in \mathbb{Z}} \subset \Omega$, we then obtain the desired bound for the distance of the eigenvalues of A from the domain Δ.

We first present the following Lemma concerning the distance of the points of the curve $\mathcal{C}$ from the curve $\mathcal{C}_\kappa$.

Lemma 4. *For all $y_0 > 0$ there exists a constant $a_0 > 0$ such that for all points $(x, y) \in \mathcal{C}$ with $y \geq y_0$ we have* $\operatorname{dist}((x, y), \mathcal{C}_\kappa) \geq a_0 y^{-\alpha}$.

Proof. Let $(x, y) \in \mathcal{C}$ with $y \geq y_0 > 0$. Denote by $(x_1, y_1) = (x, \kappa^{\frac{1}{\alpha}} y)$ and $(x_2, y_2) = (\kappa x, y)$ the points on the curve $\mathcal{C}_\kappa$ (see Figure 2).

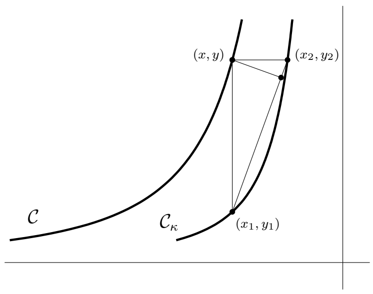

Figure 2: The distance between the curves $\mathcal{C}$ and $\mathcal{C}_\kappa$.

Since the tangent of curve $\mathcal{C}_\kappa$ has positive slope for all $x < 0$, the minimal distance from $\mathcal{C}_\kappa$ to (x, y) must be from a point between (x_1, y_1) and (x_2, y_2) on $\mathcal{C}_\kappa$. Because of this, the distance of (x, y) from $\mathcal{C}_\kappa$ must be larger than the distance of (x, y) from the line between (x_1, y_1) and (x_2, y_2).

Choose a constant $b > 0$ such that

$$b = \frac{c_\sigma^2 (1 - \kappa)^2}{(1 - \kappa^{1/\alpha})^2 y_0^{2(\alpha+1)}}.$$

Since $y \geq y_0$, we then have

$$\begin{aligned}& b \geq \frac{c_\sigma^2 (1 - \kappa)^2}{(1 - \kappa^{1/\alpha})^2 y^{2(\alpha+1)}}\\ \Rightarrow\quad & b(1 - \kappa^{1/\alpha})^2 y^2 \geq (1 - \kappa)^2 c_\sigma^2 y^{-2\alpha}.\end{aligned}$$

For the distance $|r|$ of the line between (x_1, y_1) and (x_2, y_2)

we now have

$$
\begin{aligned}
|r| &= \frac{|(x_2 - x_1)(y_1 - y) - (x_1 - x)(y_2 - y_1)|}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \\
&= \frac{(1-\kappa)c_\sigma y^{-\alpha} \cdot (1 - \kappa^{1/\alpha})y}{\sqrt{(1-\kappa)^2 c_\sigma^2 y^{-2\alpha} + (1 - \kappa^{1/\alpha})^2 y^2}} \\
&\geq \frac{(1-\kappa)c_\sigma (1 - \kappa^{1/\alpha}) y \cdot y^{-\alpha}}{\sqrt{b(1 - \kappa^{1/\alpha})^2 y^2 + (1 - \kappa^{1/\alpha})^2 y^2}} = \frac{c_\sigma (1-\kappa)}{\sqrt{1+b}} y^{-\alpha}.
\end{aligned}
$$

Thus if we choose $a_0 = \frac{c_\sigma(1-\kappa)}{\sqrt{1+b}}$ we have for all $(x,y) \in \mathcal{C}$ with $y \geq y_0$ that $\operatorname{dist}((x,y), \mathcal{C}_\kappa) \geq |r| \geq a_0 y^{-\alpha}$. $\square$

The next Lemma shows that we have an estimate of the desired type for the sets Ω_1 and Ω_2.

Lemma 5. *Let the constant $a_0 > 0$ be as in Lemma 4 with $y_0 = c_\sigma^{\frac{1}{\alpha}} \delta_\sigma^{-\frac{1}{\alpha}}$. Then for all $\lambda \in \Omega_1 \cup \Omega_2$ we have $\operatorname{dist}(\lambda, \Delta) \geq a_0 |\lambda|^{-\alpha}$.*

Proof. Assume that $\lambda \in \Omega_1$ and define points $\mu_1 = (\operatorname{Re}\lambda, (-\frac{1}{\kappa c_\sigma} \operatorname{Re}\lambda)^{-1/\alpha})$, $\mu_2 = (-\kappa c_\sigma (\operatorname{Im}\lambda)^{-\alpha}, \operatorname{Im}\lambda)$ and $\mu_b = (-\delta_\sigma, c_\sigma^{\frac{1}{\alpha}} \delta_\sigma^{-\frac{1}{\alpha}})$. Let μ_0 be the point on the curve $\mathcal{C}_\kappa$ with the minimum distance from the point λ and let λ_0 be the intersection of the line between λ and μ_0 and the curve $\mathcal{C}$ (see Figure 3).

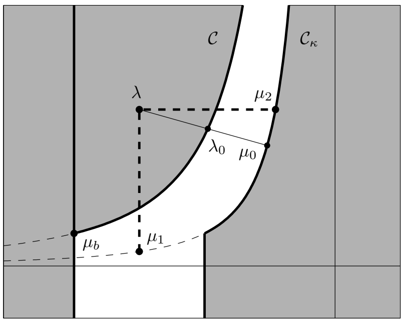

Figure 3: Points λ, λ_0 and μ_0.

Since the tangent of the curve $\mathcal{C}_\kappa$ has positive slope for all $x < 0$, the point μ_0 must be between points μ_1 and μ_2 on $\mathcal{C}_\kappa$. Because of this, we have $\operatorname{Im}\mu_b \leq \operatorname{Im}\lambda_0 \leq \operatorname{Im}\lambda$.

Clearly the distance of λ_0 from Δ is greater or equal to the distance of λ_0 from the curve $\mathcal{C}_\kappa$. Using Lemma 4 we see that for our choice of the constant a_0 we have $\operatorname{dist}(\lambda_0, \mathcal{C}_\kappa) \geq a_0 (\operatorname{Im}\lambda_0)^{-\alpha}$, since $\operatorname{Im}\lambda_0 \geq \operatorname{Im}\mu_b$.

Combining the previous estimates we see that

$$
\begin{aligned}
\operatorname{dist}(\lambda, \Delta) &\geq \operatorname{dist}(\lambda_0, \Delta) \geq \operatorname{dist}(\lambda_0, \mathcal{C}_\kappa) \geq a_0 (\operatorname{Im}\lambda_0)^{-\alpha} \\
&\geq a_0 (\operatorname{Im}\lambda)^{-\alpha} \geq a_0 |\lambda|^{-\alpha}.
\end{aligned}
$$

From symmetry we see that the same bound also holds for $\lambda \in \Omega_2$ and thus we have $\operatorname{dist}(\lambda, \Delta) \geq a_0 |\lambda|^{-\alpha}$ for all $\lambda \in \Omega_1 \cup \Omega_2$. $\square$

Theorem 6 finally concludes that we have the desired estimate for the distance of the eigenvalues $\{\lambda_k\}_k$ of A from the domain Δ.

Theorem 6. *There exists a constant $a_\sigma > 0$ such that $\operatorname{dist}(\lambda_k, \Delta) \geq a_\sigma |\lambda_k|^{-\alpha}$ for all $k \in \mathbb{Z}$.*

Proof. For any $\lambda \in \Omega_3$ we have $\operatorname{dist}(\lambda, \Delta) \geq (1-\kappa)\delta_\sigma$ and $|\lambda|^{-\alpha} \leq \delta_\sigma^{-\alpha}$ since $|\lambda| \geq \delta_\sigma$. Using these estimates we have

$$
\begin{aligned}
\operatorname{dist}(\lambda, \Delta) &\geq (1-\kappa)\delta_\sigma = (1-\kappa)\delta_\sigma^{1+\alpha}\delta_\sigma^{-\alpha} \\
&\geq (1-\kappa)\delta_\sigma^{1+\alpha} |\lambda|^{-\alpha}.
\end{aligned}
$$

Since $\{\lambda_k\}_{k \in \mathbb{Z}} \subset \Omega$, combining this estimate with Lemma 5 completes the proof. $\square$

3.3. *The Proofs of Theorems 1 and 2*

Proof of Theorem 1. Lemma 3 implies that it is sufficient to show that for any $m \in \mathbb{N}$ and $\alpha_{\max} \geq \alpha + 1$ we can find $\delta > 0$ such that $|\omega(\lambda; A, B)| > 0$ for all $\lambda \in \Delta$ whenever $\alpha + 1 \leq \beta + \gamma \leq \alpha_{\max}$ and B is A-bounded and of rank m with $\{b_j\}_j \subset \mathcal{D}_\beta$ and $\{g_j\}_j \subset \mathcal{D}_\gamma^*$ such that we have $\|b_j\|_\beta \cdot \|g_k\|_{*,\gamma} < \delta$ for all $j, k \in \{1, \ldots, m\}$.

Let $m \in \mathbb{N}$ and $\alpha_{\max} \geq \alpha + 1$. For $\lambda \in \Delta \subset \rho(A)$ the Weinstein-Aronszajn determinant $\omega(\lambda; A, B)$ is given by

$$
\omega(\lambda; A, B) = \det\left(\delta_{jk} - \langle AR(\lambda, A)b_j, g_k \rangle\right)_{jk} = 1 + h(\lambda),
$$

where $h(\lambda)$ is a linear combination of products of terms $\langle AR(\lambda, A)b_j, g_k \rangle$. Because of this and because

$$
\sup_{\lambda \in \Delta} |h(\lambda)| < 1 \qquad \Rightarrow \qquad \sup_{\lambda \in \Delta} |\omega(\lambda; A, B)| > 0,
$$

it is sufficient to show that for all $j, k \in \{1, \ldots, m\}$ and for any $\varepsilon > 0$ we can find $\delta' > 0$ such that $\sup_{\lambda \in \Delta} |\langle AR(\lambda, A)b_j, g_k \rangle| < \varepsilon$ if $\alpha + 1 \leq \beta + \gamma \leq \alpha_{\max}$, $b_j \in \mathcal{D}_\beta$, $g_k \in \mathcal{D}_\gamma^*$ and $\|b_j\|_\beta \cdot \|g_k\|_{*,\gamma} < \delta'$.

Let $\varepsilon > 0$ be arbitrary. Choose $0 < r_\sigma \leq 1$ such that $|\lambda_k| \geq r_\sigma$ for all $k \in \mathbb{Z}$ and

$$
\delta' = \frac{\varepsilon \cdot a_\sigma}{r_\sigma^{\alpha + 1 - \alpha_{\max}}} > 0,
$$

where $a_\sigma > 0$ is as in Theorem 6.

