Robust Output Regulation of A Thermoelastic System*

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ABSTRACT

In this paper, we consider the robust output regulation problem for the thermoelastic system where the boundary control input and the output are located at the left end of the wave equation and the disturbance input is located at the right end of the heat equation. We formulate the system as a boundary control system and prove that it is impedance passive and well-posed. We design a finite-dimensional controller based on the internal model principle and show that the controller can achieve robust output tracking and disturbance rejection. The numerical simulations demonstrate that the finite-dimensional dynamic error feedback controller can make the output track the reference signal asymptotically and the state of the closed-loop system is bounded.

1. Introduction

During the past decades, due to the extensive application of output regulation in aerospace, engineering, and other fields, the output regulation problem of distributed parameter systems has been widely studied. The objective of the output regulation problem is to design a controller to make the output of the system track the reference signal and reject the disturbance signal asymptotically.

The study of the output regulation problem for finite-dimensional systems begun as early as the 1970s [9, 6, 10]. The internal model principle was introduced in [10] for finite-dimensional systems. The principle is summarized as that a robust output regulation problem is solvable if and only if the controller contains a multiple copy of the dynamics of the exosystem. Later, the output regulation theory was extended to infinite-dimensional systems. In [3, 19, 20], the output regulation problems for the infinite-dimensional system are considered using state space methods, and in [18, 33], the robust output regulation problems are analyzed in the frequency domain.

We consider the robust output regulation problem for the thermoelastic system [5, 21, 2, 25, 1, 14], which models the interaction between temperature and elastic system for $0 \le x \le 1$ and $t \ge 0$:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) - \dfrac{\partial^2 w}{\partial x^2}(x,t) + \gamma \dfrac{\partial \theta}{\partial x}(x,t) = 0, \\ \dfrac{\partial \theta}{\partial t}(x,t) - k\dfrac{\partial^2 \theta}{\partial x^2}(x,t) + \gamma \dfrac{\partial^2 w}{\partial x \partial t}(x,t) = 0, \\ -\dfrac{\partial w}{\partial x}(0,t) = u(t),\ w(1,t) = 0, \\ \theta(0,t) = 0,\ \dfrac{\partial \theta}{\partial x}(1,t) = d(t), \end{cases} \tag{1}$$

where $w(x,t)$ represents the displacement, $\theta(x,t)$ represents the absolute temperature, $k > 0$ is the thermal conductivity, $\gamma > 0$ is the coupling constant, $u(t) \in \mathbb{C}$ is the control input, $d(t) \in \mathbb{C}$ is the disturbance input. The output signal is

$$y(t) = \frac{\partial w}{\partial t}(0,t).$$

The disturbance $d(t) \in \mathbb{C}$ and the reference signal $y_{ref}(t) \in \mathbb{C}$ are assumed to be generated by the exosystem

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0, \tag{2a}$$
$$d(t) = Ev(t), \tag{2b}$$
$$y_{ref}(t) = Fv(t), \tag{2c}$$

where $S \in \mathbb{C}^{2q\times 2q}$ has purely imaginary eigenvalues, $E \in \mathbb{C}^{1\times 2q}$, $F \in \mathbb{C}^{1\times 2q}$. The eigenvalues of S are given by

$$\sigma(S) = \{i\omega_1, -i\omega_1, \cdots, i\omega_q, -i\omega_q\} \subset \mathbb{C}. \tag{3}$$

We assume $\omega_1, \omega_2, \cdots, \omega_q$ are known and $0 < \omega_1 < \omega_2 < \cdots < \omega_q$, E, F and the initial value $v_0 \in \mathbb{C}^{2q\times 1}$ are unknown. Our contributions focus on the following aspects:

- We formulate the thermoelastic system as a boundary control system and prove this system is an impedance passive system.
- We prove the disturbance input operator, the control input operator and the output operator are admissible, and from the boundedness of the transfer functions, we prove that the open-loop system is well-posed.
- We design a finite-dimensional dynamic error feedback controller for (1) based on [33] and prove that the controller we designed solves the robust output regulation problem.

In the past decade, output regulation has been studied in detail for several different types of PDEs. The state feedback regulator obtained by using the solvability of regulator equations and the backstepping method is considered in [7, 8, 38, 11] for parabolic PDEs with spatially varying parameters, coupled linear parabolic partial integro-differential

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equations (PIDEs), first-order hyperbolic PIDE systems, anti-stable coupled wave equations. In addition, the adaptive error feedback output regulation of the one-dimensional wave equation, beam equation, heat equation are presented in [17, 16, 15], and the controllers are designed based on the construction of some auxiliary systems and the adaptive control approach. The internal model principle has been used to design controllers for robust error feedback output regulation for a heat equation and a Schrödinger equation in [13, 24], as well as for a general class of impedance passive distributed port-Hamiltonian systems in [22, 31]. These works are about the output regulation of particular PDE systems, and the abstract theory of output regulation has also been developed in several articles during the past decades [3, 32, 29]. The internal model principle for infinite-dimensional systems with unbounded control and observation is introduced in [32], and the authors concentrate on characterizing the solvability of the robust output regulation problem. The internal model principle has been used to design controllers for robust output regulation of several different classes of infinite-dimensional systems in [18, 33, 23, 20, 30, 22, 31]. In particular the controller designs in [33, 30] are applicable for well-posed and regular linear systems, which cover large classes of PDE models with boundary control and observation.

The organization of this paper is as follows. In Section 2, we formulate the thermoelastic system as a boundary control system. In Section 3, we prove that the open-loop system is well-posed. In Section 4, we design an error feedback controller and show that the designed controller solves the robust output regulation problem. Numerical simulations are presented in Section 5 to illustrate the theoretical results. Some conclusions are given in Section 6.

1.1. Notation

For the Banach spaces X and Y and the linear operator $A : X \to Y$, $L(X,Y)$ denotes the space of bounded linear operators from X to Y. The domain, spectrum and resolvent of a linear operator A are denoted by $D(A), \sigma(A)$ and $\rho(A)$, respectively. The resolvent operator of a linear operator A is denoted by $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm $\|z\|_{X_{-1}} = \|(\lambda_0 - A)^{-1} z\|_X$ with $\lambda_0 \in \rho(A)$ and $z \in X$. For any $\alpha \geq 0$, we denote $\mathbb{C}_\alpha = \{s \in \mathbb{C} | \mathrm{Re} s > \alpha\}$.

