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Robustness of Stability of C_0-Semigroups

Master of Science Thesis

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ABSTRACT

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This thesis is concerned with strongly continuous semigroups of linear operators, called C_0-semigroups, on Banach and Hilbert spaces. In particular, we are interested in how exponential stability is preserved under additive perturbations of the infinitesimal generator of the C_0-semigroup. We consider both bounded and relatively bounded perturbations.

The problem is divided into two parts. We will first look for conditions under which the perturbed generator remains an infinitesimal generator of a C_0-semigroup. Subsequently, we will impose additional conditions for the stability of the perturbed C_0-semigroup.

To answer the first part of the problem, we present a variety theoretical results found in the literature. These results provide conditions under which the perturbed operator generates a C_0-semigroup.

As the first approach to the second part of the problem, we introduce additional conditions under which the perturbed C_0-semigroup is exponentially stable. This is done by applying conditions for the stability of a C_0-semigroup to the case of the perturbed C_0-semigroup.

As a second approach, we provide conditions under which the stability of the perturbed C_0-semigroup can be determined from the spectrum of the perturbed operator. Some of these results require certain special properties from the C_0-semigroup and some of them are applicable to the case of general C_0-semigroups.

TIIVISTELMÄ

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Tässä diplomityössä käsitellään vahvasti jatkuvia puoliryhmiä, eli C_0-puoliryhmiä, Banach- ja Hilbert-avaruuksissa. Erityisesti olemme kiinnostuneita siitä miten niiden stabiilisuusominaisuudet muuttuvat infinitesimaaliseen generaattoriin kohdistuvien häiriöiden vaikutuksesta.
Vahvasti jatkuvien puoliryhmien voidaan ajatella olevan eksponenttifunktion e^{At} yleistys äärettömulotteisiin avaruuksiin. Suljettu operaattori Banach-avaruudessa voi generoida samankaltaisen vahvasti jatkuvan kuvauksen $t \mapsto T(t)$ puoliavoimelta väliltä $[0, \infty)$ avaruuteen $\mathcal{L}(X)$. Eksponenttifunktion tavoin tämä kuvaus toteuttaa ehdot $T(s+t) = T(s)T(t)$ ja $T(0) = I$ kun $s, t \geq 0$. Toisaalta äärettömulotteisesta tilanteesta löytyy myös paljon eroavaisuuksia eksponenttifunktion tapaukseen verrattuna. Esimerkiksi yleisessä Banach-avaruudessa jokainen lineaarinen operaattori ei generoi vahvasti jatkuvaa puoliryhmää.
Selvitämme seuraavaksi mitä puoliryhmän eksponentiaalinen stabiilisuus tarkoittaa. Vahvasti jatkuvat puoliryhmät liittyvät läheisesti abstraktien Cauchy-ongelmien teoriaan: Jos A generoi vahvasti jatkuvan puoliryhmän $T(t)$ Banach-avaruudessa X, niin yhtälön $\dot{x}(t) = Ax(t)$ alkuehdolla $x(0) = x_0 \in X$ ratkaisuksi saadaan $x(t) = T(t)x_0$. Ratkaisua kutsutaan stabiiliksi, jos se lähestyy nollaa t:n kasvaessa. Tässä työssä olemme kiinnostuneita erityisesti tapauksesta, jossa kaikilla alkutiloilla $x_0 \in X$ yhtälön ratkaisun normi lähenee nollaa eksponentiaalisella nopeudella t:n kasvaessa. Puoliryhmän $T(t)$ ominaisuuksien avulla ilmaistuna tämä tarkoittaa sitä, että on olemassa reaaliset vakiot $M \geq 1$ ja $\omega > 0$ siten että $\|T(t)\| \leq Me^{-\omega t}$ pätee kaikilla $t \geq 0$.
Koska kaikilla alkutiloilla x_0 vahvasti jatkuva puoliryhmä $T(t)$ määrää ratkaisun $x(t)$ käyttäytymisen, voimme nähdä yhtälön ratkaisujen stabiilisuuden puoliryhmän $T(t)$

ominaisuutena. Tämän vuoksi kutsumme vahvasti jatkuvaa puoliryhmää $T(t)$ eksponentiaalisesti stabiiliksi, jos on olemassa reaaliset vakiot $M \geq 1$ ja $\omega > 0$ siten että $\|T(t)\| \leq Me^{-\omega t}$ pätee kaikilla $t \geq 0$.

Tässä työssä tarkastelemme seuraavaa ongelmaa: Oletetaan, että operaattori A, jonka määrittelyjoukko on $\mathcal{D}(A)$, generoi eksponentiaalisesti stabiilin vahvasti jatkuvan puoliryhmän Banach- tai Hilbert-avaruudessa X ja B, jonka määrittelyjoukko on $\mathcal{D}(B)$, on lineaarinen operaattori avaruudessa X. Millä ehdoilla operaattori $A + B$ generoi eksponentiaalisesti stabiilin vahvasti jatkuvan puoliryhmän avaruudessa X?

Ongelma voidaan jakaa kahteen osaan. Koska jokainen lineaarinen operaattori ei generoi vahvasti jatkuvaa puoliryhmää, on meidän ensin tarkasteltava milloin operaattori $A + B$ on jonkin C_0-puoliryhmän infinitesimaalinen generaattori. Tämän jälkeen etsimme lisäehtoja sille, että häiritty puoliryhmä on eksponentiaalisesti stabiili.

Ongelman ensimmäiseen osaan vastataksemme esittelemme erilaisia ehtoja sille, että myös häiritty operaattori generoi vahvasti jatkuvan puoliryhmän avaruudessa X. Tätä aihetta on tutkittu 1950-luvulta lähtien ja teoriaa on kehitetty sekä rajoitetuille että ei-rajoitetuille häiriöille. Jo vuonna 1953 R.S. Phillips julkaisi tuloksia, jotka antavat tyhjentävän vastauksen rajoitettujen häiriöiden tapaukseen. Tämän jälkeen tutkimuksen pääpaino on ollut ei-rajoitettujen häiriöiden tapauksessa. Jo hyvin aikaisessa vaiheessa huomattiin, että tietyt vahvasti jatkuvien puoliryhmien luokat sietävät toisia paremmin ei-rajoitettuja häiriöitä. Tässä työssä esittämämme tulokset analyyttisten ja kontraktiivisten puoliryhmien generaattorien häiriöille esiteltiin ensimmäisen kerran jo 1950-luvun loppupuolella. Näiden häiriötulosten lisäksi esittelemme vielä kolme erilaista kokoelmaa ehtoja yleisen vahvasti jatkuvan puoliryhmän generaattorin häiritsemiselle siten, että myös häiritty operaattori generoi vahvasti jatkuvan puoliryhmän avaruudessa X. Nämä ehdot voidaan järjestää C_0-puoliryhmän generoinnin säilyttävien häiriöiden luokiksi $\mathcal{S}_{t_0}^{\mathrm{DS}}$, $\mathcal{S}_{t_0}^{\mathrm{MV}}$ ja $\mathcal{S}^{\mathrm{KW}}$. Eli, jos A generoi vahvasti jatkuvan puoliryhmän avaruudessa X ja operaattori B kuuluu johonkin vastaavista luokista $\mathcal{S}_{t_0}^{\mathrm{DS}}$, $\mathcal{S}_{t_0}^{\mathrm{MV}}$ tai $\mathcal{S}^{\mathrm{KW}}$, niin myös häiritty operaattori generoi vahvasti jatkuvan puoliryhmän avaruudessa X. Ensimmäinen luokista seuraa W. Deschin ja W. Schappacherin johtamista ehdoista. Toisen luokan ehdot ovat alunperin I. Miyaderan julkaisemia ja myöhemmin J. Voigtin eteenpäin kehittämiä. Tässä työssä käyttämämme yhtenäistetty lähestymistapa näiden kahden häiriöluokan käsittelyyn seuraa K-J. Engelin ja R. Nagelin esittelemää menettelyä. Kolmas luokka seuraa C. Kaiserin ja L. Weisin esittelemistä ehdoista.

Ongelmamme toiseen osaan löytyy kirjallisuudesta vain muutamia suoria vastauksia. Lisäksi monet näistä tuloksista pätevät vain tietyille vahvasti jatkuvien puoliryhmien luokille tai tietyn tyyppisille häiriöille. Tässä diplomityössä käytämme useampaa erilaista lähestymistapaa.

Ensimmäisessä lähestymistavassa aloitamme esittelemällä erilaisia kirjallisuudesta löytyviä tapoja määrittää vahvasti jatkuvan puoliryhmän stabiilisuus. Vaikka puoliryhmän stabiilisuuden määrääminen ääretönulotteisessa avaruudessa on monimutkaisempaa kuin äärellisulotteisessa tapauksessa, löytyy tähän useita erilaisia menetelmiä. Soveltamalla joitakin näistä ehdoista häirityn puoliryhmän tapaukseen voimme johtaa riittäviä ehtoja tämän puoliryhmän eksponentiaaliselle stabiilisuudelle. Johdamme ehtoja ensin tapaukselle jossa operaattori B on A-rajoitettu ja häiritty operaattori $A+B$, jonka määrittelyjoukko on $\mathcal{D}(A+B)=\mathcal{D}(A)$, generoi vahvasti jatkuvan puoliryhmän. Lisäksi esittelemme nämä ehdot erikseen tapauksissa $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ ja $B \in \mathcal{S}^{\mathrm{KW}}$. Yleinen ehtomme on sovellettavissa myös rajoitettujen häiriöiden tapaukseen. Tässä tapauksessa näemme myös, että rajoitettujen häiriöiden tunnettu ja yksinkertainen ehto eksponentiaaliselle stabiilisuudelle seuraa tässä työssä johtamistamme ehdoista. Yleistä ehtoamme ei voida käyttää jos $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, mutta käsittelemällä tapauksen erikseen voimme johtaa vastaavan tuloksen myös näille häiriöille. Lopuksi esittelemme vielä samankaltaisen mutta erillisen tuloksen kontraktiivisten puoliryhmien häiriöille.

Toisena lähestymistapana etsimme osittaista ratkaisua eksponentiaalisen stabiilisuuden säilymiselle. Äärellisulotteisissa avaruuksissa puoliryhmän, siis eksponenttifunktion e^{At}, stabiilisuusominaisuuksien määrittäminen onnistuu määrittämällä matriisin A ominaisarvot. Puoliryhmä on tällöin stabiili, jos kaikkien ominaisarvojen reaaliosat ovat aidosti negatiivisia. Joissakin erikoistapauksissa myös ääretönulotteisen avaruuden puoliryhmän stabiilisuus voidaan päätellä sen infinitesimaalisen generaattorin spektristä. Joillekin vahvasti jatkuvien puoliryhmien luokille nimittäin pätee, että puoliryhmä on eksponentiaalisesti stabiili täsmälleen silloin kun sen generaattorin spektrin alkioiden reaaliosat ovat ylhäältäpäin rajoitettuja jollakin negatiivisella luvulla. Tässä työssä etsimme ehtoja sille, että häirityn puoliryhmän stabiilisuus voidaan päätellä häirityn operaattorin spektristä. Näitä ehtoja on kahta tyyppiä. Ensimmäisissä näistä hyödynnetään suoraan sitä, että jos vahvasti jatkuvalla puoliryhmällä on tiettyjä säännöllisyysominaisuuksia, voimme päätellä onko puoliryhmä eksponentiaalisesti stabiili tarkastelemalla sen generaattorin spektriä. Jos alkuperäisellä puoliryhmällämme on tällainen ominaisuus ja rajoitumme tarkastelemaan häiriöitä jotka säilyttävät tämän säännöllisyysominaisuuden, voimme tällöin päätellä häirityn puoliryhmän stabiilisuuden häirityn operaattorin spektristä. Tämän käsittelytavan lisäksi voimme saavuttaa vastaavanlaisen tilanteen myös yleisempien puoliryhmien tapauksessa. Jos alkuperäinen vahvasti jatkuva puoliryhmämme on eksponentiaalisesti stabiili, on olemassa suoraan häiriöön kohdistuvia ehtoja, joiden toteutuessa häirityn puoliryhmän stabiilisuus voidaan päätellä häirityn operaattorin spektristä.

Teoriaa voidaan käyttää esimerkiksi tutkittaessa miten abstraktin Cauchy-ongelman $\dot{x}(t)=Ax(t)$ ratkaisujen stabiilisuus muuttuu, jos tämä korvataan yhtälöllä $\dot{x}(t)=(A+\Delta A)x(t)$ alkuehdon pysyessä samana. Tällaisessa tapauksessa häiriön ΔA voi aiheuttaa esimerkiksi operaattorin A korvaaminen sen äärellisulotteisella approksimaatiolla. Jos A on rajoittamaton operaattori, on selvää ettei operaattori ΔA välttämättä ole rajoitettu. Toisaalta esimerkiksi säätöteoriassa tulee monesti vastaan tilanteita, joissa operaattorit ovat rajoitettuja tai kompakteja. Tämän vuoksi on hyödyllistä etsiä myös tuloksia, jotka pätevät joissain erityistapauksissa.

Preface

This Master of Science thesis was written at the Department of Mathematics of Tampere University of Technology. The main part of the work was done during the autumn of 2006 and the following winter.

I would like to thank Professor Seppo Pohjolainen for providing the interesting subject of this thesis and for the valuable guidance throughout the process. I am also grateful to Senior Researcher Timo Hämäläinen for the great amount of helpful advice I have received during the making of this thesis. Special thanks to the staff at the Department of Mathematics for the enjoyable working enviroment.

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Contents

Symbols

$\langle \cdot, \cdot \rangle$	inner product or dual pair $\langle x, x^* \rangle = x^*(x)$
$\|\cdot\|$	norm
$\|\cdot\|_A$	graph norm of operator A
$\varnothing$	empty set
$X \subset Y$	X is a subset of Y
$X \supset Y$	X is a superset of Y
$A\|_Y$	restriction of operator A to Y
$\overline{A}$	extension of operator A
A^{-1}	inverse of operator A
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^+$	$\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0 \}$
$\mathcal{D}(A)$	domain of operator A
F_α	Favard space of order $\alpha \in \mathbb{R}$
$\mathcal{F}$	Fourier transform
i	imaginary unit $\sqrt{-1}$
$\operatorname{Im} \lambda$	imaginary part of $\lambda \in \mathbb{C}$
$\mathcal{K}(X)$	compact operators on X
$\ell^p(\mathbb{C})$	$\{ (x_n) \subset \mathbb{C} \mid \sum_{n=1}^\infty \|x_n\|^p < \infty \}$
$\|\cdot\|_{\ell^p(\mathbb{C})}$	$\|(x_n)\|_{\ell^p(\mathbb{C})} = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p}$
$\mathcal{L}(X, Y)$	bounded linear opeartors from X to Y
$\mathcal{L}(X)$	bounded linear operators on X
$L^p(I, Y)$	p-integrable functions from I to Y
$\mathbb{N}$	the set of natural numbers
$\omega_0(T(t))$	growth bound of a C_0-semigroup
$\omega_{\text{ess}}(T(t))$	essential growth bound of a C_0-semigroup
$\omega_{\text{crit}}(T(t))$	critical growth bound of a C_0-semigroup
$\mathbb{R}$	the set of real numbers

$\mathbb{R}^+$	the set of non-negative real numbers
$r(A)$	spectral radius of operator A
$\operatorname{ran} A$	range of operator A
$\operatorname{Re} \lambda$	real part of $\lambda \in \mathbb{C}$
$R(\lambda, A) = (\lambda I - A)^{-1}$	resolvent operator of operator A at $\lambda \in \mathbb{C}$
$\rho(A)$	resolvent set of operator A
$\sigma(A)$	spectrum of operator A
$s(A)$	spectral bound of operator A
$\mathcal{S}_{t_0}^{\mathrm{DS}}$	class of Desch-Schappacher perturbations
$\mathcal{S}_{t_0}^{\mathrm{MV}}$	class of Miyadera-Voigt perturbations
$\mathcal{S}^{\mathrm{KW}}$	perturbations satisfying conditions of Kaiser and Weis
$t \to 0^+$	limit from the right
X^*	Dual space of X, $\mathcal{L}(X, \mathbb{C})$
X_n	Sobolev space of order $n \in \mathbb{Z}$
$X/_M$	quotient space of X modulo M
$\mathcal{X}$	strongly continuous functions from $[0, \infty)$ to $\mathcal{L}(X)$
$\mathcal{X}_{t_0}$	strongly continuous functions from $[0, t_0]$ to $\mathcal{L}(X)$
$\mathbb{Z}$	the set of integers

Chapter 1

Introduction

In this thesis we study strongly continuous semigroups of bounded linear operators, or C_0-semigroups, and how their properties change under certain kinds of perturbations. We will start by introducing the operator semigroups and relating them to familiar structures of finite-dimensional spaces. A more mathematical formulation of the concepts involved will be given in chapter 2. In this chapter we will also formulate the main problem, give a brief account of how it has been addressed before and outline the approach used in this thesis.

In a way, a semigroup of linear operators is a generalization of an exponential function. Recall that for a matrix $A \in \mathbb{C}^{n\times n}$ the mapping

$$t \mapsto T(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \tag{1.1}$$

produces an $n \times n$-matrix for every $t \geq 0$. From the basic properties of the exponential function we also know that this mapping satisfies $T(t+s) = T(t)T(s)$ and $T(0) = I$.

In a general infinite-dimensional Banach space X a closed and possibly unbounded operator

$$A : X \supset \mathcal{D}(A) \to X$$

can give rise to a similar structure. If this is the case, then the *strongly continuous semigroup of bounded linear operators generated by* A is a strongly continuous mapping

$$T(\cdot) : [0, \infty) \to \mathcal{L}(X)$$

satisfying $T(t+s) = T(t)T(s)$ and $T(0) = I$. However, unlike in the finite-dimensional case, it is in general not possible to find a formula like the one in (1.1) for $T(t)$ in infinite-dimensional spaces. A more detailed introduction to the properties of strongly continuous semigroups and their generators is given in chapter 2. We will now describe what is meant by the *stability* of a semigroup.

Semigroups of linear operators are heavily related to the behaviour of dynamical systems. Let us start with the homogeneous abstract Cauchy equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \tag{1.2}$$

on X where $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup. As in finite-dimensional case, the solution of this equation can be given in terms of the semigroup $T(t)$:

$$x(t) = T(t)x_0$$

If the initial value x_0 belongs to $\mathcal{D}(A)$, the domain of the operator A, this is the classical solution of the equation. If this is not the case, the solution is a mild (or weak) solution of the equation.

The solution $x(t)$ of this equation is called stable if it approaches zero as t grows. There are several different ways this can happen. In particular we are interested in the case where the norm of the solution approach zero at an exponential rate with t for all initial values $x_0 \in X$. Expressed in terms of the properties of the semigroup, this means that we are able to find real constants $M \geq 1$ and $\omega > 0$ such that for all $t \geq 0$

$$\|T(t)\| \leq Me^{-\omega t}. \tag{1.3}$$

Since for all initial values x_0 the behaviour of $x(t)$ is determined by the strongly continuous semigroup, we can see that the stability of the solutions is a property of the semigroup. To this end, we will call a strongly continuous semigroup $T(t)$ *exponentially stable* if there exist real constants $M \geq 1$ and $\omega > 0$ such that (1.3) holds for all $t \geq 0$.

In this thesis we consider the preservation of the exponential stability of the semigroup when its generator A is subjected to an additive perturbation. When considering abstract Cauchy equations, this means that equation (1.2) is replaced with an equation of form

$$\dot{x}(t) = (A + \Delta A)\, x(t), \qquad x(0) = x_0. \tag{1.4}$$

This can happen for example when the operator is not exactly known or when discretization error occurs in simulations. This kind of situation is also often encountered in *control theory* where we have a *system* described by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

Here $y(t)$ is called the *output* and $u(t)$ the *control* of the system. If we apply a *feedback* of form $u(t) = Ky(t)$, the *state* $x(t)$ will be given by the abstract Cauchy equation

$$\dot{x}(t) = (A + BKC)x(t), \qquad x(0) = x_0.$$

Obviously the operator BKC can be seen as a perturbation.

Our aim is to consider a wide range of perturbations. If A is an unbounded operator and the perturbation in the abstract Cauchy equation (1.4) is caused by replacing it with its finite-dimensional approximation, it is clear that the perturbing operator ΔA does not have to be bounded. On the other hand, in the case of feedback considered above it is common that some of the operators B, K and C are finite-dimensional and bounded. In some of these cases the perturbing operator BKC can become compact. This motivates us not only to consider the most general perturbations, but also to find particular results for certain special classes of perturbations.

Considering the abstract Cauchy problem (1.4) also gives rise to a question under what conditions the perturbed equation has a solution. This can, however, be answered using the theory of strongly continuous semigroups: It turns out that as long as the operator $A + \Delta A$ generates a strongly continuous semigroup, the perturbed equation has a unique solution for all initial values $x_0 \in X$ [8, Cor II.6.9.].

We will now formulate the main problem of the thesis:

Assume that an operator A generates an exponentially stable strongly continuous semigroup $T(t)$ on a Banach or Hilbert space X and consider a perturbing operator $B : \mathcal{D}(B) \to X$ with $\mathcal{D}(B) \subset X$. Under what conditions on the operators A and B does the perturbed operator $A + B$ with domain $\mathcal{D}(A + B)$ generate an exponentially stable strongly continuous semigroup on X?

The problem can be divided into two main parts:

- Under what assumptions does $A + B$ generate a strongly continuous semigroup on X?
- What additional conditions are needed for this semigroup to be stable?

Answering the first part of the problem is pretty straightforward. This particular question has been studied actively since the 1950's and a fair amount of theory has been developed for both bounded and relatively bounded perturbations. As early as in 1953, Phillips [21] presented results which give a thorough answer to the problem in the case $B \in \mathcal{L}(X)$. Since then the main emphasis has been in the study of relatively bounded perturbations. In chapter 4 we will introduce different types of conditions for the perturbed operator to be a generator of a C_0-semigroup. At the beginning of the chapter we will also look at the developement of the theory in more detail.

Literature presents only few answers to the second part of the problem and in this thesis we will consider different approaches. We will first introduce the theory on the stability of strongly continuous semigroups. Subsequently, we will apply some of these criteria to the case of perturbed semigroups in order to formulate an answer to the second part of the problem. We will first derive sufficient conditions for the preservation of stability when the perturbed operator generates a C_0-semigroup on X. We will also formulate these conditions for certain classes of perturbations for which the perturbed operator again generates a semigroup on X. We will also derive separate sufficient conditions for certain perturbations for which the general conditions are not applicable. We will also compare the results obtained with this method to existing ones by Pritchard and Townley [23]. These results are presented in Section 5.1 and they are by the author.

As a second approach to the problem we look for a partial answer to the question. In finite-dimensional spaces the semigroup of linear operators (the exponential function), is exponentially stable exactly when the spectrum of its generator is contained in the open left half-plane of the complex plane $\mathbb{C}$. Even though the case is far more complicated in infinite-dimensional spaces, we will learn in chapter 3 that if the semigroup has certain regularity properties we can achieve a characterization which is similar to the one in finite-dimensional case. More precisely, for some classes of regular semigroups it holds that the strongly continuous semigroup is exponentially stable if the real part of the spectrum of the generator is bounded from above by a negative constant. A more detailed description of what we mean by a "regular semigroup" is given in section 2.1.3. Using this theory we will in section 5.2 formulate conditions under which the exponential stability of the perturbed semigroup is determined by the spectrum of the perturbed operator. In the same section we will also characterize some perturbations which lead to the same situation without additional assumptions on the unperturbed semigroup.

Pandolfi and Zwart [19] have considered this problem in Hilbert spaces for relatively bounded perturbations satisfying certain special assumptions. They use the fact that the exponential stability of a strongly continuous semigroup can be characterized by the existence of a positive self-adjoint solution to a certain *Lyapunov equation* (see [5] or [4, Thm 5.1.3]). We will present the main results of this theory in section 5.3.

This thesis is arranged as follows:

Chapter 2 introduces mathematical concepts used in this thesis. We will give a detailed formulation of the strongly continuous semigroups of linear operators and introduce their most important properties. We will also introduce theory on Sobolev towers and Favard spaces, both of which will be used throughout the thesis.

Chapter 3 discusses the stability of strongly continuous semigroups. This is more complicated in infinite-dimensional spaces than in finite-dimensional ones. The first part of the chapter discusses characterization of general strongly continuous semigroups on Banach and Hilbert spaces. The latter part of the chapter shows that in case the strongly continuous semigroups has certain special properties, its stability is completely determined by the spectrum of its infinitesimal generator.

Chapter 4 studies conditions under which the perturbed operator again generates a strongly continuous semigroup. Before considering more general perturbations, we will give a brief account of the theory on bounded perturbations and perturbations of certain special classes of semigroups. The rest of the chapter is used to introduce three classes of relatively bounded perturbations relating to general strongly continuous semigroups.

Chapter 5 contains derivation of conditions for the stability of the perturbed semigroup. The first part of the chapter presents direct conditions for the stability of a general strongly continuous semigroup. In the second part, we present conditions under which the stability of the perturbed semigroup is determined by the spectrum of the perturbed generator. Finally, we present conditions obtained by Pandolfi and

Zwart for the stability of the perturbed strongly continuous semigroups under certain type of perturbations.

Chapter 6 contains concluding remarks.

Appendix A lists some helpful results from functional analysis and integration theory.

Chapter 2

Mathematical Background

In this chapter we will present some mathematical results which we will use later in the thesis. For the purposes of this thesis, the most important part is Section 2.1 where we define a strongly continuous semigroup of linear operators. More thorough introductions to this topic can be found for example in [20, 4, 8]. We will also use some more advanced concepts related to strongly continuous semigroups. The theory on *the essential growth bound* [8, 15] and *the critical growth bound* [17, 3] are needed when considering particular classes of perturbations in Section 5.2. Sobolev towers and Favard spaces are used for a unified treatment of two classes of unbounded perturbations in chapter 4. The introduction to this theory follows the one given in [8].

2.1 Semigroup Theory

In this section we will define the strongly continuous semigroups mathematically and introduce their most important properties. As we already stated, this structure can be seen as a generalization of the exponential function from finite-dimensional spaces to Banach spaces. Because of this, it is useful to compare the properties of the strongly continuous semigroups introduced here to the corresponding properties of exponential functions. The following two definitions formulate the concepts of strongly continuous semigroups and their generators.

Definition 2.1. A *semigroup* of bounded linear operators is an operator-valued function $T(t)$ from $\mathbb{R}^+$ to $\mathcal{L}(X)$ having the properties

(i) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$,

(ii) $T(0) = I$.

Furthermore, if the function satisfies

(iii) $\|T(t)x_0 - x_0\| \rightarrow 0$ as $t \rightarrow 0^+$, $\forall x_0 \in X$,

it is called a *strongly continuous semigroup*. The term strongly continuous semigroup is often abbreviated as C_0-*semigroup*. ■

Definition 2.2. The *infinitesimal generator* A of a C_0-semigroup on a Banach space X is defined by

$$\begin{aligned} Ax &= \lim_{t \rightarrow 0+} \frac{T(t)x - x}{t} \\ \mathcal{D}(A) &= \{ x \in X \mid \lim_{t \rightarrow 0+} \frac{T(t)x - x}{t} \text{ exists} \}. \end{aligned}$$

■

The infinitesimal generator of a C_0-semigroup $T(t)$ is sometimes simply called the generator and we say that "A generates $T(t)$".

It is now easy to see that if $X = \mathbb{C}^n$, then for a matrix $A \in \mathbb{C}^{n \times n}$ the exponential function

$$T(t) = e^{At}$$

is a C_0-semigroup and its infinitesimal generator is the matrix A (which is an operator on X).