Let $\beta, \gamma \geq 0$ be such that $\alpha + 1 \leq \beta + \gamma \leq \alpha_{\max}$ and let $b \in \mathcal{D}_\beta$ and $g \in \mathcal{D}_\gamma^*$ be such that $\|b\|_\beta \cdot \|g\|_{*,\gamma} < \delta'$. For brevity denote $b^k = \langle b, \psi_k \rangle$ and $g^k = \langle \phi_k, g \rangle$ for $k \in \mathbb{Z}$. Now since for any $\lambda \in \mathbb{C}$ with $|\lambda| \geq r_\sigma$

$$
\begin{aligned}
|\lambda|^{\alpha+1} &= |\lambda|^{\alpha+1-\beta-\gamma} |\lambda|^{\beta+\gamma} \leq r_\sigma^{\alpha+1-\beta-\gamma} |\lambda|^{\beta+\gamma} \\
&\leq r_\sigma^{\alpha+1-\alpha_{\max}} |\lambda|^{\beta+\gamma},
\end{aligned}
$$

we have using Theorem 6 that

$$\begin{aligned}
&\sup_{\lambda\in\Delta} |\langle AR(\lambda,A)b,g\rangle| \le \sum_{k\in\mathbb{Z}} \sup_{\lambda\in\Delta} \frac{|\lambda_k||b^k||g^k|}{|\lambda-\lambda_k|} \\
&= \sum_{k\in\mathbb{Z}} \frac{|\lambda_k||b^k||g^k|}{\operatorname{dist}(\lambda_k,\Delta)} \le \sum_{k\in\mathbb{Z}} \frac{1}{a_\sigma}|\lambda_k|^{\alpha+1}|b^k||g^k| \\
&\le \frac{r_\sigma^{\alpha+1-\alpha_{\max}}}{a_\sigma} \sum_{k\in\mathbb{Z}} |\lambda_k|^{\beta+\gamma}|b^k||g^k| \\
&\le \frac{r_\sigma^{\alpha+1-\alpha_{\max}}}{a_\sigma} \left(\sum_{k\in\mathbb{Z}}|\lambda_k|^{2\beta}|b^k|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}}|\lambda_k|^{2\gamma}|g^k|^2\right)^{\frac{1}{2}} \\
&= \frac{r_\sigma^{\alpha+1-\alpha_{\max}}}{a_\sigma} \|b\|_\beta \cdot \|g\|_{*,\gamma} < \varepsilon
\end{aligned}$$

This concludes the proof of the Theorem. □

The proof of Theorem 2 is similar to the proof of Theorem 1. The main difference is that since the perturbation B is a bounded operator the Weinstein-Aronszajn determinant can be given using the representation (4).

Proof of Theorem 2. If B is bounded and of rank m, for any $\lambda \in \Delta \subset \rho(A)$ the Weinstein-Aronszajn determinant is given by [18, Sec IV.6]

$$\omega(\lambda; A, B) = \det\left(\delta_{jk} - \langle R(\lambda,A)\tilde{b}_j, \tilde{g}_k\rangle\right)_{jk}$$

and similarly as in the proof of Theorem 1 we have $\omega(\lambda; A, B) = 1 + h(\lambda)$, where $h(\lambda)$ is a linear combination of products of terms $\langle R(\lambda,A)\tilde{b}_j, \tilde{g}_k\rangle$. Using Lemma 3 and following the reasoning of the proof of Theorem 1 we see that it suffices to show that for all $j,k \in \{1,\dots,m\}$ and for any $\varepsilon > 0$ we can find $\delta' > 0$ such that $\sup_{\lambda\in\Delta}|\langle R(\lambda,A)\tilde{b}_j, \tilde{g}_k\rangle| < \varepsilon$ if $\alpha \le \beta+\gamma \le \alpha_{\max}$, $\tilde{b}_j \in \mathcal{D}_\beta$, $\tilde{g}_k \in \mathcal{D}_\gamma^*$ and $\|\tilde{b}_j\|_\beta \cdot \|\tilde{g}_k\|_{*,\gamma} < \delta'$.

Let $\varepsilon > 0$ be arbitrary. Again choose $0 < r_\sigma \le 1$ such that $|\lambda_k| \ge r_\sigma$ for all $k \in \mathbb{Z}$, let $a_\sigma > 0$ be as in Theorem 6 and choose

$$\delta' = \frac{\varepsilon \cdot a_\sigma}{r_\sigma^{\alpha-\alpha_{\max}}} > 0.$$

Let $\beta,\gamma \ge 0$ be such that $\alpha \le \beta+\gamma \le \alpha_{\max}$ and let $\tilde{b} \in \mathcal{D}_\beta$ and $\tilde{g} \in \mathcal{D}_\gamma^*$ be such that $\|\tilde{b}\|_\beta \cdot \|\tilde{g}\|_{*,\gamma} < \delta'$. For brevity denote $b^k = \langle \tilde{b}, \psi_k\rangle$ and $g^k = \langle \phi_k, \tilde{g}\rangle$ for $k \in \mathbb{Z}$. Since for any $\lambda \in \mathbb{C}$ with $|\lambda| \ge r_\sigma$

$$|\lambda|^\alpha = |\lambda|^{\alpha-\beta-\gamma}|\lambda|^{\beta+\gamma} \le r_\sigma^{\alpha-\beta-\gamma}|\lambda|^{\beta+\gamma} \le r_\sigma^{\alpha-\alpha_{\max}}|\lambda|^{\beta+\gamma},$$

similarly as in the proof of Theorem 1 we have using Theorem 6 that

$$\begin{aligned}
&\sup_{\lambda\in\Delta} |\langle R(\lambda,A)\tilde{b},\tilde{g}\rangle| \le \sum_{k\in\mathbb{Z}} \frac{|b^k||g^k|}{\operatorname{dist}(\lambda_k,\Delta)} \\
&\le \sum_{k\in\mathbb{Z}} \frac{1}{a_\sigma}|\lambda_k|^{\alpha}|b^k||g^k| \le \frac{r_\sigma^{\alpha-\alpha_{\max}}}{a_\sigma} \sum_{k\in\mathbb{Z}} |\lambda_k|^{\beta+\gamma}|b^k||g^k| \\
&\le \frac{r_\sigma^{\alpha-\alpha_{\max}}}{a_\sigma} \|b\|_\beta \cdot \|g\|_{*,\gamma} < \varepsilon.
\end{aligned}$$

This concludes the proof of the Theorem. □

Remark 7. If $\sigma(A)$ consists only of isolated eigenvalues with finite multiplicities, the proof of Theorem 1 can be greatly simplified. In this case the construction of the domain Δ is unnecessary. This follows from a more general form of the first Weinstein-Aronszajn formula [18, Thm IV.6.2], which under these assumptions holds in the whole of $\mathbb{C}$. As in the previous proof we could in this case determine bounds for $\|b_j\|_\beta$ and $\|g_k\|_{*,\gamma}$ such that for some constant $c > 0$ we have $\omega(\lambda; A, B) \ge c$ for all $\lambda \in \mathbb{C}^+$. The First Weinstein-Aronszajn Formula would then imply that $\mathbb{C}^+ \subset \rho(A+B)$ and further, since the Weinstein-Aronszajn determinant is continuous on $\rho(A)$, that $i\mathbb{R} \subset \rho(A+B)$. This would conclude that $\sigma(A+B) \subset \mathbb{C}^-$.

Remark 8. From the proofs of Theorems 1 and 2 it is easy to see that if $|\lambda_k| \ge 1$ for all $k \in \mathbb{Z}$ we can then choose $r_\sigma = 1$ and the bound $\delta > 0$ is independent of the value of $\alpha_{\max}$.

4. Results on The Perturbation of Strong and Polynomial Stability

In this Section we present conditions for the preservation of the strong and polynomial stabilities of the C_0-semigroup. The following Theorem concerning the preservation of the strong stability under a bounded finite rank perturbation B is the first main result of this Section. The proof of this result uses a result by Casarino and Piazzera [19] concerning the preservation of uniform boundedness of a C_0-semigroup.

Theorem 9. *Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$. The following hold.*

(a) *For every $m \in \mathbb{N}$ there exists $\delta > 0$ such that if $B \in \mathcal{L}(X)$ is of rank m with $\{\tilde{b}_j\}_j \subset \mathcal{D}_\alpha$ such that $\|\tilde{b}_j\|_\alpha \cdot \|\tilde{g}_k\|_X < \delta$ for all $j,k \in \{1,\dots,m\}$, then the semigroup generated by $A+B$ is strongly stable;*

(b) *For every $m \in \mathbb{N}$ there exists $\delta > 0$ such that if $B \in \mathcal{L}(X)$ is of rank m with $\{\tilde{g}_j\}_j \subset \mathcal{D}_\alpha^*$ such that $\|\tilde{b}_j\|_X \cdot \|\tilde{g}_k\|_{*,\alpha} < \delta$ for all $j,k \in \{1,\dots,m\}$, then the semigroup generated by $A+B$ is strongly stable.*

The proof of the Theorem is based on the following Lemma.

Lemma 10. *Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$. For every $m \in \mathbb{N}$ there exists $\delta > 0$ such that if $B \in \mathcal{L}(X)$ is of rank m with $\{\tilde{g}_j\}_j \subset \mathcal{D}_\alpha^*$ such that $\|\tilde{b}_j\|_X \cdot \|\tilde{g}_j\|_{*,\alpha} < \delta$ for all $j \in \{1,\dots,m\}$, then the C_0-semigroup generated by $A+B$ is uniformly bounded.*

Proof. Let $m \in \mathbb{N}$ and choose $\delta = \frac{a_\sigma\sqrt{m_\sigma}}{2m}$ where $a_\sigma > 0$ is as in Theorem 6. Assume $B \in \mathcal{L}(X)$ is of rank m with

$\{\tilde{b}_j\}_j \subset X$ and $\{\tilde{g}_j\}_j \subset \mathcal{D}_\alpha^*$ such that $\|\tilde{b}_j\|_X \cdot \|\tilde{g}_j\|_{*,\alpha} < \delta$ for all $j \in \{1, \dots, m\}$. We will show that

$$\int_0^\infty \|BT(t)x\| dt \leq \frac{1}{2}\|x\|, \qquad \forall x \in X.$$

We then have from [19, Cor. 2.2] that the C_0-semigroup generated by $A + B$ is uniformly bounded. Let $x \in X$ and for brevity denote $x^k = \langle x, \psi_k \rangle$ and $g_j^k = \langle \phi_k, \tilde{g}_j \rangle$ for $k \in \mathbb{Z}$. Now

$$\begin{aligned}\int_0^\infty |\langle T(t)x, \tilde{g}_j \rangle| dt &\leq \int_0^\infty \sum_{k\in\mathbb{Z}} |e^{\lambda_k t} x^k g_j^k| dt \\ &= \int_0^\infty \sum_{k\in\mathbb{Z}} e^{\operatorname{Re}\lambda_k t} |x^k||g_j^k| dt\end{aligned}$$

For all $k \in \mathbb{Z}$ we have $|\operatorname{Re}\lambda_k| \geq \operatorname{dist}(\lambda_k, \Delta) \geq a_\sigma |\lambda_k|^{-\alpha}$ and thus

$$\int_0^\infty e^{\operatorname{Re}\lambda_k t} dt = \left[\frac{e^{\operatorname{Re}\lambda_k t}}{\operatorname{Re}\lambda_k}\right]_{t=0}^\infty = \frac{1}{|\operatorname{Re}\lambda_k|} \leq \frac{|\lambda_k|^\alpha}{a_\sigma}.$$