2. The boundary control system

In this section, we formulate the thermoelastic system as a boundary control system [34, 4, 28, 36] and prove that this system is impedance passive [28, Section 3]. We define

$$H_R^1(0,1) = \{f \in H^1(0,1) | f(1) = 0\},$$

and the state space

$$X = H_R^1(0,1) \times L^2(0,1) \times L^2(0,1)$$

with the inner product

$$\begin{aligned}&\left\langle (f_1, g_1, h_1)^\top, (f_2, g_2, h_2)^\top \right\rangle \\ &= \int_0^1 \left[\frac{df_1}{dx}(x) \overline{\frac{df_2}{dx}(x)} + g_1(x)\overline{g_2(x)} + h_1(x)\overline{h_2(x)} \right] dx,\end{aligned}$$

where $(f_i, g_i, h_i)^\top \in X, i = 1, 2$. Choosing the state variable $z(t) = (w(\cdot,t), \eta(\cdot,t), \theta(\cdot,t))^\top$, the system (1) can be expressed as a boundary control and observation system [4]

$$\dot{z}(t) = \mathfrak{A}_0 z(t), \quad z(0) = z_0 \in Z, \tag{4a}$$
$$\mathfrak{B} z(t) = u(t), \tag{4b}$$
$$\mathfrak{B}_d z(t) = d(t), \tag{4c}$$
$$y(t) = \mathfrak{C} z(t), \tag{4d}$$

where

$$\mathfrak{A}_0 = \begin{pmatrix} 0 & I & 0 \\ \frac{d^2}{dx^2} & 0 & -\gamma \frac{d}{dx} \\ 0 & -\gamma \frac{d}{dx} & k \frac{d^2}{dx^2} \end{pmatrix},$$

$$\mathfrak{B} z = -\frac{dw}{dx}(0,t),$$

$$\mathfrak{B}_d z = \frac{d\theta}{dx}(1,t),$$

$$\mathfrak{C} z = \eta(0,t),$$

and

$$\begin{aligned}D(\mathfrak{A}_0) = Z := \big\{ &(w, \eta, \theta)^\top \in (H^2(0,1) \cap H_R^1(0,1)) \\ &\times H_R^1(0,1) \times H^2(0,1) | \theta(0) = 0 \big\},\end{aligned}$$

the norm on Z is defined by $\|z\|_Z = \sqrt{\|\mathfrak{A}_0 z\|_X^2 + \|z\|^2}$, and $\mathfrak{A}_0 \in \mathcal{L}(Z, X)$, $\mathfrak{B}, \mathfrak{B}_d, \mathfrak{C} \in \mathcal{L}(Z, \mathbb{C})$.

We define the operator $A = \mathfrak{A}_0|_{D(A)}$ with $D(A) = \mathcal{N}(\mathfrak{B}) \cap \mathcal{N}(\mathfrak{B}_d)$. It has been shown in [21] that A generates an exponentially stable semigroup $T(t)$ on X.

Theorem 2.1. *([21, Corollary 3.2]) The operator A generates an exponentially stable semigroup $T(t)$ on X, i.e., there exist constants M_1, $\mu_1 > 0$ such that*

$$\|T(t)\| \leq M_1 e^{-\mu_1 t}, \quad \forall\, t > 0.$$

The operator $\begin{pmatrix} \mathfrak{B} \\ \mathfrak{B}_d \end{pmatrix} \in \mathcal{L}(Z, \mathbb{C}^2)$ is surjective, and thus by [36, Proposition 10.1.2] there exist unique $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ and $B_d \in \mathcal{L}(\mathbb{C}, X_{-1})$ such that $\mathfrak{A}_0 z = Az + B\mathfrak{B}z + B_d \mathfrak{B}_d z$ for all $z \in D(\mathfrak{A}_0)$. We also define $C = \mathfrak{C}|_{D(A)} \in \mathcal{L}(D(A), \mathbb{C})$, and $\mathfrak{A} := \mathfrak{A}_0|_{D(\mathfrak{A})} : D(\mathfrak{A}) \subset X \to X$ with $D(\mathfrak{A}) = \mathcal{N}(\mathfrak{B}_d)$.

Lemma 2.2. *If $d(t) = 0$, the system (4) is an impedance passive boundary control and observation system in the sense that*

$$\mathrm{Re}\, \langle \mathfrak{A} z, z \rangle \leq \mathrm{Re}\, \langle \mathfrak{B} z, \mathfrak{C} z \rangle_{\mathbb{C}}, \quad z \in D(\mathfrak{A}).$$

PROOF. Using integration by parts, we obtain

$$
\begin{aligned}
&\operatorname{Re}\langle \mathfrak{A}z,\ z\rangle \\
&= \operatorname{Re}\left\langle \left(v, \frac{d^2w}{dx^2} - \gamma\frac{d\theta}{dx}, k\frac{d^2\theta}{dx^2} - \gamma\frac{dv}{dx}\right)^{\top}, (w, v, \theta)^{\top}\right\rangle \\
&= \operatorname{Re}\left\langle \frac{dv}{dx}, \frac{dw}{dx}\right\rangle_{L^2(0,1)} + \operatorname{Re}\left\langle \frac{d^2w}{dx^2} - \gamma\frac{d\theta}{dx}, v\right\rangle_{L^2(0,1)} \\
&\quad + \operatorname{Re}\left\langle k\frac{d^2\theta}{dx^2} - \gamma\frac{dv}{dx}, \theta\right\rangle_{L^2(0,1)} \\
&= \operatorname{Re}\left(\mathfrak{B}z\overline{\mathfrak{C}z}\right) - k\left\|\frac{d\theta}{dx}\right\|^2_{L^2(0,1)}
\end{aligned}
$$

for every $z \in D(\mathfrak{A})$. This implies

$$\operatorname{Re}\langle \mathfrak{A}z,\ z\rangle \leq \operatorname{Re}\langle \mathfrak{B}z,\ \mathfrak{C}z\rangle_{\mathbb{C}}, \quad z \in D(\mathfrak{A}).$$

Hence, the system (4) with $d(t) = 0$ is impedance passive. □

3. Well-posedness of the system (4)

In this section, we prove that (4) defines a well-posed linear system.

3.1. Computing the transfer function

In this subsection, we apply the Laplace transform to obtain the transfer functions from the control input to the output and from the disturbance input to the output respectively, we also show that the transfer functions are bounded.

Lemma 3.1. *The transfer function $P_u(s)$ of (4) (from the control input $u(t)$ to the output $y(t)$) is bounded on a vertical line in $\mathbb{C}_0$ in the sense that*

$$\sup_{\operatorname{Re}s=\alpha} \| P_u(s)\| < \infty, \quad \text{for some } \alpha > 0.$$

PROOF. In (4), we have shown that (1) can be written as a boundary control system $(\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$, and by [4, Theorem 2.9] we obtain the transfer function $P_u(s)$ (from the input $u(t)$ to the output $y(t)$) by Laplace transform (with respect to t). For $d(t) = 0$, taking the Laplace transform on both sides of (1) gives

$$s^2\hat{w}(x,s) - \hat{w}''(x,s) + \gamma\hat{\theta}'(x,s) = 0, \tag{5a}$$
$$s\hat{\theta}(x,s) - k\hat{\theta}''(x,s) + \gamma s\hat{w}'(x,s) = 0, \tag{5b}$$
$$-\hat{w}'(0,s) = \hat{u}(s),\ \hat{w}(1,s) = 0, \tag{5c}$$
$$\hat{\theta}(0,s) = 0,\ \hat{\theta}'(1,s) = 0, \tag{5d}$$
$$\hat{y}(s) = s\hat{w}(0,s). \tag{5e}$$