Before stating properties of C_0-semigroups, we will consider their asymptotic behaviour. Strongly continuous semigroups have a property that the growth of their norm is at most exponential with respect to t. This means that for any C_0-semigroup $T(t)$ we can find real constants $M \geq 1$ and ω such that

$$\|T(t)\| \leq Me^{\omega t}$$

holds for all $t \geq 0$ [8, Prop I.5.5]. Of course, since this estimate is only an upper bound for the growth, we are naturally interested in finding a bound which is as strict as possible. Because of this, we will define the growth bound of a C_0-semigroup as follows.

Definition 2.3. The *growth bound* $\omega_0(T(t))$ of a C_0-semigroup $T(t)$ is defined as

$$\omega_0(T(t)) = \inf\{\, \omega \in \mathbb{R} \;\big|\; \exists M \geq 1 : \; \|T(t)\| \leq Me^{\omega t}, \;\; \forall t \geq 0 \,\}.$$

■

As we already mentioned, we are interested in the kind of stability where the norm of the C_0-semigroup decays exponentially with respect to t. We will now give a precise definition of this property. This definition also immediately leads us to the characterization of this kind of stability in terms of the growth bound of the C_0-semigroup.

Definition 2.4. A C_0-semigroup $T(t)$ on a Banach space X is *exponentially stable* if there exist positive constants M and ω such that

$$\|T(t)\| \leq Me^{-\omega t}, \quad \text{for all } t \geq 0.$$

In other words, for the growth bound of the C_0-semigroup holds $\omega_0(T(t)) < 0$. ■

In finite-dimensional spaces, the "growth bound" of an exponential function is equal to the largest real part of the generator's eigenvalues. This does not hold in the infinite-dimensional space, but we have the following relation between the growth bound of a C_0-semigroup and the spectral bound of its generator

Theorem 2.5. *Let A generate a C_0-semigroup $T(t)$ on a Banach space X. Then the following holds:*

$$-\infty \leq s(A) \leq \omega_0(T(t)) < \infty.$$

Also, the spectral radius of the C_0-semigroup is given by

$$r(T(t)) = e^{\omega_0(T(t))t}, \quad \textit{for all } t \geq 0.$$

Proof. [8, Prop IV.2.2] □

The following theorem gives some properties of C_0-semigroups and their infinitesimal generators. These results are frequently used throughout the thesis.

Theorem 2.6. *Let $T(t)$ be a C_0-semigroup on a Banach space X with infinitesimal generator A. The following results hold:*

(i) $\|T(t)\|$ *is bounded on every finite subinterval of* $[0, \infty)$,

(ii) $T(t)$ *is strongly continuous for all* $t \in [0, \infty)$,

(iii) A *is a closed densely defined linear operator,*

(iv) If $\lambda \in \mathbb{C}$ *and* $\operatorname{Re}\lambda > \omega_0(T(t))$, *then* $\lambda \in \rho(A)$ *and* $R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x dt$ *for all* $x \in X$,

(v) If $x_0 \in \mathcal{D}(A)$, *then* $T(t)x_0 \in \mathcal{D}(A)$ *for all* $t \geq 0$,

(vi) $\frac{d^n}{dt^n}(T(t)x_0) = A^nT(t)x_0 = T(t)A^nx_0$ *for* $x_0 \in \mathcal{D}(A^n)$, $t > 0$.

Proof. [20],[8],[4] □

Although expressing the C_0-semigroup in terms of its infinitesimal generator is generally not as straight-forward as it was in finite-dimensional spaces, we can determine when a closed linear operator on a Banach space is a generator of a C_0-semigroup. The following well-known theorem presents a complete characterization of the generators of C_0-semigroups on a Banach space.

Theorem 2.7 (Hille-Yosida)**.** *A closed, densely defined, linear operator A on a Banach space X is the infinitesimal generator of a strongly continuous semigroup $T(t)$ if and only if there exist real numbers M and ω such that for all λ with* $\operatorname{Re}\lambda > \omega$ *it follows that* $\lambda \in \rho(A)$ *and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad \textit{for all } n \in \mathbb{N}.$$

The C_0-semigroup then satisfies

$$\|T(t)\| \leq Me^{\omega t}.$$

Proof. [20], [8], [4] □

We will now define another concept frequently used in this thesis. Let the operator A with domain $\mathcal{D}(A)$ be the generator of a C_0-semigroup $T(t)$ on a Banach space X. If $\beta \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ with $\alpha > 0$, then the operator $B = \alpha A + \beta I$ with domain $\mathcal{D}(B) = \mathcal{D}(A)$ is the generator of a *rescaled* C_0-semigroup $e^{\beta t}T(\alpha t)$ on X [8, Par II.2.2]. Clearly, since for all $\omega > \omega_0(T(t))$

$$\|e^{\beta t}T(\alpha t)\| = e^{\operatorname{Re}\beta t}\|T(\alpha t)\| \leq Me^{\operatorname{Re}\beta t}e^{\omega\alpha t} = Me^{(\alpha\omega+\operatorname{Re}\beta)t},$$

the rescaled C_0-semigroup has the growth bound $\omega_0(e^{\beta t}T(\alpha t)) = \alpha\omega_0(T(t)) + \operatorname{Re}\beta$.

Before moving on, we will give a simple example of an exponentially stable C_0-semigroup.

Example 2.8. Consider the space $X = \ell^2(\mathbb{C})$ and an unbounded operator A

$$A(x_k) = (-kx_k), \qquad \mathcal{D}(A) = \left\{ x \in X \;\middle|\; Ax \in X \right\}.$$

The operator A generates a C_0-semigroup $T(t)$ on X with

$$T(t)x = (e^{-kt}x_k), \qquad x \in X.$$

Since we have for all $x \in X$ with $\|x\| = 1$

$$\|T(t)x\|^2 = \sum_{k=1}^{\infty} |e^{-kt}x_k|^2 = \sum_{k=1}^{\infty} e^{-2kt}|x_k|^2 \leq \sum_{k=1}^{\infty} e^{-2t}|x_k|^2 = e^{-2t}\|x\| = e^{-2t},$$

the C_0-semigroup satisfies $\|T(t)\| \leq Me^{-\omega t}$ with $M = 1$ and $\omega = 1$.

In sections 2.1.1 and 2.1.2 we will define quantities which can be used together with the spectral bound of a generator to determine the stability of a C_0-semigroup.

2.1.1 The Essential Growth Bound

Let A be a bounded linear operator on a Banach space X. The operator A is called a *Fredholm operator* if

$$\dim \ker A < \infty \quad \text{and} \quad \dim \left(X\big/ \operatorname{ran} A \right) < \infty.$$

For an operator $B \in \mathcal{L}(X)$ a *Fredholm domain* is defined as

$$\rho_F(B) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \lambda I - B \text{ is a Fredholm operator} \right\}$$

and the *essential spectrum of* B is defined as its complement,

$$\sigma_{\text{ess}}(B) = \mathbb{C} \setminus \rho_F(B).$$

Consider the quotient space $\mathcal{L}(X)\big/\mathcal{K}(X)$ where $\mathcal{K}(X) \subset \mathcal{L}(X)$ is the set of all compact operators on X. For an operator

$$\widehat{A} \in \mathcal{L}(X)\big/\mathcal{K}(X)$$

where $\widehat{A} = A + K$, $K \in \mathcal{K}(X)$, we have the quotient norm (see definition A.1)

$$\|\widehat{A}\| = \operatorname{dist}(A, \mathcal{K}(X)) = \inf\left\{ \|A - K\| \;\middle|\; K \in \mathcal{K}(X) \right\}.$$

Now the Fredholm domain and the essential spectrum are given by [8, Par IV.1.20]

$$\begin{aligned} \rho_F(A) &= \rho(\widehat{A}) \\ \sigma_{\text{ess}}(A) &= \sigma(\widehat{A}). \end{aligned}$$

We also define the *essential norm* by $\|A\|_{\text{ess}} = \|\widehat{A}\|$. Finally, the *essential growth bound* of a C_0-semigroup $T(t)$ is defined as

$$\omega_{\text{ess}}(T(t)) = \inf\left\{ \omega \in \mathbb{R} \;\middle|\; \exists M \geq 1 : \; \|T(t)\|_{\text{ess}} \leq M e^{\omega t}, \;\; \forall t \geq 0 \right\}.$$

The following theorem states the relation between the growth bound and the essential growth bound of a C_0-semigroup.

Theorem 2.9. *Let $T(t)$ be a C_0-semigroup on a Banach space with generator A. Then the growth bound of $T(t)$ is given by*

$$\omega_0(T(t)) = \max\{\omega_{ess}(T(t)), s(A)\}.$$

Proof. [8, Cor IV.2.11] □

2.1.2 The Critical Growth Bound

We will go on to define the critical growth bound introduced in [17]. Define first a (not necessarily strongly continuous) semigroup $\tilde{T}(t)$ in the space $\tilde{X} = l^\infty(X)$ by

$$\tilde{T}(t)\tilde{x} = (T(t)x_n), \qquad \tilde{x} = (x_n) \in \tilde{X}.$$

For this semigroup, consider the subspace of strong continuity

$$\tilde{X}_{T(t)} = \{\, \tilde{x} \in \tilde{X} \mid \lim_{t\to 0+} \|\tilde{T}(t)\tilde{x} - \tilde{x}\| = 0 \,\}.$$

Denote by $\hat{T}(t)$ the quotient semigroup induced by $\tilde{T}(t)$ in the space $\hat{X} = {}^{\tilde{X}}\!/_{\tilde{X}_{T(t)}}$:

$$\hat{T}(t)\hat{x} = \tilde{T}(t)\tilde{x} + \tilde{X}_{T(t)} \;\text{ where }\; \hat{x} = \tilde{x} + \tilde{X}_{T(t)}.$$

The *critical growth bound* of a C_0-semigroup $T(t)$ is now defined as

$$\omega_{\text{crit}}(T(t)) = \omega_0(\hat{T}(t)).$$

Similarly to the case of the essential growth bound in Section 2.1.1, we have the following result which states the relation between the growth bound and the critical growth bound of a C_0-semigroup $T(t)$.

Theorem 2.10. *Let A be the generator of a C_0-semigroup $T(t)$. Then the growth bound of $T(t)$ is given by*

$$\omega_0(T(t)) = \max\{s(A), \omega_{crit}(T(t))\}.$$

Proof. [17] □

2.1.3 Special Classes of Semigroups

In this section we will define C_0-semigroup classes based on certain special properties. We will see in chapter 3 that this classification will be helpful when determining the stability of a C_0-semigroup.

Definition 2.11. A C_0-semigroup $T(t)$ is called a *C_0-semigroup of contractions* if $\|T(t)\| \leq 1$ holds for all $t \geq 0$. ■

Definition 2.12. A C_0-semigroup $T(t)$ is called *eventually compact* if $T(t)$ is a compact operator for $t > t_0$ for some $t_0 \geq 0$. The C_0-semigroup $T(t)$ is called *immediately compact* if we can choose $t_0 = 0$. ■

Definition 2.13. A C_0-semigroup $T(t)$ is called *eventually differentiable* if for some $t_0 \geq 0$ and for every $x \in X$ the maps $t \mapsto T(t)x$ are differentiable for $t > t_0$. The C_0-semigroup $T(t)$ is called *immediately differentiable* if we can choose $t_0 = 0$. ■

Definition 2.14. A C_0-semigroup $T(t)$ is called *eventually norm-continuous* if $T(t)$ is norm-continuous for $t > t_0$, that is,

$$\lim_{h \to 0+} \|T(t+h) - T(t)\| = 0, \quad \text{for } t > t_0.$$

The C_0-semigroup is called *immediately norm-continuous* if we can choose $t_0 = 0$. ■

In the previous definition, the limit is only required to be zero when approaching 0 from the positive side. However, in this case the properties of the C_0-semigroups imply that $T(s)$ converges uniformly to $T(t)$ whenever $s \to t$. The same holds for the the next definition.

Definition 2.15. A C_0-semigroup is called *uniformly continuous* if $T(t)$ is uniformly continuous for $t \geq 0$, that is,

$$\lim_{h \to 0+} \|T(t+h) - T(t)\| = 0, \quad \forall t \geq 0.$$

■

Uniformly continuous C_0-semigroups are a relatively restricted class, because they are exactly those C_0-semigroups whose infinitesimal generators are bounded linear operators on X [20, Thm 1.1.2].

As the last special class, we will define analytic semigroups. The name comes from the property that the mapping

$$t \mapsto T(t)$$

can be continued from the positive real axis to an analytic function on a certain part of the complex plane. We denote a sector in $\mathbb{C}$ by

$$\Sigma_\delta = \left\{ \lambda \in \mathbb{C} \;\middle|\; |\arg \lambda| < \delta \right\} \setminus \{0\}.$$

Definition 2.16. A family of linear operators $T(z)$, $z \in \Sigma_\delta \cup \{0\}$, is called an *analytic semigroup* if

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Sigma_\delta$,

(ii) The map $z \to T(z)$ is analytic in Σ_δ,

(iii) For all $x \in X$ and $0 < \delta' < \delta$

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_{\delta'}}} T(z)x = x.$$

Analytic semigroup $T(z)$ is called a *bounded analytic semigroup* if $\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ for all $0 < \delta < \delta'$. ■

We will need the the following characterization of analytic semigroups later in the thesis.

Theorem 2.17. *For an operator A on a Banach space X the following properties are equivalent:*

(i) A generates a bounded analytic semigroup $T(z)$ on X.

(ii) A generates a bounded C_0-semigroup on X and there exists a constant $M > 0$ such that

$$\|R(r + is, A)\| \leq \frac{M}{|s|}$$

for all $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$.

Proof. [8, Thm II.4.6], [20, Thm 2.5.2] □

Throughout the thesis we will use terms such as ”regular C_0-semigroup” or speak of ”regularity properties” of the C_0-semigroup. By this we mean that the C_0-semigroup in question belongs to some of the special classes of semigroups introduced in definitions 2.12, 2.13, 2.14 or 2.16.

2.2 Interpolation and Extrapolation Spaces

We will now introduce Sobolev towers and Favard spaces. These concepts allow unified treatment of certain classes of perturbations and they help us simplify notation throughout the thesis. The spaces are constructed by extending and reducing the original Banach space X and the set of spaces is always related to a specific C_0-semigroup on X.

We assume that A generates an exponentially stable C_0-semigroup on a Hilbert space X. If this is not the case, we can carry out the construction by considering a rescaled C_0-semigroup $e^{-\omega t}T(t)$ generated by the operator $A - \omega I$ for some $\omega > \omega_0(T(t))$ (see Section 2.1).

We will present the construction of Sobolev spaces X_n and Favard spaces F_α for all $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ even though we will mainly use these spaces for values $n = -1, 0, 1$ and $\alpha = 0, 1$.

A more detailed account of the concepts presents here can be found in [8].

2.2.1 Sobolev Towers

Define $X_0 = X$, $T_0(t) = T(t)$, $A_0 = A$ and $\|x\|_n = \|A^n x\|$. Now the Sobolev space of order $n \in \mathbb{N}$ is defined as

$$X_n = (\mathcal{D}(A^n), \|\cdot\|_n).$$

With this definition, X_n are Banach spaces for all $n \in \mathbb{N}$ [8, Prop II.5.2]. We define the C_0-semigroup $T_n(t)$ as the restriction of $T(t)$ to X_n,

$$T_n(t)x = T(t)x \quad \text{for} \quad x \in \left\{ x \in X_n \;\middle|\; T(t)x \in X_n \right\}.$$

It turns out that the generator of the C_0-semigroup $T_n(t)$ is given by the part of A in X_n,

$$\begin{aligned} A_n x &= Ax \quad \text{for } x \in \mathcal{D}(A_n) \\ \mathcal{D}(A_n) &= \left\{ x \in X_n \mid Ax \in X_n \right\} = \mathcal{D}(A^{n+1}) = X_{n+1} \end{aligned}$$

For negative integers n, the norms are defined recursively as $\|x\|_{-n} = \|A^{-1}_{-n+1}x\|_{-n+1}$. The Sobolev space of order $-n$ is now defined as completion of Sobolev space of order $-n+1$ with respect to the norm $\|\cdot\|_{-n}$.

It can now be shown that the space X_n with norm $\|\cdot\|_n$ is a Banach space for all $n \in \mathbb{Z}$ [8, Thm II.5.5]. For $n \in \mathbb{N}$, the C_0-semigroup $T_{-n}(t)$ is the continuous extension of $T_{-n+1}(t)$ to the space X_{-n} and the generator A_{-n} of $T_{-n}(t)$ is then the unique continuous extension of

$$A_{-n+1} : X_{-n+2} \to X_{-n+1}$$

to an isometry $A_{-n} : X_{-n+1} \to X_{-n}$. We will now introduce some important properties which will be useful for us later in the thesis.

The first important result is that the C_0-semigroups $T_n(t)$ are bounded similar for $n \in \mathbb{Z}$ (corollary II.5.3 and theorem II.5.5 in [8]) and thus their growth bounds coincide.

For comparison of perturbations and spaces we will need to know the relationships between certain norms. The results are stated in the next lemma.

Lemma 2.18. *For two elements of the resolvent, $\lambda, \mu \in \rho(A)$, the following hold*

(i) The norm $\|\cdot\|_1$ is equivalent to the graph norm of A (see definition A.2),

(ii) The norms defined by $\|(\lambda I - A)x\|$ and $\|(\mu I - A)x\|$ on X_1 are equivalent,

(iii) The norms defined by $\|R(\lambda, A)x\|$ and $\|R(\mu, A)x\|$ on X are equivalent.

Proof. (i): Let $x \in \mathcal{D}(A)$. Then

$$\|x\|_1^2 = \|Ax\|^2 \leq \|x\|_A^2 = \|Ax\|^2 + \|x\|^2 = \|Ax\|^2 + \|A^{-1}Ax\|^2 \leq (1 + \|A^{-1}\|^2)\|x\|_1^2$$

(ii): Let $x \in X_1$ and $\lambda, \mu \in \rho(A)$. Using the resolvent equation (equation (A.1)) we get

$$\begin{aligned}
\|(\lambda I - A)x\| &= \|(\lambda I - A)(\mu I - A)R(\mu, A)x\| = \|(\mu I - A)(\lambda I - A)R(\mu, A)x\| \\
&\overset{res.eq.}{=} \|(\mu I - A)(\lambda I - A)R(\lambda, A)(I + (\lambda - \mu)R(\mu, A))x\| \\
&= \|(\mu I - A)(I + (\lambda - \mu)R(\mu, A))x\| \\
&= \|(I + (\lambda - \mu)R(\mu, A))(\mu I - A)x\| \\
&\leq (1 + |\lambda - \mu|\|R(\mu, A)\|)\|(\mu I - A)x\|
\end{aligned}$$

Likewise,

$$\|(\mu I - A)x\| \leq (1 + |\mu - \lambda|\|R(\lambda, A)\|)\|(\lambda I - A)x\|$$

(iii): Let $x \in X$ and $\lambda, \mu \in \rho(A)$. Using the resolvent equation again, we get

$$\begin{aligned}
\|R(\lambda, A)x\| &= \|R(\lambda, A)(\mu I - A)R(\mu, A)x\| = \\
&\overset{res.eq.}{=} \|(I + (\mu - \lambda)R(\lambda, A))R(\mu, A)(\mu I - A)R(\mu, A))x\| \\
&= \|(I + (\mu - \lambda)R(\lambda, A))R(\mu, A))x\| \\
&\leq (1 + |\mu - \lambda|\|R(\lambda, A)\|)\|R(\mu, A)x\|
\end{aligned}$$

Similarly,

$$\|R(\mu, A)x\| \leq (1 + |\lambda - \mu|\|R(\mu, A)\|)\|R(\lambda, A)x\|$$

□

To clarify the meaning of these concepts, we present the following simple example.

Example 2.19. Let $X = \ell^2(\mathbb{C})$. Then $x \in X$ is a sequence $x = (x_k)$ with $x_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. Let $Ax = (-kx_k)$ with domain

$$\mathcal{D}(A) = \left\{ x \in X \mid Ax \in X \right\} = \left\{ (x_k) \in X \mid \sum_{k=1}^{\infty} k^2 |x_k|^2 < \infty \right\}.$$

We saw in example 2.8 that A generates a C_0-semigroup on X. Now for $n \in \mathbb{Z}$, the Sobolev space $(X_n, \|\cdot\|_n)$ is defined by

$$\begin{aligned}
X_n &= \left\{ (x_k) \in X \mid \sum_{k=1}^{\infty} k^{2n} |x_k|^2 < \infty \right\} \\
\|x\|_n &= \sqrt{\sum_{k=1}^{\infty} k^{2n} |x_k|^2}
\end{aligned}$$

2.2.2 Favard Spaces

Finally, we will briefly introduce Favard spaces. This concept helps us formulate some perturbation results in chapter 4.

For $0 < \alpha \leq 1$ we define the Favard space of order α as $(F_\alpha, \|\cdot\|_{F_\alpha})$, where

$$F_\alpha = \left\{ x \in X \mid \sup_{t>0} \left\| \frac{1}{t^\alpha}(T(t)x - x) \right\| < \infty \right\}$$

and

$$\|x\|_{F_\alpha} = \sup_{t>0} \left\| \frac{1}{t^\alpha}(T(t)x - x) \right\|.$$

For a general $\alpha \in \mathbb{R}$, we define F_α as follows: Choose $k \in \mathbb{Z}$ and $0 < \gamma \leq 1$ so that $\alpha = k + \gamma$. The Favard space F_α is then defined as the Favard space of order γ associated to the C_0-semigroup $T_k(t)$.

It follows directly from the previous definition that if n is an integer, the Favard space of order n is given by

$$F_n = \left\{ x \in X_{n-1} \mid \sup_{t>0} \| \frac{1}{t}(T_{n-1}(t)x - x) \|_{n-1} < \infty \right\}$$

and the corresponding norm is defined as

$$\|x\|_{F_n} = \sup_{t>0} \| \frac{1}{t}(T_{n-1}(t)x - x) \|_{n-1}.$$

We list a couple of useful properties of Favard spaces in the following lemma.

Lemma 2.20. *Let A be a generator of a C_0-semigroup on a Banach space X. Then the following properties hold.*

(i) Let $n \in \mathbb{Z}$. Then $A_{n-1}F_{n+1} = F_n$.

(ii) If X is reflexive, then $F_1 = \mathcal{D}(A)$. In particular this holds if X is a Hilbert space.

(iii) For all $n \in \mathbb{Z}$ we have $X_n \subset F_n$.

Proof. part (i):

$A_{n-1}F_{n+1} \subset F_n$:

Let $x \in F_{n+1}$. This means that $x \in X_n$ and

$$\sup_{t>0} \| \frac{1}{t}(T_n(t)x - x)\|_n < \infty$$

Since $A_{n-1} : X_n \to X_{n-1}$ and $x \in X_n = \mathcal{D}(A_{n-1})$, we have $A_{n-1}x \in X_{n-1}$. Now

$$\begin{aligned} & \sup_{t>0} \| \frac{1}{t}(T_{n-1}(t)A_{n-1}x - A_{n-1}x)\|_{n-1} = \sup_{t>0} \| A_{n-1}^{-1}A_{n-1}\frac{1}{t}(\overbrace{T_{n-1}(t)x}^{\in \mathcal{D}(A_{n-1})} - x)\|_n \\ = & \sup_{t>0} \| \frac{1}{t}(T_n(t)x - x)\|_n < \infty \end{aligned}$$

and thus $A_{n-1}x \in F_n$. Since $x \in F_{n+1}$ was arbitrary, we have $A_{n-1}F_{n+1} \subset F_n$.

$A_{n-1}F_{n+1} \supset F_n$:

Let $x \in F_n$. This means that $x \in X_{n-1}$ and

$$\sup_{t>0} \| \frac{1}{t}(T_{n-1}(t)x - x)\|_{n-1} < \infty$$

We need to show that $x = A_{n-1}y$ for some $y \in F_{n+1}$. Since A_{n-1} is invertible, $y = A_{n-1}^{-1}x$. Since $A_{n-1}^{-1} : X_{n-1} \to X_n$, we have that $y \in X_n$. It follows that

$$\begin{aligned} & \sup_{t>0} \| \frac{1}{t}(T_n(t)\overbrace{A_{n-1}^{-1}x}^{\in X_n} - A_{n-1}^{-1}x)\|_n = \sup_{t>0} \| A_{n-1}\frac{1}{t}(T_{n-1}(t)A_{n-1}^{-1}x - A_{n-1}^{-1}x)\|_{n-1} \\ = & \sup_{t>0} \| \frac{1}{t}(T_{n-1}(t)A_{n-1}A_{n-1}^{-1}x - A_{n-1}A_{n-1}^{-1}x)\|_{n-1} \\ = & \sup_{t>0} \| \frac{1}{t}(T_{n-1}(t)x - x)\|_{n-1} < \infty. \end{aligned}$$

This means that $y \in F_{n+1}$ and thus $F_n \subset A_{n-1}F_{n+1}$.

part (ii): [8, Cor II.5.21]

part (iii): Follows directly from the definition. □

Chapter 3

Stability of C_0-Semigroups

In chapter 2 we saw that the exponential stability of a C_0-semigroup $T(t)$ means that we can find real constants $M \geq 1$ and $\omega > 0$ such that for all $t \geq 0$

$$\|T(t)\| \leq Me^{-\omega t}.$$

In this chapter we will study different characterizations of this kind of stability of a C_0-semigroup. As we already mentioned, the problem is more complicated in infinite-dimensional spaces than in finite-dimensional ones. In the general case, the stability of a general C_0-semigroup isn't always characterized by the spectrum of its generator. However, for certain special classes of semigroups this holds even in Banach and Hilbert spaces. Even though these are special cases, some of the semigroups most frequently encountered in applications belong to some of these classes. For example, the second order partial differential operator $\frac{d^2}{dz^2}$ with homogeneous Dirichlet boundary conditions $x(0) = x(1) = 0$ and an appropriate domain generates an analytic semigroup on $X = L^2([0,1], \mathbb{C})$ [8, Ex II.4.8].

In section 3.1 we will present various means of characterizing exponential stability of C_0-semigroups without additional assumptions on its behaviour. The stability of classes of regular C_0-semigroups will be studied separately in section 3.2.

3.1 Criteria for Exponential Stability

In this section we will present some necessary can sufficient conditions for exponential stability of a general C_0-semigroup. Some of these involve the generator and some the C_0-semigroup. As mentioned earlier, we will present the theory in its most general form as opposed to restricting our attention to Hilbert spaces.

For our purposes in this thesis, the most useful characterization of exponential stability will be the one presented in theorem 3.4. In a Hilbert space this result allows us to characterize the stability of the C_0-semigroup generated by A through the behaviour of the resolvent $R(\cdot, A)$ on the open right half-plane of the complex plane $\mathbb{C}$.

We will first state a few properties equivalent to exponential stability. This result should demonstrate the effect of the semigroup property $T(t+s) = T(s)T(t)$ on the asymptotic behaviour $\|T(t)\|$.

