Dual Look at Robust Regulation: Frequency Domain and State Space Approaches

(Invited Paper)

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***Abstract*— In this paper robust output regulation of distributed parameter systems with infinite-dimensional exosystems is discussed. We divide the problem into two parts, namely robust stabilization and robust regulation, and focus on the latter. Our aim is to give a unified treatment of the problem in time and frequency domains by using blocking zeros.**

I. Introduction

In this paper we discuss robust output regulation problem. Robustness is of fundamental importance, since when modeling real world phenomena the models are inevitably subject to some inaccuracies. To tackle this problem, one should design a controller that is able to regulate all the systems (in some sense) close to the model, i.e. to design a robustly regulating controller.

We focus on the frequency domain setting, but we motivate our study by presenting the related state-space theory [1] and compare the results in both domains in a unifying manner. Especially, we generalize the transmission zero condition in [2] to infinite-dimensional setting. It provides a natural link between state-space and frequency domain considerations.

The problem of robust regulation have been extensively studied in various setups by several authors. For finite-dimensional results we refer to Francis and Wonham [3] and Davison [4]. These results were generalized for infinite-dimensional systems with finite-dimensional exosystems by Pohjolainen [5], Hämäläinen and Pohjolainen [6] and Rebarber and Weiss [7]. Robust regulation problem with infinite-dimensional exosystem was considered in [1], [8] and [9].

In [1] Paunonen and Pohjolainen divided the robust output regulation problem into two parts, namely robust stabilization and robust regulation, see also [10] and [11]. When solving the regulation part one tries to find the essential features of the controller that guarantee regulation under the assumption that the controller stabilizes the system. These features are characterized by the internal model principle, first stated for finite-dimensional systems by Francis and Wonham [3]. Roughly speaking the internal model principle states that a robustly regulating controller must contain a reduplicated model of the dynamics to be controlled.

In the infinite-dimensional case the problem of robust stabilization is difficult. In fact an infinite-dimensional closed loop system cannot be exponentially stabilized, in general, if the exosystem is infinite-dimensional [11]. Thus one should consider some weaker type of stability. However, unlike exponential stability, strong stability of a system can be destroyed by an arbitrarily small perturbation of the system parameters. On the other hand, some perturbations can make the system even more stable. This underlines the importance of the separation of the stabilization part and the regulation part.

In the frequency domain the problem of robust regulation is well understood for rational transfer functions [12]. In [13] robust regulation with a generator having an infinite number of poles on the imaginary axis, was considered in H^∞-setting. It was shown, that for the problem to be solvable the system's transfer function should not vanish at infinity. This is a major limitation and in sharp contrast to the state-space results, e.g. [9, Example 17] where a robustly regulating controller was found for a plant with transfer function that approaches zero at infinity with rate $\mathcal{O}(\frac{1}{s\sqrt{s}})$. In the state-space setting one can compensate the vanishing transfer function by setting some smoothness assumptions on the reference signals. This is not possible in frequency domain if H^∞-stability is considered. Thus, like exponential stability in state-space setting, H^∞-stability in frequency domain is a too strong stability type. In this paper we propose a new ring for stable transfer functions.

The role of transmission zeros in regulation is also well known in the finite-dimensional case. In [2] Francis and Wonham show, that a controller is robustly regulating if and only if the transfer function of the extended system from the exosystems state to the error has a blocking zero of suitable multiplicity at the poles of the exosystem. In this paper the above result is generalized for infinite-dimensional systems. To the authors knowledge, the result is new for in infinite-dimensional setting.

The paper is organized as follows. In Section II notations and preliminary results are introduced. In Section III we give a blocking zero condition for robust regulation. Section IV is dedicated for frequency domain results. We consider a weaker type of stability than H^∞ stability in order to facilitate the possibility for systems to be strictly proper. The blocking zero condition is discussed in frequency domain terms. Finally in Section V we summarize the results and give some directions for future research.

II. Notations and preliminary results

The sets of integers, complex numbers, real numbers and imaginary numbers are denoted $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{R}$ and $\mathrm{i}\mathbb{R}$. The set of complex numbers, with real part greater than $\beta \in \mathbb{R}$ is denoted $\mathbb{C}_\beta$. Notation H^∞_β is used for the set of complex functions analytic and uniformly bounded in $\mathbb{C}_\beta$, and if $\beta = 0$ shorthand notation H^∞ is used. The set of all matrices over a set Γ is denoted $\mathrm{M}(\Gamma)$. Domain of a function f is denoted $\mathcal{D}(f)$. The spectrum of a linear operator L is denoted $\sigma(L)$.

Definition 1: Let $0 < l \in \mathbb{Z}$. Assume, that $P : \mathcal{D}(P) \subseteq \mathbb{C} \to \mathbb{C}^{n\times m}$ is meromorphic in an open set $\Gamma \subseteq \mathbb{C}$.