Since

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \int_0^\infty e^{\operatorname{Re}\lambda_k t} dt |x^k||g_j^k| \leq \frac{1}{a_\sigma} \sum_{k\in\mathbb{Z}} |\lambda_k|^\alpha |x^k||g_j^k| \\ \leq\ & \frac{1}{a_\sigma} \left(\sum_{k\in\mathbb{Z}} |x^k|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^{2\alpha} |g_j^k|^2\right)^{\frac{1}{2}} \\ =\ & \frac{1}{a_\sigma\sqrt{m_\sigma}} \|x\| \cdot \|\tilde{g}_j\|_{*,\alpha},\end{aligned}$$

we can conclude using the Dominated Convergence Theorem that

$$\begin{aligned}&\int_0^\infty |\langle T(t)x, \tilde{g}_j \rangle| dt \leq \int_0^\infty \sum_{k\in\mathbb{Z}} e^{\operatorname{Re}\lambda_k t} |x^k||g^k| dt \\ =\ & \sum_{k\in\mathbb{Z}} \int_0^\infty e^{\operatorname{Re}\lambda_k t} dt |x^k||g^k| \leq \frac{1}{a_\sigma\sqrt{m_\sigma}} \|x\| \cdot \|\tilde{g}_j\|_{*,\alpha}.\end{aligned}$$

Using this we get

$$\begin{aligned}\int_0^\infty \|BT(t)x\| dt &\leq \int_0^\infty \sum_{j=1}^m |\langle T(t)x, \tilde{g}_j \rangle| \|b_j\| dt \\ &= \sum_{j=1}^m \|\tilde{b}_j\| \int_0^\infty |\langle T(t)x, \tilde{g}_j \rangle| dt \\ &\leq \sum_{j=1}^m \frac{\|\tilde{b}_j\| \cdot \|\tilde{g}_j\|_{*,\alpha}}{a_\sigma\sqrt{m_\sigma}} \|x\| < \frac{1}{2}\|x\|.\end{aligned}$$

This concludes the proof of the Lemma. □

Proof of Theorem 9. The ABLV Theorem [7] states that if $\sigma(A+B) \subset \mathbb{C}^-$ and if $A+B$ generates a uniformly bounded C_0-semigroup, then this C_0-semigroup in strongly stable. Because of this the part (b) of Theorem 9 follows from Theorem 2 and Lemma 10.

If $B \in \mathcal{L}(X)$ is of form (4), then its adjoint is given by

$$B^* = \sum_{j=1}^m \langle \cdot, \tilde{b}_j \rangle \tilde{g}_j.$$

Clearly the operator A^* satisfies the geometric assumption for some $\alpha > 0$ whenever A does. Because of this and because an operator on a Hilbert space generates a uniformly bounded C_0-semigroup if and only if its adjoint does, the sufficiency of the part (a) of Theorem 9 follows from Theorem 2 and Lemma 10 applied to the operators A^* and B^*. □

The second main result of the Section concerns the preservation of the polynomial stability of the semigroup. It is given in the following Theorem.

Theorem 11. *Assume that the conditions of Theorem 1 or Theorem 2 are satisfied and that there exists an isomorphism $T \in \mathcal{L}(X)$ such that*

$$T(A+B)T^{-1} = \begin{bmatrix} A_e & \\ & A_n \end{bmatrix}$$

where A_e generates an exponentially stable semigroup and A_n is a normal operator. The perturbed semigroup $S(t)$ is then polynomially stable and there exists a constant $C_\sigma > 0$ such that

$$\|S(t)(A+B)^{-1}\| \leq \frac{C_\sigma}{t^{1/\alpha}}$$

for all $t \geq 0$.

Proof. Denote by $T_e(t)$ and $T_n(t)$ the semigroups generated by A_e and A_n, respectively. The proofs of Theorems 1 and 2 guarantee that $\sigma(A_n) \subset \sigma(A+B) \subset \mathbb{C} \setminus \Delta$. The construction of Δ then implies that the spectrum of A_n satisfies Assumption 1 for the same $\alpha > 0$ as the unperturbed operator and for some constants $c_\sigma, y_\sigma > 0$. We thus have from [4, Prop. 4.1] that $T_n(t)$ is polynomially stable and that there exists a constant $C_n > 0$ such that for all $t > 0$

$$\|T_n(t)A_n^{-1}\| \leq \frac{C_n}{t^{1/\alpha}}.$$

Using this we can see that there exists a constant $C_\sigma > 0$ such that for all $t > 0$

$$\begin{aligned}&\|S(t)(A+B)^{-1}\| = \left\| T \begin{bmatrix} T_e(t)A_e^{-1} & \\ & T_n(t)A_n^{-1} \end{bmatrix} T^{-1} \right\| \\ \leq\ & \|T\| \cdot \|T^{-1}\| \cdot \max\left\{ \|T_e(t)\| \cdot \|A_e^{-1}\|, \frac{C_n}{t^{1/\alpha}} \right\} \leq \frac{C_\sigma}{t^{1/\alpha}}.\end{aligned}$$

□

Remark 12. The conclusion of Theorem 11 of course also holds if $T(A+B)T^{-1} = A_n$, i.e. if the eigenvectors of $A+B$ form a Riesz basis of X.

5. Perturbation of a Polynomially Stabilized Wave Equation

In this Section we consider an example of application of the perturbation results presented earlier. To this end we consider a one-dimensional wave equation on the interval $(0,1)$. We first use one-dimensional state feedback to stabilize this equation polynomially in such a way that the system operator of the stabilized equation is a Riesz-spectral operator whose spectrum satisfies the geometric assumption. We then consider perturbations of this stabilized equation. We will complete the stabilization part of the example in such a way that it is ultimately possible to compute concrete bounds $\delta > 0$ for the perturbations such that the spectrum of the perturbed equation remains in the open left half-plane of $\mathbb{C}$ and such that the perturbed equation is strongly and polynomially stable.

Consider the following one-dimensional wave equation on $(0,1)$.

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)u(t) \tag{5a}$$

$$w(0,t) = w(1,t) = 0 \tag{5b}$$

$$w(z,0) = w_0, \quad \frac{\partial w}{\partial t}(z,0) = w_1, \tag{5c}$$

where $b_0(z) = \sqrt{3}(1-z)$. It is well-known that this equation can be written as a first order linear system on a Hilbert space in the following way. Define the operator $A_0 : \mathcal{D}(A_0) \subset L^2(0,1) \to L^2(0,1)$ to be $A_0 = -\frac{d^2}{dz^2}$ with domain

$$\mathcal{D}(A_0) = \big\{x \in L^2(0,1) \;\big|\; x, x' \text{ abs. cont. } x'' \in L^2(0,1),\; x(0) = x(1) = 0\big\}.$$

The operator A_0 has a positive self-adjoint square root $A_0^{1/2}$ and the space $X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$ equipped with an inner product $\langle x, y\rangle_X = \langle A_0^{1/2}x_1, A_0^{1/2}y_1\rangle_{L^2} + \langle x_2, y_2\rangle_{L^2}$ is a Hilbert space. Choosing

$$x = \begin{bmatrix} w \\ \frac{dw}{dt}\end{bmatrix}, \; A = \begin{bmatrix} 0 & I \\ -A_0 & 0\end{bmatrix}, \; \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}),$$

$$Bu = bu = \begin{bmatrix} 0 \\ b_0\end{bmatrix} u, \quad x_0 = \begin{bmatrix} w_0 \\ w_1\end{bmatrix}$$

the wave equation (5) can be written as

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0. \tag{6}$$

The eigenvalues of A are $\lambda_k = ik\pi$ for $k \in \mathbb{Z} \setminus \{0\}$ and the corresponding eigenvectors $\varphi_k(z) = \frac{1}{\lambda_k}\begin{bmatrix} \sin(k\pi z) \\ \lambda_k \sin(k\pi z)\end{bmatrix}$ form an orthonormal basis of X.

The next Theorem gives us a stabilizing feedback law $u = Kx = \langle x, h\rangle$ for the system (6) and establishes the fact that the stabilized system operator $A+BK$ is a Riesz spectral operator. Then, since the spectrum of $A+BK$ satisfies Assumption 1, the stabilized semigroup is polynomially stable. In addition the Theorem gives us the constants $m_\sigma > 0$ and $M_\sigma > 0$ related to the Riesz basis of eigenvectors of $A+BK$. These constants are essential in computing explicit perturbation bounds later in the Section.

The Theorem incorporates a parameter $0 < \nu \leq 1$, which allows us to control how much we shift the spectrum of A from the imaginary axis. We will show later in the Section that this parameter can be used to control the size of the norm $\|K\|$ of the feedback. This is required in our approach of using perturbation theory of normal operators to study the strong and polynomial stabilities of the perturbed semigroup.

Theorem 13. *Choose $K = \langle \cdot, h\rangle \in \mathcal{L}(X, \mathbb{C})$ with*

$$h = -\frac{\nu\pi^2}{\sqrt{3}} \sum_{k \neq 0} \frac{\overline{\alpha_k}}{k}\varphi_k, \quad \textit{where} \quad \alpha_k = \prod_{l \neq 0, k}\left(1 + i\frac{\nu}{l^2(l-k)}\right)$$

and $0 < \nu \leq 1$. Then $A+BK$ is a strongly stable Riesz-spectral operator with $\sigma(A+BK) = \{-\nu\frac{\pi}{k^2} + ik\pi\}_{k\neq 0}$. For $\nu = \frac{1}{10}$ the Riesz basis $\{\phi_k\}_{k\neq 0}$ of eigenfunctions of the operator $A+BK$ satisfies

$$\frac{1}{M_\sigma}\sum_{k\neq 0} |\langle x, \phi_k\rangle|^2 \leq \|x\|_X^2 \leq \frac{1}{m_\sigma}\sum_{k \neq 0}|\langle x, \phi_k\rangle|^2 \tag{7}$$

with $m_\sigma = \frac{3}{5}$ and $M_\sigma = \frac{5}{3}$.