Theorem 3.1. *Let A generate a C_0-semigroup $T(t)$ on a Banach space X. The following properties are equivalent*

(i) *$T(t)$ is exponentially stable*

(ii) $\lim_{t\to\infty}\|T(t)\| = 0$

(iii) *There exists a $t_0 > 0$ such that $\|T(t_0)\| < 1$.*

Proof.
(i) $\Rightarrow$ (ii): We can make a direct estimate $\lim_{t\to\infty}\|T(t)\| \le \lim_{t\to\infty} Me^{-\omega t} = 0$.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): Let $t \ge 0$. Choose $n \in \mathbb{N}_0$ and $t_1 \in [0, t_0)$ such that $t = nt_0 + t_1$. Since theorem 2.6 tells us that $\|T(t)\|$ is bounded on every finite subinterval of $[0, \infty)$, we can choose M_1 such that

$$M_1 = \sup_{t\in[0,t_0)} \|T(t)\|.$$

By assumption we have $\|T(t_0)\| \leq q$ for some $q < 1$. Since

$$q = e^{-\omega t_0} \qquad \Leftrightarrow \qquad \omega = -\frac{\ln q}{t_0} > 0,$$

we can make an estimate

$$\begin{aligned}
\|T(t)\| &= \|T(nt_0 + t_1)\| = \|T(t_0)^n T(t_1)\| \leq \|T(t_0)\|^n \|T(t_1)\| \leq M_1 q^n \\
&= M_1 e^{-\omega n t_0} = M_1 e^{\omega t_1} e^{-\omega t_1} e^{-\omega n t_0} = M_1 e^{\omega t_1} e^{-\omega(nt_0 + t_1)} \\
&\leq M_1 e^{\omega t_0} e^{-\omega t} = M e^{-\omega t}
\end{aligned}$$

□

The following theorem gives a characterization for exponential stability through integrability of the mappings $t \mapsto T(t)x$ for $x \in X$. It was first proved by Datko for the case $p = 2$ and later extended by Pazy for other values of p. The property (3.1) is often referred to as *L^p-stability*.

Theorem 3.2. *Let A generate a C_0-semigroup $T(t)$ on a Banach space X. The semigroup $T(t)$ is exponentially stable if and only if for one/all $p \in [1, \infty)$*

$$\int_0^\infty \|T(t)x\|^p dt < \infty \qquad \textit{for all } x \in X. \tag{3.1}$$

This is equivalent with condition

$$T(\cdot)x \in L^p([0, \infty), X) \qquad \textit{for all } x \in X$$

Proof. It is easy to see that exponential stability of $T(t)$ implies (3.1). The proof of the converse implication can be found in [20, Thm 4.1 p. 116], [8, Thm V.1.8] or [15, Thm 3.28]. □

In a Hilbert space a weaker property, *weak L^p-stability* (the condition in (3.2)), is equivalent to exponential stability. This was first proved by Huang Falun for the case $p = 1$ and later extended by Weiss in [28] for other values of p.

Theorem 3.3. *Let A generate a C_0-semigroup $T(t)$ on a Hilbert space X. The semigroup $T(t)$ is exponentially stable if and only if for some $p \in [1, \infty)$*

$$\int_0^\infty |\langle T(t)x, y\rangle|^p dt < \infty \quad \forall x, y \in X. \tag{3.2}$$

Proof. [28] □

The next theorem tells us that in a Hilbert space the exponential stability can be determined from the behaviour of the resolvent operator $R(\lambda, A)$ on the right half-plane of $\mathbb{C}$. The proof of this theorem uses Plancherel's Theorem, ([8, Thm C.14]) which only holds in Hilbert spaces. This theorem is due to Gearhart, Prüss and Greiner.

Theorem 3.4. *Let A generate a C_0-semigroup $T(t)$ on a Hilbert space X. The semigroup $T(t)$ is exponentially stable if and only if $\{\, \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0 \,\} \subset \rho(A)$ and*

$$\sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| < \infty. \tag{3.3}$$

Proof. [8, Thm V.1.11], [15, Thm 3.35] □

Remark 3.5. It follows from the properties of the resolvent operator that if (3.3) holds, then

$$\sup_{\operatorname{Re}\lambda \geq 0} \|R(\lambda, A)\| = \sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| < \infty \quad \text{and} \quad \{\, \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0 \,\} \subset \rho(A).$$

To see this, let

$$M = \sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\|.$$

Using lemma A.18 we see that for every $\lambda \in \mathbb{C}^+$

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|} \geq \frac{1}{M}$$

and thus $\{\, \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0 \,\} \subset \rho(A)$. It now suffices to show that $\|R(\lambda i, A)\| \leq M$ for all $\lambda \in \mathbb{R}$. For this purpose, choose $\varepsilon > 0$. From the properties of the resolvent operator (see [14, Thm IV.3.15]) it follows that for every $\lambda \in \mathbb{R}$ there exists a real constant $\delta_\lambda > 0$ such that

$$\|R(\lambda i, A) - R(\lambda i + \delta_\lambda, A)\| < \varepsilon.$$

Because $\mathrm{Re}(\lambda i + \delta_\lambda) > 0$, we can make an estimate

$$\begin{aligned} \|R(\lambda i, A)\| &= \|R(\lambda i, A) - R(\lambda i + \delta_\lambda, A) + R(\lambda i + \delta_\lambda, A)\| \\ &\leq \|R(\lambda i, A) - R(\lambda i + \delta_\lambda, A)\| + \|R(\lambda i + \delta_\lambda, A)\| < \varepsilon + M. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this implies that $\|R(\lambda i, A)\| \leq M$ for all $\lambda \in \mathbb{R}$.

If we already know that the C_0-semigroup is uniformly bounded, it suffices to know the behaviour of the resolvent operator on the imaginary axis. We can use this condition for example when dealing with C_0-semigroups of contractions. We will see in section 4.3 that under certain assumptions when a C_0-semigroup of contractions is perturbed, the perturbed semigroup is also contractive and hence also uniformly bounded.

Corollary 3.6. *Let A generate a C_0-semigroup $T(t)$ on a Hilbert space X. If $T(t)$ is uniformly bounded, $i\mathbb{R} \subset \rho(A)$ and*

$$\sup_{\lambda \in \mathbb{R}} \|R(i\lambda, A)\| < \infty$$

holds, then $T(t)$ is exponentially stable.

Proof. [15, Cor 3.36] □

The next theorem is due to Datko [5]. It characterizes the exponential stability of a C_0-semigroup in a Hilbert space by the existence of a certain type of solution to a *Lyapunov equation*. This result in a Hilbert space is a generalization of a corresponding result by Lyapunov in finite-dimensional spaces. This theorem is used in the theory presented in section 5.3

Theorem 3.7. *Let A generate a C_0-semigroup $T(t)$ on a Hilbert space X. The semigroup $T(t)$ is exponentially stable if and only if the Lyapunov equation*

$$\langle Ax, \Pi y \rangle + \langle x, \Pi A y \rangle = -\langle x, y \rangle \qquad \forall x, y \in \mathcal{D}(A) \tag{3.4}$$

has a self-adjoint positive solution $\Pi \in \mathcal{L}(X)$.

Proof. We will first show that Lyapunov equation in (3.4) is equivalent to

$$\langle Ax, \Pi x\rangle + \langle x, \Pi Ax\rangle = -\langle x, x\rangle \quad \forall x \in \mathcal{D}(A). \tag{3.5}$$

By this we mean that a self-adjoint positive operator $\Pi \in \mathcal{L}(X)$ is a solution of (3.4) if and only if it is a solution of (3.5). The actual theorem is proved in [4, Thm 5.1.3] for the Lyapunov equation of form (3.5).

It is clear that if operator $\Pi \in \mathcal{L}(X)$ is a solution of (3.4) then it is also a solution of (3.5) since we can choose $y = x$.

Now, let $x, y \in \mathcal{D}(A)$ and assume $\Pi \in \mathcal{L}(X)$ satisfies (3.5). For all $\alpha \in \mathbb{C}$ we have $x + \alpha y \in \mathcal{D}(A)$ and

$$\begin{aligned}
& \langle A(x+\alpha y), \Pi(x+\alpha y)\rangle + \langle x+\alpha y, \Pi A(x+\alpha y)\rangle \\
= \; & \langle Ax, \Pi x\rangle + \alpha\langle Ay, \Pi x\rangle + \overline{\alpha}\langle Ax, \Pi y\rangle + |\alpha|^2\langle Ay, \Pi y\rangle \\
& + \langle x, \Pi Ax\rangle + \alpha\langle y, \Pi Ax\rangle + \overline{\alpha}\langle x, \Pi Ay\rangle + |\alpha|^2\langle y, \Pi Ay\rangle \\
= \; & -\langle x, x\rangle + \alpha\left(\langle Ay, \Pi x\rangle + \langle y, \Pi Ax\rangle\right) + \overline{\alpha}\left(\langle Ax, \Pi y\rangle + \langle x, \Pi Ay\rangle\right) - |\alpha|^2\langle y, y\rangle.
\end{aligned}$$

By our assumption, this is equal to

$$-\langle x+\alpha y, x+\alpha y\rangle = -\langle x, x\rangle - \alpha\langle y, x\rangle - \overline{\alpha}\langle x, y\rangle - |\alpha|^2\langle y, y\rangle.$$

Combining these we get

$$\begin{aligned}
& \alpha\left(\langle Ay, \Pi x\rangle + \langle y, \Pi Ax\rangle\right) + \overline{\alpha}\left(\langle Ax, \Pi y\rangle + \langle x, \Pi Ay\rangle\right) = -\alpha\langle y, x\rangle - \overline{\alpha}\langle x, y\rangle \\
\Leftrightarrow \; & \alpha\underbrace{\left(\langle Ay, \Pi x\rangle + \langle y, \Pi Ax\rangle + \langle y, x\rangle\right)}_{=\, c_{yx}} + \overline{\alpha}\underbrace{\left(\langle Ax, \Pi y\rangle + \langle x, \Pi Ay\rangle + \langle x, y\rangle\right)}_{=\, c_{xy}} = 0.
\end{aligned}$$

This holds for all $\alpha \in \mathbb{C}$. In particular, this holds for $\alpha = 1$ and $\alpha = i$. This implies

$$\left.\begin{aligned} c_{yx} + c_{xy} &= 0 \\ i(c_{yx} - c_{xy}) &= 0 \end{aligned}\right\} \quad \Rightarrow \quad c_{xy} = c_{yx} = 0$$

and thus

$$\langle Ax, \Pi y\rangle + \langle x, \Pi Ay\rangle = -\langle x, y\rangle \quad \forall x, y \in \mathcal{D}(A).$$

As we already mentioned, the theorem is proved in [4, Thm 5.1.3]. □

3.2 Stability of Regular Semigroups

The special property which guarantees that the spectrum (and hence also the stability properties) of a C_0-semigroup is determined by the spectrum of its generator is called the *Spectral Mapping Theorem* (SMT) which claims

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \quad \forall t \geq 0. \tag{3.6}$$

If the Spectral Mapping Theorem holds for a given C_0-semigroup $T(t)$, we are able to determine the growth bound of $T(t)$ directly from the spectral bound of its generator [8, Lem V.1.9]. This result is presented in the next lemma.

Lemma 3.8. *Let $T(t)$ be a C_0-semigroup with generator A on a Banach space X. If the Spectral Mapping Theorem* (3.6) *holds, then the growth bound of the $T(t)$ equals the spectral bound of its generator, that is,*

$$s(A) = \omega_0(T(t)).$$

We refer to this property as the spectrum determined growth condition.

Proof. The proof is taken from [8, Lem V.1.9]. The growth bound of a C_0-semigroup can expressed as [8, Prop IV.2.2]

$$\omega_0(T(t)) = \frac{1}{t}\log r(T(t)), \quad \text{for all } t > 0$$

By theorem 2.5 we have $-\infty \leq s(A) \leq \omega_0(T(t))$, and thus the equality holds if $\omega_0(T(t)) = -\infty$. Assume that $\omega_0 > -\infty$. Then

$$\begin{aligned} \omega_0(T(t)) &= \frac{1}{t}\log r(T(t)) = \frac{1}{t}\log\sup\{\, |\mu| \mid \mu \in \sigma(T(t))\,\} \\ &= \frac{1}{t}\log\sup\{\, |e^{t\lambda}| \mid \lambda \in \sigma(A)\,\} = \frac{1}{t}\log\sup\{\, e^{t\operatorname{Re}\lambda} \mid \lambda \in \sigma(A)\,\} \\ &= \sup\{\, \frac{1}{t}\log e^{t\operatorname{Re}\lambda} \mid \lambda \in \sigma(A)\,\} = \sup\{\, \operatorname{Re}\lambda \mid \lambda \in \sigma(A)\,\} = s(A) \end{aligned}$$

□

Our motivation for the study of this theory is that if the C_0-semigroup has certain regularity properties, then the spectrum determined growth condition holds automatically. On Banach spaces we have the following result for eventually norm-continuous C_0-semigroups.

Theorem 3.9. *Let $T(t)$ be an eventually norm-continuous C_0-semigroup with generator A on a Banach space X. Then the Spectral Mapping Theorem* (3.6) *holds and thus $s(A) = \omega_0(T(t))$.*

Proof. First part is given in the proof of [8, Thm IV.3.10] and $s(A) = \omega_0(T(t))$ follows directly from lemma 3.8. □

By considering specific subclasses of eventually norm-continuous semigroups, we obtain a corresponding result for other regular semigroups as well. The diagram in figure 3.1 illustrates the relations between special classes of C_0-semigroups [8, p. 119].

Analytic	$\Rightarrow$	Immediately differentiable	$\Rightarrow$	Eventually differentiable
		$\Downarrow$		$\Downarrow$
		Immediately norm cont.	$\Rightarrow$	Eventually norm cont.
		$\Uparrow$		$\Uparrow$
		Immediately compact	$\Rightarrow$	Eventually compact

Figure 3.1: Relations between semigroup classes

These relations combined with theorem 3.9 lead to the following result.

Corollary 3.10. *The* spectrum determined growth condition *holds for the following classes of C_0-semigroups on a Banach space X:*

- *Analytic semigroups*
- *Eventually compact C_0-semigroups*
- *Eventually differentiable C_0-semigroups*
- *Eventually norm-continuous C_0-semigroups*
- *Uniformly continuous C_0-semigroups*

Chapter 4

Robustness of C_0-Semigroup Generation

Let a linear operator A be an infinitesimal generator of a C_0-semigroup $T(t)$ on a Banach or Hilbert space X and let B be a linear operator on X. In this chapter we will introduce conditions under which the the perturbed operator $A+B$ is a generator of a C_0-semigroup on X. This problem has been studied since the early 1950's and theory exists for both bounded and unbounded perturbations.

Throughout this chapter, we will assume that the unperturbed C_0-semigroup $T(t)$ is exponentially stable. This assumption simplifies certain matters, but is not essential to the development of the theory. The results can be extended to a more general case by considering the rescaled C_0-semigroup generated by the operator $A - \omega I$ for some $\omega > \omega_0(T(t))$ (see rescaled semigroups in section 2.1).

We will start with theory for bounded perturbations in section 4.1. We will see from the beginning that considering these perturbations is very straightforward: The perturbed operator remains a generator for all bounded perturbations and we even get an upper bound for the growth of the perturbed semigroup in terms of the norm of the perturbing operator. These results were formulated as early as 1953 by Phillips [21]. Although the results are exhaustive, the estimate of the growth bound given by this theory is generally not optimal. Because of this, we will go on to analyse the preservation of certain regularity properties under bounded perturbations. This theory will be used in section 5.2 where we derive spectral conditions for the stability of the perturbed C_0-semigroup.

Particular perturbation theory for analytic and contractive semigroups is presented in sections 4.2 and 4.3, respectively. We will see that these particular classes of semigroups can deal with perturbations with a degree of unboundedness. In a way this means that if we impose more conditions on the unperturbed C_0-semigroup, we can relax the ones on the perturbing operator. Perturbation of these particular classes of semigroups was first considered by Hille and Phillips [11].

Sections 4.4, 4.5 and 4.6 deal with general unbounded perturbations. This is meant in the sense that we do not need any regularity assumptions for the unperturbed C_0-semigroup $T(t)$. We will characterize the classes of perturbations considered in the first two of these sections using abstract Volterra operators as was done by Engel and Nagel in [8]. This treatment allows us to deal with unboundedness by considering bounded operators between Sobolev spaces of different orders. The class of perturbations considered in section 4.4 results from the perturbation theorems of Desch and Schappacher. The unboundedness of these perturbations is handled by considering operators belonging to $\mathcal{L}(X, X_{-1})$. The class considered in section 4.5 follows from theory presented by Miyadera and later extended by Voigt. Similarly to the previous class of perturbations, the unboundedness of the perturbations is dealt with by considering perturbations belonging to $\mathcal{L}(X_1, X)$. As extensions of the class of bounded operators on X, the classes $\mathcal{L}(X, X_{-1})$ and $\mathcal{L}(X_1, X)$ of operators are natural and we will in remark 4.19 see that an operator B belongs to $\mathcal{L}(X_1, X)$ if and only if it is A-bounded.

The perturbation theorem presented in section 4.6 is relatively new compared to the other theory considered in this chapter. The theorem allows unboundedness of a closed perturbing operator B by imposing special conditions on operators $BR(\lambda, A) : X \to X$ and $R(\lambda, A)B : \mathcal{D}(B) \to X$. The conditions are very simple and the theorem is independent of the other results presented in this chapter. However, unlike most of the theory presented in this chapter, the results are only applicable in Hibert spaces.

4.1 Bounded Perturbations

In this section we consider the case where the perturbing operator B is a bounded operator on X. The following is a well-known result formulated by R. S. Phillips in [21].

Theorem 4.1. *Let A be an infinitesimal generator of a C_0-semigroup $T(t)$ on a Banach space X, satisfying $\|T(t)\| \leq Me^{\omega t}$. If $B \in \mathcal{L}(X)$, then $A + B$ generates a C_0-semigroup $S(t)$ on X such that*

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$$

For every $x \in X$ we have

$$\begin{aligned} S(t)x &= T(t)x + \int_0^t T(t-s)BS(s)xds \\ T(t)x &= S(t)x + \int_0^t S(s)BT(t-s)xds \end{aligned}$$

Proof. [20, Sec 3.1], [4, Thm 3.2.1], [8, Sec III.1] □

The previous theorem shows us that the perturbed operator $A + B$ is a generator of a C_0-semigroup on X for all bounded perturbations B. The theorem only expresses the perturbed C_0-semigroup implicitly, but we can also derive a formula for it using a *Dyson-Phillips* -series.

Theorem 4.2. *The semigroup $S(t)$ in theorem 4.1 can be expressed as*

$$S(t) = \sum_{n=0}^{\infty} S_n(t) \tag{4.1}$$

where $S_0(t) = T(t)$ and for $n \geq 1$

$$S_n(t)x = \int_0^t T(t-s)BS_{n-1}(s)xds \qquad \forall x \in X \tag{4.2}$$

Proof. [8, Thm III.1.10] □

The series in (4.1) converges in uniform operator topology on compact intervals of $[0, \infty)$ [8, p. 163]. We will now present a simple example of a bounded perturbation

Example 4.3. Consider again the space $X = \ell^2(\mathbb{C})$ and an unbounded operator A such that

$$A(x_k) = (-kx_k), \quad \mathcal{D}(A) = \left\{ x \in X \mid Ax \in X \right\}.$$

The operator A generates a C_0-semigroup $T(t)$ on X with

$$T(t)x = (e^{-kt}x_k), \qquad x \in X.$$

We already saw in example 2.8 that the semigroup satisfies $\|T(t)\| \leq Me^{-\omega t}$ with $M = 1$ and $\omega = 1$. Consider now a perturbation B such that for $x \in X$ we have $Bx = (\beta x_k)$ with some $\beta \in \mathbb{C}$. Now, for $x \in X$ with $\|x\| = 1$, we have

$$\|Bx\|^2 = \sum_{k=1}^{\infty} |\beta x_k|^2 = |\beta|^2\|x\|^2 = |\beta|^2.$$

This means that $B \in \mathcal{L}(X)$ with norm $\|B\| = |\beta|$. Theorem 4.1 now tells us that operator $A + B$ for which $(A + B)x = ((-k + \beta)x_k)$ for all $x \in \mathcal{D}(A + B) = \mathcal{D}(A)$ generates C_0-semigroup $S(t)$ satisfying

$$\|S(t)\| \leq Me^{(-\omega+M|\beta|)t} = e^{-(1-|\beta|)t}.$$

It can also be seen from the last expression that $S(t)$ is exponentially stable whenever $|\beta| < 1$.

Theorem 4.1 guarantees that the growth bound of the perturbed C_0-semigroup is at most $-\omega + M\|B\|$. In some cases this can be used to determine the stability of the perturbed C_0-semigroup. However, this bound is not always optimal and the growth bound of the perturbed C_0-semigroup can even be smaller than the growth bound of the unperturbed C_0-semigroup. The possibility of obtaining sharper bounds motivates us to further address the preservation of exponential stability in the case of bounded perturbations.

For the rest of the section we will consider the regularity properties of the perturbed C_0-semigroup. For some classes of C_0-semigroups, the regularity properties are preserved under all bounded perturbations. These results are summarized in the following theorem.

Theorem 4.4. *Let A be a generator of an (analytic, immediately compact, immediately norm-continuous) C_0-semigroup on a Banach space X and let $B \in \mathcal{L}(X)$. Then $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates an (analytic, immediately compact, immediately norm-continuous) C_0-semigroup on X.*

Proof. See [20, Cor 3.2.2] for preservation of analyticity and [8, Thm III.1.16] for the preservation of the other regularity properties. □

Furthermore, if certain additional conditions are satisfied, also other regularity properties of the C_0-semigroup are inherited by the perturbed C_0-semigroup [16]:

Theorem 4.5. *Let $\mathcal{X}$ denote the space of all strongly continuous functions from $[0, \infty)$ to $\mathcal{L}(X)$ and let $B \in \mathcal{L}(X)$. Define an abstract Volterra operator on space $\mathcal{X}$ by*

$$(VF)(t)x = \int_0^t T(t-s)BF(s)xds, \quad F \in \mathcal{X},\ t \geq 0 \text{ and } x \in X.$$

If A generates an eventually (differentiable, compact, norm-continuous) C_0-semigroup and for some $n \in \mathbb{N}$

$$\operatorname{ran} V^n \quad \subset \quad \Big\{ F \in \mathcal{X} \;\Big|\; F \text{ is immediately (differentiable, compact, norm-continuous) for } t \geq 0 \Big\}$$

then the C_0-semigroup generated by $A + B$ is also eventually (differentiable, compact, norm-continuous).

Proof. [16, Thms 6.1, 6.3 & 6.5]. □

The previous results can be used with the theory presented in section 3.2 to derive sufficient conditions for the stability of the perturbed C_0-semigroup. This is done in section 5.2.

We will now move on to consider unbounded perturbations.

4.2 Perturbation of Analytic Semigroups

In this section we will present some results concerning analytic semigroups and unbounded perturbations. It should be noted that in all the cases the perturbed semigroups remain analytic. As stated earlier, this can be helpful when analysing the stability of the perturbed C_0-semigroup. Later in sections 4.4, 4.5 and 4.6 we will see that analytic semigroups remain analytic under even more general unbounded perturbations.

Theorem 4.6. *Let A be the generator of an analytic semigroup. Let B be a closed linear operator and let B be A-bounded with A-bound a_0. There exists a constant $\delta > 0$ such that if $0 \leq a_0 < \delta$ then $A + B$ is the generator of an analytic semigroup.*

Proof. [20, Thm 2.1 p. 80], [8, Thm III.2.10] □

The previous theorem also gives rise to the following corollary. For the definition of the powers A^α for $0 < \alpha < 1$, see [20, Sec 2.6].

Corollary 4.7. *Let A be the generator of an analytic semigroup. Let B be closed and suppose that for some $0 < \alpha < 1$, $\mathcal{D}(B) \supset \mathcal{D}(A^\alpha)$. Then $A + B$ is the generator of an analytic semigroup.*

Proof. [20, Cor 2.4 p. 81] □

We will note here that the conditions in previous corollary require that B is A-bounded. Since A is a closed operator as a generator of a C_0-semigroup and B is closed, this follows from lemma A.9 and the fact that $\mathcal{D}(A) \subset \mathcal{D}(A^\alpha) \subset \mathcal{D}(B)$ for $0 < \alpha < 1$.

The last result for analytic semigroups concerns perturbing with A-compact operators defined in section A.2.

Theorem 4.8. *Let A be the generator of an analytic semigroup and let the operator B be A-compact. Then the operator $A+cB$ with domain $\mathcal{D}(A+cB) = \mathcal{D}(A)$ generates an analytic semigroup on X for all $c \in \mathbb{C}$.*

Proof. See [8, Cor III.2.17] for the case where B is closable or X is reflexive and [6] for the general case. □

The following example makes use of the theory presented in this section.

Example 4.9. Consider Hilbert space $X = L^2([0,1], \mathbb{C})$ and the operator

$$\begin{aligned} A &= \frac{d^2}{dz^2}, \\ \mathcal{D}(A) &= \left\{ x \in X \;\middle|\; x, \frac{d}{dz}x, \text{ abs. cont., } \frac{d^2}{dz^2}x \in X,\ x(0) = x(1) = 0 \right\}. \end{aligned}$$

It is shown in [8, Ex II.4.8] that A generates an analytic semigroup on X. Now consider an unbounded perturbation

$$\begin{aligned} B &= \frac{d}{dz}, \\ \mathcal{D}(B) &= \left\{ x \in X \;\middle|\; x \text{ abs. cont., } \frac{d}{dz}x \in X \right\}. \end{aligned}$$

Operator B is closed and A-bounded with A-bound $a_0 = 0$ [8, Ex III.2.2]. Thus, by theorem 4.6 we know that operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates an analytic semigroup on X.

4.3 Perturbation of Semigroups of Contractions

Before moving on to perturbation of more general C_0-semigroups we will consider the case of C_0-semigroups of contractions. This special class has a few simple and well-known results stated here. They all consider A-bounded and dissipative perturbing operators. Similarly to the case of perturbation of analytic semigroups, all the results presented here guarantee that the perturbed C_0-semigroup is again contractive. Even though contractive C_0-semigroups are not regular in the same sense as analytic or norm-continuous semigroups, they are automatically uniformly bounded. This is an advantage for us since we have a particular result concerning exponential stability of uniformly bounded C_0-semigroups (see corollary 3.6).

Theorem 4.10. *Let A generate a C_0-semigroup of contractions and let B be dissipative with $\mathcal{D}(B) \supset \mathcal{D}(A)$. If B is A-bounded with A-bound $a_0 < 1$ then $A + B$, $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates a C_0-semigroup of contractions.*

Proof. [8, Thm III.3.7], [20, Cor 3.3 p. 82] □

If the underlying space is a reflexive Banach space, we do not need the A-bound of B to be less than one. In particular this holds if X is a Hilbert space, since the Riesz representation theorem states that every Hilbert space is reflexive.

Theorem 4.11. *Let X be a reflexive Banach space (or a Hilbert space) and let A be the generator of a C_0-semigroup of contractions. Let B be dissipative with $\mathcal{D}(B) \supset \mathcal{D}(A)$ and*

$$\|Bx\| \leq \|Ax\| + b\|x\| \quad \text{for} \quad x \in \mathcal{D}(A)$$

where $b \geq 0$. Then then the closure $\overline{A + B}$ of $A + B$ is the generator of a C_0-semigroup of contractions.

Proof. [8, Cor III.2.9], [20, Cor 3.5 p. 84] □

We will now give an example of an unbounded perturbation of a contractive C_0-semigroup.