If $P(s)$ is analytic at $s_0 \in \mathcal{D}(P)$ and $\lim_{s\to s_0}(s - s_0)^{-l}P(s)$ exists, then we say that $P(s)$ has a blocking zero of order at least l at s_0. If at $s_0 \in \Gamma$ limit $\lim_{s\to s_0}(s - s_0)^{l}P(s)$ does not exists or differs from zero, then we say that $P(s)$ has a full pole of order at least l at s_0.

A. State space notations

In state space the plant is given by

$$\dot{x}(t) = Ax(t) + Bu(t) + w_s(t) \qquad x(0) = x_0 \in X, \tag{1a}$$
$$y(t) = Cx(t) + Du(t) + w_m(t), \tag{1b}$$

where the state-space X is a Hilbert-space, input and output spaces U and Y are finite-dimensional complex spaces and B, C and D are linear bounded operators. The disturbance signals w_s and w_m are defined below. The system operator A is the generator of a C_0-semigroup.

Next we define the exosystem. For this we consider a Hilbert space W with orthonormal basis vectors ϕ^l_k, where $k \in \mathbb{Z}$ and $l = 1, \dots, n_k$. Constants n_k are uniformly bounded. We define operators

$$S_k = i\omega_k \left\langle \cdot, \phi^1_k \right\rangle + \sum_{l=2}^{n_k} \left\langle \cdot, \phi^l_k \right\rangle (i\omega_k \phi^l_k + \phi^{l-1}_k), \tag{2a}$$

and

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \tag{2b}$$

with domain $\mathcal{D}(S) = \{v \in W \mid \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty\}$. The exosystem generating the reference and disturbance signals is given by

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W, \tag{3a}$$
$$y_r(t) = F_s v(t), \tag{3b}$$
$$w_m(t) = F_m v(t), \tag{3c}$$
$$w_s(t) = Ev(t), \tag{3d}$$

where F_s, F_m and E are linear bounded operators. Define $e(t) = y(t) - y_r(t)$ and $F = F_m - F_s$. We consider the following controller

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t) \qquad z(0) = z_0 \in Z, \tag{4a}$$
$$u(t) = Kx(t), \tag{4b}$$

where $\mathcal{G}_1$ is a generator of a C_0-semigroup and $\mathcal{G}_2$ and K are linear bounded operators. Combining (1), (3) and (4) we find the closed-loop system to be

$$\dot{x}_e(t) = A_e x_e(t) + B_e \begin{pmatrix} w_s(t) \\ w_m(t) - y_{ref}(t) \end{pmatrix},$$
$$e(t) = C_e x_e(t) + D_e \begin{pmatrix} w_s(t) \\ w_m(t) - y_{ref}(t) \end{pmatrix},$$

where $x_e(0) = x_{e0} \in X_e = X \times Z$, $C_e = \begin{pmatrix} C & DK \end{pmatrix}$, $D_e = \begin{pmatrix} 0 & I_Y \end{pmatrix}$, $A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix}$ and $B_e = \begin{pmatrix} I & 0 \\ 0 & \mathcal{G}_2 \end{pmatrix}$. The transfer function is given by

$$P_e(s) = C_e R(s, A_e) B_e + D_e, \qquad s \in \rho(A_e).$$

We next present the robust output regulation problem. However, since the precise definition requires even more definitions, we settle only to describe the problem. Detailed definition can be found in [1] or [10]. The robust output regulation problem is to find such controller parameters, that

1) A_e generates a strongly stable C_0-semigroup.
2) Regulation error e approaches zero asymptotically for all $x_{e0} \in X_e$ and all sufficiently smooth reference signals.
3) If the operators A, B, C, D, E and F are perturbed in such a way that the perturbed closed loop system (A'_e, B'_e, C'_e, D'_e) is strongly stable and the Sylvester equation $\Sigma S = A'_e \Sigma + B'_e \begin{pmatrix} E' \\ F' \end{pmatrix}$ has a solution then 2) holds for the perturbed system.

We have used a vague term 'sufficiently smooth reference signal' above. The smoothness of the reference signals is related to the existence of the solution to the Sylvester equations. It can be controlled either by restricting the set of initial states of the exosystem [1] or by restricting the set of allowed operators F_m, F_s and E [9], [14].

The *conditional robustness* [10] can be written as a condition that for any operators A, B, C, D, E, and F we have

$$\Sigma S = A_e \Sigma + B_e \begin{pmatrix} E \\ F \end{pmatrix} \Rightarrow C_e \Sigma + D_e \begin{pmatrix} E \\ F \end{pmatrix} = 0. \tag{5}$$

This property guarantees that if for some perturbations of the parameters of the plant the closed-loop system is strongly stable, then the regulator error decays to zero asymptotically.