Proof. Let $0 < \nu \leq 1$ and denote $\mu_k = -\nu\frac{\pi}{k^2} + ik\pi$ for $k \neq 0$. Using

$$\langle b, \varphi_k\rangle_X = \langle b_0, \sin(k\pi\cdot)\rangle_{L^2} = \frac{\sqrt{3}}{k\pi}$$

we see that for any $\lambda \in \mathbb{C}$ with $\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{\pi}{3}$ and for any $l \neq 0$

$$\begin{aligned}\sum_{k\neq 0}\frac{|\langle b, \varphi_k\rangle|^2}{|\lambda - \lambda_k|^2} &\leq \frac{3}{\pi^2 \operatorname{dist}(\lambda, \sigma(A))^2}\sum_{k\neq 0}\frac{1}{k^2} \\ &\leq \frac{18}{\pi^4}\cdot\frac{\pi^2}{3} = \frac{9}{\pi^2}\end{aligned} \tag{8a}$$

$$\sum_{\substack{k\neq 0\\ k\neq l}}\frac{|\langle b, \varphi_k\rangle|^2}{|\lambda_k - \lambda_l|^2} \leq \frac{3}{\pi^2}\sum_{\substack{k\neq 0\\ k\neq l}}\frac{1}{k^2\pi^2} = \frac{3}{\pi^4}\cdot\frac{\pi^2}{3} = \frac{1}{\pi^2} \tag{8b}$$

$$\begin{aligned}\sum_{k\neq 0}\frac{|\mu_k - \lambda_k|^2}{|\langle b, \varphi_k\rangle|^2} &= \sum_{k\neq 0}\frac{\nu^2\frac{\pi^2}{k^4}}{\frac{3}{\pi^2 k^2}} = \nu^2\frac{\pi^4}{3}\sum_{k\neq 0}\frac{1}{k^2} \\ &= \nu^2\frac{\pi^4}{3}\cdot\frac{\pi^2}{3} = \nu^2\frac{\pi^6}{3}.\end{aligned} \tag{8c}$$

We thus have from [6, Thm. 1] that there exists a feedback $K = \langle\cdot, h\rangle \in \mathcal{L}(X, \mathbb{C})$ such that $\sigma(A+BK) = \{\mu_k\}_{k\neq 0}$ and that $h \in X$ is given by

$$h = \sum_{k\neq 0} \frac{\overline{\mu_k} - \overline{\lambda_k}}{\langle \varphi_k, b\rangle} \overline{\alpha_k} \varphi_k = \sum_{k\neq 0} \frac{-\nu \frac{\pi}{k^2}}{\frac{\sqrt{3}}{k\pi}} \overline{\alpha_k} = -\frac{\nu\pi^2}{\sqrt{3}} \sum_{k\neq 0} \frac{\overline{\alpha_k}}{k} \varphi_k,$$

$$\alpha_k = \prod_{l\neq 0,k} \frac{\lambda_k - \mu_l}{\lambda_k - \lambda_l} = \prod_{l\neq 0,k} \frac{i(k-l) + \frac{\nu}{l^2}}{i(k-l)}$$
$$= \prod_{l\neq 0,k} \left(1 + i\frac{\nu}{l^2(l-k)}\right).$$

It remains to prove that the eigenfunctions of $A+BK$ form a Riesz basis of X. It is fairly easy to verify that for every $k \neq 0$ we have $\langle R(\mu_k, A)b, h\rangle = 1$ since μ_k is an eigenvalue of $A + \langle \cdot, h\rangle b$ and that $R(\mu_k, A)b$ is an eigenfunction of $A + \langle \cdot, h\rangle b$ corresponding to μ_k. We will show that the set $\{\phi_k\}_{k\neq 0}$ of eigenfunctions of the operator $A + BK$ forms a Riesz basis of X when

$$\phi_k = \frac{\mu_k - \lambda_k}{b^k} R(\mu_k, A)b = \frac{\pi^2}{\sqrt{3}} \frac{-\frac{\nu}{k^2}}{\frac{1}{k}} \sum_{l\neq 0} \frac{\langle b, \varphi_l\rangle}{\mu_k - \lambda_l} \varphi_l$$
$$= -\frac{\nu}{k} \sum_{l\neq 0} \frac{\frac{1}{l}}{-\frac{\nu}{k^2} - i(l-k)} \varphi_l = \sum_{l\neq 0} \frac{\nu k}{\nu l - ik^2 l(k-l)} \varphi_l.$$

Define a mapping $T : X \to X$ so that $\phi_k = T\varphi_k$ for all $k \neq 0$. It is well-known (see for example [16, Exer 2.21]) that the set $\{\phi_k\}_{k\neq 0}$ is a Riesz basis if T is a bounded isomorphism on X. Furthermore, in this case (7) is satisfied for any m_σ and M_σ such that $\|T\|^2 \leq \frac{1}{m_\sigma}$ and $\|T^{-1}\|^2 \leq M_\sigma$. If we denote

$$c_{kl} = \frac{\nu k}{\nu l - ik^2 l(k-l)},$$

then since $c_{kk} = 1$ for $k \neq 0$ we have that for any $x \in X$

$$\|Tx\| = \|\sum_{k\neq 0} \langle x, \varphi_k\rangle T\varphi_k\| = \|\sum_{k\neq 0} \langle x, \varphi_k\rangle (\varphi_k + \sum_{l\neq 0,k} c_{kl}\varphi_l)\|$$
$$\leq \|x\| + \sum_{k\neq 0} |\langle x, \varphi_k\rangle| \cdot \|\sum_{l\neq 0,k} c_{kl}\varphi_l\|$$
$$= \|x\| + \|x\| (\sum_{k\neq 0} \sum_{l\neq 0,k} |c_{kl}|^2)^{\frac{1}{2}}.$$

Because

$$|c_{kl}|^2 = \frac{\nu^2 k^2}{\nu^2 l^2 + k^4 l^2 (k-l)^2},$$

we have $|c_{-k,-l}| = |c_{kl}|$ and a straight-forward estimation gives

$$\sum_{k\neq 0} \sum_{l\neq 0,k} |c_{kl}|^2 = \sum_{k\neq 0} \left(\sum_{l=-\infty}^{-1} |c_{kl}|^2 + \sum_{l=1}^{k-1} |c_{kl}|^2 + \sum_{l=k+1}^{\infty} |c_{kl}|^2 \right)$$
$$\leq 6\nu^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l=1}^{\infty} \frac{1}{l^4} \leq 6\nu^2 \frac{\pi^2}{6} \cdot \frac{\pi^4}{90} = \frac{\pi^6}{90}\nu^2.$$

This concludes that $T \in \mathcal{L}(X)$ and $\|T\| \leq 1 + \frac{\pi^6}{90}\nu^2$. For $\nu = \frac{1}{10}$ we have $\|T\|^2 \leq \left(1 + \frac{\pi^2}{90\cdot 100}\right)^2 < \frac{5}{3}$. Thus we can choose $m_\sigma = \frac{3}{5}$.

It remains to find a bound for $\|T^{-1}\|$. This is equivalent to finding $\gamma > 0$ such that $\|Tx\| \geq \gamma\|x\|$ for all $x \in X$. Since $T \in \mathcal{L}(X)$, this is satisfied for all $x \in X$ if it is satisfied for every x in a dense subset of X. To this end define

$$X_F = \left\{ x \in X \;\middle|\; \exists N \in \mathbb{N} : \langle x, \varphi_k\rangle = 0 \text{ whenever } |k| > N \right\}.$$

This subset is dense in X. Let $x \in X_F$ and let $N \in \mathbb{N}$ be such that $\langle x, \varphi_k\rangle = 0$ whenever $|k| > N$. We then have

$$\|Tx\|^2 = \|\sum_{k\neq 0} \langle x, \varphi_k\rangle T\varphi_k\|^2 = \|\sum_{k\neq 0} \langle x, \varphi_k\rangle \sum_{l\neq 0} c_{kl}\varphi_l\|^2$$
$$= \sum_{l\neq 0} |\sum_{k\neq 0} \langle x, \varphi_k\rangle c_{kl}|^2 = \sum_{l\neq 0} |\sum_{\substack{k=-N \\ k\neq 0}}^{N} \langle x, \varphi_k\rangle c_{kl}|^2$$
$$\geq \sum_{\substack{l=-N \\ l\neq 0}}^{N} |\sum_{\substack{k=-N \\ k\neq 0}}^{N} \langle x, \varphi_k\rangle c_{kl}|^2 = \|T_N((\langle x, \varphi_k\rangle)_k\|_{2N}^2$$
$$\geq (\sigma_s^N)^2 \cdot \|((\langle x, \varphi_k\rangle)_k\|_{2N}^2 = (\sigma_s^N)^2 \cdot \|x\|^2$$

where $\|\cdot\|_{2N}$ is the Euclidean norm in $\mathbb{C}^{2N}$, T_N denotes a $2N \times 2N$ matrix $T_N = (c_{kl})_{kl}$ where $k, l \in \{-N, \dots, N\} \setminus \{0\}$ and σ_s^N denotes the smallest singular value of T_N. This estimate implies that if we choose $\gamma > 0$ such that

$$\gamma \leq \inf_{N\in\mathbb{N}} \sigma_s^N,$$

then $\|Tx\|^2 \geq \gamma^2\|x\|^2$ for all $x \in X_F$ and hence also for all $x \in X$.

For $N \in \mathbb{N}$ let $H_N = \frac{1}{2}(T_N + T_N^*)$ be the Hermitian part of T_N. We now have from [20, Cor 3.1.5] that $\sigma_s^N \geq \lambda_{\min}(H_N)$, where $\lambda_{\min}(H_N)$ denotes the smallest eigenvalue of H_N. Thus it suffices to find a uniform lower bound $\gamma > 0$ for the eigenvalues of matrices H_N.

In order to find this uniform lower bound we will use a generalization of the Gerschgorin Circle Theorem for block matrices [15]. Fix $N \in \mathbb{N}$ and let $n \in \{1, \dots, N\}$. Recall that $h_{kk} = \frac{1}{2}(c_{kk} + \overline{c_{kk}}) = 1$ for $k \neq 0$. We can partition the matrix H_N into

$$\begin{bmatrix} 1 & \cdots & h_{-N,-n} & H_{-N,0} & h_{-N,n} & \cdots & h_{-N,N} \\ \vdots & \ddots & & \vdots & & & \vdots \\ h_{-n-1,-N} & & 1 & H_{-1,0} & & & h_{-n-1,N} \\ H_{0,-N} & \cdots & H_{0,-1} & H_{00}^n & H_{01} & \cdots & H_{0N} \\ h_{n+1,-N} & & & H_{10} & 1 & & h_{n+1,N} \\ \vdots & & & \vdots & & \ddots & \vdots \\ h_{N,-N} & \cdots & h_{N,-n-1} & H_{N0} & h_{N,n+1} & \cdots & 1 \end{bmatrix}.$$

Since the center block $H_{00}^n \in \mathbb{C}^{2n\times 2n}$ is Hermitian and invertible, we have from [15, Thm. 2] that each eigenvalue

λ of H_N satisfies

$$\operatorname{dist}(\lambda, \sigma(H_{00}^n)) \leq \sum_{\substack{k=-N \\ k\neq 0}}^{N} \|H_{0,k}\|_2 \tag{9a}$$

or

$$|\lambda - 1| \leq \|H_{l,0}\|_2 + \sum_{\substack{n<|k|\leq N \\ k\neq l}} |h_{lk}| \tag{9b}$$

for at least one $j \in \{-N, \ldots, -n-1\} \cup \{n+1, \ldots, N\}$.