Example 4.12. Consider again the Hilbert space $X = \ell^2(\mathbb{C})$ and the unbounded operator A such that

$$A(x_k) = (-kx_k), \quad \mathcal{D}(A) = \left\{ x \in X \;\middle|\; Ax \in X \right\}.$$

The operator A generates a C_0-semigroup $T(t)$ on X with

$$T(t)x = (e^{-kt}x_k), \quad x \in X$$

We showed in example 2.8 that the semigroup satisfies $\|T(t)\| \leq e^{-\omega t} \leq 1$ and hence it is contractive. Consider now a perturbing operator B such that for $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta \leq 0$ and $|\beta| < 1$,

$$Bx = (\beta k x_k), \quad \mathcal{D}(B) = \left\{ x \in X \;\middle|\; Bx \in X \right\}$$

Since β is a constant, we clearly have $\mathcal{D}(A) \subset \mathcal{D}(B)$. For $x \in \mathcal{D}(B)$, we have

$$\|Bx\| = \|\beta Ax\| = |\beta|\|Ax\|$$

and hence B is A-bounded with A-bound $a_0 = |\beta| < 1$. For every $x \in \mathcal{D}(B)$

$$\operatorname{Re}\langle Bx, x\rangle = \operatorname{Re}\left(\sum_{k=1}^{\infty} \beta k |x_k|^2\right) = \operatorname{Re}\beta \cdot \underbrace{\sum_{k=1}^{\infty} k|x_k|^2}_{\geq 0} \leq 0$$

This means that B is a dissipative operator. Theorem 4.10 now tells us that $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates a C_0-semigroup of contractions.

The perturbations considered in the rest of the chapter do not need any additional assumptions concerning the regularity properties of the unperturbed C_0-semigroup.

4.4 Desch-Schappacher Perturbations

The perturbation results presented here were formulated by Desch and Schappacher. The perturbations considered are bounded linear operators from a Banach space X to the extrapolation space X_{-1}. We will follow the manner of representation proposed by Engel and Nagel [8]. This approach using abstract Volterra operators results in simple characterization of the considered class of perturbations and provides a nice link between the theory in this section and the next one.

We begin by defining the abstract Volterra operator related to the perturbing operator $B \in \mathcal{L}(X, X_{-1})$. Denote by $\mathcal{X}_{t_0}$ the space of all strongly continuous, $\mathcal{L}(X)$-valued functions,

$$\mathcal{X}_{t_0} = C\left([0, t_0], \mathcal{L}_s(X)\right), \quad \text{with norm} \quad \|F\|_\infty = \sup_{0 \leq s \leq t_0} \|F(s)\|_{\mathcal{L}(X)}.$$

The space $(\mathcal{X}_{t_0}, \|\cdot\|_\infty)$ is a Banach space [8, Prop A.7]. For an operator $B \in \mathcal{L}(X, X_{-1})$ we define an *abstract Volterra operator* $V_B : \mathcal{X}_{t_0} \to \mathcal{X}_{t_0}$ by

$$(V_B F)(t)x = \int_0^t T_{-1}(t-s)BF(s)xds \quad \text{for all } t \in [0, t_0],\ F \in \mathcal{X}_{t_0} \text{ and } x \in X$$

It is clear from the definition that $(V_B F)(t) \in \mathcal{L}(X, X_{-1})$ for all $t \in [0, t_0]$. The set of *Desch-Schappacher perturbations* $\mathcal{S}_{t_0}^{\mathrm{DS}}$ is then defined by

$$\mathcal{S}_{t_0}^{\mathrm{DS}} = \left\{ B \in \mathcal{L}(X, X_{-1}) \;\middle|\; V_B \in \mathcal{L}(\mathcal{X}_{t_0}),\ \|V_B\| < 1 \right\}$$

For this class of perturbations we have the following result:

Theorem 4.13. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X. If $B \in \mathcal{S}_{t_0}^{DS}$ for some $t_0 > 0$, then the operator*

$$(A_{-1} + B)|_X, \quad \mathcal{D}\left((A_{-1} + B)|_X\right) = \left\{ x \in X \;\middle|\; A_{-1}x + Bx \in X \right\}$$

generates a C_0-semigroup on X.

Proof. [8, Thm III.3.1] □

The C_0-semigroup $S(t)$ generated by $(A_{-1} + B)|_X$ is then given by the *variation of parameters* -formula

$$S(t)x = T(t)x + \int_0^t T_{-1}(t-s)BS(s)xds, \quad \text{for all } t \geq 0 \text{ and } x \in X$$

or the *Dyson-Phillips* -series

$$S(t) = \sum_{n=0}^{\infty} S_n(t)$$

where $S_0(t) = T(t)$ and for $n \geq 1$

$$S_n(t)x = \int_0^t T_{-1}(t-s)BS_{n-1}(s)xds, \quad \text{for all } \ t \geq 0 \text{ and } x \in X$$

The series representation of $S(t)$ converges uniformly in $\mathcal{L}(X)$ on compact intervals of $\mathbb{R}^+$ [8, Cor III.3.2].

The theory of Desch-Schappacher perturbations can be used for example in the case when the boundary conditions of a generator of a C_0-semigroup are subjected to perturbations (see [8, Ex III.3.5]). However, also "simpler" perturbations belong to this class. The following lemma states that bounded perturbations are also Desch-Schappacher perturbations.

Lemma 4.14. *Let A be a generator of a C_0-semigroup $T(t)$ on a Banach space X. If $B \in \mathcal{L}(X)$, then $B \in \mathcal{S}_{t_0}^{DS}$.*

Proof. Operator $B \in \mathcal{L}(X)$ can be seen as an operator $B : X \to X_{-1}$. Since for $x \in X$ we have

$$\|Bx\|_{-1} = \|A^{-1}Bx\| \leq \|A^{-1}\|\|B\|\|x\|$$

we see that $B \in \mathcal{L}(X, X_{-1})$. Let $x \in X$. Since $\|T_{-1}(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$, we have for the abstract Volterra operator

$$\begin{aligned}
\|(V_B F)(t)x\| &= \|\int_0^t T_{-1}(t-s)BF(s)xds\| \leq \int_0^t \|T_{-1}(t-s)BF(s)x\|ds \\
&\leq \int_0^t \|T_{-1}(t-s)\| \cdot \|B\| \cdot \|F\|_\infty \|x\| ds \\
&= \|B\| \cdot \|F\|_\infty \|x\| \int_0^t \|T_{-1}(s)\| ds \\
&\leq M\|B\| \cdot \|F\|_\infty \|x\| \int_0^t e^{-\omega s} ds = \frac{M}{\omega}(1-e^{-\omega t})\|B\| \cdot \|F\|_\infty \|x\|
\end{aligned}$$

We see that V_B is bounded for every $t_0 \geq 0$. Finally, we will show that we can choose $t_0 > 0$ such that $\|V_B\| < 1$. We have

$$\begin{aligned} \tfrac{M}{\omega}(1-e^{-\omega t})\|B\| &< 1 \\ \Leftrightarrow \qquad 1-e^{-\omega t} &< \tfrac{\omega}{M\|B\|} \\ \Leftrightarrow \qquad e^{-\omega t} &> 1-\tfrac{\omega}{M\|B\|} \end{aligned}$$

If $\frac{\omega}{M\|B\|} \geq 1$, then the last inequality holds for all $t > 0$ and $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ for all $t_0 > 0$. If $\frac{\omega}{M\|B\|} < 1$, then

$$\begin{aligned} e^{-\omega t} &> 1-\tfrac{\omega}{M\|B\|} \\ \Leftrightarrow \qquad -\omega t &> \ln\left(1-\tfrac{\omega}{M\|B\|}\right) \\ \Leftrightarrow \qquad t &< -\tfrac{1}{\omega}\ln\left(1-\tfrac{\omega}{M\|B\|}\right) \end{aligned}$$

and $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ for all $0 < t_0 < -\frac{1}{\omega}\ln\left(1-\frac{\omega}{M\|B\|}\right)$. □

In the previous lemma, the form of the perturbing operator can be simplified. Since B is a bounded operator on X, we know that Bx belongs to X for all $x \in X$. Thus, $A_{-1}x + Bx$ belongs to X if and only if $A_{-1}x \in X$. Because this is satisfied for $x \in X_1$ and for these values $A_{-1}x = Ax$, the perturbed operator assumes a familiar form.

$$(A_{-1}+B)\,|_X = A + B, \qquad \mathcal{D}((A_{-1}+B)\,|_X) = \mathcal{D}(A)$$

We will now present some sufficient conditions for a perturbation $B \in \mathcal{L}(X, X_{-1})$ to be a Desch-Schappacher perturbation.

Corollary 4.15. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$. Moreover, assume that there exists $t_0 > 0$ and $q \in [0,1)$ such that*

- $\int_0^{t_0} T_{-1}(t-s)Bf(s)ds \in X$ *and*
- $\|\int_0^{t_0} T_{-1}(t-s)Bf(s)ds\| < q\|f\|_\infty$

for all continuous functions $f \in C([0,t_0],X)$. Then $B \in \mathcal{S}_{t_0}^{DS}$.

Proof. [8, Cor III.3.3] □

Corollary 4.16. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$. Moreover, assume that there exist $t_0 > 0$ and $p \in [1, \infty)$ such that*

$$\int_0^{t_0} T_{-1}(t_0 - s)Bf(s)ds \in X$$

for all functions $f \in L^p([0, t_0], X)$. Then $B \in \mathcal{S}_{t_0}^{DS}$.

Proof. [8, Cor III.3.4] □

Corollary 4.17. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$ satisfy $\operatorname{ran}(B) \subset F_0$. Then $B \in \mathcal{S}_{t_0}^{DS}$ for some $t_0 > 0$.*

Proof. [8, Cor III.3.6] □

The following proposition states that analyticity of the semigroup is preserved under Desch-Schappacher perturbations. This result will be used in section 5.2 where we consider spectral conditions for the exponential stability of the perturbed C_0-semigroup. The result was given as an exercise in [8, Exer. III.3.8.(2)] and the proof is by the author.

Proposition 4.18. *Let A be a generator of an analytic semigroup on X and let $B \in \mathcal{S}_{t_0}^{DS}$. Then $(A_{-1} + B)|_X$ generates an analytic semigroup on X.*

Proof. Let A generate an analytic semigroup $T(z)$ on X and let $\omega_1 \in \mathbb{R}$ be such that $\omega_1 > \omega_0(T(t))$. Then $A - \omega_1 I$ generates a bounded analytic semigroup on X and by theorem 2.17 there exist a constant $M_1 > 0$ and such that for all $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$

$$\begin{aligned} \|R(r + is, A - \omega_1 I)\| &\leq \frac{M_1}{|s|} \\ \Leftrightarrow \quad \|R(r + \omega_1 + is, A)\| &\leq \frac{M_1}{|s|}. \end{aligned} \tag{4.3}$$

Let $M_2 > 0$ and ω_2 be real constants such that

$$\|T_{-1}(t)\| \leq M_2 e^{\omega_2 t}.$$

Then for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_2$ we have $\lambda \in \rho(A_{-1})$ and for all $x \in X$ we can express $R(\lambda, A_{-1})x$ as an integral (see theorem 2.6). Using this fact we get for $x \in X$

$$\begin{aligned} R(\lambda, A_{-1})Bx &= \int_0^\infty e^{-\lambda s}T_{-1}(s)Bxds = \sum_{n=0}^\infty \int_{nt_0}^{(n+1)t_0} e^{-\lambda s}T_{-1}(s)Bxds \\ &= \sum_{n=0}^\infty \int_{nt_0}^{(n+1)t_0} e^{-\lambda nt_0}e^{-\lambda(s-nt_0)}T_{-1}(nt_0)T_{-1}(s-nt_0)Bxds \\ &= \sum_{n=0}^\infty e^{-\lambda nt_0}T_{-1}(nt_0)\int_0^{t_0} e^{-\lambda s}T_{-1}(s)Bxds \\ &= \sum_{n=0}^\infty e^{-\lambda nt_0}T_{-1}(nt_0)\int_0^{t_0} T_{-1}(t_0-s)BIe^{-\lambda(t_0-s)}xds \\ &= \sum_{n=0}^\infty e^{-\lambda nt_0}T_{-1}(nt_0)[V_BF_\lambda](t_0)x. \end{aligned}$$

Here we have denoted $F_\lambda(t) = Ie^{-\lambda(t_0-t)}$. Clearly we have $F_\lambda \in \mathcal{X}_{t_0}$ with $\|F_\lambda\|_\infty = \sup_{t\in[0,t_0]}|e^{-\lambda(t_0-t)}| = 1$. We can now estimate the norm of $R(\lambda, A_{-1})x$ by

$$\begin{aligned} \|R(\lambda, A_{-1})Bx\| &= \|\sum_{n=0}^\infty e^{-\lambda nt_0}T_{-1}(nt_0)[V_BF_\lambda](t_0)x\| \\ &\le \sum_{n=0}^\infty |e^{-\lambda nt_0}|\|T_{-1}(nt_0)\| \overbrace{\|V_B\|}^{<1} \overbrace{\|F_\lambda\|_\infty}^{=1}\|x\| \\ &\le \|V_B\|\|x\| + M_2\|x\|\sum_{n=1}^\infty e^{(\omega_2-\operatorname{Re}\lambda)nt_0} \\ &\le \|V_B\|\|x\| + \frac{M_2e^{(\omega_2-\operatorname{Re}\lambda)t_0}}{1-e^{(\omega_2-\operatorname{Re}\lambda)t_0}}\|x\|. \end{aligned}$$

Because $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, the Volterra operator V_B is bounded with $\|V_B\| < 1$. Since the last term is a decreasing function of λ, we have for some $\omega_3 > \omega_2$ and $0 < q < 1$ that

$$\|R(\lambda, A_{-1})B\| \le q < 1 \tag{4.4}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_3$.

Since $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, operator $(A_{-1} + B)|_X$ generates a C_0-semigroup $S(t)$ with growth bound $\omega_0(S(t))$ on X. Now choose

$$\omega > \max\{\omega_0(S(t)), \omega_1, \omega_3\}.$$

Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$. Theorem A.17 and equation (4.4) now tell us that for the spectral bound of $R(\lambda, A_{-1})B$

$$r(R(\lambda, A_{-1})B) \leq \|R(\lambda, A_{-1})B\| < 1.$$

Therefore we have $1 \in \rho(R(\lambda, A_{-1})B)$ and we can make an estimate

$$\begin{aligned} \|R(1, R(\lambda, A_{-1})B)\| &= \|\sum_{n=0}^{\infty}(R(\lambda, A_{-1})B)^n\| \leq \sum_{n=0}^{\infty}\|R(\lambda, A_{-1})B\|^n \\ &= \frac{1}{1-\|R(\lambda, A_{-1})B\|} \leq \frac{1}{1-q}. \end{aligned} \tag{4.5}$$

Since $\operatorname{Re}\lambda > \omega > \omega_2 > \omega_0(T_{-1}(t)) = \omega_0(T(t))$ (see section 2.2.1), we also have $\lambda \in \rho(A)$. The identity

$$(\lambda I - (A_{-1} + B)|_X) = (\lambda I - A)(I - R(\lambda, A_{-1})B)$$

implies that $\lambda \in \rho((A_{-1} + B)|_X)$ and

$$R(\lambda, (A_{-1} + B)|_X) = R(1, R(\lambda, A_{-1})B)R(\lambda, A).$$

Finally, since $\omega > \omega_0(S(t))$, the operator $(A_{-1} + B)|_X - \omega I$ generates a bounded C_0-semigroup on X and for $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$ we get using (4.3) and (4.5)

$$\begin{aligned} \|R(r + is, (A_{-1} + B)|_X) - \omega I\| &= \|R(r + \omega + is, (A_{-1} + B)|_X)\| \\ &= \|R(1, R(r + \omega + is, A_{-1})B)R(r + \omega + is, A)\| \\ &\leq \|R(1, R(r + \omega + is, A_{-1})B)\|\|R(r + \omega + is, A)\| \\ &\leq \frac{1}{1-q}\frac{M_1}{|s|} = \frac{M}{|s|}. \end{aligned}$$

By theorem 2.17 operator $(A_{-1} + B)|_X - \omega I$ then generates a bounded analytic semigroup $e^{-\omega \operatorname{Re} z}S(z)$ on X and the operator

$$(A_{-1} + B)|_X = (A_{-1} + B)|_X - \omega I + \omega I$$

generates an analytic semigroup $e^{\omega \operatorname{Re} z}e^{-\omega \operatorname{Re} z}S(z) = S(z)$ on X. $\square$

4.5 Miyadera-Voigt Perturbations

The perturbation results presented in this section were first considered by Miyadera in 1960's. The original results include the first part of corollary 4.22. These results were later extended by Voigt. More details on the developement of the theory can be found in [8]. As in the previous section, we formulate the class of perturbations using the abstract Volterra operators as proposed by Engel and Nagel [8]. The perturbations considered here are bounded linear operators from the Sobolev space X_1 to X. Before defining the class of perturbations, we note that the operators belonging to $\mathcal{L}(X_1, X)$ are exactly the A-bounded operators.

Remark 4.19. Property $B \in \mathcal{L}(X_1, X)$ is equivalent to B being A-bounded: If $B \in \mathcal{L}(X_1, X)$, we have

$$\|Bx\| \leq M\|x\|_1 = M\|Ax\| \quad \forall x \in X_1 = \mathcal{D}(A)$$

and thus we can choose $a = M$ and $b = 0$ in definition A.8. On the other hand, if B is A-bounded, we have $\mathcal{D}(B) \supset \mathcal{D}(A)$ and for some $a, b \geq 0$ and for all $x \in \mathcal{D}(A) = X_1$,

$$\|Bx\| \leq a\|Ax\| + b\|x\| = a\|Ax\| + b\|A^{-1}Ax\| \leq \left(a + b\|A^{-1}\|\right)\|x\|_1$$

and so $B \in \mathcal{L}(X_1, X)$.

We will now define the class of Miyadera-Voigt perturbations. Consider again the Banach space $(\mathcal{X}_{t_0}, \|\cdot\|_\infty)$, where

$$\mathcal{X}_{t_0} = C\left([0, t_0], \mathcal{L}_s(X)\right), \quad \|F\|_\infty = \sup_{0 \leq s \leq t_0} \|F(s)\|_{\mathcal{L}(X)}.$$

For a given operator $B \in \mathcal{L}(X_1, X)$ define the *abstract Volterra operator* $V_B^* : \mathcal{X}_{t_0} \to \mathcal{X}_{t_0}$ by

$$(V_B^* F)(t)x = \int_0^t F(s)BT(t-s)x ds \;\text{ for all } t \in [0, t_0],\; F \in \mathcal{X}_{t_0} \text{ and } x \in X_1$$

We can see that $(V_B^* F)(t) \in \mathcal{L}(X_1, X)$ for all $t \in [0, t_0]$. We denote by $\overline{V_B^*}$ the operator giving the extensions $\overline{(V_B^* F)(t)} : X \to X$ of operators $(V_B^* F)(t)$.

We define the set of *Miyadera-Voigt perturbations* $\mathcal{S}_{t_0}^{\mathrm{MV}}$ by

$$\mathcal{S}_{t_0}^{\mathrm{MV}} = \left\{\, B \in \mathcal{L}(X_1, X) \;\middle|\; \overline{V_B^*} \in \mathcal{L}(\mathcal{X}_{t_0}),\; \|\overline{V_B^*}\| < 1 \,\right\}.$$

We have the following result for the Miyadera-Voigt perturbations:

Theorem 4.20. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X. If $B \in \mathcal{S}_{t_0}^{MV}$ for some $t_0 > 0$, then the operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates a C_0-semigroup on X.*

Proof. [8, Thm III.3.14] □

The C_0-semigroup generated by $A + B$ is then given by the *variation of parameters* -formula

$$S(t)x = T(t)x + \int_0^t S(s)BT(t-s)xds, \ x \in \mathcal{D}(A)$$

or the *abstract Dyson-Phillips series*

$$S(t) = \sum_{n=0}^{\infty} (V^n T)(t), \quad \text{for } t \geq 0, \ V = \overline{V_B^*}$$

The following lemma states that bounded perturbations can be seen as Miyadera-Voigt perturbations.

Lemma 4.21. *Let A be a generator of a C_0-semigroup $T(t)$ on a Banach space X. If $B \in \mathcal{L}(X)$, then for the restriction $B' : X_1 \to X$ we have $B' \in \mathcal{S}_{t_0}^{MV}$.*

Proof. Let $x \in X_1$. Then we have

$$\|B'x\| = \|B'A^{-1}Ax\| \leq \|B\|\|A^{-1}\|\|x\|_1$$

and we see that $B \in \mathcal{L}(X_1, X)$. Let $t_0 > 0$ be arbitrary. Next we will show that for all $t \in [0, t_0]$ and $F \in \mathcal{X}_{t_0}$ the extension $\overline{(V_{B'}^* F)(t)} = (V_B^* F)(t)$ is a bounded operator on X. Let $x \in X$. Since $\|T(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$, we have

$$\begin{aligned}
\|(V_B^* F)(t)x\| &= \|\int_0^t F(s)BT(t-s)xds\| \leq \int_0^t \|F(s)BT(t-s)x\|ds \\
&\leq \|B\| \cdot \|F\|_\infty \|x\| \int_0^t \|T(s)\|ds \\
&\leq M\|B\| \cdot \|F\|_\infty \|x\| \int_0^t e^{-\omega s}ds = \frac{M}{\omega}(1 - e^{-\omega t})\|B\| \cdot \|F\|_\infty \|x\|
\end{aligned}$$

and thus for all $t \geq 0$ and $F \in \mathcal{X}_{t_0}$, we have $(V_B^* F)(t) \in \mathcal{L}(X)$. Finally, we will show that we can choose $t_0 > 0$ such that $\|\overline{V_{B'}^*}\| < 1$. From the previous inequalities we can also see that

$$\|\overline{V_{B'}^*}\| \leq \frac{M}{\omega}(1 - e^{-\omega t})\|B\|.$$

Now,

$$\begin{array}{rrcl} & \frac{M}{\omega}(1 - e^{-\omega t})\|B\| & < & 1 \\ \Leftrightarrow & 1 - e^{-\omega t} & < & \frac{\omega}{M\|B\|} \\ \Leftrightarrow & e^{-\omega t} & > & 1 - \frac{\omega}{M\|B\|} \end{array}$$

If $\frac{\omega}{M\|B\|} \geq 1$, then the last inequality holds for all $t > 0$ and hence $\|\overline{V_{B'}^*}\| < 1$ for all $t_0 > 0$. If $\frac{\omega}{M\|B\|} < 1$, then

$$\begin{array}{rrcl} & e^{-\omega t} & > & 1 - \frac{\omega}{M\|B\|} \\ \Leftrightarrow & -\omega t & > & \ln\left(1 - \frac{\omega}{M\|B\|}\right) \\ \Leftrightarrow & t & < & -\frac{1}{\omega}\ln\left(1 - \frac{\omega}{M\|B\|}\right) \end{array}$$

and that $\|\overline{V_{B'}^*}\| < 1$ for all $t_0 < -\frac{1}{\omega}\ln\left(1 - \frac{\omega}{M\|B\|}\right)$. This shows that we can choose $t_0 > 0$ such that $B' \in \mathcal{S}_{t_0}^{\mathrm{MV}}$. □

We will now give sufficient conditions for an operator $B \in \mathcal{L}(X_1, X)$ to be a Miyadera-Voigt perturbation. The first part of the following corollary is the original condition formulated by Miyadera in 1960's.

Corollary 4.22. *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X and let $B \in \mathcal{L}(X_1, X)$ satisfy*

$$\int_0^{t_0} \|BT(s)x\| ds \leq q\|x\| \qquad \forall x \in \mathcal{D}(A) \tag{4.6}$$

for some $0 \leq q < 1$. Then $B \in \mathcal{S}_{t_0}^{MV}$, and therefore the operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ generates a C_0-semigroup $S(t)$ on X. Moreover, $S(t)$ satisfies

- *$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x ds$ and*
- *$\int_0^{t_0} \|BS(s)x\| ds \leq \frac{q}{1-q}\|x\|$ for $x \in \mathcal{D}(A)$ and $t \geq 0$,*

where q and t_0 are given by (4.6). If, in addition, (B, X_1) is closable in X and $(B, \mathcal{D}(B))$ denotes its closure, then we have $T(t)x, S(t)x \in \mathcal{D}(B)$ for almost all $t \geq 0$ and all $x \in X$. Finally, the functions $BT(\cdot)x$ and $BS(\cdot)x$ are locally integrable, and

$$\begin{aligned} S(t)x &= T(t)x + \int_0^t S(s)BT(t-s)x ds \\ S(t)x &= T(t)x + \int_0^t T(t-s)BS(s)x ds \end{aligned}$$

hold for all $x \in X$ and $t \geq 0$.

Proof. [8, Cor III.3.16] □

The following result is an extension of the previous corollary and it is due to Voigt. The result was given as an exercise in [8, Exer III.3.17.(2)] and the part of the proof presented here is by the author.

Corollary 4.23. *The conclusion of corollary 4.22 holds if $(B, \mathcal{D}(B))$ is closed in X, there exists a $T(t)$-invariant dense subspace $D \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ such that the map $t \mapsto BT(t)x$ is continuous for all $x \in D$ and there exist constants $t_0 > 0$ and $0 \leq q < 1$ such that*

$$\int_0^{t_0} \|BT(s)x\| ds \leq q\|x\| \quad \forall x \in D$$

Proof. We will prove only that $A + B$ generates a C_0-semigroup on X, since the rest of the results can be proved by following the latter part of the proof of [8, Cor III.3.16].

We first prove that $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$. To do this, we will show the following:

(i) For all $F \in \mathcal{X}_{t_0}$ the operator $(V_B^* F)(t) : X_1 \to X$ can be extended to a bounded operator $\overline{(V_B^* F)(t)} : X \to X$.

(ii) The mapping $t \mapsto \overline{(V_B^* F)(t)}$ is strongly continuous for all $F \in \mathcal{X}_{t_0}$.

(iii) The operator $\overline{V_B^*}$ is bounded and satisfies $\|\overline{V_B^*}\| < 1$

Part (i):
Let $F \in \mathcal{X}_{t_0}$ and $x \in D$

$$\begin{aligned} \|(V_B^*F)(t)x\| &= \|\int_0^t F(r)BT(t-r)xdr\| \leq \int_0^t \|F(r)BT(t-r)x\|dr \\ &\leq \|F\|_\infty \int_0^t \|BT(r)x\|dr \leq \|F\|_\infty \int_0^{t_0} \|BT(r)x\|dr \\ &\leq q\|F\|_\infty \cdot \|x\| \end{aligned}$$

Since D is dense in X, theorem A.14 tells us that $(V_B^*F)(t)$ can be extended to a bounded operator $\overline{(V_B^*F)(t)}$ on X for all $F \in \mathcal{X}_{t_0}$ and $t \in [0, t_0]$.

Part (2):
To show that the mapping $t \mapsto \overline{(V_B^*F)(t)}$ is strongly continuous for all $F \in \mathcal{X}_{t_0}$, we will show that the mapping $t \mapsto \overline{(V_B^*F)(t)}x = (V^*F)(t)x$ from $[0, t_0]$ to X is continuous $\forall x \in D$. Because D is dense in X, this is an equivalent property [8, Lem I.5.2].