The nature of the studied robustness is the same as in [1] and [10], i.e., conditional robustness which requires that the robust regulation is achieved for any reference and disturbance signals generated by an exosystem with arbitrary operators E and F. This follows (roughly) from the fact that for any Laplace transformable functions w_s, w_m and y_{ref} we have

$$\hat{e}(i\omega_k) = P_e(i\omega_k) \begin{pmatrix} \hat{w}_s \\ \hat{w}_m - \hat{y}_{ref} \end{pmatrix} = 0 \qquad \forall k \in \mathbb{Z}.$$

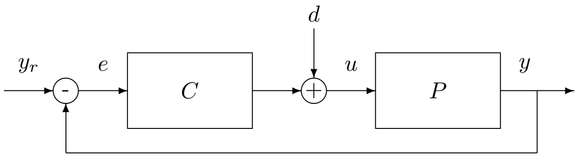

Fig. 1. The closed loop system

B. Frequency domain notations

Let $\mathcal{S}$ be a commutative ring with an multiplicative unit and no zero divisors. We call $\mathcal{S}$ the ring of stable functions. The set of all transfer functions $\mathcal{G}$ is the field of fractions of $\mathcal{S}$. We say that a matrix $M \in \mathrm{M}(\mathcal{G})$ is *stable* if $M \in \mathrm{M}(\mathcal{S})$.

We say that $N \in \mathrm{M}(\mathcal{S})$ and $D \in \mathrm{M}(\mathcal{S})$ are *right [left] coprime*, if there exists matrices $U \in \mathrm{M}(\mathcal{S})$ and $V \in \mathrm{M}(\mathcal{S})$ such, that $UN + VD = I$ $[NU + VD = I]$. We call a pair (N, D) *a right [left] coprime factorization* of a $P \in \mathrm{M}(\mathcal{G})$ if $\det(D) \neq 0$, $P = ND^{-1}$ $[P = D^{-1}N]$ and N and D are right [left] coprime. The set of all matrices with both coprime factorizations is denoted $\mathcal{C}$.

In this paper we consider the closed loop system depicted in Figure 1 and denote it $\Sigma(P, C)$. The closed loop transfer function P_c is given by

$$\begin{pmatrix} e \\ u \end{pmatrix} = \begin{pmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ C(I+PC)^{-1} & I - C(I+PC)^{-1}P \end{pmatrix} \begin{pmatrix} y_r \\ d \end{pmatrix}$$

We say that the system $\Sigma(P, C)$ is *stable* if $\det(I + PC) \neq 0$ and P_c is stable. If $\Sigma(P, C)$ is stable, we say that C is a stabilizing controller for P.

The following result is well known, but for completeness we provide a short proof for it.

Lemma 1: Assume, that $P \in \mathrm{M}(\mathcal{G})$ has right [left] coprime factorization (N_P, D_P) $[(D_P, N_P)]$. $C \in \mathrm{M}(\mathcal{G})$ is a stabilizing controller if and only if C has such a left [right] coprime factorization (D_C, N_C) $[(N_C, D_C)]$, that $N_C N_P + D_C D_P = I$ $[N_P N_C + D_P D_C = I]$.

Proof: Sufficiency follows from [15, Lemma 3.1]. Necessity follows by showing, that choosing $N_C = D_P^{-1}(I + CP)^{-1}C$ and $D_C = ((I + CP)D_P)^{-1}$ gives the desired left-coprime factorization of C. The left analog is shown in a similar manner. ∎

The signals $y_r(s)$ we want to regulate are the analytic continuations of Laplace transforms of the reference signals in (3). Direct calculation gives

$$\begin{aligned} y_r(s) &= F_s(sI - S)^{-1}v_0 \\ &= \sum_{k\in\mathbb{Z}} \left(\sum_{l=1}^{n_k} \langle v_0, \phi_k^l \rangle \sum_{j=1}^{l} \frac{1}{(s - \mathrm{i}\omega_k)^{l+1-j}} F_s \phi_k^j \right). \end{aligned}$$

Here v_0 and F_s are arbitrary, so the references and disturbance signals are of form $y_r(s) = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} (s - \mathrm{i}\omega_k)^{-l} a_{kl}$ where (a_{kl}) is an absolutely summable sequence of vectors. The disturbance signals $d(s) = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} (s - \mathrm{i}\omega_k)^{-l} b_{kl}$ are generated in the same way.

As in state space we want to control the smoothness of the reference signals in some way. In state space, this was done by restricting the operators F and E or the initial state v_0. Here we just restrict the sequences (a_{kl}) and (b_{kl}). Let $f = (f_k)_{k\in\mathbb{Z}}$ be a bounded sequence of strictly positive real numbers. We define the set of appropriate reference and disturbance signals by setting

$$Y_r(f) = \{y_r(s) \mid \exists M > 0 : \|a_{kl}\| < M f_k\}$$

and

$$D(f) = \{d(s) \mid \exists M > 0 : \|b_{kl}\| < M f_k\}.$$

We say that C is *a regulating controller* for P if $\det(I + PC) \neq 0$ and $(I + PC)^{-1}y_r$ and $(I + PC)^{-1}Pd$ are stable for all $y_r \in Y_r(f)$ and $d \in D(f)$.