A direct computation yields that for all $k \neq l$ we have

$$\begin{aligned}
|h_{kl}|^2 &= \frac{\nu^2}{4} \cdot \frac{\nu^2(k^4 + 2k^2l^2 + l^4) + 4k^4l^4(k-l)^2}{(\nu^2 l^2 + k^4 l^2 (k-l)^2)(\nu^2 k^2 + l^4 k^2 (k-l)^2)} \\
&\leq \frac{\nu^2}{4} \cdot \frac{\nu^2(k^4 + 2k^2l^2 + l^4) + 4k^4l^4(k-l)^2}{k^4 l^2 (k-l)^2 l^4 k^2 (l-k)^2} \\
&\leq \frac{\nu^2(\nu^2+1)}{k^2 l^2 (k-l)^2}.
\end{aligned}$$

Using this we can estimate and the Hölder's inequality for $p = 10$ and $q = \frac{p}{p-1} = \frac{10}{9}$ we have

$$\begin{aligned}
&\sum_{\substack{k=-N \\ k\neq 0}}^{N} \|H_{0,k}\|_2 \leq \sum_{|k|>n} \Big(\sum_{\substack{l=-n \\ l\neq 0}}^{n} |h_{lk}|^2\Big)^{\frac{1}{2}} \\
&\leq \nu\sqrt{1+\nu^2} \sum_{|k|>n} \frac{1}{|k|} \Big(\sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{1}{l^2(k-l)^2}\Big)^{\frac{1}{2}} \\
&\leq 2\nu\sqrt{1+\nu^2} \sum_{k=n+1}^{\infty} \frac{1}{k(k-n)} \Big(\sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{1}{l^2}\Big)^{\frac{1}{2}} \\
&\leq \frac{2\pi\nu\sqrt{1+\nu^2}}{\sqrt{3}} \left(\sum_{k=n+1}^{\infty} \frac{1}{k^{10}}\right)^{\frac{1}{10}} \left(\sum_{k=1}^{\infty} \frac{1}{k^{\frac{10}{9}}}\right)^{\frac{9}{10}} =: R_1^n.
\end{aligned}$$

On the other hand, if $l \in \{-N, \ldots, -n-1\} \cup \{n+1, \ldots, N\}$, we have $\|H_{l,0}\|_2 = \max_{0<|k|\leq n} |h_{lk}|$ and thus

$$\begin{aligned}
&\|H_{l,0}\|_2 + \sum_{\substack{n<|k|\leq N \\ k\neq l}} |h_{lk}| \leq \sum_{k\neq 0,l} |h_{lk}| \\
&\leq \nu\sqrt{1+\nu^2} \sum_{k\neq 0,l} \frac{1}{|k|\cdot|l|\cdot|k-l|} \\
&\leq \frac{\nu\sqrt{1+\nu^2}}{|l|} \Big(\sum_{k\neq 0,l} \frac{1}{k^2}\Big)^{\frac{1}{2}} \Big(\sum_{k\neq 0,l} \frac{1}{(k-l)^2}\Big)^{\frac{1}{2}} \\
&\leq \frac{\nu\sqrt{1+\nu^2}}{n} \cdot \frac{\pi^2}{3} =: R_2^n.
\end{aligned}$$

The conditions (9) now imply that for any $n \leq N$ the spectrum of H_N belongs to a Gerschgorin set

$$G_n = \overline{B(1, R_2^n)} \;\cup \bigcup_{\mu\in\sigma(H_{00}^n)} \overline{B(\mu, R_1^n)}$$

where R_1^n and R_2^n are decreasing functions of n. The sets G_n are independent of the value of N. This implies that for any $n \in \mathbb{N}$ the union of the spectra of the matrices $\{H_N\}_{N\in\mathbb{N}}$ is contained in the set

$$\Sigma_{\text{tot}}^n = G_n \;\cup\; \bigcup_{N\leq n} \sigma(H_N).$$

Thus we can choose the required lower bound to be any $\gamma > 0$ such that $\gamma \leq \inf\{\operatorname{Re}\lambda \mid \lambda \in \Sigma_{\text{tot}}^n\}$ for some $n \in \mathbb{N}$ for which the infimum is positive. It is also clear that this bound can be computed explicitly. Figure 4 shows the set Σ_{tot}^n for $\nu = \frac{1}{10}$ and $n = 50$. In this case we have $R_1^n \approx 0.06558$ and $R_2^n \approx 0.006613$. The light and dark grey areas denote the two parts of the Gerschgorin set G_n and the black dots denote the spectra of the matrices H_N for $N \leq n$.

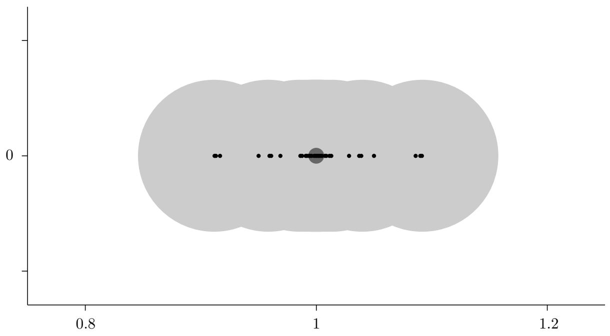

Figure 4: The set Σ_{tot}^n for $n = 50$.

This computation shows that if we choose $\gamma^2 = \frac{3}{5}$, we then have $\|Tx\|^2 \geq \gamma^2\|x\|^2$ for all $x \in X$. This concludes that $T^{-1} \in \mathcal{L}(X)$, $\|T^{-1}\|^2 \leq \frac{5}{3}$ and finally that we can choose $M_\sigma = \frac{5}{3}$. ☐

It is now clear that the spectrum of the stabilized operator $A + BK$ satisfies the geometric assumption for values $\alpha = 2$, $c_\sigma = \nu\pi^3$ and $y_\sigma = \pi$ and we can choose $\delta_\sigma = \nu\pi$ (Section 3.2) and $r_\sigma = 1$ (proof of Theorem 1 in Section 3.3). We can now use the perturbation results presented earlier to study the perturbation of the polynomially stabilized wave equation. The perturbations we consider are of form

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(z,t) &= \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)u(t) \\
&\quad + d_0(z)\left(\langle w(\cdot,t), g_1\rangle_{L^2} + \langle \frac{\partial w}{\partial t}(\cdot,t), g_2\rangle_{L^2}\right)
\end{aligned}$$

where $d_0, g_2 \in \mathcal{D}(A_0)$ and $g_1 \in L^2(0,1)$. If we denote

$$d = \begin{bmatrix} 0 \\ d_0 \end{bmatrix}, \qquad g = \begin{bmatrix} A_0^{-1} g_1 \\ g_2 \end{bmatrix},$$

then the perturbed equation can be written as

$$\dot{x} = (A + BK)x + \langle x, g\rangle_X d, \qquad x(0) = x_0,$$

with $d, g \in \mathcal{D}(A_0) \times \mathcal{D}(A_0) \subset \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}) = \mathcal{D}(A)$ and $A+BK$ is a Riesz-spectral operator. Since A is skew-adjoint and since $\mathcal{D}(A+BK) = \mathcal{D}(A)$, Theorem 2 implies that there exists a constant $\delta > 0$ such that the spectrum of the operator $(A+BK) + \langle \cdot, g\rangle_X d$ belongs to the open left half-plane of $\mathbb{C}$ whenever

$$\|d\|_1 < \sqrt{\delta} \qquad \text{and} \qquad \|g\|_{*,1} < \sqrt{\delta}.$$

As in Section 3 we can see that

$$\|d\|_1 \le \frac{1}{\sqrt{m_\sigma}} \|(A+BK)d\|_X,$$
$$\|g\|_{*,1} \le \sqrt{M_\sigma} \|(A+BK)^* g\|_X.$$

Using this we can conclude that if we choose $\delta > 0$ as above, then the spectrum of the perturbed system operator $(A+BK) + \langle \cdot, g\rangle_X d$ remains $\mathbb{C}^-$ whenever

$$\|(A+BK)d\|_X < \sqrt{m_\sigma \delta}, \quad \|(A+BK)^* g\|_X < \sqrt{\frac{\delta}{M_\sigma}}.$$

From the proof of Thereom 2 we see that we can choose any $\delta > 0$ such that $\delta \le a_\sigma$ and from the construction of the domain Δ in Section 3.2 we obtain

$$\begin{aligned}
a_\sigma &= \min\left\{ \frac{(1-\kappa)(1-\kappa^{\frac{1}{\alpha}}) c_\sigma^{\frac{\alpha+1}{\alpha}}}{\sqrt{(1-\kappa^{\frac{1}{\alpha}})^2 c_\sigma^{\frac{2}{\alpha}} + \delta_\sigma^{\frac{\alpha+1}{\alpha}} (1-\kappa)^2}}, (1-\kappa)\delta_\sigma^{1+\alpha} \right\} \\
&= \min\left\{ \frac{(1-\kappa)(1-\sqrt{\kappa}) \nu^{\frac{3}{2}} \pi^{\frac{9}{2}}}{\sqrt{(1-\sqrt{\kappa})^2 \nu \pi^3 + \pi^{\frac{3}{2}} (1-\kappa)^2}}, (1-\kappa)\pi^3 \right\} \\
&= (1-\kappa)\pi^3 \cdot \min\left\{ \frac{\nu\sqrt{\nu}\pi^{\frac{3}{4}}}{\sqrt{\nu\pi^{\frac{3}{2}} + (1+\sqrt{\kappa})^2}}, 1 \right\} \\
&= \frac{\nu^{\frac{3}{2}} (1-\kappa) \pi^{\frac{15}{4}}}{\sqrt{\nu\pi^{\frac{3}{2}} + (1+\sqrt{\kappa})^2}}
\end{aligned}$$

and where $0 < \nu \le 1$ and $\kappa \in (0,1)$. Clearly a_σ is a decreasing function of κ and if we choose $\kappa = \frac{1}{100}$ we get for $\nu = \frac{1}{10}$

$$a_\sigma = \frac{0.99 \cdot \left(\frac{1}{10}\right)^{\frac{3}{2}} \pi^{\frac{15}{4}}}{\sqrt{\frac{1}{10}\pi^{\frac{3}{2}} + 1.21}} > \frac{5}{3}.$$

Thus we can choose $\delta = \frac{5}{3}$. Computing $\sqrt{m_\sigma \delta} = 1$ and $\sqrt{\delta / M_\sigma} = 1$ we see that the condition for the perturbation of the spectrum becomes

$$\|(A+BK)d\| < 1, \qquad \|(A+BK)^* g\|_X < 1.$$

We clearly have $\|B\| = \|b\|_X = \|b_0\|_{L^2} = 1$ and

$$\begin{aligned}
\|K\|^2 = \|h\|_X^2 = \|-\frac{\nu\pi^2}{\sqrt{3}} \sum_{k\neq 0} \frac{\overline{\alpha_k}}{k} \varphi_k\|_X^2 = \frac{\nu^2\pi^4}{3} \sum_{k\neq 0} \frac{|\alpha_k|^2}{k^2} \\
\le \frac{\nu^2\pi^6}{9} \cdot \sup_{k\neq 0} |\alpha_k|^2
\end{aligned}$$

since the set $\{\varphi_k\}$ is orthonormal. For $k \neq 0$

$$|\alpha_k|^2 = \prod_{l\neq 0,k} |1 + i\frac{\nu}{l^2(l-k)}|^2 = \prod_{l \neq 0,k} \left(1 + \frac{\nu^2}{l^4(l-k)^2}\right).$$

Since $\log(1+x) \le x$ for any $x \ge 0$, we have

$$\begin{aligned}
\log|\alpha_k|^2 &= \sum_{l\neq 0,k} \log\left(1 + \frac{\nu^2}{l^4(l-k)^2}\right) \le \sum_{l\neq 0,k} \frac{\nu^2}{l^4(l-k)^2} \\
&\le 2\nu^2 \left(\sum_{l=1}^\infty \frac{1}{l^8}\right)^{\frac{1}{2}} \left(\sum_{l=1}^\infty \frac{1}{l^4}\right)^{\frac{1}{2}} = \frac{2\nu^2\pi^5}{9\sqrt{1050}}.
\end{aligned}$$