Let $x \in D$ and $t, s \in [0, t_0]$ with $s \leq t$. Then

$$\begin{aligned} &\|(V_B^*F)(t)x - (V_B^*F)(s)x\| \\ = \; &\|\int_0^t F(r)BT(t-r)xdr - \int_0^s F(r)BT(s-r)xdr\| \\ = \; &\|\int_0^s (F(r)BT(t-r)x - F(r)BT(s-r)x)\,dr + \int_s^t F(r)BT(t-r)xdr\| \\ \leq \; &\|\int_0^s (F(r)BT(t-r)x - F(r)BT(s-r)x)\,dr\| + \|\int_s^t F(r)BT(t-r)xdr\| \\ \leq \; &\int_0^s \|F(r)BT(t-r)x - F(r)BT(s-r)x\|dr + \int_s^t \|F(r)BT(t-r)x\|dr. \end{aligned}$$

For the first term we have an estimate

$$\begin{aligned}
& \int_0^s \|F(r)BT(t-r)x - F(r)BT(s-r)x\|dr \\
= & \int_0^s \|F(r)BT(s-r)T(t-s)x - F(r)BT(s-r)x\|dr \\
= & \int_0^s \|F(r)BT(s-r)\left(T(t-s)x - x\right)\|dr \\
\leq & \|F\|_\infty \int_0^s \|BT(s-r)\left(T(t-s)x - x\right)\|dr \\
= & \|F\|_\infty \int_0^s \|BT(r)\left(T(t-s)x - x\right)\|dr \\
\leq & \|F\|_\infty \int_0^{t_0} \|BT(r)\left(T(t-s)x - x\right)\|dr \\
\leq & q\|F\|_\infty \|T(t-s)x - x\| \to 0, \quad \text{when } t \to s+ .
\end{aligned}$$

We used the fact that $T(t-s)x - x \in D$ which follows from $T(t)$-invariance of D. The convergence to 0 is due to the strong continuity of $T(t)$. We can estimate the second term by

$$\int_s^t \|F(r)BT(t-r)x\|dr \leq \|F\|_\infty \int_s^t \|BT(t-r)x\|dr = \|F\|_\infty \int_0^{t-s} \|BT(r)x\|dr$$

Since by our assumption $\|BT(\cdot)x\| \in L^1([0,t_0],\mathbb{R}^+)$ for all $x \in D$, corollary A.21 tells us that

$$\int_0^{t-s} \|BT(r)x\|dr \;\to\; 0$$

as $t \to s+$. This shows that the mapping $t \mapsto (V_B^* F)(t)x$ is continuous $\forall x \in D$ and hence the mapping $t \mapsto (V_B^* F)(t)$ is strongly continuous for all $F \in \mathcal{X}_{t_0}$.

Part (iii):
We want to show that operator $\overline{V_B^*}$ is bounded with $\|\overline{V_B^*}\| < 1$. This follows from Part (i) with $\left\|\overline{(V_B^* F)(t)}\right\| \leq q\|F\|_\infty$.

Together the parts (i)-(iii) imply that $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ and hence $A + B$ is a generator of a C_0-semigroup $S(t)$ on X by theorem 4.20.

The rest of the proof follows the proof of [8, Cor III.3.16]. □

The next sufficient condition tells us that $B \in \mathcal{L}(X_1, X)$ is a Miadera-Voigt perturbation if its range is contained in the Favard space of order 1. The result was presented as an exercise in [8, Exer III.3.23.(iii)] and the proof is by the author.

Corollary 4.24. *Let A be a generator on X with $\rho(A) \neq \varnothing$. If $B \in \mathcal{L}(X_1, X)$ and* $\operatorname{ran} B \subset F_1$*, then the operator $A + B$ with domain $\mathcal{D}(A)$ is a generator on X.*

To prove this theorem, we need the following result concerning *multiplicative perturbations* (see [8, Sec III.3.d] for more information).

Lemma 4.25. *Let A be the generator of a C_0-semigroup on a Banach space X. If A is invertible and $B \in \mathcal{S}_{t_0}^{DS}$, then the operator*

$$A + A_{-1}^{-1}BA, \qquad \mathcal{D}(A + A_{-1}^{-1}BA) = \mathcal{D}(A)$$

is a generator of a C_0-semigroup on X.

Proof. [8, Cor III.3.22] □

Proof of corollary 4.24. If $0 \notin \rho(A)$, we can choose a real positive constant $\omega > \omega_0(T(t))$ and consider

$$A + B = (A - \omega I) + (B + \omega I) = A' + B'$$

Now $0 \in \rho(A')$. We will show that $B \in \mathcal{L}(X_1', X')$ and $\operatorname{ran} B \subset F_1'$ where X', X_1' and F_1' are corresponding Sobolev and Favard spaces associated to the operator $A - \omega I$.

Since we know from section 2.2.1 that $B \in \mathcal{L}(X_1, X)$ and the norm $\|\cdot\|_1$ associated to operator A' is equivalent to the graph norm $\|\cdot\|_{A'}$, we have

$$\begin{aligned} \|B'x\| &= \|(B + \omega I)x\| \leq M\|Ax\| + |\omega|\|x\| \leq M\|(A - \omega I)x\| + (M+1)|\omega|\|x\| \\ &= M\|(A - \omega I)x\| + (M+1)|\omega|\|R(\omega, A)(\omega I - A)x\| \\ &\leq (M + (M+1)|\omega|\|R(\omega, A)\|)\,\|A'x\|. \end{aligned}$$

This means that $B' \in \mathcal{L}(X_1', X')$. Finally, we will show that $\operatorname{ran} B' \subset F_1'$. Let $x \in \mathcal{D}(B')$. Since $\mathcal{D}(B') = \mathcal{D}(B + \omega I) = \mathcal{D}(B) = X_1$, we have that $x \in X_1$. We can make an estimate

$$\begin{aligned}
&\sup_{t>0}\|\frac{1}{t}(e^{-\omega t}T(t)-I)B'x\| = \sup_{t>0}\|\frac{1}{t}(e^{-\omega t}T(t)-Ie^{-\omega t}+Ie^{-\omega t}-I)B'x\| \\
\leq\ & \sup_{t>0} e^{-\omega t}\|\frac{1}{t}(T(t)-I)(B+\omega I)x\| + \sup_{t>0}\|\frac{1}{t}(e^{-\omega t}-1)B'x\| \\
\leq\ & \sup_{t>0}\|\frac{1}{t}(T(t)-I)Bx\| + \omega\sup_{t>0}\|\frac{1}{t}(T(t)-I)x\| + \|B'x\|\sup_{t>0}\frac{1-e^{-\omega t}}{t} \\
=\ & \sup_{t>0}\|\frac{1}{t}(T(t)-I)Bx\| + \omega\sup_{t>0}\|\frac{1}{t}(T(t)-I)x\| + \omega\|B'x\| < \infty
\end{aligned}$$

The first term is finite, because $\operatorname{ran} B \subset F_1$ and the second term is finite because $x \in X_1 \subset F_1$ (see lemma 2.20). This means that $\operatorname{ran} B' \subset F_1'$.

Assume now that $0 \in \rho(A)$. Consider a perturbation $C = A_{-1}BA^{-1} : X \to X_{-1}$. If we can show that $C \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, we get by lemma 4.25 that the operator

$$A + A_{-1}^{-1}CA = A + A_{-1}^{-1}A_{-1}BA^{-1}A = A + B, \quad \mathcal{D}(A+B) = \mathcal{D}(A)$$

is a generator of a C_0-semigroup on X. By corollary 4.17 it suffices to show that $C \in \mathcal{L}(X, X_{-1})$ and $\operatorname{ran} C \subset F_0$.

Since $B \in \mathcal{L}(X_1, X)$, we have $\|Bx\| \leq M\|x\|_1$ for all $x \in X_1$. Let $x \in X$. Then

$$\|A_{-1}BA^{-1}x\|_{-1} = \|A_{-1}^{-1}A_{-1}BA^{-1}x\| = \|BA^{-1}x\| \leq M\|AA^{-1}x\| = M\|x\|$$

This implies that $A_{-1}BA^{-1} \in \mathcal{L}(X, X_{-1})$.

We have from lemma 2.20 that $A_{-1}F_1 = F_0$. Thus

$$\operatorname{ran} A_{-1}BA^{-1} = A_{-1}\operatorname{ran} BA^{-1} \subset A_{-1}F_1 = F_0$$

since $\operatorname{ran} B \subset F_1$. □

Since by part (ii) of lemma 2.20 we know that for a Hilbert space X the Favard space of order 1 and the domain of A coincide, $F_1 = \mathcal{D}(A)$, we get the following result as a direct consequence of corollary 4.24.

Corollary 4.26. *Let A be a generator on a Hilbert space X with $\rho(A) \neq \varnothing$. If $B \in \mathcal{L}(X_1, X)$ and $\operatorname{ran} B \subset \mathcal{D}(A)$, then the operator $A + B$ with domain $\mathcal{D}(A)$ is a generator on X.*

Corollary 4.24 also leads to a simple proof of the following result concerning perturbations $B \in \mathcal{L}(X_1)$ [8, Cor III.1.5]. The proof is by the author.

Corollary 4.27. *Let A generate a C_0-semigroup $T(t)$ on a Banach space X. If $B \in \mathcal{L}(X_1)$ where $X_1 = (\mathcal{D}(A), \|\cdot\|_1)$, then the operator $A+B$ with domain $\mathcal{D}(A)$ generates a C_0-semigroup on X.*

Proof. We will show that B satisfies the conditions of corollary 4.24. Since A is a generator of a C_0-semigroup, we know that $\rho(A) \neq \varnothing$ since by theorem 2.6 for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(T(t))$ we have $\lambda \in \rho(A)$. We can now assume that $0 \in \rho(A)$ (otherwise we can consider $A - \lambda I$ with $\lambda > \omega_0(T(t))$).

Using part (iii) of lemma 2.20 we see that $\operatorname{ran} B \subset X_1 \subset F_1$.

It remains to show that $B \in \mathcal{L}(X_1, X)$. Since $B \in \mathcal{L}(X_1)$, we know that for some $M > 0$ we have $\|Bx\|_1 \leq M\|x\|_1$. Let $x \in X_1$. Now

$$\|Bx\| = \|A^{-1}ABx\| \leq \|A^{-1}\|\|ABx\| = \|A^{-1}\|\|Bx\|_1 \leq \|A^{-1}\|M\|x\|_1$$

and thus $B \in \mathcal{L}(X_1, X)$. By corollary 4.24 the operator $A + B$ with domain $\mathcal{D}(A)$ generates a C_0-semigroup on X. □

The following example shows a perturbation which is bounded on X_1 but not on X. The example is taken from [8, Ex III.1.6].

Example 4.28. Let $Af = f'$ with domain $C_0^1(\mathbb{R})$ on $X = C_0(\mathbb{R})$. Let $h \in \mathbb{C}_0^1(\mathbb{R})$ and define the operator B by

$$Bf = f'(0) \cdot h, \qquad f \in C_0^1(\mathbb{R})$$

Now $B \in \mathcal{L}(X_1)$, since for $f \in C_0^1(\mathbb{R})$ we have

$$\begin{aligned}\|Bf\|_1 &= \|A(f'(0) \cdot h)\| = \|f'(0) \cdot h'\| = \sup_{t\in\mathbb{R}}|f'(0) \cdot h'(t)| \leq \sup_{t\in\mathbb{R}}|f'(t)| \cdot \overbrace{\sup_{t\in\mathbb{R}}|h'(t)|}^{=M} \\ &= M\|Af\| = M\|f\|_1.\end{aligned}$$

By corollary 4.27 the operator $A + B$ with domain $C_0^1(\mathbb{R})$ generates a C_0-semigroup on $C_0(\mathbb{R})$.

We have the following result concerning preservation of regularity properties under Miyadera-Voigt pertubations. As in the case of Desch-Schappacher perturbations, the result was given in an exercise in [8, Exer III.3.17.(1)] and the proof is by the author.

Proposition 4.29. *Let A generate an analytic semigroup on X and let $B \in \mathcal{S}_{t_0}^{MV}$. Then $A + B$ generates an analytic semigroup on X.*

Proof. Let A generate an analytic semigroup $T(z)$ on X and choose $\omega_1 \in \mathbb{R}$ such that $\omega_1 > \omega_0(T(t))$. Then the operator $A - \omega_1 I$ generates a bounded analytic semigroup on X and by theorem 2.17 there exists a constant $M_1 > 0$ such that for all $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$

$$\begin{aligned} \|R(r + is, A - \omega_1 I)\| &\leq \frac{M_1}{|s|} \\ \Leftrightarrow \quad \|R(r + \omega_1 + is, A)\| &\leq \frac{M_1}{|s|} \end{aligned} \tag{4.7}$$

Let $M_2 > 0$, ω_2 be real constants such that

$$\|T(t)\| \leq M_2 e^{\omega_2 t}.$$

Then for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega_2$ we have $\lambda \in \rho(A)$ and theorem 2.6 tells us that for all $x \in X$ we can express $R(\lambda, A)x$ as an integral. Therefore for all $x \in \mathcal{D}(A)$

$$\begin{aligned} BR(\lambda, A)x &= B \int_0^\infty e^{-\lambda s} T(s) x ds = \sum_{n=0}^\infty \int_{nt_0}^{(n+1)t_0} e^{-\lambda s} BT(s) x ds \\ &= \sum_{n=0}^\infty \int_{nt_0}^{(n+1)t_0} e^{-\lambda n t_0} e^{-\lambda(s - nt_0)} BT(s - nt_0) T(nt_0) x ds \\ &= \sum_{n=0}^\infty e^{-\lambda n t_0} \int_0^{t_0} e^{-\lambda s} BT(s) T(nt_0) x ds \\ &= \sum_{n=0}^\infty e^{-\lambda n t_0} \int_0^{t_0} I e^{-\lambda(t_0 - s)} BT(t_0 - s) T(nt_0) x ds \\ &= \sum_{n=0}^\infty e^{-\lambda n t_0} \left[\overline{V_B^*} F_\lambda \right] (t_0) T(nt_0) x. \end{aligned}$$

Here $F_\lambda(t) = I e^{-\lambda(t_0 - t)}$ and $F_\lambda \in \mathcal{X}_{t_0}$ with $\|F_\lambda\|_\infty = \sup_{t \in [0, t_0]} |e^{-\lambda(t_0 - t)}| = 1$. For simplicity, we will denote $V = \overline{V_B^*}$. We can estimate the norm by

$$\begin{aligned}
\|BR(\lambda, A)x\| &= \|\sum_{n=0}^{\infty} e^{-\lambda n t_0}[VF_\lambda](t_0)T(nt_0)x\| \\
&\leq \left[\sum_{n=0}^{\infty} |e^{-\lambda n t_0}| \|V\| \|F_\lambda\|_\infty \|T(nt_0)\|\right] \|x\| \\
&\leq \left[\|V\| + M_2 \sum_{n=1}^{\infty} e^{(\omega_2 - \operatorname{Re}\lambda)nt_0}\right] \|x\| \leq \left[\|V\| + \frac{M_2 e^{(\omega_2 - \operatorname{Re}\lambda)t_0}}{1 - e^{(\omega_2 - \operatorname{Re}\lambda)t_0}}\right] \|x\|.
\end{aligned}$$

Because $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ the operator V is bounded with $\|V\| < 1$. Since the last term goes to zero as the real part of λ goes to infinity, we have for some $\omega_3 > \omega_2$ and $0 < q < 1$ that

$$\|BR(\lambda, A)x\| \leq q\|x\|, \qquad \forall x \in \mathcal{D}(A)$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_3$. Since $\mathcal{D}(A)$ is dense in X and $BR(\lambda, A) \in \mathcal{L}(X)$, the uniqueness of the extension in theorem A.14 tells us that

$$\|BR(\lambda, A)\| \leq q < 1, \qquad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda > \omega_3. \tag{4.8}$$

Since $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, the perturbed operator $A + B$ generates a C_0-semigroup $S(t)$ with growth bound $\omega_0(S(t))$ on X. Choose

$$\omega > \max\{\omega_0(S(t)), \omega_1, \omega_3\}.$$

Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$. Now we have by equation (4.8) and theorem A.17 that the spectral radius of $BR(\lambda, A)$ satisfies

$$r(BR(\lambda, A)) \leq \|BR(\lambda, A)\| < 1.$$

Therefore $1 \in \rho(BR(\lambda, A))$ and we can make an estimate

$$\begin{aligned}
\|R(1, BR(\lambda, A)\| &= \|\sum_{n=0}^{\infty} (BR(\lambda, A))^n\| \leq \sum_{n=0}^{\infty} \|BR(\lambda, A)\|^n \\
&= \frac{1}{1 - \|BR(\lambda, A)\|} \leq \frac{1}{1-q}.
\end{aligned} \tag{4.9}$$

Since $\operatorname{Re}\lambda > \omega > \omega_2$, we also have $\lambda \in \rho(A)$. The identity

$$(\lambda I - A - B) = (I - BR(\lambda, A))(\lambda I - A)$$

then implies that $\lambda \in \rho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A)R(1, BR(\lambda, A)).$$

Since $\omega > \omega_0(S(t))$, we know that $A + B - \omega I$ generates a bounded C_0-semigroup on X. Finally, for $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$ we get using (4.7) and (4.9) that

$$\begin{aligned}
\|R(r + is, A + B - \omega I)\| &= \|R(r + \omega + is, A + B)\| \\
&= \|R(r + \omega + is, A)R(1, BR(r + \omega + is, A))\| \\
&\leq \|R(r + \omega + is, A)\| \|R(1, BR(r + \omega + is, A))\| \\
&\leq \frac{M_1}{|s|} \frac{1}{1 - q} = \frac{M}{|s|}.
\end{aligned}$$

By theorem 2.17 the operator $A + B - \omega I$ then generates an analytic semigroup $e^{-\omega \operatorname{Re} z} S(z)$ on X and $A + B = (A + B - \omega I) + \omega I$ generates an analytic semigroup $e^{\omega \operatorname{Re} z} e^{-\omega \operatorname{Re} z} S(z) = S(z)$ on X. □

4.5.1 Class $\mathscr{P}$ Perturbations

First results concerning unbounded perturbations and general C_0-semigroups were presented as early as in the 1950's. One of these concerns a class of perturbations called *class $\mathscr{P}$ perturbations*. Basic theory can be found for example in [7] or [9]. We will show here that class $\mathscr{P}$ perturbations can in fact be seen as Miyadera-Voigt perturbations. The proof is by the author, but the result is not new.

Definition 4.30. An operator B with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$ is a *class $\mathscr{P}$ perturbation* if B is closed and for every $t > 0$ there exists a constant $K(t)$ such that

$$\|BT(t)x\| \leq K(t)\|x\| \quad \text{for} \;\; x \in \mathcal{D}(A)$$

and $K(t)$ can be chosen so that

$$\int_0^1 K(t)dt < \infty.$$

■

To prove that class $\mathscr{P}$ perturbations are Miyadera-Voigt perturbations, we will show that they satisfy conditions of corollary 4.22. Let B be a class $\mathscr{P}$ perturbation.

We will first have to show that $B \in \mathcal{L}(X_1, X)$. By remark 4.19 it suffices to show that B is A-bounded. Since A is closed as a generator of a C_0-semigroup and B is closed with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$, this follows directly from lemma A.9.

It remains to show that we can choose $t_0 > 0$ so that

$$\int_0^{t_0} \|BT(t)x\| \leq q\|x\| \quad \forall x \in \mathcal{D}(A)$$

for some $0 \leq q < 1$. Because for $x \in \mathcal{D}(A)$ we have

$$\int_0^{t_0} \|BT(t)x\|dt \leq \int_0^{t_0} K(t)dt\|x\|,$$

it suffices to show that we can choose $t_0 > 0$ such that $\int_0^{t_0} K(t)dt < 1$. Now, since

$$\int_0^1 K(t)dt < \infty$$

we know that $K(t) \in L^1([0,1]; \mathbb{R}^+)$. Now lemma A.20 guarantees that we can find $\delta > 0$ such that for all $a < b$ with $b - a < \delta$ we have

$$\int_a^b K(t)dt < 1.$$

Thus, choosing any $0 < t_0 < \delta$ we have

$$\int_0^{t_0} K(t)dt < 1.$$

This shows that the operator B satisfies the conditions of corollary 4.22 and thus $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ for some $t_0 > 0$.

4.6 The Perturbation Theorem of Kaiser and Weis

In their article [13], Kaiser and Weis presented sufficient conditions for an operator $A + B$ in Hilbert space to generate a semigroup that is strongly continuous on $(0, \infty)$. Batty later proved in [2] that the semigroup $S(t)$ generated by $A + B$ approaches the identity operator in the strong operator topology as $t \to 0^+$ and thus $S(t)$ is a C_0-semigroup. The main result is presented in theorem 4.31. The theorem presented here is a slightly modified version of the one presented in the article by Batty [2]: The original version required the condition (4.11) to hold for all $x \in \mathcal{D}(B)$. We prove here that it is sufficient that this holds for all $x \in \mathcal{D}(A)$.

Theorem 4.31. *Let A be a generator of a C_0-semigroup $T(t)$ on a Hilbert space X and let B be a closed operator on X with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$. Assume that there exist constants $0 < q < 1$ and $\lambda_0 \in \mathbb{R}$ such that*

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re} \lambda \geq \lambda_0 \right\} \subset \rho(A)$$

and

$$\begin{aligned} \|BR(\lambda, A)\| &\leq q && (4.10) \\ \|R(\lambda, A)Bx\| &\leq q\|x\| \qquad \forall x \in \mathcal{D}(A) && (4.11) \end{aligned}$$

whenever $\operatorname{Re} \lambda \geq \lambda_0$. Then $A + B$ generates a C_0-semigroup on X.

The proof presented here follows [2, Thm 1] and is based on the following result from [24, Thm 1.1].

Lemma 4.32. *A linear operator A on a Hilbert space X is a infinitesimal generator of a C_0-semigroup $T(t)$ satisfying*

$$\|T(t)\| \leq M e^{\omega_0 t}, \quad \forall t \geq 0,$$

for some $M \geq 1$, $\omega_0 \in \mathbb{R}$ if and only if

(i) A is a closed densely defined operator

(ii) $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega_0\} \subset \rho(A)$ *and for any* $\lambda = \omega + is \in \mathbb{C}$ *with* $\omega > \omega_0$ *the resolvent estimates*

$$\sup_{\omega>\omega_0} (\omega - \omega_0) \int_{-\infty}^{\infty} \|R(\omega + is, A)x\|^2 ds < \infty, \qquad \forall x \in X$$

and

$$\sup_{\omega>\omega_0} (\omega - \omega_0) \int_{-\infty}^{\infty} \|R(\omega + is, A^*)x\|^2 ds < \infty, \qquad \forall x \in X$$

are satisfied.

Proof of theorem 4.31. We can assume that $\|T(t)\| \le M$ for all $t \ge 0$ and that $\lambda_0 \le 0$ (if this is not the case, we consider the rescaled semigroup generated by $A - \omega I$ where $\omega > \max\{\omega_0(T(t)), \lambda_0\}$. See section 2.1 for details).

Let $x \in \mathcal{D}(A)$. Since $0 \in \rho(A)$, we have from (4.10) that

$$\|Bx\| \;=\; \|BA^{-1}Ax\| \le q\|Ax\|$$

and thus the A-bound of B must be less than 1. Thus the operator $A + B$ is closed [14, Thm IV.1.1]. Because $\mathcal{D}(A)$ is dense in X (see theorem 2.6), the operator $A + B$ is also densely defined.

Let $x \in X$ and $a > 0$. Consider the mapping

$$f(t) = \begin{cases} e^{-at}T(t)x & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Now, since by theorem 3.2 we have $f \in L^2(\mathbb{R}, X)$ and

$$R(a + is, A)x = \int_0^{\infty} e^{-(a+is)r} T(r)x dr = \int_{-\infty}^{\infty} e^{-isr} f(r)x dr = (\mathcal{F}f)(s)$$

where $\mathcal{F}$ denotes the Fourier transform. By Plancherel's Theorem [8, Thm C.14] $\|\mathcal{F}f\|_2 = \sqrt{2\pi}\|f\|_2$ and thus

$$\begin{aligned} \int_{-\infty}^{\infty} \|R(a + is, A)x\|^2 ds &= \int_{-\infty}^{\infty} \|(\mathcal{F}f)(s)\|^2 ds = \|\mathcal{F}f\|_2^2 = 2\pi\|f\|_2^2 \\ &= \int_{-\infty}^{\infty} \|f(t)\|^2 dt = \int_0^{\infty} e^{-2at}\|T(t)x\|^2 dt \le \frac{M^2\pi}{a}\|x\|^2 \end{aligned}$$

Similarly, considering the mapping

$$f(t) = \begin{cases} e^{-at}T(t)^*x & t \geq 0 \\ 0 & t < 0 \end{cases}$$

we get an estimate

$$\int_{-\infty}^{\infty} \|R(a+is, A)^*x\| ds \leq \frac{M^2\pi}{a}\|x\|^2$$

Denote by $R|_{\mathcal{D}(A)}$ the restriction of $R(a+is, A)B$ to $\mathcal{D}(A)$. Because ran $R(a+is, A)B \subset \mathcal{D}(A)$ and (4.11) holds, $R|_{\mathcal{D}(A)}$ is a bounded operator on $\mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in X, we know by theorem A.14 that $R|_{\mathcal{D}(A)}$ can be extended to a bounded operator $\overline{R} \in \mathcal{L}(X)$ with $\|\overline{R}\| \leq q$. This immediately tells us that $r(\overline{R}) \leq \|\overline{R}\| \leq q < 1$ and thus $1 \in \rho(\overline{R}) = \rho(R|_{\mathcal{D}(A)})$ (see lemma A.19).

Since $\mathcal{D}(A+B) = \mathcal{D}(A)$, we have the identity

$$\begin{aligned} (a+is)I - A - B &= ((a+is)I - A)(I - R(a+is, A)B) \\ &= ((a+is)I - A)(I - R|_{\mathcal{D}(A)}). \end{aligned}$$

This tells us that $a + is \in \rho(A+B)$ and

$$\begin{aligned} R(a+is, A+B)x &= R(1, R|_{\mathcal{D}(A)})R(a+is, A)x \\ &= \sum_{n=0}^{\infty} \left(R|_{\mathcal{D}(A)}\right)^n R(a+is, A)x. \end{aligned}$$

This identity allows us to estimate

$$\begin{aligned} \int_{-\infty}^{\infty} \|R(a+is, A+B)x\|^2 ds &\leq \int_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \|R|_{\mathcal{D}(A)}\|^n \|R(a+is, A)x\|\right]^2 ds \\ &\leq \int_{-\infty}^{\infty} \frac{\|R(a+is, A)x\|^2}{(1-q)^2} ds \\ &\leq \frac{M^2\pi}{a(1-q)^2}\|x\|^2 \end{aligned}$$

Since condition (4.10) implies that $\|R(\lambda, A)^* B^* x'\| \leq q\|x'\|$ for all $x' \in \mathcal{D}(B^*)$ and $\mathcal{D}(B^*)$ is dense in X (B is closed), we can make another estimate

$$\int_{-\infty}^{\infty} \|R(a+is, A+B)^* x\|^2 ds \leq \frac{M^2 \pi}{a(1-q)^2} \|x\|^2$$

This shows that $A+B$ satisfies the assumptions of lemma 4.32 and thus $A+B$ generates a C_0-semigroup on X. □

We will introduce for the perturbations satisfying assumptions of theorem 4.31 a notation similar to the one we have adapted from Engel and Nagel [8] for Desch-Schappacher ($\mathcal{S}_{t_0}^{\mathrm{DS}}$) and Miyadera-Voigt ($\mathcal{S}_{t_0}^{\mathrm{MV}}$) perturbations. This will help us discuss the stability of the pertubed semigroup later in the thesis.

Definition 4.33. Let A generate a C_0-semigroup on a Hilbert space X and let B be a closed operator on X with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$. We say that $B \in \mathcal{S}^{\mathrm{KW}}$ if there exist constants $0 < q < 1$ and $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \lambda_0$

$$\begin{aligned} \|BR(\lambda, A)\| &\leq q \\ \|R(\lambda, A)Bx\| &\leq q\|x\| \quad \forall x \in \mathcal{D}(A) \end{aligned}$$

■

Definition 4.33 does not require that the set $\{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \lambda_0 \}$ belongs to the resolvent set of A. However, since A is a generator of a C_0-semigroup $T(t)$ on X, we know from theorem 2.6 that for all $\omega > \omega_0(T(t))$ we have $\lambda \in \rho(A)$ whenever $\operatorname{Re}\lambda \geq \omega$. If we choose $\max\{\lambda_0, \omega\}$ as the new λ_0, then all the assumptions of theorem 4.31 are satisfied.