Let a plant P and a controller C be given. We say that C *robustly regulates* P if

1) C is a stabilizing and regulating controller for P.
2) If C is a stabilizing controller for a plant G, then C is also regulating.

The robust regulation problem usually needs some topology. The definition above is quite different. It is close to the one we have in state space, where we have two assumptions, the strong stability of the closed loop system and the existence of solution to the Sylvester equation, under which the regulation property should hold. In Chapter IV we will discuss the role of the existence of the solution in frequency domain terms.

Note that the above results are independent of the ring of stable transfer functions. We next define a new ring of stable transfer functions and all the stability considerations in frequency domain are done respect to this ring unless otherwise mentioned. In general it is very hard to describe strong stability in frequency domain. However, for certain C_0-semigroups, polynomial stability can be characterized in terms of the polynomial boundedness of the resolvent operator on the imaginary axis [16, Lemma 2.3]. These C_0-semigroup are uniformly bounded and the imaginary axis is in the resolvent of its generator. Our ring of stable transfer functions is based on this observation and is defined to be

$$\mathcal{P} = \{f \ \mid \ \forall \beta > 0 : f \in H_\beta^\infty \text{ and } \exists \alpha > 0 : f \in \mathcal{A}(\alpha)\}.$$

The set $\mathcal{A}(\alpha)$ is the set of complex functions that are analytic in some open set containing the closed right half plane $\overline{\mathbb{C}}_0$ and are α-polynomially bounded on the imaginary axis, i.e. $|f(i\omega)| < M(1 + |\omega|)^\alpha$ for some $M > 0$.

III. Blocking zero condition in state space

In this section we show that the conditional robustness of the controller can be characterized by the closed-loop system transfer function having blocking zeros of high enough orders at the frequencies $i\omega_k$ of the exosystem. In the condition presented in the following theorem the higher order terms are directly linked to the derivatives of $P_e(s)$ (which is holomorphic at $i\omega_k$ for $k \in \mathbb{Z}$) since for all $l \in \{2, \dots, n_k\}$

$$C_e R(i\omega_k, A_e)^l B_e = \frac{(-1)^{l-1}}{(l-1)!} \left[\frac{d^{l-1}}{ds^{l-1}} P_e(s) \right]_{s=i\omega_k}.$$

Theorem 1: Assume that S is a block diagonal operator as in (2) and that $\sigma(A_e) \cap \sigma(S) = \varnothing$. Then condition (5) is satisfied if and only if for all $k \in \mathbb{Z}$ and $j \in \{2, \dots, n_k\}$

$$P_e(i\omega_k) = 0, \tag{6a}$$
$$C_e R(i\omega_k, A_e)^j B_e = 0, \tag{6b}$$

i.e. the transfer function $P_e(s)$ has blocking zeros at $i\omega_k$ for all $k \in \mathbb{Z}$ and

$$P^{(j)}(i\omega_k) = 0, \qquad \forall k \in \mathbb{Z},\ j \in \{0, \dots, n_k - 1\},$$

where $P^{(j)}(s) = \frac{d^j}{ds^j} P(s)$.

Proof: We will first prove sufficiency of the blocking zero condition. Assume (6) is satisfied. If the Sylvester equation $\Sigma S = A_e \Sigma + B_e \begin{pmatrix} E \\ F \end{pmatrix}$ has a solution, then applying the both sides of the equation to ϕ_k^l we obtain for all $k \in \mathbb{Z}$ and $l \in \{1, \dots, n_k\}$

$$\Sigma \phi_k^l = \sum_{j=1}^{l} (-1)^{l-j} R(i\omega_k, A_e)^{l+1-j} B_e \begin{pmatrix} E \\ F \end{pmatrix} \phi_k^j.$$

We now use shorthand notation $\tilde{P}_e^{(l)} = C_e R(i\omega_k, A_e)^l B_e$. We now have for any $k \in \mathbb{Z}$ and $l \in \{1, \dots, n_k\}$

$$\begin{aligned} &\left[C_e \Sigma + D_e \begin{pmatrix} E \\ F \end{pmatrix} \right] \phi_k^l \\ &= P_e(i\omega_k) \begin{pmatrix} E \\ F \end{pmatrix} \phi_k^l + \sum_{j=1}^{l-1} (-1)^{l-j} \tilde{P}_e^{(l+1-j)} \begin{pmatrix} E \\ F \end{pmatrix} \phi_k^j = 0, \end{aligned}$$

since (6) are satisfied. Since $\left\{ \phi_k^l \;\middle|\; k \in \mathbb{Z},\ l = 1, \dots, n_k \right\}$ form a basis of W, this concludes that $C_e \Sigma + D_e = 0$ and thus (5) is satisfied.