Define

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \hat{w}(x,s) \\ \dfrac{d\hat{w}}{dx}(x,s) \end{pmatrix}, \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \hat{\theta}(x,s) \\ \dfrac{d\hat{\theta}}{dx}(x,s) \end{pmatrix},$$

then (5) can be written as

$$\frac{d}{dx}\begin{pmatrix} w_1 \\ w_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ s^2 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & \frac{s\gamma}{k} & \frac{s}{k} & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ \theta_1 \\ \theta_2 \end{pmatrix}, \tag{6a}$$
$$-w_2(0,s) = \hat{u}(s),\ w_1(1,s) = 0, \tag{6b}$$
$$\theta_1(0,s) = 0,\ \theta_2(1,s) = 0, \tag{6c}$$
$$\hat{y}(s) = sw_1(0,s). \tag{6d}$$

Solving (6), we obtain

$$\hat{y}(s) = sw_1(0,s) = s\frac{\hat{A}(s)}{\hat{B}(s)}\hat{u}(s),$$

where

$$
\begin{aligned}
\hat{A}(s) = &\left(r_1(s)r_2(s) + s^2\right)\left(r_1(s) - r_2(s)\right)\left(e^{r_1(s)+r_2(s)}\right. \\
&\left.-e^{-r_1(s)-r_2(s)}\right) + \left(r_1(s)r_2(s) - s^2\right)\left(r_1(s)\right. \\
&\left.+r_2(s)\right)\left(-e^{r_1(s)-r_2(s)} + e^{r_2(s)-r_1(s)}\right),
\end{aligned} \tag{7}
$$

$$\hat{B}(s) = s^2\left(r_1^2(s) - r_2^2(s)\right)\left(e^{r_1(s)} + e^{-r_1(s)}\right)\left(e^{r_2(s)} + e^{-r_2(s)}\right), \tag{8}$$

$$r_1(s) = \sqrt{\frac{ks^2 + \gamma^2 s + s + \sqrt{(ks^2 + \gamma^2 s + s)^2 - 4s^3k}}{2k}}, \tag{9}$$

$$r_2(s) = \sqrt{\frac{ks^2 + \gamma^2 s + s - \sqrt{(ks^2 + \gamma^2 s + s)^2 - 4s^3k}}{2k}}. \tag{10}$$

Thus the transfer function is given by

$$P_u(s) = s\frac{\hat{A}(s)}{\hat{B}(s)}. \tag{11}$$

From [14, Section 3.6], we have the following estimations for (9) and (10):

$$r_1(s) = s\left(1 + \frac{\gamma^2}{2ks} + \mathcal{O}\left(|s|^{-2}\right)\right),\ |s| \to +\infty, \tag{12}$$
$$r_2(s) = \frac{\sqrt{s}}{\sqrt{k}}\left(1 - \frac{\gamma^2}{2ks} + \mathcal{O}\left(|s|^{-2}\right)\right),\ |s| \to +\infty. \tag{13}$$

When $|s| \to +\infty$, it can be seen from (12) and (13) that the highest power terms of $r_1(s)$ and $r_2(s)$ with respect to s are s and $\sqrt{s}$ repectively. If we consider the highest power terms and ignore the lower order terms of s in (7) and (8), then the highest power terms of $\hat{A}(s)$ and $\hat{B}(s)$ are $s^3\left(e^{s+\sqrt{s}} + e^{s-\sqrt{s}}\right)$ and $s^4\left(e^{s+\sqrt{s}} + e^{s-\sqrt{s}}\right)$ respectively. So, for any fixed $\alpha > 0$ and $\operatorname{Re}s = \alpha$, we have

$$\limsup_{|s|\to+\infty}\left|s\frac{\hat{A}(s)}{\hat{B}(s)}\right| \leq \limsup_{|s|\to+\infty}\left|s\frac{s^3\left(e^{s+\sqrt{s}} + e^{s-\sqrt{s}}\right)}{s^4\left(e^{s+\sqrt{s}} + e^{s-\sqrt{s}}\right)}\right| = 1.$$

Thus, for any fixed $\alpha > 0$, we have $\sup_{\operatorname{Re}s=\alpha} \| P_u(s)\| < \infty$. □

Then, we calculate the transfer function from the disturbance input to the output and show that it is bounded.

Lemma 3.2. *The transfer function $P_d(s)$ (from the disturbance input $d(t)$ to the output $y(t)$) is bounded on a right half-plane $\mathbb{C}_\alpha$ for some $\alpha \geq 0$ in the sense that*

$$\sup_{\mathrm{Re}s>\alpha} \| P_d(s)\| < \infty.$$

PROOF. For $u(t) = 0$, taking the Laplace transform on both sides of the system (1), we get

$$\begin{aligned}
&s^2\hat{w}(x,s) - \hat{w}''(x,s) + \gamma\hat{\theta}'(x,s) = 0,\\
&s\hat{\theta}(x,s) - k\hat{\theta}''(x,s) + \gamma s\hat{w}'(x,s) = 0,\\
&\hat{w}'(0,s) = 0,\ \hat{w}(1,s) = 0,\\
&\hat{\theta}(0,s) = 0,\ \hat{\theta}'(1,s) = \hat{d}(s),\\
&\hat{y}(s) = s\hat{w}(0,s).
\end{aligned}$$

By simple calculation, we can obtain the transfer function $P_d(s)$ as follows:

$$P_d(s) = s\frac{\hat{C}(s)}{\hat{D}(s)},$$

where

$$\begin{aligned}
\hat{C}(s) &= -4k\gamma\left(e^{r_1(s)} + e^{-r_1(s)} - e^{r_2(s)} - e^{-r_2(s)}\right),\\
\hat{D}(s) &= \sqrt{(ks+\gamma^2+1)^2 - 4ks}\left(e^{r_1(s)} + e^{-r_1(s)}\right)\\
&\quad\cdot\left(e^{r_2(s)} + e^{-r_2(s)}\right),
\end{aligned}$$

and $r_1(s), r_2(s)$ are given by (9) and (10). Similarly as in Lemma 3.1, for any $\alpha \geq 0$ and $s \in \mathbb{C}_\alpha$, $\hat{C}(s)$ and $\hat{D}(s)$ can be regarded as e^s and $s\left(e^{s+\sqrt{s}} + e^{s-\sqrt{s}}\right)$ respectively. Then for any $\alpha \geq 0$ and for all $s \in \mathbb{C}_\alpha$ we obtain

$$\limsup_{|s|\to+\infty}\left|s\frac{\hat{C}(s)}{\hat{D}(s)}\right| \leq \limsup_{|s|\to+\infty}\left|s\frac{e^s}{s\left(e^{s+\sqrt{s}} + e^{s-\sqrt{s}}\right)}\right| < \infty.$$