It is now clear from theorem 4.31 that if A is a generator of a C_0-semigroup on a Hilbert space X and $B \in \mathcal{S}^{\mathrm{KW}}$, then the operator $A+B$ with domain $\mathcal{D}(A+B) = \mathcal{D}(A)$ is a generator of a C_0-semigroup on X.

It should be noted that conditions (4.10) and (4.11) are not completely unfamiliar to us. We saw in the proof of proposition 4.29 that if $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, then there exists a $\lambda_1 \in \mathbb{R}$ such that

$$\|BR(\lambda, A)\| \leq q < 1, \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq \lambda_1. \tag{4.12}$$

In other words, all Miyadera-Voigt perturbations satisfy condition (4.10) for some λ_0. Relating Desch-Schappacher perturbations to the ones considered in this section is not as straightforward because the perturbed operators differ in form. Recall from section 4.4 that if A generates a C_0-semigroup on X, then for $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ the perturbed operator is constructed as $(A_{-1} + B)|_X$. Also, B is defined on all of X. In the proof of proposition 4.18 we found that for a perturbation $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ there exists a $\lambda_1 \in \mathbb{R}$ such that

$$\|R(\lambda, A_{-1})B\| \leq q < 1, \qquad \text{for all} \quad \lambda \in \mathbb{C} \quad \text{with} \quad \operatorname{Re}\lambda \geq \lambda_1. \tag{4.13}$$

As we mentioned, the forms of the perturbations in this case differ too much for us to state as strong an implication as in the case of Miyadera-Voigt perturbations. Nevertheless, the property (4.13) has similar features compared to condition (4.11).

There are also strong differences between perturbations of class $\mathcal{S}^{\mathrm{KW}}$ compared to the classes $\mathcal{S}_{t_0}^{\mathrm{DS}}$ and $\mathcal{S}_{t_0}^{\mathrm{MV}}$. Perturbations of class $\mathcal{S}^{\mathrm{KW}}$ are required to be closed operators on a Hilbert space, while perturbations of classes $\mathcal{S}_{t_0}^{\mathrm{DS}}$ and $\mathcal{S}_{t_0}^{\mathrm{MV}}$ are not required to be closed and it suffices that the underlying space is a Banach space.

Similarly to the case of Miyadera-Voigt perturbations in section 4.5, we can prove the following result concerning the preservation of regularity properties for perturbations in $\mathcal{S}^{\mathrm{KW}}$. The result and proof are by the author.

Proposition 4.34. *Let A generate an analytic semigroup on X and let $B \in \mathcal{S}^{KW}$. Then $A + B$ generates an analytic semigroup on X.*

Proof. Let A generate an analytic semigroup $T(z)$ on X and choose $\omega_1 \in \mathbb{R}$ such that $\omega_1 > \omega_0(T(t))$. Then the operator $A - \omega_1 I$ generates a bounded analytic semigroup on X and by theorem 2.17 there exists a constant $M_1 > 0$ such that for all $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$

$$\begin{aligned} \|R(r + is, A - \omega_1 I)\| &\leq \frac{M_1}{|s|} \\ \Leftrightarrow \qquad \|R(r + \omega_1 + is, A)\| &\leq \frac{M_1}{|s|} \end{aligned} \tag{4.14}$$

Since B satisfies the assumptions of theorem 4.31, we know that $A + B$ is a generator of a C_0-semigroup $S(t)$ on X.

Furthermore, from our assumptions we know that there exists a $\lambda_0 \in \mathbb{R}$ such that for some $0 < q < 1$ we have $\|BR(\lambda, A)\| \leq q$ and $\lambda \in \rho(A)$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$. Choose

$$\omega > \max\{\omega_0(S(t)), \omega_1, \lambda_0\}$$

Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > \omega$. Since $\operatorname{Re}\lambda > \omega > \lambda_0$, we have $\lambda \in \rho(A)$ and theorem A.17 implies that the spectral radius of $BR(\lambda, A)$ satisfies

$$r(BR(\lambda, A)) \le \|BR(\lambda, A)\| < 1.$$

Therefore $1 \in \rho(BR(\lambda, A))$ and we can make an estimate

$$\begin{aligned} \|R(1, BR(\lambda, A)\| &= \|\sum_{n=0}^{\infty}(BR(\lambda, A))^n\| \le \sum_{n=0}^{\infty}\|BR(\lambda, A)\|^n \\ &= \frac{1}{1 - \|BR(\lambda, A)\|} \le \frac{1}{1-q}. \end{aligned} \tag{4.15}$$

The identity

$$(\lambda I - A - B) = (I - BR(\lambda, A))(\lambda I - A)$$

now implies that $\lambda \in \rho(A + B)$ and

$$R(\lambda, A + B) = R(\lambda, A)R(1, BR(\lambda, A)).$$

Since $\omega > \omega_0(S(t))$, we know that $A + B - \omega I$ generates a bounded C_0-semigroup on X. Finally, for $r, s \in \mathbb{R}$ with $r > 0$ and $s \neq 0$ we get using (4.14) and (4.15) that

$$\begin{aligned} \|R(r + is, A + B - \omega I)\| &= \|R(r + \omega + is, A + B)\| \\ &= \|R(r + \omega + is, A)R(1, BR(r + \omega + is, A))\| \\ &\le \|R(r + \omega + is, A)\|\|R(1, BR(r + \omega + is, A))\| \\ &\le \frac{M_1}{|s|}\frac{1}{1-q} = \frac{M}{|s|}. \end{aligned}$$

By theorem 2.17 the operator $A + B - \omega I$ then generates an analytic semigroup $e^{-\omega \operatorname{Re} z}S(z)$ on X and $A + B = (A + B - \omega I) + \omega I$ generates an analytic semigroup $e^{\omega \operatorname{Re} z}e^{-\omega \operatorname{Re} z}S(z) = S(z)$ on X. □

Chapter 5

Stability Criteria for Perturbed C_0-Semigroups

In this chapter we apply the stability criteria studied in chapter 3 to the case of perturbed C_0-semigroups. Our aim is to find sufficient conditions for the stability of the C_0-semigroup generated by the perturbed operator. We will consider different types of perturbations separately.

In case of bounded perturbations we can easily derive such a sufficient condition. Let A generate an exponentially stable C_0-semigroup $T(t)$ satisfying

$$\|T(t)\| \leq Me^{-\omega t}$$

for some $M \geq 1$ and $\omega > 0$. Theorem 4.1 now tells us that for any $B \in \mathcal{L}(X)$ the operator $A + B$ generates a C_0-semigroup $S(t)$ satisfying

$$\|S(t)\| \leq Me^{(-\omega + M\|B\|)t}$$

This gives rise to the following simple sufficient condition for the stability of the perturbed semigroup.

Corollary 5.1. *Let A generate a C_0-semigroup $T(t)$ on a Banach space X statisfying $\|T(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$. Let $B \in \mathcal{L}(X)$. Then $A + B$ generates an exponentially stable C_0-semigroup on X if $\|B\| < \frac{\omega}{M}$.*

If we have a perturbation of form $A + \varepsilon B$ where $B \in \mathcal{L}(X)$ and $\varepsilon \in \mathbb{C}$, the previous corollary states that the perturbed C_0-semigroup is exponentially stable for all ε satisfying

$$|\varepsilon| < \frac{\omega}{M\|B\|}.$$

Even though the result of corollary 5.1 seems like a crude estimate, it turns out to be the sharpest bound we can achieve if we need to cover all C_0-semigroups and bounded perturbations. To see this, consider again the case of example 4.9. Applying corollary 5.1 we see that $A + B$ generates an exponentially stable C_0-semigroup whenever $\|B\| = |\beta| < 1$. If we choose $\beta = 1$, we can see that the perturbed operator becomes

$$(A + B)(x_k) = ((-k + 1)x_k)$$

and thus it generates a C_0-semigroup $S(t)$ with

$$S(t)(x_k) = (e^{(-k+1)t}x_k).$$

Now choose $e_1 \in \ell^2(\mathbb{C})$ such that $e_1 = (1, 0, 0, \ldots)$. We see that $S(t)e_1 = e_1$ does not decay as t grows and thus the C_0-semigroup $S(t)$ is not stable. This means that there exists a perturbation B with norm $\|B\| = \frac{\omega}{M}$ such that the C_0-semigroup generated by $A + B$ is not exponentially stable.

As we saw in chapter 4, generation results are far more complicated when the perturbing operator is not bounded. In addition, it turns out that obtaining conditions for the exponential stability of the perturbed C_0-semigroup is more tricky. One of the obvious reasons is that we do not have any definite way to measure the "size" of the perturbation since $\|B\|$ is not defined. Since we are only considering relatively bounded perturbations, the first intuitive attempt would be to use the A-bound of B instead of $\|B\|$. However, this will not work because the spectrum of the operator A is not guaranteed to stay in the left half-plane of the complex plane $\mathbb{C}$ even for perturbations with small A-bound. We will use a different approach. We will formulate conditions for the stability of the perturbed C_0-semigroup using the norms of the operators

$$BR(\lambda, A) \quad \text{or} \quad R(\lambda, A_{-1})B$$

on the open half-plane of the complex plane $\mathbb{C}$.

The results presented in section 5.1 are based on the application of theorem 3.4 and corollary 3.6 to the case of perturbed C_0-semigroups. We will first present sufficient conditions for the stability of the perturbed C_0-semigroup when the perturbing operator is a relatively bounded operator for which the perturbed operator is generator on X. We will then proceed to formulate this condition particularly for the perturbations belonging to classes $\mathcal{S}_{t_0}^{\mathrm{MV}}$ or $\mathcal{S}^{\mathrm{KW}}$. The differences in Desch-Schappacher perturbations compared to the ones of classes $\mathcal{S}_{t_0}^{\mathrm{MV}}$ and $\mathcal{S}^{\mathrm{KW}}$ make separate treatment necessary. We will, however, be able to derive a similar sufficient condition for this class of perturbations. Finally, we will formulate a separate condition for the perturbation of exponentially stable C_0-semigroups of contractions. This case is dealt with separately, because the results concerning perturbation of contractive C_0-semigroups allows us to weaken the conditions for the exponential stability of the perturbed semigroup.

Some of these conditions are similar to the ones presented by Thieme for locally Lipschitz continuous *integrated semigroups* and positive perturbations in [26]. However, the assumptions used in the article are relatively restrictive compared to the case of general C_0-semigroups.

Also the results presented by Pritchard and Townley in [23] are related to the conditions presented in section 5.1. We analyze this relationship in section 5.1.6.

In section 5.2 we will present situations where we are able to determine the stability of the perturbed C_0-semigroup directly from the spectrum of the operator $A + B$. More precisely, we present conditions under which the negative spectral bound of the operator $A + B$ determines the stability of the perturbed C_0-semigroup. Since we lack proper perturbation theory for the spectral bound of an unbounded operator, the theory will be of use mainly when the spectrum of $A + B$ is known or can be obtained easily.

5.1 Conditions On The Resolvent

In this section we consider unbounded perturbations considered in chapter 4 and discuss the stability of the perturbed C_0-semigroup. Our results are based on application of theorem 3.4 and corollary 3.6. Because of the conditions in these results, we will throughout this section assume that X is a Hilbert space.

Like we already stated, we will first formulate sufficient conditions for the exponential stability of the perturbed C_0-semigroup when the perturbation is a relatively bounded operator. This is done in section 5.1.1. We will go on to formulate this condition for classes $\mathcal{S}_{t_0}^{\mathrm{MV}}$ and $\mathcal{S}^{\mathrm{KW}}$ in sections 5.1.2 and 5.1.3. These sections also contain results concerning the perturbations of form εB, where B is a perturbation of the class mentioned and ε is a real or complex parameter. The conditions derived in section 5.1.1 are not applicable to perturbations of class $\mathcal{S}_{t_0}^{\mathrm{DS}}$. A similar condition for exponential stability under perturbations of this class is derived in section 5.1.4. Finally, we will in section 5.1.5 formulate separate conditions for perturbation of exponentially stable C_0-semigroups of contractions.

The results obtained in this section are compared to the existing ones for bounded perturbations in section 5.1.6.

All the results and proofs presented in this section are by the author.

5.1.1 Conditions for General Perturbed C_0-Semigroups

In this section, we will present conditions for the stability of the perturbed C_0-semigroup. The first result applies to the case of relatively bounded perturbations for which the perturbed operator is a generator of a C_0-semigroup on X. The following is the main result of this section.

Proposition 5.2. *Let A generate an exponentially stable C_0-semigroup $T(t)$ on a Hilbert space X and let $B : X \supset \mathcal{D}(B) \to X$ be an A-bounded perturbation such that the operator $A+B$ with domain $\mathcal{D}(A+B) = \mathcal{D}(A)$ generates a C_0-semigroup $S(t)$ on X. If there exists a real constant $0 < q < 1$ such that*

$$\|BR(\lambda, A)\| \leq q \qquad \text{for all } \lambda \in \mathbb{C}^+,$$

then $S(t)$ is exponentially stable.

Proof. We will show that the operator $A+B$ satisfies conditions of theorem 3.4. By theorem A.17 we now know that

$$r(BR(\lambda, A)) \leq \|BR(\lambda, A)\| \leq q < 1$$

and thus $1 \in \rho(BR(\lambda, A))$ and $R(1, BR(\lambda, A)) \in \mathcal{L}(X)$. Since $T(t)$ is exponentially stable, we have $\lambda \in \rho(A)$ for all $\lambda \in \mathbb{C}^+$.

The identity

$$\lambda I - A - B = (I - BR(\lambda, A))(\lambda I - A)$$

implies that $\lambda \in \rho(A + B)$ whenever $\lambda \in \rho(A)$ and $1 \in \rho(BR(\lambda, A))$. This means that

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda > 0 \right\} \subset \rho(A + B)$$

and for all $\lambda \in \mathbb{C}^+$ the resolvent can be expressed as

$$R(\lambda, A + B) = R(\lambda, A)R(1, BR(\lambda, A)).$$

We can make an estimate

$$\begin{aligned}
\sup_{\operatorname{Re}\lambda>0} \|R(1, BR(\lambda, A))\| &= \sup_{\operatorname{Re}\lambda>0} \|\sum_{n=0}^{\infty} (BR(\lambda, A))^n\| \le \sup_{\operatorname{Re}\lambda>0} \sum_{n=0}^{\infty} \|BR(\lambda, A)\|^n \\
&= \sup_{\operatorname{Re}\lambda>0} \frac{1}{1 - \|BR(\lambda, A)\|} \le \sup_{\operatorname{Re}\lambda>0} \frac{1}{1-q} \\
&= \frac{1}{1-q} < \infty
\end{aligned}$$

and thus

$$\begin{aligned}
\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A + B)\| &\le \sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)R(1, BR(\lambda, A))\| \\
&\le \sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\|\|R(1, BR(\lambda, A))\| \\
&\le \underbrace{\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\|}_{<\infty} \underbrace{\sup_{\operatorname{Re}\lambda>0} \|R(1, BR(\lambda, A))\|}_{<\infty} < \infty.
\end{aligned}$$

Theorem 3.4 now tells us that the C_0-semigroup generated by $A + B$ is exponentially stable. □

Since bounded perturbations are also relatively bounded, we can see that the conditions of proposition 5.2 are clearly satisfied for perturbations $B \in \mathcal{L}(X)$. This case is further addressed in section 5.1.6.

Before moving on, we will note that if A is a generator of an exponentially stable C_0-semigroup on X and B is a relatively bounded perturbation, the operator $BR(\lambda, A)$ is always bounded for all $\lambda \in \mathbb{C}^+$. This follows directly from the A-boundedness of B and the fact that $R(\lambda, A)$ is bounded for all $\lambda \in \mathbb{C}^+$. In addition, the condition presented in proposition 5.2 then requires that $BR(\cdot, A)$ is uniformly bounded on $\mathbb{C}^+$ with supremum less than 1.

5.1.2 Miyadera-Voigt perturbations

In this section we will consider Miyadera-Voigt perturbations. We will first reformulate proposition 5.2 in the case $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$.

Corollary 5.3. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space X. If $B \in \mathcal{S}_{t_0}^{MV}$ and for some $0 < q < 1$*

$$\|BR(\lambda, A)\| \leq q \qquad \text{for all } \lambda \in \mathbb{C}^+,$$

then the C_0-semigroup generated by $A + B$ is exponentially stable.

Proof. Because $B \in \mathcal{L}(X_1, X)$, remark 4.19 tells us that B is A-bounded. Since the operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ is a generator of a C_0-semigroup on X, the assumptions of proposition 5.2 are satisfied. Therefore the C_0-semigroup generated by $A + B$ is exponentially stable. □

We saw in the proof of proposition 4.29 that if A generates an exponentially stable C_0-semigroup and $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, then $\|BR(\cdot, A)\|$ is uniformly bounded on $\mathbb{C}^+$. We can use this to obtain the following result concerning perturbations of form εB.

Proposition 5.4. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space X. If $B \in \mathcal{S}_{t_0}^{MV}$, we can choose $0 < \varepsilon_0 \leq 1$ such that $A + \varepsilon B$ generates an exponentially stable C_0-semigroup for all $\varepsilon \in \mathbb{C}$ with $0 < |\varepsilon| \leq \varepsilon_0$.*

Proof. Let A generate an exponentially stable C_0-semigroup $T(t)$ on X and let $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$. Due to corollary 5.3, it suffices to show that there exists $0 < \varepsilon_0 \leq 1$ such that for all $0 < |\varepsilon| \leq \varepsilon_0$ we have $\varepsilon B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ and for some $0 < q < 1$

$$\|\varepsilon BR(\lambda, A)\| \leq q \quad \text{for all } \lambda \in \mathbb{C}^+.$$

We will first show that for all $0 < |\varepsilon| \leq 1$, we have $\varepsilon B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$. Recall from section 4.5 that

$$\mathcal{S}_{t_0}^{\mathrm{MV}} = \left\{ B \in \mathcal{L}(X_1, X) \;\middle|\; \overline{V_B^*} \in \mathcal{L}(\mathcal{X}_{t_0}),\ \left\|\overline{V_B^*}\right\| < 1 \right\}.$$

where

$$(V_B^* F)(t)x = \int_0^t F(s)BT(t-s)x ds \quad \text{for} \quad t \in [0, t_0], \; x \in X_1$$

Since $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, we have $B \in \mathcal{L}(X_1, X)$ and $\|Bx\| \le M_1 \|x\|_1$ for some $M_1 > 0$. Now

$$\|\varepsilon Bx\| = |\varepsilon| \|Bx\| \le |\varepsilon| M_1 \|x\|_1$$

and thus $\varepsilon B \in \mathcal{L}(X_1, X)$. For all $x \in X_1$ and $t \in [0, t_0]$ we have

$$(V_{\varepsilon B}^* F)(t)x = \int_0^t F(s)\varepsilon BT(t-s)x ds = \varepsilon \int_0^t F(s)BT(t-s)x ds = \varepsilon (V_B^* F)(t)x$$

Since V_B^* is closable, clearly also $V_{\varepsilon B}^*$ is closable, $\overline{V_{\varepsilon B}^*} \in \mathcal{L}(\mathcal{X}_{t_0})$ and

$$\|\overline{V_{\varepsilon B}^*}\| = |\varepsilon| \|\overline{V_B^*}\| < |\varepsilon| \le 1$$

This means that $\varepsilon B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$.

Now, since $T(t)$ is exponentially stable, we have for some $M \ge 1$ and $\omega > 0$ that

$$\|T(t)\| \le M e^{-\omega t}.$$

As in proof of proposition 4.29, we can achieve an estimate

$$\|BR(\lambda, A)x\| \le \|\overline{V_B^*}\| + \frac{M e^{(-\omega - \operatorname{Re}\lambda)t_0}}{1 - e^{(-\omega - \operatorname{Re}\lambda)t_0}}.$$

Since we are considering $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, and the second term is a decreasing function of $\operatorname{Re}\lambda$, we can estimate

$$\|BR(\lambda, A)\| \le \|\overline{V_B^*}\| + \frac{M e^{(-\omega - \operatorname{Re}\lambda)t_0}}{1 - e^{(-\omega - \operatorname{Re}\lambda)t_0}} \le \|\overline{V_B^*}\| + \frac{M e^{-\omega t_0}}{1 - e^{-\omega t_0}} < 1 + \frac{M e^{-\omega t_0}}{1 - e^{-\omega t_0}}.$$

Let $0 < q < 1$ be a constant and choose

$$\varepsilon_0 = \frac{q}{1 + \frac{Me^{-\omega t_0}}{1-e^{-\omega t_0}}} = \frac{q(1 - e^{-\omega t_0})}{1 + (M-1)e^{-\omega t_0}}.$$

Now, for all $\varepsilon \in \mathbb{C}$ with $0 < |\varepsilon| \leq \varepsilon_0$

$$\|\varepsilon BR(\lambda, A)\| = |\varepsilon|\|BR(\lambda, A)\| < \frac{q}{1 + \frac{Me^{-\omega t_0}}{1-e^{-\omega t_0}}} \left(1 + \frac{Me^{-\omega t_0}}{1 - e^{-\omega t_0}}\right) = q.$$

Corollary 5.3 now tells us that with this choise of ε_0 the C_0-semigroup generated by $A + \varepsilon B$ is exponentially stable for all $0 < |\varepsilon| \leq \varepsilon_0$. □

5.1.3 The Perturbation Theorem of Kaiser and Weis

We will now consider perturbations of class $\mathcal{S}^{\mathrm{KW}}$ and formulate the conditions of proposition 5.2 in this particular case. The following corollary states the main result.

Corollary 5.5. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space X and let B be a closed operator on X with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$. If there exist constants $0 < q < 1$ and $\mu \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}^+$*

$$\|BR(\lambda, A)\| \leq q$$

and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \mu$

$$\|R(\lambda, A)Bx\| \leq q\|x\| \quad \forall x \in \mathcal{D}(A),$$

then $A + B$ generates an exponentially stable C_0-semigroup on X.

Proof. We will first show that $A + B$ generates a C_0-semigroup on X by showing that the assumptions of theorem 4.31 are satisfied. Since A generates an exponentially stable C_0-semigroup $T(t)$ on X, we know from theorem 2.6 that

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda \geq 0 \right\} \subset \rho(A).$$

If $\mu > 0$, choose $\lambda_0 = \mu$. Otherwise we can choose λ_0 to be any positive real number. This shows that the conditions of theorem 4.31 are satisfied and thus $A + B$ generates a C_0-semigroup $S(t)$ on X.

Because the closedness of B and $\mathcal{D}(B) \supset \mathcal{D}(A)$ imply that B is A-bounded (see lemma A.9), the assumptions of proposition 5.2 are satisfied. This shows that the C_0-semigroup generated by $A+B$ is exponentially stable. □

We will again present a direct consequence of the previous result concerning perturbed operators of the form εB.

Corollary 5.6. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space X and let B be a closed operator on X with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$. If there exist constants $M_1, M_2 > 0$ and $\mu \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}^+$*

$$\|BR(\lambda, A)\| \leq M_1$$

and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \mu$

$$\|R(\lambda, A)Bx\| \leq M_2\|x\| \quad \forall x \in \mathcal{D}(B), \tag{5.1}$$

then there exists a constant $\varepsilon_0 > 0$ such that $A + \varepsilon B$ generates an exponentially stable C_0-semigroup on X for all $0 < \varepsilon \leq \varepsilon_0$.

Proof. Choose

$$\varepsilon_0 \leq q \cdot \min\left\{\frac{1}{M_1}, \frac{1}{M_2}\right\}$$

for some $0 < q < 1$. Clearly $\mathcal{D}(\varepsilon B) = \mathcal{D}(B)$ and εB is closed. Because for all $\lambda \in \mathbb{C}^+$

$$\|\varepsilon BR(\lambda, A)\| = |\varepsilon|\|BR(\lambda, A)\| \leq q$$

and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \mu$

$$\|R(\lambda, A)\varepsilon B\| = |\varepsilon|\|R(\lambda, A)B\| \leq q,$$

the operator εB satisfies conditions of corollary 5.5 for all $0 < |\varepsilon| \leq \varepsilon_0$. This shows that $A + B$ generates an exponentially stable C_0-semigroup on X. □

We will now present an example of an application for the theory presented in this section. This example is a modification of the one in [13, Sec 6].

Example 5.7. Let $X = L^2(\mathbb{R},\mathbb{C})$ and choose $k \in \mathbb{N}$. Consider an operator A_0 such that

$$A_0 x = i x^{(2k)}$$

where $x^{(2k)}$ denotes the $2k$th distributional derivative of x. Let the domain of this operator be

$$\mathcal{D}(A_0) = \left\{ x \in X \;\middle|\; x^{(i)} \text{ abs. cont. for } i \in \{1,\dots,2k-1\},\ x^{(2k)} \in L^2(\mathbb{R},\mathbb{C}) \right\}.$$

Operator A_0 then generates a C_0-semigroup $T_0(t)$ on X [1, Sec 8.1] and $\sigma(A_0) \subset i\mathbb{R}$. Choose a real constant ω such that $\omega > \omega_0(T_0(t))$. Since $\omega_0(T_0(t)) \geq s(A_0)$, we know that $\omega > 0$. The operator $A_0 - \omega I$ with

$$(A_0 - \omega I)x = i x^{(2k)} - \omega x, \qquad \mathcal{D}(A) = \mathcal{D}(A_0)$$

now generates an exponentially stable C_0-semigroup $T(t)$ on X.

Consider a perturbing operator B on X such that

$$Bx = V \cdot x^{(l)}$$

with domain

$$\mathcal{D}(B) = \left\{ x \in X \;\middle|\; V \cdot x^{(i)} \text{ abs. cont. for } i \in \{1,\dots,l-1\},\ V \cdot x^{(l)} \in X \right\}$$

where $V \in L^2(\mathbb{R},\mathbb{C})$ is a potential function and $l \in \mathbb{N}_0$ such that $l < k$. It is shown in [13] that for $\lambda \in \mathbb{C} \setminus (i\mathbb{R})$

$$\|BR(\lambda, A_0)\| \leq \frac{\|V\|_2}{2c(\operatorname{Re}\lambda)^{1-l/(2k)-1/(4k)}} \tag{5.2}$$

for some real constant $c > 0$. Since by our assumption we have that $1 - \frac{l}{2k} - \frac{1}{4k} > 0$, this immediately implies that for all $\lambda \in \mathbb{C}^+$ we have

$$\begin{aligned}
\|BR(\lambda, A)\| &= \|BR(\lambda, A_0 - \omega I)\| = \|BR(\lambda + \omega, A_0)\| \\
&\leq \frac{\|V\|_2}{2c(\operatorname{Re}\lambda + \operatorname{Re}\omega)^{1-l/(2k)-1/(4k)}} \leq \frac{\|V\|_2}{2c(\operatorname{Re}\omega)^{1-l/(2k)-1/(4k)}}.
\end{aligned}$$

This means that for any potential function $V \in L^2(\mathbb{R}, \mathbb{C})$ whose norm satisfies

$$\|V\|_2 < 2c(\operatorname{Re}\omega)^{1-l/(2k)-1/(4k)}, \tag{5.3}$$

we have for some $0 < q < 1$ and for all $\lambda \in \mathbb{C}^+$ that $\|BR(\lambda, A)\| \leq q$.