To prove the necessity, assume the condition (5) is satisfied. Let $k \in \mathbb{Z}$ and let $\{x_E^j\}_{j=1}^{n_k} \subset X$ and $\{y_F^j\}_{j=1}^{n_k} \subset Y$ be arbitrary. Choose the operators $E \in \mathcal{L}(W, X)$ and $F \in \mathcal{L}(W, Y)$ as

$$E = \sum_{j=1}^{n_k} \langle \cdot, \phi_k^j \rangle x_E^j, \qquad \text{and} \qquad F = \sum_{j=1}^{n_k} \langle \cdot, \phi_k^j \rangle y_F^j.$$

We then have from [1, Lem. 3.2] that the Sylvester equation $\Sigma S = A_e \Sigma + B_e \begin{pmatrix} E \\ F \end{pmatrix}$ has a solution $\Sigma \in \mathcal{L}(W, X_e)$, and for all $k \in \mathbb{Z}$

$$\Sigma \phi_k^{n_k} = \sum_{j=1}^{n_k} (-1)^{n_k - j} R(i\omega_k, A_e)^{n_k + 1 - j} B_e \begin{pmatrix} E \\ F \end{pmatrix} \phi_k^j.$$

Again denote $\tilde{P}_e^{(l)} = C_e R(i\omega_k, A_e)^l B_e$. The condition (5) now in particular implies that

$$\begin{aligned} 0 &= \left[C_e \Sigma + D_e \begin{pmatrix} E \\ F \end{pmatrix} \right] \phi_k^{n_k} \\ &= P_e(i\omega_k) \begin{pmatrix} E \\ F \end{pmatrix} \phi_k^{n_k} + \sum_{j=1}^{n_k - 1} (-1)^{n_k - j} \tilde{P}_e^{(n_k + 1 - j)} \begin{pmatrix} E \\ F \end{pmatrix} \phi_k^j \\ &= P_e(i\omega_k) \begin{pmatrix} x_E^{n_k} \\ y_F^{n_k} \end{pmatrix} + \sum_{j=1}^{n_k - 1} (-1)^{n_k - j} \tilde{P}_e^{(n_k + 1 - j)} \begin{pmatrix} x_E^j \\ y_F^j \end{pmatrix} \\ &= P_e(i\omega_k) \begin{pmatrix} x_E^{n_k} \\ y_F^{n_k} \end{pmatrix} + \sum_{l=2}^{n_k} (-1)^{l-1} \tilde{P}_e^{(l)} \begin{pmatrix} x_E^{n_k + 1 - l} \\ y_F^{n_k + 1 - l} \end{pmatrix} \end{aligned}$$

Since for all $j \in \{1, \dots, n_k\}$ the elements $x_E^j \in X$ and $y_F^j \in Y$ were arbitrary, we must have $P_e(i\omega_k)$ and $C_e R(i\omega_k, A_e)^j B_e = 0$ for all $j \in \{2, \dots, n_k\}$. Since $k \in \mathbb{Z}$ was arbitrary, this concludes that the blocking zero condition (6) is satisfied. ∎

For a diagonal exosystem the condition (6) simplifies to the following form.

Corollary 1: Assume $S = \operatorname{diag}(i\omega_k)_{k \in \mathbb{Z}}$ and $\sigma(A_e) \cap \sigma(S) = \varnothing$. Then the condition (5) is satisfied if and only if

$$P_e(i\omega_k) = 0, \qquad \forall k \in \mathbb{Z},$$

i.e. the transfer function $P_e(s)$ has blocking zeros at $i\omega_k$ for all $k \in \mathbb{Z}$.

Writing out Taylor series of $P_e(s)$ at $i\omega_k$ for arbitrary $k \in \mathbb{Z}$ and using Theorem 1, we find the following result.

Corollary 2: Assume that S is a block diagonal operator as in (2) and that $\sigma(A_e) \cap \sigma(S) = \varnothing$. The condition (5) is satisfied if and only if $P_e(s)$ has blocking zeros of order at least n_k at $i\omega_k$ for all $k \in \mathbb{Z}$.

IV. Robust Regulation in Frequency Domain

In the next example we show why H^∞ is not a good choice for the ring of stable transfer functions. There we see, that if the reference signals have an infinite number of poles approaching infinity, then the there cannot exist a stabilizing and regulating controller if the plant transfer function approaches zero at infinity.