Thus the transfer function $P_d(s)$ is bounded on the right half-plane $\mathbb{C}_\alpha$. □

3.2. Well-posedness of system (4)

In this subsection, we will prove that the system (4) is well-posed. We first give the definition of the disturbance input operator. A direct computation shows that the adjoint operator of A satisfies

$$\begin{cases}
A^*(f,g,h)^\top = (-g, -f'' + \gamma h', \gamma g' + kh'')^\top,\\
D(A^*) = \{(f,g,h)^\top \in (H^2(0,1)\cap H^1_R(0,1))\times\\
H^1_R(0,1)\times H^2(0,1) | f'(0) = h(0) = h'(1) = 0\}.
\end{cases}$$

If we take $(w,\eta,\theta)^\top \in D(\mathfrak{A}_0)$, $(f,g,h)^\top \in D(A^*)$, then we obtain

$$\begin{aligned}
&\left\langle \mathfrak{A}_0(w,\eta,\theta)^\top, (f,g,h)^\top\right\rangle - \left\langle (w,\eta,\theta)^\top, A^*(f,g,h)^\top\right\rangle\\
&= -w'(0)g(0) + k\theta'(1)h(1).
\end{aligned}$$

By [28, Remark 10.1.6] we therefore have that the adjoint of the disturbance input operator B_d is given by

$$B_d^*(f,g,h)^\top = kh(1),$$

thus, the disturbance input operator B_d is

$$B_d = (0, 0, k\delta(\cdot - 1))^\top,$$

where $\delta(\cdot)$ is the Dirac distribution.

Next, we prove that the disturbance input operator is admissible for the semigroup $T(t)$.

Lemma 3.3. *The operator B_d is an admissible disturbance input operator for the semigroup $T(t)$.*

PROOF. From Theorem 2.1, we know that A generates a semigroup $T(t)$ in X, thus A^* generates a semigroup $T^*(t)$. Then we will prove that B_d^* is admissible with respect to $T^*(t)$, which will imply the claim. A direct computation shows that for $(f_1, f_2, f_3)^\top \in X$, we have

$$\begin{aligned}
&B_d^* {A^*}^{-1}(f_1, f_2, f_3)^\top\\
&\quad = -\int_0^1\int_\xi^1 f_3(\tau)d\tau d\xi + \gamma\int_0^1 f_1(\xi)d\xi,
\end{aligned} \tag{14}$$

and thus $B_d^* {A^*}^{-1}$ is bounded from X to $\mathbb{C}$. In [12], the eigenvalues of A and the eigenfunctions with respect to the eigenvalues are given, from [26, p.26] we know A^* have the same eigenvalues as A with the same algebraic multiplicity for the conjugate eigenvalues. Then from $A^*W = \lambda W$, where $W = (f,g,h)$ is the eigenfunction of A^* with respect to the eigenvalue λ, we obtain two families of eigenpairs of A^*:

$$\left\{(\lambda_m, K_m), (\overline{\lambda_m}, \overline{K_m})\right\}_{m=1}^\infty, \left\{(\sigma_m, H_m)\right\}_{m=1}^\infty,$$

where $K_m = (f_m, -\lambda_m f_m, h_m)$, $\overline{K_m} = (\overline{f_m}, -\overline{\lambda_m f_m}, \overline{h_m})$, $H_m = (f_m^H, -\sigma_m f_m^H, h_m^H)$ and σ_m are real. Based on [12, Theorem 5] the eigenpairs $(\lambda_m, K_m), (\sigma_m, H_m)$ have asymptotic expressions

$$\begin{cases}
\lambda_m = -\dfrac{\gamma^2}{2k} + \left(m - \dfrac{1}{2}\right)\pi i + \mathcal{O}\left(m^{-1}\right),\\
K_m(x) = \begin{pmatrix} \phi_m(x) \\ -i\sin\left(m - \dfrac{1}{2}\right)\pi x \\ 0 \end{pmatrix} + \mathcal{O}\left(m^{-1}\right),
\end{cases}$$

where $\phi_m'(x) = \cos\left(m - \frac{1}{2}\right)\pi x$,

$$\begin{cases}
\sigma_m = -k\left(\left(m - \dfrac{1}{2}\right)\pi\right)^2 + \dfrac{\gamma^2}{k} + \mathcal{O}\left(m^{-2}\right),\\
H_m(x) = \begin{pmatrix} 0 \\ 0 \\ i\sin\left(m - \dfrac{1}{2}\right)\pi x \end{pmatrix} + \mathcal{O}\left(m^{-1}\right).
\end{cases} \tag{15}$$

From [12, Lemma 3], we know that A^{-1} is compact in X and $\sigma(A)$ consists of isolated eigenvalues only. Moreover, there is a sequence of generalized eigenfunctions of A, which forms a Riesz basis for the state space X, and all eigenvalues of A with sufficiently large modulus are algebraically simple. This also implies that there is a sequence of generalized eigenfunctions of A^*, which forms a Riesz basis for X, and all eigenvalues of A^* with sufficiently large modulus are algebraically simple, that is there exits an integer $N>0$ such that $\lambda_m, \overline{\lambda_m}, \sigma_m$ are algebraically simple if $m \geq N$. For $m<N$, we assume the algebraic multiplicities of λ_m, σ_m are n_{m1}, n_{m3} respectively, and we assume $K_{m,n_{m1}} = \left(f_{m,n_{m1}}, -\lambda_m f_{m,n_{m1}}, h_{m,n_{m1}}\right)$, $H_{m,n_{m3}} = \left(f^H_{m,n_{m3}}, -\sigma_m f^H_{m,n_{m3}}, h^H_{m,n_{m3}}\right)$ are generalized eigenfunctions of A^* with respect to λ_m, σ_m. Moreover, we assume $K_m = \left(f_m, -\lambda_m f_m, h_m\right)$, $H_m = \left(f^H_m, -\sigma_m f^H_m, h^H_m\right)$ are the normalized eigenfunctions of A^* corresponding to λ_m, σ_m with $m \geq N$ respectively. Then, all linearly independent generalized eigenfunctions of A^* are

$$\left\{\left\{K_{m,j}, \overline{K_{m,j}}\right\}_{j=1}^{n_{m1}}\right\}_{m<N} \cup \left\{K_m, \overline{K_m}\right\}_{m\geq N} \bigcup \left\{\left\{H_{m,j}\right\}_{j=1}^{n_{m3}}\right\}_{m<N} \cup \left\{H_m\right\}_{m\geq N}.$$