This allows us to conclude that the bound for the norm $\|K\|$ of the feedback depends continuously on ν and can be made arbitrarily small. For $\nu = \frac{1}{10}$ we have

$$\|K\| \le \frac{\nu\pi^3}{3} e^{\frac{\nu^2\pi^5}{9\sqrt{1050}}} < \frac{\pi}{3}.$$

We have $A_0^{-1/2} = \sum_{k=1}^\infty \frac{1}{k\pi} \langle \cdot, \sin(k\pi \cdot)\rangle_{L^2} \sin(k\pi\cdot)$ and thus $\|A_0^{-1/2}\| = \frac{1}{\pi}$. Using this we have for the norms of the elements d, g, Ad and A^*g that $\|d\|_X = \|d_0\|_{L^2}$ and

$$\begin{aligned}
\|g\|_X^2 &= \|A_0^{1/2} A_0^{-1} g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2 \le \frac{1}{\pi^2}\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2 \\
\|Ad\|_X^2 &= \left\| \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ d_0 \end{bmatrix} \right\|_X^2 = \langle A_0^{1/2} d_0, A_0^{1/2} d_0 \rangle_{L^2} \\
&= \langle A_0 d_0, d_0\rangle_{L^2} = \int_0^1 -d_0''(z) \cdot \overline{d_0(z)} dz \\
&= \int_0^1 d_0'(z) \cdot \overline{d_0'(z)} dz = \|d_0'\|_{L^2}^2 \\
\|A^*g\|_X^2 &= \|Ag\|_X^2 = \left\| \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} A_0^{-1} g_1 \\ g_2 \end{bmatrix} \right\|_X^2 \\
&= \left\| \begin{bmatrix} g_2 \\ -g_1 \end{bmatrix} \right\|_X^2 = \langle A_0^{1/2} g_2, A_0^{1/2} g_2\rangle_{L^2} + \langle -g_1, -g_1 \rangle \\
&= \|g_2'\|_{L^2}^2 + \|g_1\|_{L^2}^2
\end{aligned}$$

and we can estimate

$$\begin{aligned}
\|(A+BK)d\|_X &\le \|Ad\|_X + \|B\|\|K\|\|d\| \\
&\le \|d_0'\|_{L^2} + \frac{\pi}{3}\|d_0\|_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
\|(A+BK)^*g\|_X &\le \|A^*g\|_X + \|K^*\|\|B^*\|\|g\|_X \\
&\le \sqrt{\|g_1\|_{L^2}^2 + \|g_2'\|_{L^2}^2} + \frac{\pi}{3}\sqrt{\frac{1}{\pi^2}\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2}.
\end{aligned}$$

Combining these estimates we see that if $d_0, g_2 \in \mathcal{D}(A_0)$ and $g_1 \in L^2(0,1)$ are such that the L^2-norms of d_0, d_0', g_1, g_2 and g_2' are small enough, then the spectrum of the operator $A + BK + \langle \cdot, g\rangle d$ is contained in the open left half-plane of $\mathbb{C}$. For example for functions

$$d_0(z) = c_1 \sin(\pi z)\cos(2\pi z),$$
$$g_1(z) = c_2(\chi_{[0,.25]}(z) + i\chi_{[0,.5]}(z)),$$
$$g_2(z) = c_3 e^{-\tan(\pi z + \frac{\pi}{2})^2}$$

this is true whenever $c_1, c_2, c_3 \leq \frac{1}{5}$.

To use Theorem 9 to study the preservation of the strong stability of the C_0-semigroup we would now need that either $d \in \mathcal{D}(A + BK)^2$ or $g \in \mathcal{D}((A + BK)^*)^2$. However, this is a difficult requirement to satisfy because $\mathcal{R}(B) \not\subset \mathcal{D}(A)$. In order to overcome this difficulty we could choose B in such a way that $\mathcal{R}(B) \subset \mathcal{D}(A)$. This can be done by simply choosing a smoother function b_0, more precisely any $b_0 \in \mathcal{D}(A_0^{1/2})$. However, if we do this, then in order to satisfy the conditions (8) for the existence of the bounded feedback completing the pole placement we would need to choose the target set $\{\mu_k\}_k$ of eigenvalues in such a way that its members approach the imaginary axis faster. This, in turn, would lead to larger α in the geometric assumption and consequently to a requirement that $d \in \mathcal{D}(A + BK)^\alpha$ or $g \in \mathcal{D}((A + BK)^*)^\alpha$ for $\alpha > 2$. Because of this, the conditions for the preservation of the strong stability presented in Section 4 are not very useful in this example.

We can, however, use Theorem 11 to determine conditions for the uniform boundedness and the stability of the semigroup generated by the perturbed operator. The next Theorem concludes that if the L^2-norms of the functions d_0, g_1 and g_2 are small enough, then the semigroup generated by $A + BK + \langle \cdot, g\rangle_X d$ is polynomially stable.

Theorem 14. *If in addition to the bounds presented above the norms of the functions d_0, g_1 and g_2 satisfy*

$$\|d_0\|_{L^2}^2 \cdot \left(\|g_1\|_{L^2}^2 + \pi^2 \|g_2\|_{L^2}^2\right) < \frac{\pi^4}{36},$$

the perturbed operator $A + BK + \langle \cdot, g\rangle_X d$ generates a strongly and polynomially stable C_0-semigroup.

Proof. Writing

$$A + (BK + \langle \cdot, g\rangle_X d)$$

we can consider the perturbed operator as a perturbation of the operator A. The operator A is normal, has compact resolvent, its eigenvalues $\lambda_k = ik\pi$ lie on the imaginary axis and have a uniform gap

$$\inf_{k \neq l} |\lambda_k - \lambda_l| = \pi > 0.$$

Theorem 6.2 and Remark 6.3 in [21] state that the generalized eigenvectors of $A + BK + \langle \cdot, g\rangle_X d$ form a Riesz basis of X and all but a finite number of its eigenvalues are simple provided that

$$\|BK + \langle \cdot, g\rangle_X d\| < \frac{\pi}{2}.$$

This is satisfied, since we have using our assumption that

$$\|BK + \langle \cdot, g\rangle_X d\| \leq \|B\|\|K\| + \|g\|_X \|d\|_X$$
$$\leq \frac{\pi}{3} + \|d_0\|_{L^2} \cdot \sqrt{\frac{1}{\pi^2}\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2} < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}.$$

Since $\sigma(A + BK + \langle \cdot, g\rangle_X d) \subset \mathbb{C}^-$ the fact that the generalized eigenvectors $A + BK + \langle \cdot, g\rangle_X d$ form a Riesz basis of X and the infinite part of the spectrum consists of simple eigenvalues concludes that it generates a uniformly bounded semigroup. This in turn concludes that the semigroup generated by $A + BK + \langle \cdot, g\rangle_X d$ is strongly stable.

The above conclusion also means that the perturbed operator is of the form of Theorem 11. Thus the perturbed semigroup $S(t)$ is polynomially stable and there exists a constant $C_\sigma > 0$ such that for all $t > 0$

$$\|S(t)(A + BK + \langle \cdot, g\rangle_X d)^{-1}\| \leq \frac{C_\sigma}{t^{1/\alpha}}.$$

□

6. Perturbation of a Robust Controller

In this Section we consider an example related to the robust output regulation of distributed parameter systems with infinite-dimensional exosystems [1, 2]. The results in [1, 2] state that under certain conditions the robust controller achieves output regulation regardless of the perturbations to the systems parameters provided that the closed-loop remains strongly stable. We consider a closed-loop system consiting of a plant and a robust observer-based controller and study the preservation of its strong stability under finite rank perturbations of the infinite-dimensional internal model in the controller. Although the results on robustness of output regulation in [1, 2] are not directly applicable to perturbations of the internal model, the preservation of the stability of the closed-loop system is still an interesting problem. Also, these types of perturbations can be considered in the context of practical output regulation [22], where the aim is to regulate the reference signal with a given finite accuracy. Even more importantly, this example illustrates how the perturbation of composite systems can be handled with the techniques presented in this paper.

Since in this case the unperturbed operator is not obtained by feedback stabilization of a skew-adjoint operator, the methods presented in [10] cannot be applied to this example.

For simplicity we consider a stable single input, single output system and rank one perturbations of the internal model.

The aim of this example is to demonstrate the application of the perturbation results to problems of control theory and robust output regulation. Because of this, we do not compute the bounds for the perturbations explicitly, as was done in the previous example. Instead, we simplify things by choosing the parameters in such a way that we can reuse the computations carried out in Section 5.

Consider a linear time-invariant distributed parameter system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) \in X \\ y &= Cx + Du \end{aligned}$$

on a Hilbert space X, where x is the state of the system, u the control, y the output. The operator A generates an exponentially stable analytic semigroup and $B \in \mathcal{L}(\mathbb{C}, X)$, $C \in \mathcal{L}(X, \mathbb{C})$ and $D \in \mathbb{C} \setminus \{0\}$. For $s \in \rho(A)$ we denote by $P(s) = CR(s, A)B + D$ the transfer function of the plant.

We consider signals generated by an infinite-dimensional exosystem

$$\begin{aligned} \dot{v} &= Sv, \qquad v(0) \in W, \\ y_{ref} &= Fv \end{aligned}$$

on a separable Hilbert space $W = \overline{\mathrm{span}}\{\varphi_k\}_{k \neq 0}$. The system operator S of the exosystem is chosen to be

$$\begin{aligned} Sv &= \sum_{k\neq 0} ik\pi \langle v, \varphi_k \rangle \varphi_k, \\ v \in \mathcal{D}(S) &= \{\, v \in W \mid \sum_{k \neq 0} k^2 |\langle v, \varphi_k \rangle|^2 < \infty \,\}. \end{aligned}$$

We denote by $T_S(t)$ the group generated by S and choose $F = \sum_{k\neq 0} \frac{1}{k} \langle \cdot, \varphi_k \rangle$. An example of a reference signal generated by this exosystem can be given by choosing the initial state $v(0) \in W$ of the exosystem to be

$$v(0) = -\frac{1}{2\pi} \sum_{k\neq 0} \frac{(e^{ik\pi} - 1)^2}{k} \varphi_k.$$

The corresponding reference signal is then given by

$$\begin{aligned} y_{ref}(t) = Fv(t) = FT_S(t)v(0) &= \sum_{k \neq 0} e^{ik\pi t} \langle v(0), \varphi_k \rangle F\varphi_k \\ &= -\frac{1}{2\pi} \sum_{k \neq 0} \frac{(e^{ik\pi} - 1)^2}{k^2} e^{ik\pi t}. \end{aligned}$$

This is precisely the Fourier series representation of the sawtooth signal depicted in Figure 5.

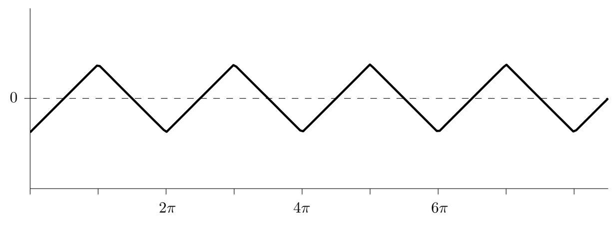

Figure 5: Example of a signal generated by the exosystem.