It can also be shown [13, Proof of Prop 6.1] that the estimate in (5.2) holds for A_0^* and B^* instead of A_0 and B. This means that also for all $x \in \mathcal{D}(B)$ we have

$$\|R(\lambda, A)Bx\| = \|R(\lambda + \omega, A_0)Bx\| \leq \frac{\|V\|_2}{2c(\operatorname{Re}\lambda + \operatorname{Re}\omega)^{1-l/(2k)-1/(4k)}}\|x\|$$

and for all potential functions satisfying (5.3) we have

$$\|R(\lambda, A)Bx\| \leq q\|x\|$$

for all $\lambda \in \mathbb{C}^+$. This shows that the conditions of corollary 5.5 are satisfied for any real $\mu > 0$ and thus the operator $A + B$,

$$(A + B)x = ix^{(2k)} + V \cdot x^{(l)} - \omega x, \qquad \mathcal{D}(A + B) = \mathcal{D}(A),$$

generates an exponentially stable C_0-semigroup on X.

5.1.4 Desch-Schappacher perturbations

Results similar to the ones in previous sections can also be formulated for Desch-Schappacher perturbations, but because of the differences this case must be dealt with separately. The following proposition states the main result of this section.

Proposition 5.8. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space X. If $B \in \mathcal{S}_{t_0}^{DS}$ and for some $0 < q < 1$*

$$\|R(\lambda, A_{-1})B\| \leq q \qquad \text{for all } \lambda \in \mathbb{C}^+,$$

then the C_0-semigroup generated by $(A_{-1} + B)|_X$ on X is exponentially stable.

Proof. By theorem A.17 we now know that

$$r(R(\lambda, A_{-1})B) \leq \|R(\lambda, A_{-1})B\| \leq q < 1$$

and thus $1 \in \rho(R(\lambda, A_{-1})B)$ and $R(1, R(\lambda, A_{-1})B) \in \mathcal{L}(X)$. Since $T(t)$ is exponentially stable, we have $\lambda \in \rho(A)$ for all $\lambda \in \mathbb{C}^+$.

The identity

$$\lambda I - (A_{-1} + B)|_X = (\lambda I - A)(I - R(\lambda, A_{-1})B)$$

now implies that $\lambda \in \rho((A+B)|_X)$ whenever $\lambda \in \rho(A)$ and $1 \in \rho(R(\lambda, A_{-1})B)$. This means that

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda > 0 \right\} \subset \rho((A+B)|_X)$$

and the resolvent can be expressed as

$$R(\lambda, (A_{-1} + B)|_X) = R(1, R(\lambda, A_{-1})B)R(\lambda, A).$$

We have an estimate

$$\begin{aligned}
\sup_{\operatorname{Re}\lambda>0} \|R(1, R(\lambda, A_{-1})B)\| &= \sup_{\operatorname{Re}\lambda>0} \|\sum_{n=0}^{\infty} (R(\lambda, A_{-1})B)^n\| \leq \sup_{\operatorname{Re}\lambda>0} \sum_{n=0}^{\infty} \|R(\lambda, A_{-1})B\|^n \\
&= \sup_{\operatorname{Re}\lambda>0} \frac{1}{1 - \|R(\lambda, A_{-1})B\|} \leq \sup_{\operatorname{Re}\lambda>0} \frac{1}{1-q} \\
&= \frac{1}{1-q} < \infty
\end{aligned}$$

and thus

$$\begin{aligned}
\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, (A_{-1} + B)|_X)\| &\leq \sup_{\operatorname{Re}\lambda>0} \|R(1, R(\lambda, A_{-1})B)R(\lambda, A)\| \\
&\leq \sup_{\operatorname{Re}\lambda>0} \|R(1, R(\lambda, A_{-1})B)\| \|R(\lambda, A)\| \\
&\leq \underbrace{\sup_{\operatorname{Re}\lambda>0} \|R(1, R(\lambda, A_{-1})B)\|}_{<\,\infty} \underbrace{\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\|}_{<\,\infty} < \infty.
\end{aligned}$$

Theorem 3.4 now tells us that the C_0-semigroup generated by $(A+B)|_X$ is exponentially stable. $\square$

We will again note that if A generates an exponentially stable C_0-semigroup $T(t)$ and $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, then $R(\lambda, A_{-1})B$ is a bounded linear operator on X for all $\lambda \in \mathbb{C}^+$. Because this is not as evident as the boundedness of $BR(\lambda, A)$ in section 5.1.1, we will prove it here.

Let $\lambda \in \mathbb{C}^+$. From the exponential stability of $T(t)$ and section 2.2.1 we know that $\omega_0(T_{-1}(t)) = \omega_0(T(t))$. This also implies that $0, \lambda \in \rho(A_{-1})$. Thus, by part (ii) of lemma 2.18, the norms defined by $\|A_{-1}^{-1}x\|$ and $\|R(\lambda, A_{-1})x\|$ are equivalent. Because of this and the fact that $B \in \mathcal{L}(X, X_{-1})$, we have that for some real positive constants M and C

$$\|R(\lambda, A_{-1})Bx\| \leq C\|A_{-1}^{-1}Bx\| = C\|Bx\|_{-1} \leq CM\|x\| \quad \forall x \in X.$$

This means that $R(\lambda, A_{-1})B \in \mathcal{L}(X)$.

As a direct consequence of proposition 5.8, we get the following result concerning the stability of an operator $(A_{-1} + \varepsilon B)|_X$ where $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ and ε is a constant.

Proposition 5.9. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space X. If $B \in \mathcal{S}_{t_0}^{DS}$, we can choose $0 < \varepsilon_0 \leq 1$ so that $(A_{-1} + \varepsilon B)|_X$ generates an exponentially stable C_0-semigroup for all $\varepsilon \in \mathbb{C}$ with $0 < |\varepsilon| \leq \varepsilon_0$.*

Proof. Due to proposition 5.8, it suffices to show that there exists $0 < \varepsilon_0 \leq 1$ such that for all $0 < |\varepsilon| \leq \varepsilon_0$ we have $\varepsilon B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ and for some $0 < q < 1$

$$\|R(\lambda, A_{-1})\varepsilon B\| \leq q \quad \text{for all } \lambda \in \mathbb{C}^+.$$

We will first show that for all $0 < |\varepsilon| \leq 1$, we have $\varepsilon B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$. Recall from section 4.4 that

$$\mathcal{S}_{t_0}^{\mathrm{DS}} = \left\{ B \in \mathcal{L}(X, X_{-1}) \;\middle|\; V_B \in \mathcal{L}(\mathcal{X}_{t_0}),\ \|V_B\| < 1 \right\}.$$

where

$$(V_B F)(t)x = \int_0^t T_{-1}(t-s)BF(s)x ds \quad \text{for} \;\; t \in [0, t_0],\ x \in X$$

Since $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, we have $B \in \mathcal{L}(X, X_{-1})$ and $\|Bx\|_{-1} \leq M_1\|x\|$ for some $M_1 > 0$. Now

$$\|\varepsilon Bx\|_{-1} = |\varepsilon|\|Bx\|_{-1} \leq |\varepsilon|M_1\|x\|$$

and thus $\varepsilon B \in \mathcal{L}(X, X_{-1})$. Since for all $x \in X$ and $t \in [0, t_0]$

$$(V_{\varepsilon B}F)(t)x = \int_0^t T_{-1}(t-s)\varepsilon BF(s)xds = \varepsilon \int_0^t T_{-1}(t-s)BF(s)xds = \varepsilon(V_B F)(t)x$$

and $V_B \in \mathcal{L}(\mathcal{X}_{t_0})$, then also $V_{\varepsilon B} \in \mathcal{L}(\mathcal{X}_{t_0})$ and

$$\|V_{\varepsilon B}\| = |\varepsilon|\|V_B\| < |\varepsilon| \leq 1.$$

This means that $\varepsilon B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$.

Now, since $T(t)$ is exponentially stable and $\omega_0(T_{-1}(t)) = \omega_0(T(t))$, we have for some $M \geq 1$ and $\omega > 0$ that

$$\|T_{-1}(t)\| \leq Me^{-\omega t}.$$

As in proof of proposition 4.18, we can achieve an estimate

$$\|R(\lambda, A_{-1})B\| \leq \|V_B\| + \frac{Me^{(-\omega - \operatorname{Re}\lambda)t_0}}{1 - e^{(-\omega - \operatorname{Re}\lambda)t_0}}.$$

Since we are considering $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$ and the second term is a decreasing function of $\operatorname{Re}\lambda$, we get

$$\|R(\lambda, A_{-1})B\| \leq \|V_B\| + \frac{Me^{(-\omega - \operatorname{Re}\lambda)t_0}}{1 - e^{(-\omega - \operatorname{Re}\lambda)t_0}} \leq \|V_B\| + \frac{Me^{-\omega t_0}}{1 - e^{-\omega t_0}} < 1 + \frac{Me^{-\omega t_0}}{1 - e^{-\omega t_0}}.$$

Let $0 < q < 1$ and choose

$$\varepsilon_0 = \frac{q}{1 + \frac{Me^{-\omega t_0}}{1 - e^{-\omega t_0}}} = \frac{q(1 - e^{-\omega t_0})}{1 + (M-1)e^{-\omega t_0}}.$$

Now for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \leq \varepsilon_0$

$$\|R(\lambda, A_{-1})\varepsilon B\| = |\varepsilon|\|R(\lambda, A_{-1})B\| \leq \frac{q}{1 + \frac{Me^{-\omega t_0}}{1 - e^{-\omega t_0}}}\left(1 + \frac{Me^{-\omega t_0}}{1 - e^{-\omega t_0}}\right) = q.$$

Proposition 5.8 now tells us that with this choise of ε_0 the C_0-semigroup generated by $(A_{-1} + \varepsilon B)|_X$ is exponentially stable for all $0 < |\varepsilon| \leq \varepsilon_0$. □

5.1.5 C_0-semigroups of Contractions

As we saw in section 4.3, some perturbations of C_0-semigroups of contractions result again in contractive C_0-semigroups. Using corollary 3.6 concerning the stability of uniformly bounded C_0-semigroups we can derive a stability result for a particular type of C_0-semigroup. This section deals with C_0-semigroups that are both contractive and exponentially stable. This means that the C_0-semigroup $T(t)$ satisfies the norm estimate

$$\|T(t)\| \leq e^{-\omega t}$$

for some $\omega > 0$. The following proposition contains the main result of this section.

Proposition 5.10. *Let A generate a C_0-semigroup $T(t)$ on a Hilbert space X satisfying*

$$\|T(t)\| \leq e^{-\omega t}$$

for some $\omega > 0$. Let B with domain $\mathcal{D}(B) \supset \mathcal{D}(A)$ be dissipative and A-bounded with A-bound $a_0 < 1$. If there exists a constant $0 < q < 1$ such that

$$\|BR(\lambda i, A)\| \leq q$$

for all $\lambda \in \mathbb{R}$, then the operator $A + B$ generates an exponentially stable C_0-semigroup on X.

Proof. Since B satisfies the assumptions of theorem 4.10, the operator $A+B$ generates a C_0-semigroup of contractions $S(t)$ on X. We will now show that $A+B$ satisfies the assumptions of corollary 3.6 and thus the perturbed C_0-semigroup is exponentially stable.

Let $\lambda \in \mathbb{R}$. Since A is exponentially stable, we know that $\lambda i \in \rho(A)$. Since theorem A.17 now tells us that

$$r(BR(\lambda i, A)) \leq \|BR(\lambda i, A)\| < 1,$$

we have $1 \in \rho(BR(\lambda i, A))$. Furthermore, the identity

$$\lambda i I - A - B = (I - BR(\lambda i, A))(\lambda i I - A)$$

implies that $\lambda i \in \rho(A + B)$ and

$$R(\lambda i, A+B) = R(\lambda i, A)R(1, BR(\lambda i, A)).$$

We need to show that $\sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A+B)\| < \infty$. The previous identity allows us to estimate

$$\begin{aligned} \sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A+B)\| &= \sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A)R(1, BR(\lambda i, A))\| \\ &\leq \sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A)\|\|R(1, BR(\lambda i, A))\| \\ &\leq \sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A)\| \sup_{\lambda\in\mathbb{R}}\|R(1, BR(\lambda i, A))\|. \end{aligned}$$

Thus it suffices to show that $\sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A)\| < \infty$ and $\sup_{\lambda\in\mathbb{R}}\|R(1, BR(\lambda i, A))\| < \infty$.

Since $T(t)$ is exponentially stable, it follows from remark 3.5 that

$$\sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A)\| < \infty.$$

Let $\lambda \in \mathbb{R}$. To show that $\sup_{\lambda\in\mathbb{R}}\|R(1, BR(\lambda i, A))\| < \infty$, we estimate

$$\begin{aligned} \|R(1, BR(\lambda i, A))\| &= \|\sum_{n=0}^{\infty}(BR(\lambda i, A))^n\| \leq \sum_{n=0}^{\infty}\|BR(\lambda i, A)\|^n \\ &= \frac{1}{1-\|BR(\lambda i, A)\|} \leq \frac{1}{1-q} \end{aligned}$$

and thus

$$\sup_{\lambda\in\mathbb{R}}\|R(1, BR(\lambda i, A))\| \leq \sup_{\lambda\in\mathbb{R}}\frac{1}{1-q} = \frac{1}{1-q} < \infty.$$

This shows that

$$\sup_{\lambda\in\mathbb{R}}\|R(\lambda i, A+B)\| < \infty$$

and thus by corollary 3.6 the C_0-semigroup generated by $A+B$ on X is exponentially stable. □

In closing, we will present a simple example of a perturbation of an exponentially stable C_0-semigroup of contractions.

Example 5.11. Consider again the case of example 4.12. We showed that the operator $A+B$ generates a C_0-semigroup of contractions on $X = \ell^2(\mathbb{C})$. We will now show that the perturbed C_0-semigroup is exponentially stable.

Let $\lambda \in \mathbb{R}$ and $(x_k) \in X$. The resolvent operator $R(\lambda i, A)$ is given by

$$R(\lambda i, A)(x_k) = \left(\frac{x_k}{\lambda i - k}\right)$$

and thus

$$\begin{aligned} \|BR(\lambda i, A)(x_k)\|^2 &= \left\|\left(\frac{\beta k}{\lambda i - k}x_k\right)\right\|^2 = \sum_{k=1}^{\infty}|\beta|^2\frac{k^2}{|\lambda i - k|^2}|x_k|^2 \\ &= |\beta|^2\sum_{k=1}^{\infty}\frac{k^2}{\lambda^2 + k^2}|x_k|^2. \end{aligned}$$

This shows that $\|BR(\lambda i, A)\| \leq |\beta|$. Now $\frac{k^2}{\lambda^2+k^2} \to 1$ when $k \to \infty$. If we choose (x_k) such that

$$x_k = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases}$$

and let $i \to \infty$, it is easy to see that $\|BR(\lambda i, A)\| \geq |\beta|$. This means that for all $\lambda \in \mathbb{R}$

$$\|BR(\lambda i, A)\| = |\beta|.$$

Because we assumed $|\beta| < 1$, this means that the conditions of proposition 5.10 are satisfied and thus the C_0-semigroup generated by $A + B$ is exponentially stable.

5.1.6 Comparison of Results

From the definition of relative boundedness it is easy to see that all bounded operators are also relatively bounded. Furthermore, we know from theorem 4.1 that for all bounded perturbations the perturbed operator generates a C_0-semigroup on X. Because of this, the condition presented in proposition 5.2 is also applicable in the case $B \in \mathcal{L}(X)$. The following corollary is a formulation of proposition 5.2 for these perturbations.

Corollary 5.12. *Let A generate an exponentially stable C_0-semigroup $T(t)$ on a Hilbert space X and let $B \in \mathcal{L}(X)$. If there exists a real constant $0 < q < 1$ such that*

$$\|BR(\lambda, A)\| \leq q \qquad \textit{for all } \lambda \in \mathbb{C}^+, \tag{5.4}$$

then $A + B$ generates an exponentially stable C_0-semigroup on X.

Because B is bounded, we can use the inequality

$$\|BR(\lambda, A)\| \leq \|B\| \|R(\lambda, A)\| \quad \forall \lambda \in \rho(A)$$

to derive a sufficient condition for the condition (5.4) to hold. If we have for some $0 < q < 1$

$$\|B\| \leq \frac{q}{\sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\|}, \tag{5.5}$$

then for all $\lambda \in \mathbb{C}^+$ we also have

$$\|BR(\lambda, A)\| \leq \|B\| \|R(\lambda, A)\| \leq \|B\| \sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| \leq q < 1.$$

Since the unperturbed C_0-semigroup is exponentially stable, we have for some constants $M \geq 1$ and $\omega > 0$ that the estimate $\|T(t)\| \leq Me^{-\omega t}$ holds for all $t \geq 0$. The Hille-Yosida Theorem (theorem 2.7) now implies that

$$\sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| \leq \sup_{\operatorname{Re}\lambda > 0} \frac{M}{\operatorname{Re}\lambda + \omega} = \frac{M}{\omega}.$$

This means that we obtain another sufficient condition by estimating

$$\frac{q}{\sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\|} \geq q \frac{\omega}{M}.$$

The new sufficient condition is then $\|B\| \leq q\frac{\omega}{M}$. This, however, is equivalent to $\|B\| < \frac{\omega}{M}$ which is the sufficient condition presented in corollary 5.1. In this sense, the condition in corollary 5.12 is stronger than the bound in corollary 5.1.

As we stated at the beginning of this chapter, the bound in corollary 5.1 is optimal for *general* C_0-semigroups and perturbing operators $B \in \mathcal{L}(X)$. This combined with the fact that the condition of corollary 5.12 is stronger for all particular perturbations means that the conditions in corollaries 5.1 and 5.12 are equivalent for the general case of bounded perturbations.

In this sense, the sufficient condition presented in proposition 5.2 is an extension of the condition presented in corollary 5.1.

Pritchard and Townley [23] have considered the preservation of exponential stability under perturbations of form BDC where the operators are bounded operators between different spaces. The unboundedness of the perturbation can be embedded in the choise of those spaces (for example by using Sobolev spaces of different orders). The conditions for exponential stability of the perturbed semigroups formulated in the article involve the operator norm $\|D\|$. The theory itself is an infinite-dimensional generalization of the theory introduced in [12]. We will now compare the conditions for the preservation of exponential stability presented in [23] to the ones we have obtained in this section.

The operators of the perturbation BDC are defined as $B \in \mathcal{L}(Z, X^2)$, $D \in \mathcal{L}(Y, Z)$ and $C \in \mathcal{L}(X^1, Y)$ where X is a Banach space and Y and Z are Hilbert spaces. The spaces X^1 and X^2 restrict and extend the space X, respectively. If A generates an exponentially stable C_0-semigroup on X and the spaces and the operators B and C satisfy certain assumptions, Pritchard and Townley present a stability radius r_{stab} such that the operator $A + BDC$ generates an exponentially stable C_0-semigroup on X whenever $\|D\| < r_{\text{stab}}$. It is also shown that if

$$\sup_{\lambda \in \mathbb{R}} \|CR(\lambda i, A)B\| < \infty,$$

then

$$r_{\text{stab}} = \frac{1}{\sup_{\lambda \in \mathbb{R}} \|CR(\lambda i, A)B\|}.$$

We will now show that these results can be used to obtain conditions similar to the ones we have presented in this section. Let A generate an exponentially stable C_0-semigroup on a Hilbert space X. Choose $X^1 = X_1$, $X^2 = X$, $Y = Z = X$, $B = I$ and $D = \varepsilon I$ where ε is a real parameter such that $\varepsilon \geq 0$. Now the perturbing operator is εC with $C \in \mathcal{L}(X_1, X)$. If certain initial assumptions are satisfied and $\sup_{\lambda \in \mathbb{R}} \|CR(\lambda i, A)\| < \infty$, then the operator $A + \varepsilon C$ generates an exponentially stable C_0-semigroup on X whenever

$$\varepsilon < \frac{1}{\sup_{\lambda \in \mathbb{R}} \|CR(\lambda i, A)\|}.$$

This condition can also be formulated as

$$\sup_{\lambda\in\mathbb{R}}\|(\varepsilon C)R(\lambda i, A)\| = \varepsilon \cdot \sup_{\lambda\in\mathbb{R}}\|CR(\lambda i, A)\| < 1. \tag{5.6}$$

This condition is clearly very similar to the one in proposition 5.2. However, the initial assumptions turn out to be relatively restrictive for our purposes. This follows from the fact that Pritchard and Townley consider simultaneously both the perturbed operator being a generator of a C_0-semigroup and the preservation of exponential stability. We only want to derive conditions for the preservation of stability, that is, we already know that the perturbed operator generates a C_0-semigroup on X. In particular, one of the assumptions requires that for all $T > 0$ there exists a constant k such that

$$\int_0^T \|CT(t)x\|^2 dt \leq k\|x\|^2 \quad \text{for all } x \in \mathcal{D}(A).$$

This condition is stricter than is generally needed, for example, for $C \in \mathcal{S}_{t_0}^{\mathrm{MV}}$.

Naturally, the assumptions used by Pritchard and Townley can be seen as additional conditions on the perturbing operators. However, because of the nature of these assumptions, the set of conditions obtained this way would not imply the conditions in proposition 5.2.

It should also be noted that in (5.6) the supremum only needs to be taken over the imaginary axis. We saw in remark 3.5 that if $\|R(\cdot, A)\|$ is uniformly bounded on $\mathbb{C}^+$, then it is also uniformly bounded on the imaginary axis. Because the perturbing operator C is unbounded, this kind of relationship does not necessarily hold in this case. Because of this, the conditions in proposition 5.2 do not imply that

$$\sup_{\lambda\in\mathbb{R}}\|BR(\lambda i, A)\| < 1$$

holds. This means that in this particular case the conditions of Pritchard and Townley can not be obtained from our conditions. In this sense, even though similar, the conditions obtained in this section and the ones presented by Pritchard and Townley are separate.

Another difference between the results is that the ones by Pritchard and Townley are also applicable when X is a Banach space, whereas the results we have presented in this section all require X to be a Hilbert space. The spaces Y and Z need to be Hilbert spaces but in applications this is generally not very restrictive. It should also be noted that our main concern are the A-bounded perturbing operators B with $\mathcal{D}(B) \supset \mathcal{D}(A)$. The theory by Pritchard and Townley can be used for more general restricting and extending spaces X^1 and X^2.

5.2 Spectral Conditions

As we saw in section 3.2, determining the growth bound of the C_0-semigroup from the spectrum of its generator can be done in special cases, for example when the Spectral Mapping Theorem holds. To this end, we will now turn our attention to classes of regular C_0-semigroups which have this property. However, it is not always easy to determine when the perturbed operator generates a regular C_0-semigroup. In this thesis, we concentrate on the case in which we know the unperturbed semigroup has certain regularity properties which enable us to determine its growth bound from the spectrum of its generator. After that, we will restrict our attention to perturbations which preserve these regularity properties. The theory will then tell us that the growth bound of the perturbed C_0-semigroup is again determined by the spectrum of the perturbed operator. Perturbation of such C_0-semigroups is considered in sections 5.2.1 and 5.2.2. It should be noted that unlike many results presented in this thesis, the ones in those two sections do not assume that the unperturbed C_0-semigroup is exponentially stable.

When considering more general C_0-semigroups we can sometimes impose conditions directly on the perturbing operator to get results close to the spectrum determined growth condition. The underlying idea is that although the spectrum of the perturbed operator doesn't determine the growth bound of the perturbed C_0-semigroup *alone*, we can still use it to characterize the growth bound together with another quantity. We saw in sections 2.1.1 and 2.1.2 that both the essential growth bound $\omega_{\text{ess}}(T(t))$ and the critical growth bound $\omega_{\text{crit}}(T(t))$ together with the spectral bound of the generator determine the growth bound of the C_0-semigroup. Our aim is to find perturbations which do not increase either ω_{ess} or ω_{crit}. When perturbing the generator of an exponentially stable C_0-semigroup with such an operator, the growth bound of the perturbed C_0-semigroup is determined by the spectral bound of the perturbed operator and ω_{ess} or ω_{crit} of the *unperturbed* C_0-semigroup.

We shall see in section 5.2.3 that compact perturbations leave the essential growth bound unchanged. As special cases of bounded linear operators, compact operators are not very general. However, they are common in applications.

Describing perturbations which do not increase the critical growth bound requires more complicated conditions. In general, however, they do not even need to be bounded operators. These kinds of perturbations are considered in section 5.2.4.

5.2.1 Perturbation of Analytic Semigroups

Analytic semigroups are in a sense "most regular" among the regular classes of semigroups considered in this thesis since they are at the same time immediately differentiable and immediately norm-continuous (see Figure 3.1). Since we have stronger results concerning the preservation of analyticity than the other regularity properties, we will consider this class separately. We saw that bounded perturbations and perturbations of classes $\mathcal{S}_{t_0}^{\mathrm{DS}}$, $\mathcal{S}_{t_0}^{\mathrm{MV}}$ and $\mathcal{S}^{\mathrm{KW}}$ preserve this property. These results are summarized in the following corollary.

Corollary 5.13. *Let A be a generator of an analytic semigroup on a Banach space X. If $B \in \mathcal{L}(X)$ or there exists a $t_0 > 0$ such that $B \in \mathcal{S}_{t_0}^{DS}$ or $B \in \mathcal{S}_{t_0}^{MV}$, then $A + B$ generates an analytic semigroup on X and this semigroup is exponentially stable if and only if $s(A+B) < 0$. The conclusions also hold if X is a Hilbert space and $B \in \mathcal{S}^{KW}$.*

Proof. Theorem 4.4, proposition 4.18, proposition 4.29, proposition 4.34 guarantee that $A + B$ generates an analytic semigroup $S(z)$ on X. By corollary 3.10 we have $\omega_0(S(z)) = s(A + B)$. □

Without restating the results, we will also mention that similar results hold for all the perturbations considered in section 4.2. All of the cases considered there result in an analytic perturbed semigroup. Therefore, the stability of the perturbed semigroup being characterized by the negative spectral bound of the perturbed operator is a direct consequence of corollary 3.10 in each of these cases.

5.2.2 Perturbation of Other Regular C_0-Semigroups

There is very little theory concerning the preservation of other regularity properties under unbounded perturbations. However, if we only consider bounded perturbations we can use theory in sections 4.1 and 3.2 to formulate conditions under which the spectral bound of the perturbed operator determines the stability of the perturbed C_0-semigroup. The following results are direct consequences of the results presented in these sections. It should also be noted that since we are only considering bounded perturbations, we know that the perturbed operator is always a generator of a C_0-semigroup.

Corollary 5.14. *Let A generate an immediately compact or immediately norm-continuous C_0-semigroup on a Banach space X and let $B \in \mathcal{L}(X)$. Then the C_0-semigroup generated by $A+B$ on X is exponentially stable if and only if $s(A+B) < 0$.*

If the unperturbed C_0-semigroup is only eventually differentiable, compact or norm-continuous we have to pose additional conditions. As in section 4.1, define the space $\mathcal{X}$ as the space of all strongly continuous functions from $[0, \infty)$ to $\mathcal{L}(X)$ and define an abstract Volterra operator V on $\mathcal{X}$ by

$$(VF)(t)x = \int_0^t T(t-s)BF(s)xds, \quad F \in \mathcal{X},\ t \geq 0 \text{ and } x \in X.$$

Using this notation, we get the following result

Corollary 5.15. *Let A generate an eventually (differentiable, compact, norm-continuous) C_0-semigroup on a Banach space X. If $B \in \mathcal{L}(X)$ and for some $n \in \mathbb{N}$*

$$\operatorname{ran} V^n \subset \left\{ F \in \mathcal{X} \;\middle|\; F \text{ is immediately (differentiable, compact, norm-continuous) for } t \geq 0 \right\},$$

then the C_0-semigroup generated by $A + B$ on X is exponentially stable if and only if $s(A + B) < 0$.