Hence every $z_0^* = (w_0^*, w_1^*, \theta_0^*)^\top \in X$ can be written as

$$\begin{aligned} z_0^* &= (w_0^*, w_1^*, \theta_0^*)^\top \\ &= \sum_{m=1}^{N-1}\sum_{j=1}^{n_{m1}} a_{m1,j} K_{m,j} + \sum_{m=N}^{+\infty} a_{m1} K_m \\ &\quad + \sum_{m=1}^{N-1}\sum_{j=1}^{n_{m1}} a_{m2,j} \overline{K_{m,j}} + \sum_{m=N}^{+\infty} a_{m2} \overline{K_m} \\ &\quad + \sum_{m=1}^{N-1}\sum_{j=1}^{n_{m3}} a_{m3,j} H_{m,j} + \sum_{m=N}^{+\infty} a_{m3} H_m, \end{aligned}$$

where $(a_{m1,j}), (a_{m1}), (a_{m2,j}), (a_{m2}), (a_{m3,j}), (a_{m3})$ are sequences in l^2. We can assume the coefficients of z_0^* corresponding to the generalized eigenspaces with $m<N$ are zero because the restriction of B_d^* to this finite-dimensional subspace of X is a bounded operator, and therefore does not affect the admissibility of B_d^*. To this end, let $z_0^* \in D(A^*)$ be such that

$$z_0^* = \sum_{m=N}^{+\infty} a_{m1} K_m + \sum_{m=N}^{+\infty} a_{m2} \overline{K_m} + \sum_{m=N}^{+\infty} a_{m3} H_m.$$

We can obtain the following estimation for $\|z_0^*\|$:

$$\begin{aligned} &C_1 \left(\sum_{m=N}^{+\infty} |a_{m1}|^2 + \sum_{m=N}^{+\infty} |a_{m2}|^2 + \sum_{m=N}^{+\infty} |a_{m3}|^2\right) \leq \|z_0^*\|_X^2 \\ &\leq C_2 \left(\sum_{m=N}^{+\infty} |a_{m1}|^2 + \sum_{m=N}^{+\infty} |a_{m2}|^2 + \sum_{m=N}^{+\infty} |a_{m3}|^2\right), \end{aligned}$$

where C_1, C_2 are constants. For $(w^*, w_t^*, \theta^*)^\top := T^*(t) z_0^* \in D(A^*)$, we have

$$\begin{aligned} &(w^*, w_t^*, \theta^*)^\top \\ &= \sum_{m=N}^{+\infty} a_{m1} e^{\lambda_m t} K_m + \sum_{m=N}^{+\infty} a_{m2} e^{\overline{\lambda_m} t} \overline{K_m} + \sum_{m=N}^{+\infty} a_{m3} e^{\sigma_m t} H_m, \end{aligned}$$

and we obtain

$$\begin{aligned} \left\|B_d^* T^*(t) z_0^*\right\|^2 = |k\theta^*(1,t)|^2 = k^2 \Bigg| &\sum_{m=N}^{+\infty} a_{m1} e^{\lambda_m t} h_m(1) \\ &+ \sum_{m=N}^{+\infty} a_{m2} e^{\overline{\lambda_m} t} \overline{h_m(1)} + \sum_{m=N}^{+\infty} a_{m3} e^{\sigma_m t} h^H_m(1)\Bigg|^2. \end{aligned}$$

Let $T_1 > 0$. Since the semigroup $T^*(t)$ is exponentially stable, we have $|e^{\lambda_m t}| \leq 1$ for all $m \geq N$. Moreover, since $|h_m(1)| = \mathcal{O}(m^{-1})$ and $C_H := \sup_{m\geq N} |h^H_m(1)| < \infty$, the Cauchy–Schwarz inequality and (15) imply that

$$\begin{aligned} &\int_0^{T_1} \left\|B_d^* T^*(t) z_0^*\right\|^2 dt \\ &\leq k^2 \int_0^{T_1} \left[\sum_{m=N}^{+\infty} (|a_{m1}| + |a_{m2}|)\mathcal{O}(m^{-1}) + |a_{m3}| e^{\sigma_m t} C_H\right]^2 dt \\ &\leq C_3 T_1 \sum_{m=N}^{+\infty} \left(|a_{m1}|^2 + |a_{m2}|^2\right) \\ &\quad + 2k^2 C_H^2 \int_0^{T_1} \sum_{m=N}^{+\infty} \left|a_{m3}\right|^2 \sum_{m=N}^{+\infty} e^{2\sigma_m t}\, dt \\ &\leq C_3 T_1 \sum_{m=N}^{+\infty} \left(|a_{m1}|^2 + |a_{m2}|^2\right) + C_{T_1} \sum_{m=N}^{+\infty} \left|a_{m3}\right|^2 \\ &\leq \frac{\max\{C_3 T_1, C_{T_1}\}}{C_1} \|z_0^*\|_X^2, \end{aligned}$$

for some constants $C_3, C_{T_1} > 0$ depending on T_1. This together with (14) imply that the operator B_d^* is admissible for the semigroup $T^*(t)$. By using [36, Theorem 4.4.3], we obtain that B_d is admissible for the semigroup $T(t)$. □

In the last part of this section we prove that the system (4) defines a well-posed linear system in the sense of [37, Definition 3.1]. This implies that for all $T>0$ there exits $D_T > 0$ such that for all initial data $(w(\cdot,0), w_t(\cdot,0), \theta(\cdot,0))^\top \in X$ and for all $u, d \in L^2_{loc}(0,\infty)$ the weak solution and output of (1) satisfy [37, Proposition 4.7]

$$\begin{aligned} &\left\|\left(w(\cdot,t), w_t(\cdot,t), \theta(\cdot,t)\right)^\top\right\|_X^2 + \int_0^T |y(\tau)|^2 d\tau \\ &\leq D_T \left[\left\|\left(w(\cdot,0), w_t(\cdot,0), \theta(\cdot,0)\right)^\top\right\|_X^2 \right. \\ &\qquad \left. + \int_0^T |u(\tau)|^2 + |d(\tau)|^2 d\tau\right]. \end{aligned}$$

Theorem 3.4. *The boundary control system (4) defines a well-posed linear system on X with input $(u(t), d(t))^T$ and output $y(t)$. The transfer function $P_u(s)$ satisfies $\mathrm{Re} P_u(s) \geq 0$ for all $s \in \mathbb{C}_0$.*

PROOF. We begin by considering the case where $d(t) \equiv 0$, in which case $u(t)$ is the only input of (4), corresponding to the boundary control system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. We will show that this system is well-posed. We have from [27, Theorem 2.3 and Proposition 2.5] that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ defines a system node $\Sigma_{node} = (A, B, C, P_{node})$ in the sense of [37, Definition 4.1]. In particular, parts (i), (ii), and (iv) of [27, Theorem 2.3] imply that the operators A, B, and C are as in Section 2. Moreover, by [27, Theorem 2.3(iv)] the combined observation and feedthrough operator $C\&D : D(C\&D) \subset X \times \mathbb{C} \to \mathbb{C}$ is given by

$$C\&D \begin{pmatrix} z \\ u \end{pmatrix} = \mathfrak{C}z,$$

for all $(z,u)^\top \in D(C\&D) = \{(z,u)^\top \in D(\mathfrak{A}) \times \mathbb{C} \mid \mathfrak{B}z = u\}$. Since the semigroup generated by A is exponentially stable by Theorem 2.1, we have from [37, Eq. (4.4)] that the transfer function $P_{node}(s)$ of Σ_{node} satisfies

$$P_S(s) = C\&D \begin{pmatrix} (s-A)^{-1}B \\ 1 \end{pmatrix} = \mathfrak{C}(s-A)^{-1}B = P_u(s)$$

for all $s \in \mathbb{C}_0$. Together with Lemma 3.1 we thus have that $P_S(s)$ is bounded on some vertical line in $\mathbb{C}_0$.