Example 1: Set $\mathcal{S} = H^\infty$ and let $P \in \mathcal{C}$. Let C be a stabilizing and regulating controller. By Lemma 1 there exists coprime factorizations of the plant and the controller, such that $N_P N_C + D_P D_C = I$. Simple calculation reveals, that $(I + PC)^{-1} = D_C D_P$. Choose $k_0 \in \mathbb{Z}$ and set $a_{kl} = a$ if $k = k_0$ and $l = 1$ and $a_{kl} = 0$ otherwise. Since C is a regulating controller for $\Sigma(P, C)$ this means, that $(s - \mathrm{i}\omega_{k_0})^{-1}(I + P(s)C(s))^{-1} a$ is uniformly bounded in $\mathbb{C}_0$ for an arbitrary vector a. We necessarily have, that $D_C(s) D_P(s) \to 0$ and $N_P(s) N_C(s) \to I$ as $s \to \mathrm{i}\omega_{k_0}$. Thus there exists an s_0 near $\mathrm{i}\omega_{k_0}$ such, that $N_P(s_0)$ is right invertible, $\|N_P(s_0)\| < 2\|N_C(s)\|_\infty$ and $D_P(s_0)$ is invertible. Since $P(s) = N_P(s) D_P^{-1}(s)$ and $D_P(s) \in \mathrm{M}(H^\infty)$ we have shown that there exists a sequence $(s_k)_{k \in \mathbb{Z}}$ in $\mathbb{C}_0$ such, that

$$\sup_{k\in\mathbb{Z}}\sup_{\omega\in\mathbb{R}}\{f_k|\mathrm{i}\omega+1|^{-\alpha}|\omega-\omega_k|^{-n_k}\cdot\|\left[(I+P(\mathrm{i}\omega)C(\mathrm{i}\omega))^{-1}\quad -(I+P(\mathrm{i}\omega)C(\mathrm{i}\omega))^{-1}P(\mathrm{i}\omega)\right]\|\}<\infty. \tag{1}$$

$P(s_k)$ have uniformly bounded right inverses and $|s_k|\to\infty$ as $k\to\pm\infty$. This shows, for example that $P(s)$ cannot be any strictly proper rational matrix.

For the rest of this section we set $\mathcal{S}=\mathcal{P}$. The next lemma gives a sufficient condition for robust regulation. It states, that if the controller includes the dynamics generating the reference signals, then it is a robustly regulating controller. This actually is the internal model principle in frequency domain stated in [13] for stable plants. Here we do not have necessity, because it requires additional assumptions.

Lemma 2: Assume, that there exists $\theta\in\mathcal{S}$ such, that $y_r\in Y_r(f)$ and $d\in D(f)$ can be written in form $y_r=\theta^{-1}y_0$ and $d=\theta^{-1}d_0$ for some stable y_0 and d_0. Let $P\in\mathcal{C}$. If the controller C is such, that it stabilizes $\Sigma(P,C)$ and has a such right [left] coprime factorization (D_C,N_C) [(D_C,N_C)], that $\theta^{-1}D_C\in\mathrm{M}(\mathcal{S})$, then C is robustly regulating.

Proof: By Lemma 1 there exists a left coprime factorization (D_P,N_P) of P and right coprime factorization $(\tilde{D}_C,\tilde{N}_C)$ of C such, that $N_P\tilde{N}_C+D_P\tilde{D}_C=I$. Since $\tilde{D}_C=UD_C$ for some $U\in\mathrm{M}(\mathcal{S})$ [15, p. 881]. Thus $\theta^{-1}\tilde{D}_C$ is stable. Furthermore $\theta^{-1}(I+PC)^{-1}=\theta^{-1}\tilde{D}_CD_P$ and $\theta^{-1}(I+PC)^{-1}P=\theta^{-1}\tilde{D}_CN_P$ are stable. ■

Theorem 2: Assume, that there exists $\theta\in\mathcal{S}$ such, that $y_r\in Y_r(f)$ and $d\in D(f)$ can be written in form $y_r=\theta^{-1}y_0$ and $d=\theta^{-1}d_0$ for some stable y_0 and d_0. Let $P\in\mathcal{C}$. Let a plant $P\in\mathcal{G}$ with left coprime factorization $(D_P(s),N_P(s))$ be given. If there exists $J\in M(\mathcal{P})$ such, that for all $\beta>0$ there exists $u_\beta>0$, such that

$$\|(\theta(s)D_P(s)+N_P(s)J(s))x\|\geq u_\beta\|x\|, \tag{7a}$$

for all $s\in\mathbb{C}_\beta^+$ and $x\in\mathbb{C}^n$ and there exists $u_0>0$ and $\alpha>0$ such, that for all $\omega\in\mathbb{R}$ and $x\in\mathbb{C}^n$

$$\|(\theta(i\omega)D_P(i\omega)+N_P(i\omega)C(i\omega))x\|\geq\frac{u_0\|x\|}{|1+i\omega|^\alpha}, \tag{7b}$$

then the controller $C=\theta^{-1}J$ regulates robustly.

Proof: Set $G:=\alpha D_P+N_PJ$. It is easy to show, that G is invertible and analytic in some open set containing $\mathbb{C}_0$. By (7) $G^{-1}\in\mathcal{S}$. Now $N_P(\theta G^{-1})+D_PJ(G^{-1})=I$ and the claim follows by Lemma 1 and Lemma 2. ■

We next give an example where a robustly regulating controller is given for a polynomially decaying plant. Then we show, that the transfer function cannot actually vanish faster than polynomially at $i\omega_k$ as $k\to\pm\infty$.