5.2.3 Compact Perturbations

As we saw in section 2.1.1, the growth bound of a C_0-semigroup only depends on its essential growth bound and the spectral bound of its generator. It turns out that this characterization has an advantage when we are perturbing the generator of an exponentially stable C_0-semigroup with a compact operator: The essential growth bound isn't changed under a compact perturbation and thus the stability of the perturbed C_0-semigroup depends on the spectral bound of $A + B$ alone. Again we can note that since all compact operators are bounded, all the perturbed operators considered here are automatically generators of C_0-semigroups on X.

Proposition 5.16. *Let $T(t)$ be a C_0-semigroup on a Banach space X with generator A and let $B \in \mathcal{K}(X)$. Denote by $S(t)$ the semigroup generated by $A + B$. Then*

$$\omega_{ess}(T(t)) = \omega_{ess}(S(t)).$$

Proof. [8, Prop IV.2.12] □

If we recall from section 2.1.1 that we can express the growth bound of a C_0-semigroup $T(t)$ generated by A as

$$\omega_0(T(t)) = \max\{\omega_{\text{ess}}(T(t)), s(A)\},$$

the previous result implies that for a compact perturbation B the growth bound of the perturbed C_0-semigroup $S(t)$ is given by

$$\omega_0(S(t)) = \max\{\omega_{\text{ess}}(T(t)), s(A+B)\},$$

Since the exponential stability of $T(t)$ implies that $\omega_{\text{ess}}(T(t)) < 0$, the stability of the perturbed C_0-semigroup depends only on the spectrum of the perturbed operator. The result is summarized in the following corollary.

Corollary 5.17. *Let $T(t)$ be an exponentially stable C_0-semigroup with generator A on a Banach space X and let $B \in \mathcal{K}(X)$. Then $A+B$ generates an exponentially stable C_0-semigroup on X if and only if $s(A+B) < 0$.*

5.2.4 Perturbation of the Critical Growth Bound

In their article [3], Brendle, Nagel and Poland achieved results which can be used to characterize the stability of the perturbed C_0-semigroup in certain situations. The article deals with unbounded perturbations B satisfying assumption 5.18. This assumption guarantees that B is a Miyadera-Voigt perturbation, but is a stronger condition than is in general needed for $B \in \mathcal{S}_{t_0}^{\text{MV}}$. We give here an introduction to those parts of this theory which are most useful to us.

Let A generate a C_0-semigroup $T(t)$ on a Banach space X and let B be an A-bounded operator on X. We make the following assumption

Assumption 5.18. *There exists a function $q : \mathbb{R}_+ \to \mathbb{R}_+$ for which $\lim_{t\to 0+} q(t) = 0$ such that*

$$\int_0^t \|BT(s)x\| ds \le q(t)\|x\|$$

for each $x \in \mathcal{D}(A)$ and $t \ge 0$.

Corollary 5.19. *Let A generate a C_0-semigroup on a Banach space X and let B be an A-bounded operator on X satisfying assumption 5.18. Then $B \in \mathcal{S}_{t_0}^{MV}$.*

Proof. Since by assumption B is A-bounded, we have by remark 4.19 that $B \in \mathcal{L}(X_1, X)$.

Since $q(t) \geq 0$ and $\lim_{t\to 0+} q(t) = 0$, there exists a $t_0 > 0$ such that $\sup_{t\in[0,t_0]} q(t) < 1$. This implies that for all $x \in \mathcal{D}(A)$

$$\int_0^{t_0} \|BT(t)x\|dt \leq q(t_0)\|x\| \leq \sup_{t\in[0,t_0]} q(t)\|x\| = q\|x\|, \quad 0 \leq q < 1.$$

Therefore B satisfies the conditions of corollary 4.22 and $B \in \mathcal{S}_{t_0}^{\text{MV}}$. □

For the rest of the section, let B be an A-bounded operator satisfying assumption 5.18 and denote by $S(t)$ the C_0-semigroup generated by the operator $A + B$ on X. Recall from section 4.5 that $S(t)$ is given by the abstract Dyson-Phillips series

$$\begin{aligned} S(t) &= \sum_{n=0}^{\infty} (V^n T)(t), \quad t \geq 0, \quad \text{where } V = \overline{V_B^*}, \\ (V_B^* F)(t)x &= \int_0^t F(s)BT(t-s)xds, \quad \text{for } \; F \in \mathcal{X}_{t_0} = C([0,t_0], \mathcal{L}_s(X)), \; x \in X_1 \end{aligned}$$

and $\overline{V_B^*}$ denotes the operator giving extensions $\overline{(V_B^* F)(t)} : X \to X$ of operators $(V_B^* F)(t)$. The series can be written as

$$S(t) = \sum_{n=1}^{\infty} S_n(t),$$

where

$$S_0(t) = T(t), \quad S_n(t)x = \int_0^t S_{n-1}(t-s)BT(s)xds \;\; \forall x \in \mathcal{D}(A), \; t \geq 0.$$

Denote the remainder terms by

$$R_k(t) = \sum_{n=k}^{\infty} S_n(t).$$

The following theorem gives sufficient conditions for a perturbation not to increase the critical growth bound of a C_0-semigroup.

Theorem 5.20. *Let A generate a C_0-semigroup $T(t)$ on a Banach space X. Let B be an A-bounded operator satisfying assumption 5.18. If the mapping $t \mapsto R_k(t)$ is right norm continuous for some $k \in \mathbb{N}$ and $t \geq t_0$ (see definition A.13), then for the C_0-semigroup $S(t)$ generated by the operator $A + B$*

$$\omega_{crit}(S(t)) \leq \omega_{crit}(T(t)).$$

Proof. [3, Thm 4.1] □

Theorem 5.20 immediately gives us knowledge of the growth bound of the perturbed C_0-semigroup in terms of the growth bound of the original semigroup and the spectral bound of the operator $A + B$.

Corollary 5.21. *Suppose conditions of theorem 5.20 hold. Then for the growth bound of the perturbed C_0-semigroup $S(t)$ we have*

$$\omega_0(S(t)) \leq \max \{\omega_{crit}(T(t)), s(A + B)\}.$$

If the C_0-semigroup $T(t)$ is exponentially stable, its critical growth bound must be negative. This leads us to the following corollary.

Corollary 5.22. *Let A generate an exponentially stable C_0-semigroup and let B be an A-bounded operator satisfying assumption 5.18. If the mapping $t \mapsto R_k(t)$ is right norm continuous for some $k \in \mathbb{N}$ and $t \geq t_0$, then the C_0-semigroup generated by $A + B$ is exponentially stable if $s(A + B) < 0$.*

5.3 Lyapunov Equation Approach

In [19], Pandolfi and Zwart derived a sufficient condition for the stability of the perturbed C_0-semigroup for a class of unbounded perturbations. They used the fact that a C_0-semigroup generated by A in Hilbert space X is exponentially stable if and only if the Lyapunov equation

$$\langle Ax, Py\rangle + \langle x, PAy\rangle = -\langle x, y\rangle, \qquad \forall x, y \in \mathcal{D}(A)$$

has a self-adjoint positive solution $P \in \mathcal{L}(X)$ (see theorem 3.7).

Assuming that A generates an exponentially stable C_0-semigroup $T(t)$ on X, consider perturbations B satisfying the following assumptions

Assumption 5.23.

(i) $\mathcal{D}(B) \supset \mathcal{D}(A)$ *and* $A + B$ *generates a* C_0*-semigroup* $S(t)$ *on* X*.*

(ii) The number L_B *defined by*

$$L_B^2 = \sup_{x \in \mathcal{D}(A), \|x\|=1} \int_0^\infty \|BT(t)x\|^2 dt \tag{5.7}$$

is finite.

(iii) B *is* A*-bounded.*

Since $T(t)$ is exponentially stable, by theorem 3.2 the function $T(\cdot)x \in L^2([0,\infty), X)$ for all $x \in X$ and we can the define number Λ by

$$\Lambda^2 = \sup_{\|x\|=1} \int_0^\infty \|T(t)x\|^2 dt.$$

The following theorem gives a sufficient condition for the perturbed C_0-semigroup to be exponentially stable.

Theorem 5.24. *Let* A *generate an exponentially stable* C_0*-semigroup* $T(t)$ *on a Hilbert space* X *and let* B *be an operator satisfying assumption 5.23. If* L_B *(as defined in equation* (5.7)*) satisfies*

$$L_B < \frac{1}{2\Lambda},$$

then the operator $A + B$ *generates an exponentially stable* C_0*-semigroup on* X*.*

Proof. [19, Thm 2] □

The exponential stability of the unperturbed C_0-semigroup $T(t)$ allows us to find an upper bound for Λ. Choose $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t} \quad \text{for all } t \geq 0$$

and let $x \in X$ be such that $\|x\| = 1$. We now have an estimate

$$\int_0^\infty \|T(t)x\|^2 dt \leq \int_0^\infty \|T(t)\|^2 \|x\|^2 dt \leq \int_0^\infty M^2 e^{-2\omega t} dt = \frac{M^2}{2\omega}$$

and thus $\Lambda \leq \frac{M}{\sqrt{2\omega}}$. This also implies that

$$\frac{1}{2\Lambda} \geq \frac{1}{2\frac{M}{\sqrt{2\omega}}} = \frac{\sqrt{\omega}}{\sqrt{2}M}.$$

This leads to a more concrete sufficient condition for the stability of the perturbed C_0-semigroup.

Corollary 5.25. *Assume A generates an exponentially stable C_0-semigroup $T(t)$ on a Hilbert space X. Let B be an operator satisfying assumption 5.23 and let $M \geq 1$ and $\omega > 0$ be such that*

$$\|T(t)\| \leq Me^{-\omega t} \qquad \forall t \geq 0.$$

If L_B satisfies

$$L_B < \frac{\sqrt{\omega}}{\sqrt{2}M},$$

then the operator $A + B$ generates an exponentially stable C_0-semigroup on X.

Chapter 6

Conclusions

In this thesis we have studied strongly continuous semigroups of linear operators on Hilbert spaces. Our main interest has been the preservation of exponential stability under additive perturbations. We have considered both bounded and relatively bounded perturbations and presented conditions under which the perturbed operator generates an exponentially stable C_0-semigroup.

Before considering the stability of the C_0-semigroups, we needed to study which perturbations preserve the property of the operator being an infinitesimal generator of a C_0-semigroup. In order to answer this question, we introduced a wide variety of theory found in the literature.

Given that the unperturbed C_0-semigroup is exponentially stable and the perturbed operator generates a C_0-semigroup, we were able to find conditions under which the perturbed C_0-semigroup is exponentially stable. Conditions were first derived for the more general case of relatively bounded perturbations for which the perturbed operator generates a C_0-semigroup and subsequently for a class of perturbations for which the first conditions were not applicable. The condition for relatively bounded perturbations also includes bounded perturbations as a special case. Using this fact, we saw that a well-known condition for the preservation of exponential stability under a bounded perturbation follows from our conditions.

As a second approach, we formulated conditions under which the spectrum of the perturbed operator determines the stability properties of the perturbed C_0-semigroup. In most of these cases, certain degree of regularity is required from the unperturbed C_0-semigroup. Although this limits the generality of the theory, these special classes of C_0-semigroups are often encountered in applications.

Finally, we introduced sufficient conditions obtained by Pandolfi and Zwart for the preservation of exponential stability. This theory is applicable to relatively bounded perturbations satisfying certain special assumptions.

To find alternate conditions for the stability of the perturbed C_0-semigroup, one reasonable approach would be to use other conditions for exponential stability of a C_0-semigroups and apply them to the case of perturbed semigroups. Possibly useful conditions for exponential stability include the *weak L^p-stability* in theorem 3.3 and Quoc Phong's condition [22]

$$\sup_{\lambda \in \mathbb{R}, t \geq 0} \| \int_0^t e^{i\lambda s} T(s)x ds \| < \infty \quad \forall x \in X.$$

Another possibly useful condition is van Neerven's result which states that the exponential stability of a C_0-semigroup $T(t)$ in a Banach space is equivalent to the condition

$$T * f \in Z \qquad \text{for all} \quad f \in Z,$$

where Z denotes either $L^p([0,\infty), X)$ or $C_0([0,\infty), X)$ [27]. The *convolution* $T * f$ is defined by

$$(T * f)(t) = \int_0^\infty T(s) f(t-s) ds.$$

Applying some these conditions to the case of perturbed C_0-semigroups was already tried during the writing of this thesis. Most of them led to very impractical conditions, but there might be ways to carry the analysis further. For example, the conditions resulted from the application of van Neerven's results could probably be simplified by using more advanced theory related to convolution. Also, in this thesis we mainly used the Dyson-Phillips -series representation of the perturbed C_0-semigroup. The variation of parameters -formula could be more suitable when working with some of these conditions.

As we saw in section 5.1.6, the conditions presented by Pritchard and Townley in [23] can be used to derive conditions similar to the ones in section 5.1. The authors consider simultaneously both the perturbed operator being a generator of a C_0-semigroup and the preservation of exponential stability. The first natural question to ask is that if we already knew that the perturbed operator generates a C_0-semigroup, could we loosen the initial assumptions used in the article? This way, it could be possible to find new conditions for the stability of the pertubed C_0-semigroup.

One possible topic for further research would be to incorporate perturbation theory concerning the spectrum of linear operators to the results presented in section 5.2. At first sight, however, the conditions needed in order to obtain bounds for the growth of the spectral bound seem to be very restrictive. One of the directly applicable theorems states the following: If the perturbing operator B is bounded and commutes with A, then the distance between $\sigma(A)$ and $\sigma(A+B)$ does not exceed $\|B\|$ [14, Thm IV.3.6]. This implies for the spectral bounds that $s(A+B) \leq s(A) + \|B\|$.

One certainly interesting topic would be to see how our conditions change if we did not require exponential stability. The next natural form of stability would be *strong stability*, which means that for a C_0-semigroup $T(t)$ on X we have

$$\lim_{t\to\infty} \|T(t)x\| = 0$$

for all $x \in X$. There are actually two separate cases to study: One would be, given an exponentially stable unperturbed C_0-semigroup, to find conditions under which the perturbed C_0-semigroup is strongly stable. The other would be to start with a strongly stable C_0-semigroup and see how this property is preserved under perturbations.

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Appendix A

Functional Analysis and Integration Theory

The results presented in this appendix can be found in most of the literature containing introductory functional analysis, for example [14, 4, 25, 18, 7]. We will also present a couple of helpful results concerning integrable functions. This theory can be found in [10].

A.1 Normed Linear Spaces

We will first define the concept of quotient space.

Definition A.1. Let M be a subspace of a normed linear space X. Two elements $x, y \in X$ are said to be *equivalent modulo* M if $x - y \in M$. The *equivalence class of* x is defined as

$$[x] = \left\{ y \in X \;\middle|\; x - y \in M \right\}$$

The *quotient space* ${}^{X}\!/_{M}$ is defined as the set of all these equivalence classes

$$ {}^{X}\!/_{M} = \left\{ [x] \;\middle|\; x \in X \right\} $$

The quotient space is a normed linear space with *quotient norm* defined by

$$ \|[x]\| = \operatorname{dist}(x, M) = \inf_{y \in M} \|x - y\| $$

■

A.2 Operator Theory

In this section we will present some basic result of operator theory.

Definition A.2. The *graph* of an operator $A : \mathcal{D}(A) \to Y$ is defined as

$$ \operatorname{Gr}(A) = \left\{ (x, Ax) \;\middle|\; x \in \mathcal{D}(A) \right\} \subset X \times Y $$

The *graph norm* $\|\cdot\|_A$ is defined as

$$ \|x\|_A = \sqrt{\|x\|^2 + \|Ax\|^2} $$

■

Definition A.3. Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let A be a densely defined linear operator on X. For the *adjoint operator* $A^* : X \supset \mathcal{D}(A^*) \to X$ we have

$$ \mathcal{D}(A^*) = \left\{ y \in X \;\middle|\; \exists z \;\text{ such that }\; \langle Ax, y \rangle = \langle x, z \rangle, \;\; \forall x \in \mathcal{D}(A) \right\}. $$

Following the notation of the definition of $\mathcal{D}(A^*)$ the adjoint operator A^* is then defined as

$$ A^* y = z, \qquad \text{for } y \in \mathcal{D}(A^*) $$

■

Definition A.4. A linear operator A is said to be *closed* if for every convergent sequence (x_n) in $\mathcal{D}(A)$ with $x_n \to x$ and $Ax_n \to y$ we have $x \in \mathcal{D}(A)$ and $Ax = y$. ■

Theorem A.5. *The operator $A : \mathcal{D}(A) \subset X \to Y$ is closed if and only if* $\mathrm{Gr}(A)$ *is a closed subspace of $X \times Y$.*

Proof. [14, p. 164] □

Theorem A.6 (Closed Graph Theorem)**.** *Let A be a closed operator from Banach space X to Banach space Y. If $\mathcal{D}(A) = X$, then A is a bounded operator.*

Proof. [14, Thm III.5.20], [25, Thm IV.5.7], [7, Thm II.2.3] □

Definition A.7. Let X and Y be normed linear spaces. An operator $A \in \mathcal{L}(X, Y)$ is said to be *compact* if for every bounded sequence $(x_n) \subset X$ the sequence (Ax_n) has a convergent subsequence. ■

Definition A.8. Let X, Y and Z be normed spaces and let A and B be linear operators such that $A : X \to Y$, $B : X \to Z$ and $\mathcal{D}(A) \subset \mathcal{D}(B)$. The operator B is said to be *A-bounded* if there exist constants $a, b \geq 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad \forall x \in \mathcal{D}(A).$$

The *A-bound* of B is then defined as

$$a_0 = \inf\left\{ a \geq 0 \;\middle|\; \exists b \geq 0 : \|Bx\| \leq a\|Ax\| + b\|x\|, \ \forall x \in \mathcal{D}(A) \right\}$$

■

Lemma A.9. *Let X, Y and Z be Banach spaces. If the operators $A : X \supset \mathcal{D}(A) \to Y$ and $B : X \supset \mathcal{D}(B) \to Z$ are closed and $\mathcal{D}(B) \supset \mathcal{D}(A)$, then B is A-bounded.*

Proof. This proof follows [14, Remarks IV.1.4-5].

Define space $\hat{X} = (\mathcal{D}(A), \|\cdot\|_A)$ where $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in \mathcal{D}(A)$. We will first show that $\hat{X}$ is Banach space with this norm.

Let $(x_n) \subset \hat{X}$ be a Cauchy sequence. Then $\|x_n - x_m\| \leq \|x_n - x_m\|_A$ and thus (x_n) is a Cauchy sequence in X and $\lim_{n\to\infty} x_n = x \in X$. Since $\|Ax_n - Ax_m\| \leq \|x_n - x_m\|_A$, also the sequence $(Ax_n) \subset Y$ is Cauchy sequence in Y and $\lim_{n\to\infty} Ax_n = y$. This means that (x_n) and (Ax_n) are both convergent and since A is closed, this means that $x \in \mathcal{D}(A) = \hat{X}$. Thus $\hat{X}$ with norm $\|\cdot\|_A$ is a Banach space.

Define operator $\hat{B}$ as the restriction of B to $\mathcal{D}(A)$. Now $\hat{B}$ can be seen as an operator $\hat{B} : \hat{X} \to Z$. We will show that $\hat{B}$ is closed.

Let $(x_n) \subset \hat{X}$ be a sequence such that $\lim_{n\to\infty} x_n = x \in \hat{X}$ and $\lim_{n\to\infty} \hat{B}x_n = y$. Since $\|x_n - x\| \leq \|x_n - x\|_A$, also $(x_n) \subset \mathcal{D}(B)$ is convergent with respect to the norm in X. Since also $(Bx_n) = (\hat{B}x_n)$ is convergent and B is closed, we know that $\hat{B}x = Bx = y$. This shows that $\hat{B}$ is closed. Since $\mathcal{D}(\hat{B}) = \hat{X}$, theorem A.6 implies that $\hat{B} \in \mathcal{L}(\hat{X}, Z)$. This means that there exists $M > 0$ such that for all $x \in \mathcal{D}(A)$

$$\|Bx\| = \|\hat{B}x\| \leq M\|x\|_A = M\|Ax\| + M\|x\|$$

and thus B is A-bounded. □

Definition A.10. Let A be a closed operator on a Banach space X. An operator B is called *(relatively) A-compact* if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B : X_1 \to X$ is compact. Here X_1 denotes $(\mathcal{D}(A), \|\cdot\|_A)$ where $\|\cdot\|_A$ is the graph norm. ■

Definition A.11. Let X be a Hilbert space. An operator $A : X \to X$ is said to be *positive* if for all $x \in X$ with $x \neq 0$ we have $\langle Ax, x\rangle > 0$. ■

Definition A.12. A linear operator A on a Banach space X is called *dissipative* if for every $x \in \mathcal{D}(A)$ there exists a $x^* \in X^*$ (the dual space of X) such that

$$\langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2$$

and $\mathrm{Re}\langle Ax, x^*\rangle \leq 0$. In Hilbert space X the operator is dissipative if $\mathrm{Re}\langle Ax, x\rangle \leq 0$ for every $x \in \mathcal{D}(A)$. ■

Definition A.13. An operator-valued mapping $t \mapsto T(t)$, where $T(t) \in \mathcal{L}(X, Y)$, for all $t \geq 0$ is called *right norm-continuous* for $t \geq t_0$ if

$$\lim_{h\to 0+} \|T(t+h) - T(t)\| = 0, \quad \text{for } t \geq t_0.$$

The mapping is called *norm-continuous* for $t \geq t_0$ if

$$\lim_{h\to 0} \|T(t+h) - T(t)\| = 0, \quad \text{for } t \geq t_0.$$

■

Theorem A.14 (Extension principle)**.** *A bounded linear operator A on a Banach space X with domain $\mathcal{D}(A)$ can be extended to a bounded linear operator on $\overline{\mathcal{D}(A)}$ with the same norm. The extension is unique.*

Proof. [14, p. 145 & Thm III.1.16], [18, Thm 5.8.7] □

A.3 Spectral Theory

Definition A.15. Let A be a linear operator on X. The *resolvent set* $\rho(A)$ of A is defined by

$$\rho(A) = \left\{ \lambda \in \mathbb{C} \;\middle|\; R(\lambda, A) = (\lambda I - A)^{-1} \begin{array}{l}\text{exists and is a densely defined}\\ \text{bounded linear operator}\end{array} \right\}.$$

For a $\lambda \in \rho(A)$, the operator $(\lambda I - A)^{-1} = R(\lambda, A)$ is called the *resolvent* operator. The *spectrum* $\sigma(A)$ of A is defined by

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

■

Definition A.16. Let $A : \mathcal{D}(A) \subset X \to X$ be a closed operator. The *spectral radius* of A is defined as

$$r(A) = \sup\left\{ |\lambda| \;\middle|\; \lambda \in \sigma(A) \right\}$$

and the *spectral bound* of A as

$$s(A) = \sup\left\{ \operatorname{Re}\lambda \;\middle|\; \lambda \in \sigma(A) \right\}.$$

■

Theorem A.17. *Let A be a bounded operator on a Banach space X. Then the spectral radius of A satisfies*

$$r(A) \leq \|A\|.$$

Proof. [8, Cor IV.1.4] □

Lemma A.18. *Let* $\lambda \in \rho(A)$ *and* $\operatorname{dist}(\lambda, \sigma(A)) = \inf\{\, |\lambda - \mu| \mid \mu \in \sigma(A) \,\}$. *Then*

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|}$$

Proof. [7, Cor VII.3.3] □

Let $\lambda, \mu \in \rho(A)$. The following *resolvent equation* is a useful identity

$$R(\mu, A) = R(\lambda, A) + (\lambda - \mu)R(\lambda, A)R(\mu, A) \tag{A.1}$$

Lemma A.19. *Let* X *be a linear space and let* $\overline{X}$ *be its completion. If* $A \in \mathcal{L}(X)$ *and* $\overline{A} \in \mathcal{L}(\overline{X})$ *is its unique linear extension, then* $\rho(\overline{A}) = \rho(A)$ *and hence also* $\sigma(\overline{A}) = \sigma(A)$.

Proof. Since A and $\overline{A}$ are bounded operators, both $\rho(A)$ and $\rho(\overline{A})$ are not empty [14, Sec III.6.2]. We will first note that a bounded operator B on a linear space Y has a bounded inverse if and only if there exists a positive real constant α such that

$$\|Bx\| \geq \alpha\|x\|, \qquad \forall x \in Y$$

Let $\lambda \in \rho(A)$. Then there exists a real positive constant α such that $\|(\lambda I - A)x\| \geq \alpha\|x\|$ for all $x \in X$.

Let $x \in \overline{X}$. There exists a sequence $(x_n) \subset X$ such that $\lim_{n\to\infty} x_n = x$. This implies that

$$\|(\lambda I - \overline{A})x\| = \lim_{n\to\infty} \|(\lambda I - \overline{A})x_n\| = \lim_{n\to\infty} \|(\lambda I - A)x_n\| \geq \lim_{n\to\infty} \alpha\|x_n\| = \alpha\|x\|$$

and thus $\lambda I - \overline{A}$ has a bounded inverse. It remains to show that $\operatorname{ran}(\lambda I - \overline{A})$ is dense in $\overline{X}$. This follows directly from the fact that $\operatorname{ran}(\lambda I - A) \subset \operatorname{ran}(\lambda I - \overline{A})$. This shows that $\lambda \in \rho(\overline{A})$ and thus $\rho(A) \subset \rho(\overline{A})$.

Let $\lambda \in \rho(\overline{A})$. There exists a real positive constant α such that $\|(\lambda I - A)x\| \geq \alpha\|x\|$ for all $x \in \overline{X}$. This implies that the same holds for all $x \in X$ and thus $(\lambda I - A)$ has a bounded inverse. We will now show that $\operatorname{ran}(\lambda I - A)$ is dense in $\overline{X}$.

Let $y \in \overline{X}$ and $\varepsilon > 0$. Since $\operatorname{ran}(\lambda I - \overline{A})$ is dense in $\overline{X}$, there exists $x \in \overline{X}$ such that

$$\|(\lambda I - \overline{A})x - y\| < \frac{\varepsilon}{2}.$$

Choose a sequence $(x_n) \subset X$ such that $\lim_{n\to\infty} x_n = x$. There exists $m \in \mathbb{N}$ such that

$$\|x - x_m\| < \frac{\varepsilon}{2\|\lambda I - \overline{A}\|}$$

Now,

$$\begin{aligned} \|(\lambda I - A)x_m - y\| &= \|(\lambda I - A)x_m - (\lambda I - \overline{A})x + (\lambda I - \overline{A})x - y\| \\ &\leq \|(\lambda I - \overline{A})x_m - (\lambda I - \overline{A})x\| + \|(\lambda I - \overline{A})x - y\| \\ &\leq \|\lambda I - \overline{A}\| \frac{\varepsilon}{2\|\lambda I - \overline{A}\|} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and thus $\operatorname{ran}(\lambda I - A)$ is dense in $\overline{X}$. This shows that $\lambda \in \rho(A)$ and thus $\rho(\overline{A}) \subset \rho(A)$. □

A.4 Integration Theory

Lemma A.20. *Let $f \in L^1([a,b];\mathbb{R}^+)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $c, d \in [a,b]$, $c < d$ and $d - c < \delta$ then*

$$\int_c^d f(t)dt < \varepsilon$$

Proof. This is a direct consequence of [10, Thm 12.34]. □

The previous Lemma immediately leads us to the following corollary

Corollary A.21. *Let $f \in L^1([a,b];\mathbb{R}^+)$. If $c, s \in [a,b]$, then*

$$\lim_{s\to c+} \int_c^s f(t)dt = 0.$$