Let $(z,u)^\top \in D(C\&D)$. Then $z \in D(\mathfrak{A}) = \mathcal{N}(\mathfrak{B}_d)$ and $\mathfrak{B}z = u$ and we have $Az + Bu = Az + B\mathfrak{B}z = \mathfrak{A}z$ by the definition of B. Because of this, Lemma 2.2 shows that

$$\begin{aligned} \mathrm{Re}\langle Az + Bu, z\rangle_X &= \mathrm{Re}\langle \mathfrak{A}z, z\rangle_X \le \mathrm{Re}\langle \mathfrak{B}z, \mathfrak{C}z\rangle_{\mathbb{C}} \\ &= \mathrm{Re}\langle C\&D \begin{pmatrix} z \\ u \end{pmatrix}, u\rangle_{\mathbb{C}}. \end{aligned}$$

Since $(z,u)^\top \in D(C\&D)$ was arbitrary, we have from [35, Theorem 4.2(iii)] that Σ_{node} is impedance passive in the sense of [37, Definition 6.1]. Since $P_S(s)$ is bounded on a vertical line in $\mathbb{C}_0$, the system node Σ_{node} is well-posed by [35, Theorem 5.1]. In particular, B and C are admissible with respect to the semigroup $T(t)$ generated by A. Moreover, we have from [35, Theorem 4.2(iv)] that $\mathrm{Re}P_u(s) = \mathrm{Re}P_S(s) \ge 0$ for all $s \in \mathbb{C}_0$.

Finally, we consider (4) with input $(u(t), d(t))^\top$ and output $y(t)$. Lemma 3.3 shows that the disturbance input operator B_d is admissible with respect to $T(t)$ and the transfer function $P_d(s)$ (from $d(t)$ to $y(t)$) is bounded on a right half-plane in $\mathbb{C}$ by Lemma 3.2. Because of this, also the operator $(B, B_d) \in \mathcal{L}(\mathbb{C}^2, X_{-1})$ is admissible with respect to $T(t)$ and the transfer function $(P_u(s), P_d(s))$ is bounded on some vertical line in $\mathbb{C}_0$. Because of this, we have from [37, Proposition 4.9 and Proposition 4.10] that $(A, (B, B_d), C)$ is a well-posed triple in the sense of [37, Definition 4.8], which exactly means that (4) defines a well-posed linear system on X in the sense of [37, Definition 3.1]. □

4. Robust output regulation

In this section, we design a dynamic error feedback controller for the system (1). Based on [33, Theorem 1.2] we consider a controller of the form

$$\dot{z}_c(t) = J_c z_c(t) + B_c(y_{ref}(t) - y(t)), \quad z_c(0) \in Z_c, \tag{16a}$$

$$u(t) = B_c^* z_c(t) + D_c(y_{ref}(t) - y(t)), \tag{16b}$$

where $Z_c = \mathbb{C}^{2q}$ is the state space of the controller, $J_c \in \mathbb{C}^{2q\times 2q}$ is skew-symmetric, $B_c \in \mathbb{C}^{2q\times 1}$, $D_c \in \mathbb{C}$. The parameters J_c and B_c will be chosen in such a way the robust output regulation is achieved for the system (1).

From (4), (2) and the controller (16), we obtain the closed-loop system

$$\begin{cases} \dot{z}_e(t) = \mathfrak{A}_e z_e(t) + B_e v(t), \\ \mathfrak{B}_e z_e(t) = d_e v(t), \\ e(t) = \mathfrak{C}_e z_e(t) - F v(t), \end{cases}$$

where $z_e = (z, z_c)^\top$, the state space is $Z_e := X \times Z_c$, and

$$\mathfrak{A}_e = \begin{pmatrix} \mathfrak{A}_0 & 0 \\ -B_c\mathfrak{C} & J_c \end{pmatrix}, \; B_e = \begin{pmatrix} 0 \\ B_c F \end{pmatrix},$$
$$\mathfrak{B}_e = \begin{pmatrix} \mathfrak{B} + D_c\mathfrak{C} & -B_c^* \\ \mathfrak{B}_d & 0 \end{pmatrix}, \; d_e = \begin{pmatrix} D_c F \\ E \end{pmatrix},$$
$$\mathfrak{C}_e = (\mathfrak{C}, \; 0).$$

We formulate the robust output regulation problem in the following way.

The Robust Output Regulation Problem. Select the parameters (J_c, B_c) in the controller (16) in such a way that the following hold.

(1) The closed-loop semigroup generated by $A_e := \mathfrak{A}_e|_{\mathcal{N}(\mathfrak{B}_e)}$ is exponentially stable.

(2) There exists $\alpha < 0$ such that for all initial states $z_0 \in X$ and $z_{c0} \in Z_c$, and for all disturbance input $d(t)$ and reference signal $y_{ref}(t)$ of the form (2b)-(2c), such that the regulation error satisfies

$$\int_0^\infty e^{-\alpha t}\|y(t) - y_{ref}(t)\| < \infty. \tag{17}$$

(3) If the operators $(\mathfrak{A}_0, \mathfrak{B}, \mathfrak{B}_d, \mathfrak{C})$ are perturbed to $(\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_d, \tilde{\mathfrak{C}})$ in such a way that the closed-loop system remains exponentially stable, then there exists $\tilde{\alpha} < 0$ such that for all initial states $z_0 \in X$ and $z_{c0} \in Z_c$, $d(t)$ and $y_{ref}(t)$ of the form (2b)-(2c), the property (17) holds with α replaced by $\tilde{\alpha}$.

To solve the robust output regulation problem, we select the parameters of controller (16) so that (16) is a realization of the controller in [33, Theorem 1.2]. The parameters are chosen as

$$\begin{aligned} J_c &= \mathrm{blockdiag}(J_c^1, \cdots, J_c^q), \\ J_c^n &= \begin{pmatrix} 0 & \omega_n I_Y \\ -\omega_n I_Y & 0 \end{pmatrix}, \\ B_c &= \begin{pmatrix} B_c^1 \\ \vdots \\ B_c^q \end{pmatrix}, \; B_c^n = \begin{pmatrix} I_Y \\ 0 \end{pmatrix}, \; D_c = \frac{3}{4}, \end{aligned} \tag{18}$$

where, $n = 1, \cdots, q$.