Example 2: Consider the plant $P(s)=\frac{1}{s+1}$ and assume, that $\omega_k=2\pi k$ and $n_k=1$, for all $k\in\mathbb{Z}$. All the reference and disturbance signals are of form $\theta^{-1}y_0$ and $\theta^{-1}d_0$, where $\theta(s)=1-\mathrm{e}^{-s}$ and y_0 and d_0 are stable functions. Since $P(s)$ is $\mathcal{P}$-stable it has a left coprime factorization $(1,P)$. Define $g(s):=\theta(s)+P(s)$. We find the real part of $g(s)$ to be

$$\mathrm{Re}\,(g(a+bi))=1-\mathrm{e}^{-a}\cos(b)+\frac{a+1}{(a+1)^2+b^2}.$$

If $a>\beta>0$, then

$$|\mathrm{Re}\,(g(a+bi))\,|\geq 1-\mathrm{e}^{-\beta}>0$$

and if $a=0$, then

$$|\mathrm{Re}\,(g(a+bi))\,|\geq\frac{1}{1+b^2}.$$

The above equations show, that conditions (7) hold for $J=1$, so by Theorem 2 $C=\frac{1}{\theta}$ is a robust $\mathcal{P}$-regulator.

Theorem 3: Let $P\in\mathcal{C}$ have a left coprime factorization (D_P,N_P) and let C be a stabilizing and regulating controller with right coprime factorization (N_C,D_C). The numerator $N_P(i\omega_k)$ is right invertible and right inverses $N_P^r(\mathrm{i}\omega_k)$ can be chosen so that $\|N_P^r(\mathrm{i}\omega_k)\|<M|\mathrm{i}\omega_k+1|^\alpha$ for some $M>0$ and $\alpha>0$.

Proof: Fix $k\in\mathbb{Z}$. Above we have shown, that $D_C(s)D_P(s)\to 0$ as $s\to\mathrm{i}\omega_k$. Since without losing generality we can assume, that $N_PN_C+D_PD_C=I$, we have that $N_P(\mathrm{i}\omega_k)N_C(\mathrm{i}\omega_k)=I$. Thus $N_C(\mathrm{i}\omega_k)$ defines desired right inverses, since $N_C\in\mathrm{M}(\mathcal{P})$. ■

In the state space the blocking zero condition is necessary and sufficient for robust regulation. The necessity follows since we are allowed to choose operators F and E freely. The proof has an analog in frequency domain. By similar arguments as above we show, that $(s-\mathrm{i}\omega_k)^{-n_k}(I+P(s)C(s))^{-1}\in\mathrm{M}(\mathcal{P})$. This means that $(s-\mathrm{i}\omega_k)^{-n_k}(I+P(s)C(s))^{-1}$ is analytic at $\mathrm{i}\omega_k$. Thus, $(I+P(s)C(s))^{-1}$ has a blocking zero of order n_k or higher at $\mathrm{i}\omega_k$. Similar arguments hold for $(I+P(s)C(s))^{-1}P(s)$, so we have the following theorem.

Theorem 4: If C is a regulating controller for $\Sigma(P,C)$, then $\left[(I+PC)^{-1}\quad -(I+PC)^{-1}P\right]$ has a blocking zero of order n_k or higher at $\mathrm{i}\omega_k$ for all $k\in\mathbb{Z}$.

Above we have seen, that if $P\in\mathcal{C}$ have a left coprime factorization (D_P,N_P) and C is stabilizing, then we can find a right coprime factorization (N_C,D_C) of C such, that $(I+PC)^{-1}P=D_CN_P$. By Theorem 3 $(I+PC)^{-1}P$ has a blocking zero of order at least n_k if and only if D_C has a blocking zero of order n_k or greater. Combining this with Theorem 4 gives the following corollary

Corollary 3: If C is a stabilizing controller of a plant $P\in\mathcal{C}$, then $\left[(I+PC)^{-1}\quad -(I+PC)^{-1}P\right]$ has a blocking zero of order at least n_k at $\mathrm{i}\omega_k$ for all $k\in\mathbb{Z}$ if and only if C has a full pole of order at least n_k at $\mathrm{i}\omega_k$ for all $k\in\mathbb{Z}$.

The next assumption relates (a_{kl}) and (b_{kl}) and the behavior of the closed loop system near the poles of the reference and disturbance signals.

Assumption 1: There exists $\alpha>0$ such that the equation (1) at the top of the page holds.

When proving the sufficiency part of the blocking zero condition in sate space we used a solution to the Sylvester equation. This relates to the behavior of sequences (a_{kl}) and (b_{kl}) as $k \to \pm\infty$ in frequency domain. Next theorem shows, that if the plant is perturbed so that the assumption above holds, then regulation follows from stability if there exists a uniform gap between the poles of reference signals.