The robust output regulation problem consists of finding a feedback controller such that for all inital states of the original system, the signal generator and the controller the output of the plant satisfies

$$\lim_{t\to\infty} |y(t) - y_{ref}(t)| = 0$$

and such that this property is robust with respect to a class of perturbations preserving the strong stability of the closed-loop system.

The system operator of the closed-loop system consisting of the plant and the robust observer-based controller solving the robust output regulation problem is given by [1, Thm. 13]

$$A_e = \begin{bmatrix} A & BKH & BK \\ -LC & A + BKH + LC & BK \\ G_2C & G_2DKH & S + G_2DK \end{bmatrix}$$

where $L \in \mathcal{L}(X, \mathbb{C})$ is such that $A + LC$ is exponentially stable, $G_2 = g_2 \in W$, and the operator $H \in \mathcal{L}(X, W)$ is such that $H(\mathcal{D}(A)) \subset \mathcal{D}(S)$ and it is the unique solution of the Sylvester equation $SH = HA + G_2C$. We have from [1, Lem. 19] that such an operator always exists and that

$$\langle Hx, \varphi_k \rangle = \langle g_2, \varphi_k \rangle CR(i\omega_k, A)x \tag{10}$$

for all $x \in X$ and $k \neq 0$. In the controller the copy of the system operator S of the exosystem is called *the internal model* of the exosystem. In this example we consider the perturbations of this particular operator in A_e.

We will first show how to choose $G_2 \in \mathcal{L}(\mathbb{C}, W)$ and $K \in \mathcal{L}(W, \mathbb{C})$ in such a way that the closed-loop system is strongly stable. For this we will use the method presented in [1]. To do this we will need the standard assumption that the transfer function of the plant satisfies

$$\sup_{k\neq 0} |P(ik\pi)| < \infty, \qquad \inf_{k \neq 0} |P(ik\pi)| > 0.$$

6.1. Stabilization of The Closed-Loop System

Choose

$$T_e = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{bmatrix}, \qquad T_e^{-1} = \begin{bmatrix} I & 0 & 0 \\ I & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

and define $\tilde{A}_e = T_e A_e T_e^{-1}$. We then have

$$\tilde{A}_e = \begin{bmatrix} A + BKH & BK & BKH \\ G_2(C + DKH) & S + G_2DK & G_2DKH \\ 0 & 0 & A + LC \end{bmatrix}$$

and since $A + LC$ is exponentially stable, $\tilde{A}_e$ is strongly stable if

$$\tilde{A}_{e1} = \begin{bmatrix} A + BKH & BK \\ G_2(C + DKH) & S + G_2DK \end{bmatrix}$$

is strongly stable [1, Lem. 20]. Choose $T_{e1} \in \mathcal{L}(X \times W)$ such that

$$T_{e1} = \begin{bmatrix} -I & 0 \\ H & I \end{bmatrix}.$$

Now $T_{e1}^{-1} = T_{e1}$ and using the properties stated above we obtain

$$A_{e1} = T_{e1}\tilde{A}_{e1}T_{e1}^{-1} = \begin{bmatrix} A & -BK \\ 0 & S + B_1K \end{bmatrix}$$

where $B_1 = HB + G_2D \in \mathcal{L}(U, W)$. Since A is exponentially stable, A_{e1} is strongly stable if $S + B_1K$ is strongly stable [1, Lem. 20]. Using (10) we can see that for $u \in \mathbb{C}$

$$\begin{aligned} \langle B_1u, \varphi_k\rangle &= \langle HBu, \varphi_k\rangle + \langle G_2Du, \varphi_k\rangle \\ &= \langle g_2, \varphi_k\rangle CR(i\omega_k, A)Bu + \langle g_2, \varphi\rangle Du \\ &= \langle g_2, \varphi_k\rangle P(i\omega_k)u, \end{aligned}$$

where $P(s)$ is the transfer function of the plant. To simplify things, we choose $g_2 \in W$ in such a way that we can use the stabilizing feedback given in Theorem 13 to also stabilize $S + B_1K$. Our assumptions imply that if we choose $\langle g_2, \varphi_k\rangle = \frac{\sqrt{3}}{k\pi P(i\omega_k)}$, then $g_2 \in W$ and

$$\langle B_1, \varphi_k\rangle = \frac{\sqrt{3}}{k\pi P(i\omega_k)} P(i\omega_k) = \frac{\sqrt{3}}{k\pi}.$$

With this choice we can use Theorem 13 to choose a feedback operator $K = \langle \cdot, h\rangle \in \mathcal{L}(W, \mathbb{C})$ in such a way that $S + B_1K$ is a Riesz-spectral operator with eigenvalues $\{\mu_k\}_{k\neq 0} = \{-\frac{\nu\pi}{k^2} + ik\pi\}$, i.e.

$$(S + B_1K)v = \sum_{k\neq 0} \mu_k \langle v, \psi_k\rangle \phi_k,$$

$$v \in \mathcal{D}(S + B_1K) = \Big\{ v \in W \;\Big|\; \sum_{k\neq 0} |\mu_k|^2 |\langle v, \psi_k\rangle|^2 < \infty \Big\}.$$

The above reasoning now concludes that the closed-loop system is strongly stable.

6.2. Perturbation of The Internal Model

We can now consider the preservation of the strong stability of the closed-loop system under bounded rank one perturbations Δ_S of the internal model S in the system operator A_e of closed-loop system. This means that we want to consider the strong stability of the operator $A_e + \Delta_e$ given by

$$\begin{bmatrix} A & BKH & BK \\ -LC & A + BKH + LC & BK \\ G_2C & G_2DKH & S + \Delta_S + G_2DK \end{bmatrix}.$$

This is equivalent to considering the strong stability of the operator $\tilde{A}_e + \tilde{\Delta}_e = T_e(A_e + \Delta_e)T_e^{-1}$ given by

$$\begin{bmatrix} A + BKH & BK & BKH \\ G_2(C + DKH) & S + \Delta_S + G_2DK & G_2DKH \\ 0 & 0 & A + LC \end{bmatrix},$$

which is strongly stable whenever

$$\tilde{A}_{e1} + \tilde{\Delta}_{e1} = \begin{bmatrix} A + BKH & BK \\ G_2(C + DKH) & S + \Delta_S + G_2DK \end{bmatrix}$$

is strongly stable. Finally, this operator is strongly stable if and only if the operator

$$\begin{aligned} A_{e1} + \Delta_{e1} &= T_{e1}(\tilde{A}_{e1} + \tilde{\Delta}_{e1})T_{e1}^{-1} \\ &= \begin{bmatrix} A & -BK \\ \Delta_S H & S + \Delta_S + B_1K \end{bmatrix} \end{aligned}$$

is strongly stable. Since A_{e1} is not in general a Riesz-spectral operator we cannot apply Theorems 2 and 9 directly to determine the preservation of its strong stability. Instead, we can exploit the block operator structure of A_{e1} and use similar geometric methods as before to determine bounds for the perturbations such that the spectrum of $A_{e1} + \Delta_{e1}$ is contained in the open left half-plane of $\mathbb{C}$. Subsequently we can again determine additional conditions for the preservation of the strong stability as was done in Section 5.

Let $\Delta_S = \langle \cdot, \eta_S\rangle \delta_S$ where $\delta_S, \eta_S \in W$. The perturbation Δ_{e1} is a bounded rank one operator given by

$$\Delta_{e1} = \begin{bmatrix} 0 & 0 \\ \delta_S\langle H\cdot, \eta_S\rangle & \delta_S\langle \cdot, \eta_S\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \delta_S \end{bmatrix} \left\langle \cdot, \begin{bmatrix} H^*\eta_S \\ \eta_S \end{bmatrix} \right\rangle.$$

Since A generates an exponentially stable analytic semigroup, there exist constants $\omega < 0$ and $M \geq 1$ such that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ we have

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}. \tag{11}$$

The operator $S + B_1K$ satisfies Assumption 1 for $\alpha = 2$ and for some $c_\sigma, y_\sigma > 0$. We can thus construct the corresponding domain Δ such that $\mathbb{C}^+ \subset \Delta \subset \rho(S + B_1K)$ and such that the conclusion of Theorem 6 holds. Choosing $\delta_\sigma < |\omega|$ we also have that $\Delta \subset \rho(A)$. This concludes that $\Delta \subset \rho(A) \cap \rho(S + B_1K) \subset \rho(A_{e1})$ and the estimate (11) holds for all $\lambda \in \Delta$.

The Weinstein-Aronszajn determinant for the perturbed operator $A_{e1} + \Delta_{e1}$ is now given by [18, Sec IV.6]

$$\omega(\lambda; A_{e1}, \Delta_{e1}) = 1 - \left\langle R(\lambda, A_{e1}) \begin{bmatrix} 0 \\ \delta_S \end{bmatrix}, \begin{bmatrix} -H^*\eta_S \\ \eta_S \end{bmatrix} \right\rangle.$$

In order to obtain bounds for the Weinstein-Aronszajn determinant we need two estimates. First of all, we have from Theorem 6 that there exists $a_\sigma > 0$ such that

$$|\lambda - \mu_k| \geq a_\sigma |\mu_k|^{-2}$$

for all $\lambda \in \Delta$. We will also need to show that there exists another constant $\tilde{a}_\sigma > 0$ such that

$$|\lambda - \omega| \cdot |\lambda - \mu_k| \geq \tilde{a}_\sigma |\mu_k|^{-1} \tag{12}$$

for all $\lambda \in \Delta$. Due to the symmetry it is sufficient to show this for $k \geq 1$.

Let $k \geq 1$. Denote by $\lambda_0(k) \in \overline{\Delta}$ the point at which $\lambda \mapsto |\lambda - \omega| \cdot |\lambda - \mu_k|$ achieves its minimum in the closed set $\overline{\Delta}$. It can be shown that there exists $N_\sigma \in \mathbb{N}$ such that for $k > N_\sigma$ we have that $k-1 \leq \operatorname{Im}\lambda_0(k) \leq k$. For $k > N_\sigma$

$$\begin{aligned} &|\lambda-\omega|\cdot|\lambda-\mu_k| \geq |\lambda_0(k)-\omega|\cdot|\lambda_0(k)-\mu_k| \\ \geq\ &\operatorname{Im}\lambda_0(k)\cdot\frac{a_\sigma}{|\mu_k|^2} \geq \frac{k-1}{|\mu_k|}\cdot\frac{a_\sigma}{|\mu_k|} \geq \frac{a_\sigma}{2\sqrt{2}}|\mu_k|^{-1}, \end{aligned}$$

since for all $k \geq 2$ we have

$$\frac{k-1}{|\mu_k|} = \left(\frac{(k-1)^2}{\frac{1}{k^4}+k^2}\right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{2}}\cdot\frac{k-1}{k} \geq \frac{1}{2\sqrt{2}}.$$

On the other hand, for $1 \leq k \leq N_\sigma$ there exists $c > 0$ such that

$$\begin{aligned} |\lambda-\omega|\cdot|\lambda-\mu_k| &\geq \min_{1\leq k\leq N_\sigma}\left(|\lambda_0(k)-\omega|\cdot|\lambda_0(k)-\mu_k|\right) \\ &= c > 0 \end{aligned}$$

and thus

$$|\lambda-\omega|\cdot|\lambda-\mu_k| \geq c\cdot\frac{|\mu_k|}{|\mu_k|} \geq \left(c\min_{1\leq k\leq N_\sigma}|\mu_k|\right)\cdot|\mu_k|^{-1}.$$

This concludes that there exists a constant $\tilde{a}_\sigma > 0$ such that the condition (12) holds for all $\lambda \in \Delta$ and $k \neq 0$.