The robust output regulation problem is solvable if no eigenvalue of S in the exosystem is a transmission zero of the control system. Here, we make the following slightly stronger assumption on the frequencies of the reference and disturbance input signals for the transfer function P_u (from $u(t)$ to $y(t)$):

Assumption 4.1. *For $\omega_1 < \omega_2 < \cdots < \omega_q$, we assume $\mathrm{Re}P_u(\pm i\omega_n) > 0$ for all $n \in \{1, ..., q\}$.*

In order to prove that the controller (16) with the parameters (18) can solve the robust output regulation problem, we will use [33, Theorem 1.2]. Now, we state the main result in the following theorem.

Theorem 4.2. *The controller (16) with parameters (18) solves the robust output regulation problem for the system (1).*

PROOF. The transfer function $P_u(s)$ from control input to the output and the transfer function $P_d(s)$ from disturbance input to the output are bounded from Lemma 3.1 and Lemma 3.2 respectively. The transfer function satisfies $\mathrm{Re}P_u(s) \geq 0$ for all $s \in \mathbb{C}_0$ by Theorem 3.4, and $\mathrm{Re}P_u(\pm i\omega_n) \neq 0$ due to Assumption 4.1. From Theorem 3.4 we obtain that (4) is well-posed.

Now, we verify that the controller (16) with parameters (18) satisfies the assumptions in [33, Theorem 1.2]. By using the Laplace transform for (16) and from (18), we obtain the transfer function $\mathbf{C}(s)$ (from $y(t) - y_{ref}(t)$ to $u(t)$) of (16) as follows:

$$\begin{aligned}\mathbf{C}(s) &= -\left[B_c^*(s - J_c)^{-1}B_c + D_c\right], \\ &= -\left[\frac{3}{4} + \sum_{n=1}^{q} \frac{1}{2(s - i\omega_n)} + \sum_{n=1}^{q} \frac{1}{2(s + i\omega_n)}\right].\end{aligned} \tag{19}$$

From the transfer function $\mathbf{C}(s)$ in (19), we know that the controller (16) has the form of the controller in [33, Theorem 1.2]. Using the structure of J_c and B_c, it is easy to show that the pair (J_c, B_c, B_c^*) is controllable and observable, and $D_c = \frac{3}{4} \geq \frac{1}{2}$. Thus we finally have from [33, Theorem 1.2] that the controller (16) with parameters (18) solves the robust output regulation problem. □

5. Numerical simulations

In this section, we present some numerical simulations to show the effectiveness of the proposed controller. The numerical results are obtained by the finite element method.

For numerical computations, the steps of space and time are both set as 0.001. The parameters are taken as $k = 1$ and $\gamma = 0.9$. The disturbance to be rejected and the reference signal to be tracked are chosen as

$$\begin{cases} d(t) = \sin\left(\frac{3\pi}{2}t\right) - \cos\left(\frac{3\pi}{2}t\right), \\ y_{ref}(t) = \frac{1}{2}\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{3}t\right), \end{cases}$$

and they can be generated by the exosystem with

$$S = \mathrm{blockdiag}\left(\frac{3\pi}{2}\tilde{I}, \frac{\pi}{2}\tilde{I}, \frac{\pi}{3}\tilde{I}\right), \quad \tilde{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$E = (1, -1, 0, 0, 0, 0), \quad F = \left(0, 0, \frac{1}{2}, 0, 0, -1\right),$$

and

$$v_0 = (0, 1, 0, 1, 0, 1)^\top.$$

The initial states for (1) are taken as $w(x, 0) = 1/3 + \sin(x)$, $w_t(x, 0) = 2\sin(x)$, $\theta(x, 0) = \sin(2x)$. Now, we select the parameters of the controller (16): $J_c = S$, $B_c = (1, 0, 1, 0, 1, 0)^\top$, $D_c = \frac{3}{4}$. From Fig.1, it is seen that the state $w(x, t)$ of (1) with the controller (16) is bounded. Fig.2 demonstrates the state $\theta(x, t)$ of (1) with the controller (16) is bounded. Fig.3 shows that the output $y(t)$ and the reference signal $y_{ref}(t)$, we can see that $y(t) = w_t(0, t)$ is forced by the controller (16) to track $y_{ref}(t)$.

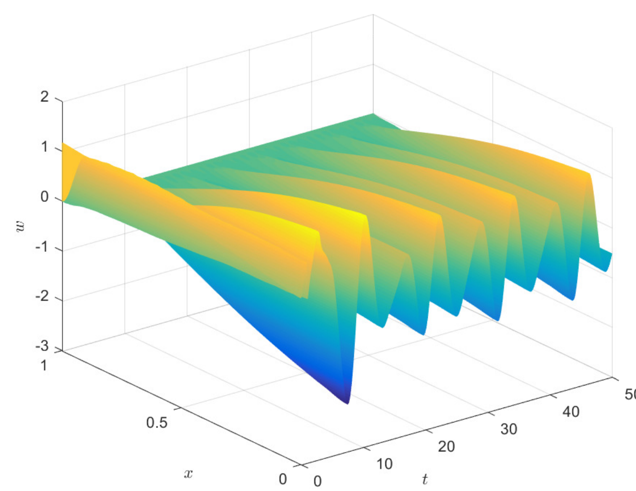

Figure 1: The state of $w(x, t)$ of the closed-loop system

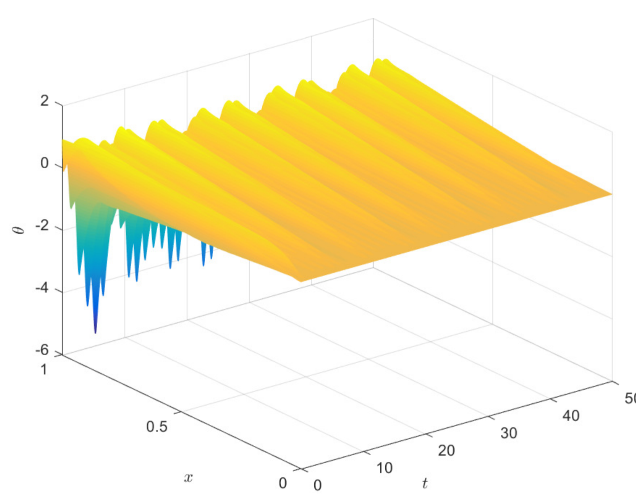

Figure 2: The state of $\theta(x, t)$ of the closed-loop system

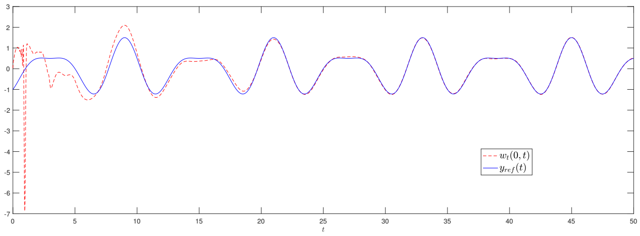

Figure 3: The output $w_t(0,t)$ and the reference signal $y_{ref}(t)$

6. Conclusion

In this paper, we studied the robust error feedback regulation for a thermoelastic system. By formulating the system as a boundary control system and proving the system is impedance passive, we proved the considered system is well-posed and constructed a controller to achieve robust output regulation by using the theory in [33]. Future research topics include the robust regulator design for distributed parameter systems with unknown frequencies in the exosystem.