Theorem 5: Let a controller C and a plant P be given. Assume, that $|\mathrm{i}\omega_k - \mathrm{i}\omega_l| > \epsilon_0 > 0$ whenever $k \neq l$. If Assumption 1 holds and $\Sigma(P, C)$ is stable then C is regulating.

Proof: Let P satisfy Assumption 1. We show, that $(I + PC)^{-1}y_r$ is stable for all y_r. That $(I + PC)^{-1}Pd$ is stable for all d is shown similarly.

Choose $0 < \epsilon < \min\left\{\frac{\epsilon_0}{2}, 1\right\}$ and denote the set s of complex numbers for which $|s - \mathrm{i}\omega_k| > \epsilon > 0$ for all $k \in \mathbb{Z}$ by C_ϵ. Define $g(s) := \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} (s - \mathrm{i}\omega_k)^{-n_k-1+l}(I + P(s)C(s))^{-1}a_{kl}$. By stability there exists $\alpha_1 > 0$ and $M_1 > 0$ such, that $\|(I + P(\mathrm{i}\omega)C(\mathrm{i}\omega))^{-1}\| < M_1|1 + \mathrm{i}\omega|^{\alpha_1}$ for all $\omega \in \mathbb{R}$.

There exists $M_2 > 0$ such, that $|s - \mathrm{i}\omega_k|^{-n_k-1+l} < M_2$ for all $k \in \mathbb{Z}$, $l = 1, \ldots, n_k$ and $s \in C_\epsilon$. For $s \in C_\epsilon$

$$\begin{aligned}\|g(s)\| &\le \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} M_2 \|(I + P(s)C(s))^{-1}\| \|a_{kl}\| \\ &= M_2 \|(I + P(s)C(s))^{-1}\| \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|a_{kl}\|.\end{aligned}$$

This is uniformly bounded in $\mathbb{C}_\beta \cap C_\epsilon$ for all $\beta > 0$ and polynomially bounded on $\mathrm{i}\mathbb{R} \cap C_e$, because $(I + P(s)C(s))^{-1}$ is in $\mathrm{M}(\mathcal{P})$. Since ϵ can be chosen arbitrarily small, $g(s)$ is uniformly bounded in any right half plane $\mathbb{C}_\beta$ for $\beta > 0$ and we need to show polynomial boundedness on $\mathrm{i}\mathbb{R}$.

Assume now, that $|\omega - \omega_{k_0}| < \epsilon$ for some fixed $k_0 \in \mathbb{Z}$. Now

$$\begin{aligned}\|g(s)\| \le &\sum_{l=1}^{n_{k_0}} \|(s - \mathrm{i}\omega_{k_0})^{-n_{k_0}-1+l}(I + P(s)C(s))^{-1}a_{k_0 l}\| \\ &+ M_k \|(I + P(s)C(s))^{-1}\| \sum_{k\neq k_0 \mathbb{Z}} \sum_{l=1}^{n_k} \|a_{kl}\|.\end{aligned}$$

The arguments above show, that a upper limit for the second term of the sum is $M_3|\mathrm{i}\omega + 1|^{\alpha_1}$, where $M_3 = M_1 M_2 \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|a_{kl}\|$. The first term is bounded above by $n_{k_0} f_{k_0} |\omega - \omega_k|^{-n_{k_0}} \|(I + P(\mathrm{i}\omega)C(\mathrm{i}\omega))^{-1}\| \le M_4|\mathrm{i}\omega + 1|^{\alpha_2}$, where constants M_4 and α_2 independent of the choice of k_0 exists by Assumption 1. Thus, $\|g(\mathrm{i}\omega)\| \le M_0|\mathrm{i}\omega + 1|^{\alpha_0}$ for all $\omega \in \mathbb{R}$, where $M_0 = M_3 + M_4$ and $\alpha_0 = \max\{\alpha_1, \alpha_2\}$. ■

V. CONLUSIONS

We have considered robust regulation problem in time and frequency domains. We have first proved that in the state space a necessary and sufficient condition for a controller to be robustly regulating is given as the blocking zero condition. Motivated by the state space theory we have presented a new ring of stable transfer functions $\mathcal{P}$, which contains transfer functions that are uniformly bounded in every right half plane $\mathbb{C}_\beta$ with $\beta > 0$ and polynomially bounded on the imaginary axis. By choosing $\mathcal{P}$ to be the ring of stable transfer functions we were able to remove the restriction that the plant cannot vanish at infinity. Finally we have discussed the blocking zero condition in frequency domain.

Future research includes elaborating the restrictions the Sylvester equation imposes on the reference signals in the frequency domain. In this article we did not use any topologies, however we would like to find a topology for $\mathcal{P}$ to define the robust regulation problem in more precise terms. A natural starting point would be the graph topology related to robust stabilization [15]. One aim is to parametrize a robustly regulating controller for a given plant.

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