Lemma 3 again implies that it is sufficient to find bounds for $\|\delta_S\|_\beta$ and $\|\eta_S\|_{*,\gamma}$ such that $|\omega(\lambda; A_{e1}, \Delta_{e1})| > 0$ for all $\lambda \in \Delta$. For $\lambda \in \Delta$ we have

$$R(\lambda, A_{e1}) = \begin{bmatrix} R(\lambda,A) & -R(\lambda,A)BKR(\lambda,S+B_1K) \\ 0 & R(\lambda,S+B_1K) \end{bmatrix}$$

and thus the inner product in the Weinstein-Aronszajn determinant is given by

$$\begin{aligned} &\left\langle R(\lambda,A_{e1})\begin{bmatrix}0\\ \delta_S\end{bmatrix}, \begin{bmatrix}H^*\eta_S\\ \eta_S\end{bmatrix}\right\rangle \\ =\ &\left\langle \begin{bmatrix}-R(\lambda,A)BKR(\lambda,S+B_1K)\delta_S \\ R(\lambda,S+B_1K)\delta_S\end{bmatrix}, \begin{bmatrix}H^*\eta_S\\ \eta_S\end{bmatrix}\right\rangle \\ =\ &\langle -R(\lambda,A)BKR(\lambda,S+B_1K)\delta_S, H^*\eta_S\rangle \\ &+\langle R(\lambda,S+B_1K)\delta_S, \eta_S\rangle. \end{aligned}$$

The second term can be estimated as in the proof of Theorem 2 by

$$\begin{aligned} &|\langle R(\lambda,S+B_1K)\delta_S,\eta_S\rangle| \leq \sum_{k\neq 0}\frac{|\langle\delta_S,\psi_k\rangle|\cdot|\langle\phi_k,\eta_S\rangle|}{|\lambda-\mu_k|} \\ \leq\ &\frac{1}{a_\sigma}\sum_{k\neq 0}|\mu_k|^2|\langle\delta_S,\psi_k\rangle|\cdot|\langle\phi_k,\eta_S\rangle| \leq \frac{1}{a_\sigma}\|\delta_S\|_1\cdot\|\eta_S\|_{*,1}. \end{aligned}$$

For the first term we can use the estimate (12) to obtain

$$\begin{aligned} &|\langle -R(\lambda,A)BKR(\lambda,S+B_1K)\delta_S, H^*\eta_S\rangle| \\ \leq\ &\|R(\lambda,A)\|\cdot\|B\|\cdot|KR(\lambda,S+B_1K)\delta_S|\cdot\|H^*\eta_S\| \\ \leq\ &\frac{M}{|\lambda-\omega|}\|B\|\cdot\|H^*\|\cdot\|\eta_S\|\cdot\left|\sum_{k\neq 0}\frac{\langle\phi_k,h\rangle\langle\delta_S,\psi_k\rangle}{\lambda-\mu_k}\right| \\ \leq\ &M\|B\|\cdot\|h\|\cdot\|\eta_S\|\cdot\sum_{k\neq 0}\frac{|\langle\phi_k,h\rangle||\langle\delta_S,\psi_k\rangle|}{|\lambda-\omega||\lambda-\mu_k|} \\ \leq\ &\frac{M}{\tilde{a}_\sigma}\|B\|\cdot\|h\|\cdot\|\eta_S\|\cdot\sum_{k\neq 0}|\mu_k||\langle\phi_k,h\rangle||\langle\delta_S,\psi_k\rangle| \\ \leq\ &\frac{M\sqrt{M_\sigma}}{\tilde{a}_\sigma}\|B\|\cdot\|h\|^2\cdot\|\eta_S\|\cdot\|\delta_S\|_1. \end{aligned}$$

These estimates show that if we have $\delta_S \in \mathcal{D}_1 = \mathcal{D}(S)$ and $\eta_S \in \mathcal{D}_1^* = \mathcal{D}(S)$, i.e.

$$\sum_{k\neq 0}k^2|\langle\delta_S,\varphi_k\rangle|^2 < \infty, \qquad \sum_{k\neq 0}k^2|\langle\eta_S,\varphi_k\rangle|^2 < \infty,$$

and if the norms $\|\delta_S\|_1$, $\|\eta_S\|$ and $\|\eta_S\|_{*,1}$ are small enough we have that $\inf_{\lambda\in\Delta}|\omega(\lambda; A_{e1}, \Delta_{e1})| > 0$. Lemma 3 then concludes that $\sigma(A_{e1}+\Delta_{e1}) \subset \mathbb{C}\setminus\Delta \subset \mathbb{C}^-$.

Under suitable assumptions the uniform boundedness of the semigroup generated by $A_{e1}+\Delta_{e1}$ can again be shown using the results in [21] as was done in Theorem 14 of Section 5. In particular this is possible if A is similar to a normal operator with compact resolvent and if its spectrum has a uniform gap and lies on a finite number of rays starting from the origin. An important case of an infinite-dimensional system satisfying these conditions is the heat equation. In this case we can write

$$A_{e1}+\Delta_{e1} = \begin{bmatrix}A & 0\\ 0 & S\end{bmatrix} + \begin{bmatrix}0 & -BK\\ 0 & B_1K\end{bmatrix} + \begin{bmatrix}0 & 0\\ \Delta_S H & \Delta_S\end{bmatrix}.$$

The results of [21] can now be applied to the first block diagonal operator. This way we can see that if the norms of the two other operators are small enough, we have that the generalized eigenvectors of $A_{e1}+\Delta_{e1}$ form a Riesz basis and all but a finite number of the corresponding eigenvalues are simple. Also — since the norm $\|K\|$ can be made arbitrarily small by a proper choice of the parameter $0 < \nu \leq 1$ — this can always be achieved if ν, $\|\delta_S\|$ and $\|\eta_S\|$ are small enough. As in the proof of Theorem 14 we can then again conclude that the perturbed semigroup is uniformly bounded and thus strongly stable. This finally concludes that under these conditions the perturbed closed-loop system remains strongly stable.

7. Coupled Systems

As a third example we will show that when applying the theory to coupled systems, the requirement of the uniform spectral gap can often be a limitation. This can be seen, for example, by consider the interconnected systems

$$\dot{x} = Ax + Cy$$
$$\dot{y} = By,$$

on Hilbert spaces X and Y. Let A and B be Riesz-spectral operators such that $A : \mathcal{D}(A) \subset X \to X$ is exponentially stable and $B : \mathcal{D}(B) \subset Y \to Y$ generates a strongly stable C_0-group satisfying Assumption 1. The composite system operator

$$A_c = \begin{bmatrix} A & C \\ & B \end{bmatrix}$$

then generates a strongly stable semigroup [1, Lem. 20]. Since A is exponentially stable and B generates a C_0-group, we have that if the growth bound $\omega_0(A) < 0$ of A is small enough, there exists an operator $H \in \mathcal{L}(Y, X)$ such that $H(\mathcal{D}(B)) \subset \mathcal{D}(A)$ and H is the unique solution of the Sylvester equation

$$AH = HB + C$$

on $\mathcal{D}(B)$ [23]. If we choose

$$T = \begin{bmatrix} I & H \\ & -I \end{bmatrix},$$

then $T^{-1} = T$ and

$$T \begin{bmatrix} A & C \\ & B \end{bmatrix} T^{-1} = \begin{bmatrix} A & \\ & B \end{bmatrix}.$$

The operator on the right-hand side is clearly a Riesz-spectral operator, and the same is thus also true for A_c.

It is clear that if the spectrum of A does not have a uniform gap, the same applies to the spectrum of A_c. Examples of these kinds of operators encountered in applications are all operators and matrices A having eigenvalues of multiplicity larger than one and operators whose spectra have a finite accumulation point. The latter can happen for example if $A = K - aI$, where K is an integral operator

$$(Kx)(t) = \int_0^1 k(t, s)x(s)ds$$

on $Y = L^2(0, 1)$ with $k(\cdot, \cdot) \in L^2((0, 1) \times (0, 1))$ such that $k(t, s) = \overline{k(s, t)}$. The operator K is self-adjoint and compact and thus has a representation

$$K = \sum_{k=1}^{\infty} \lambda_k \langle \cdot, \varphi_k \rangle \varphi_k$$

where $\{\varphi_k\}_{k=1}^{\infty}$ is orthonormal set and the eigenvalues $\{\lambda_k\} \subset \mathbb{R}$ are such that $|\lambda_1| \geq \cdots \geq |\lambda_n| \to 0$ as $n \to 0$ [24, Ch. 6]. For a large enough $a > 0$ the operator A generates an exponentially stable semigroup and its spectrum has a finite accumulation point in $\mathbb{C}^-$.

We can now consider the preservation of the strong stability of the interconnected systems when a coupling in the other direction is added to the system. For example we can consider the case where the state of the first equation is inserted into the second equation through a rank one operator $\Delta = \langle \cdot, g \rangle b$. The systems then become

$$\dot{x} = Ax + Cy$$
$$\dot{y} = \Delta x + By.$$

Expressed using composite operators, we need to study the strong stability of the operator

$$A_c + \Delta_c = \begin{bmatrix} A & C \\ \Delta & B \end{bmatrix}.$$

It is possible to study this kind of perturbation in the same way as in Section 6, but our results can also be applied directly to the perturbed operator

$$\begin{aligned} T(A_c + \Delta_c)T^{-1} &= \begin{bmatrix} A & \\ & B \end{bmatrix} + \begin{bmatrix} H\Delta & H\Delta H \\ -\Delta & -\Delta H \end{bmatrix} \\ &= \begin{bmatrix} A & \\ & B \end{bmatrix} + \left\langle \cdot, \begin{bmatrix} g \\ H^* g \end{bmatrix} \right\rangle \begin{bmatrix} Hb \\ -b \end{bmatrix}. \end{aligned}$$

As was stated above, the unperturbed operator is now a Riesz-spectral operator clearly satisfying Assumption 1, but since the spectrum of A does not have a uniform gap, neither does the spectrum of A_c.

8. Conclusions

In this paper we considered finite rank perturbations of Riesz-spectral operators generating strongly and polynomially stable C_0-semigroups. We presented conditions under which the spectrum of the perturbed operator remains in the open left half-plane of $\mathbb{C}$, conditions for the preservation of the uniform boundedness of the semigroup and conditions for the polynomial stability of the perturbed semigroup.

As the first example we considered perturbation of a polynomially stabilized wave equation. It was observed in the example that the results on the perturbation of the spectrum were applicable to the case but the results on the preservation of the uniform boundedness of the C_0-semigroup were not. This illustrates a fundamental difference between results of otherwise similar type. This difference is the fact that in the results on the perturbation of the spectrum the requirements of $b \in \mathcal{D}_\beta$ and $g \in \mathcal{D}_\gamma^*$ can be distributed between the elements b and g whereas in the results concerning the preservation of the uniform boundedness of the C_0-semigroup all of the requirements are imposed on only one of these elements.

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