References

[1] Ammar-Khodja, F., Bader, A., Benabdallah, A., 1999. Dynamic stabilization of systems via decoupling techniques. ESAIM Contr. Optim. Ca. 4, 577–593.

[2] Burns, J.A., Liu, Z.Y., Zheng, S.M., 1993. On the energy decay of a linear thermoelastic bar. J. Math. Anal. Appl. 179, 574–591.

[3] Byrnes, C., Lauko, I., Gilliam, D., Shubov, V., 2000. Output regulation for linear distributed parameter systems. IEEE Trans. Automat. Control 45, 2236–2252.

[4] Cheng, A., Morris, K., 2003. Well-posedness of boundary control systems. SIAM J. Control Optim. 42, 1244–1265.

[5] Dafermos, C.M., 1968. On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. Arch. Ration. Mech. An. 29, 241–271.

[6] Davison, E.J., 1976. The robust control of a servo-mechanism problem for linear time-invariant multivariable systems. IEEE Trans. Automat. Control 21, 25–34.

[7] Deutscher, J., 2015. A backstepping approach to the output regulation of boundary controlled parabolic PDEs. Automatica 57, 56–64.

[8] Deutscher, J., Kerschbaum, S., 2019. Output regulation for coupled linear parabolic PIDEs. Automatica 100, 360–370.

[9] Francis, B.A., Wonham, W.M., 1975. The internal model principle for linear multivariable regulators. Appl. Math. Opt. 2, 170–194.

[10] Francis, B.A., Wonham, W.M., 1976. The internal model principle of control theory. Automatica 12, 457–465.

[11] Gu, J.J., Wang, J.M., Guo, Y.P., 2017. Output regulation of anti-stable coupled wave equations via the backstepping technique. IET Control Theory A. 12, 431–445.

[12] Guo, B.Z., 2002. Further results for a one-dimensional linear thermoelastic equation with Dirichlet-Dirichlet boundary conditions. Anziam J. 43, 449–462.

[13] Guo, B.Z., Meng, T., 2020. Robust error based non-collocated output tracking control for a heat equation. Automatica 114, 108818.

[14] Guo, B.Z., Wang, J.M., 2019. Control of Wave and Beam PDEs: The Riesz Basis Approach. Springer, Cham.

[15] Guo, W., Jin, F.F., 2020. Adaptive error feedback regulator design for one-dimensional heat equation with unknown harmonic disturbance anticollocated with control. IEEE Trans. Automat. Control 65, 824–830.

[16] Guo, W., Zhou, H.C., 2019. Adaptive error feedback regulation problem for an Euler–Bernoulli beam equation with general unmatched boundary harmonic disturbance. SIAM J. Control Optim. 57, 1890–1928.

[17] Guo, W., Zhou, H.C., Krstic, M., 2018. Adaptive error feedback regulation problem for 1D wave equation. Int. J. Robust Nonlin. 28, 4309–4329.

[18] Hämäläinen, T., Pohjolainen, S., 2000. A finite-dimensional robust controller for systems in the CD-algebra. IEEE Trans. Automat. Control 45, 421–431.

[19] Hämäläinen, T., Pohjolainen, S., 2002. Robust regulation for exponentially stable boundary control systems in Hilbert space, in: 8th IEEE International Conference on Methods and Models in Automation and Robotics.

[20] Hämäläinen, T., Pohjolainen, S., 2010. Robust regulation of distributed parameter systems with infinite-dimensional exosystems. SIAM J. Control Optim. 48, 4846–4873.

[21] Hansen, S.W., 1992. Exponential energy decay in a linear thermoelastic rod. J. Math. Anal. Appl. 167, 429–442.

[22] Humaloja, J.P., Paunonen, L., 2018. Robust regulation of infinite-dimensional port-Hamiltonian systems. IEEE Trans. Automat. Control 63, 1480–1486.

[23] Immonen, E., 2007. On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control. SIAM J. Control Optim. 45, 2065–2093.

[24] Liu, J.J., Guo, B.Z., 2022. Robust tracking error feedback control for a one-dimensional Schrödinger equation. IEEE Trans. Automat. Control 67, 1120–1134.

[25] Liu, Z., Zheng, S.M., 1993. Exponential stability of semigroup associated with thermoelastic system. Q. Appl. Math. 51, 535–545.

[26] Locker, J., 2000. Spectral Theory of Non-Selfadjoint Two-Point Differential Operators. American Mathematical Society.

[27] Malinen, J., Staffans, O.J., 2006. Conservative boundary control systems. J. Differ. Equations 231, 290–312.

[28] Malinen, J., Staffans, O.J., 2007. Impedance passive and conservative boundary control systems. Complex Anal. Oper. Th. 1, 279–300.

[29] Natarajan, V., Gilliam, D.S., Weiss, G., 2014. The state feedback regulator problem for regular linear systems. IEEE Trans. Automat. Control 59, 2708–2723.

[30] Paunonen, L., 2016. Controller design for robust output regulation of regular linear systems. IEEE Trans. Automat. Control 61, 2974–2986.

[31] Paunonen, L., Le Gorrec, Y., Ramírez, H., 2021. A Lyapunov approach to robust regulation of distributed port-Hamiltonian systems. IEEE Trans. Automat. Control 66, 6041–6048.

[32] Paunonen, L., Pohjolainen, S., 2014. The internal model principle for systems with unbounded control and observation. SIAM J. Control Optim. 52, 1–38.

[33] Rebarber, R., Weiss, G., 2003. Internal model based tracking and disturbance rejection for stable well-posed systems. Automatica 39, 1555–1569.

[34] Salamon, D., 1987. Infinite dimensional linear systems with unbounded control and observation: A functional analytic approach. Trans. Amer. Math. Soc. 300, 383–383.

[35] Staffans, O.J., 2002. Passive and conservative continuous-time impedance and scattering systems. Part I: Well-posed systems. Math. Control Signals Systems 15, 291–315.

[36] Tucsnak, M., Weiss, G., 2009. Observation and Control for Operator Semigroups. Birkhäuser Basel.

[37] Tucsnak, M., Weiss, G., 2014. Well-posed systems - the LTI case and beyond. Automatica 50, 1757–1779.

[38] Xu, X., Dubljevic, S., 2017. Output regulation for a class of linear boundary controlled first-order hyperbolic PIDE systems. Automatica 85, 43–52.