A Simple Controller with a Reduced Order Internal Model in the Frequency Domain

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***Abstract*— We use frequency domain methods to study robust output regulation of a stable plant in a situation where the controller is only required to be robust with respect to a predefined class of perturbations. We present a characterization for the solvability of the control problem and design a minimal order controller that achieves robustness with respect to a given class of uncertainties. The construction of the controller is illustrated with an example.**

I. Introduction

In this paper we study the robust output regulation problem for a stable system

$$P(\cdot) \in H_\infty(\mathbb{C}_+;\mathbb{C}^{p\times m}).$$

In particular, our aim is to design an error feedback controller in such a way that the output $y(t)$ of the plant converges asymptotically to a given reference signal

$$y_{\mathit{ref}}(t) = \sum_{k=1}^{q} a_k e^{i\omega_k t}, \qquad a_k \in \mathbb{C}^p \setminus \{0\}, \tag{1}$$

and the control law tolerates small perturbations in the transfer function $P(\cdot)$ of the plant. Any reference signal that is a sum of sinusoids or a finite dimensional approximation of a general periodic signal, e.g. of a sawtooth function, may be presented in the form (1).

The well-known internal model principle [1], [2] states that the robust output regulation problem is solved by any controller that contains p copies of the every frequency $i\omega_k$ of the reference signal and for which the closed-loop system is stable. In particular, it has been shown in [3], [6], [11] that for a stable system the robust output regulation problem can be solved with an error feedback controller of the form

$$C(s) = \epsilon \sum_{k=1}^{q} \frac{C_k}{s - i\omega_k}, \tag{2}$$

where $C_k \in \mathbb{C}^{m\times p}$ are chosen in such a way that the eigenvalues of $P(i\omega_k)C_k \in \mathbb{C}^{p\times p}$ have negative real parts. With such choices of parameters there exists $\epsilon^* > 0$ such that for any $0 < \epsilon \le \epsilon^*$ solves the robust output regulation problem [3]. The internal model principle is visible in the controller (2) in the property that the matrices C_k have full ranks $\operatorname{rank} C_k = p$.

In this paper we concentrate on a situation where the controller is not required to be robust with respect to arbitrary small perturbations, but instead it is only required to tolerate uncertainties from a predefined class of perturbations. Such a situation may occur in the case of parameter uncertainties or component failures [4], [9]. Recently, in [9], [10] it has been shown that robust output regulation may be achievable with a controller with strictly less than p copies of the frequencies $i\omega_k$ of the exosystem for a predefined class of perturbations. This result leads to design of controllers with so-called *reduced order internal models* [7], [8]. To this date, robust output regulation with restricted classes of perturbations has only been studied using state space techniques. The purpose of this paper is to study the control problem and to develop methods for construction of controllers in the frequency domain.

As our first main result we present a characterization for the solvability of the robust output regulation problem for a given class $\mathcal{O}$ of perturbations. In particular, we show that if the controller (2) stabilizes the closed-loop system, then it solves the control problem if and only if

$$a_k \in \mathcal{R}(\tilde{P}(i\omega_k)C_k), \qquad \forall k \in \{1,\dots,q\} \tag{3}$$

for all perturbed plants $\tilde{P}(\cdot) \in \mathcal{O}$. Similarly as in [9], this type of characterization immediately implies that robustness with respect to certain classes $\mathcal{O}$ of perturbations can be achieved without a full internal model in the controller. Our result further shows that in the frequency domain the concept of a reduced order internal model means that some of the matrices $C_k \in \mathbb{C}^{m\times p}$ in the controller (2) satisfy $\operatorname{rank} C_k < p$.

We show that the condition (3) implies a lower bound for the ranks of the matrices C_k in the controller. In particular, if $p = m$ and $\tilde{P}(\cdot) \in \mathcal{O}$ are invertible at $i\omega_k$ and if the controller (2) is robust with respect to the class $\mathcal{O}$ of perturbations, then the lower bounds for the ranks of C_k are given by

$$\operatorname{rank} C_k \ge p_k := \dim \operatorname{span}\{\, \tilde{P}(i\omega_k)^{-1}a_k \mid \tilde{P}(\cdot) \in \mathcal{O}\,\}. \tag{4}$$

In the second part of the paper we construct a reduced order controller to solve the robust output regulation problem for a given class of admissible perturbations $\mathcal{O}$. The found controller shows that the lower bound in (4) is optimal, since our controller has ranks equal to p_k. The design procedure for the controller with a reduced order internal model is illustrated with an example in Section V.

II. The Robust Output Regulation Problem

In this section we introduce the notation used in this paper and state the robust output regulation problem.

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We denote the class of the function that bounded and analytic in the right half plane $\mathbb{C}_+ := \{\, s \in \mathbb{C} \mid \mathrm{Re}(s) > 0 \,\}$ by H_∞. The set of all matrices of arbitrary size and of all $n \times m$-matrices over a set S are denoted by $\mathcal{M}(S)$ and $\mathcal{M}^{n\times m}(S)$, respectively. We denote the rank, the range, the kernel, and the Moore-Penrose pseudoinverse of a matrix $M \in \mathcal{M}(\mathbb{C})$ by $\mathrm{rank}(M)$, $\mathcal{R}(M)$, $\mathcal{N}(M)$, and M^+, respectively.

A. Robust Output Regulation for a Class $\mathcal{O}$ of Perturbations

We assume that the class $\mathcal{O}$ of perturbations has the following properties.

- The nominal plant belongs to the class $\mathcal{O}$, i.e., $P(\cdot) \in \mathcal{O}$.
- Every $\tilde{P}(\cdot) \in \mathcal{O}$ is analytic at the points $\{i\omega_k\}_{k=1}^q$.

In this paper we assume that the error feedback controller is of the form,

$$C(s) = \sum_{k=1}^{q} \frac{C_k}{s - i\omega_k} + C_0(s) \tag{5}$$

where $C_k \in \mathbb{C}^{m\times p}$ and where $C_0(\cdot)$ is analytic at $i\omega_k$ for all $k \in \{1, \dots, q\}$. This in particular means that the poles of the controller locate at the frequencies $i\omega_k$ the reference signal (1) and that their order is at most one. The plant and controller form the closed loop depicted in Fig. (1). There $\hat{d}$ is an external disturbance. The closed loop transfer function from $(\hat{y}_{ref}, \hat{d})$ to $(\hat{e}, \hat{u})$ is

$$H(P,C) = \begin{bmatrix} (I-PC)^{-1} & (I-PC)^{-1}P \\ C(I-PC)^{-1} & I + C(I-PC)^{-1}P \end{bmatrix}.$$

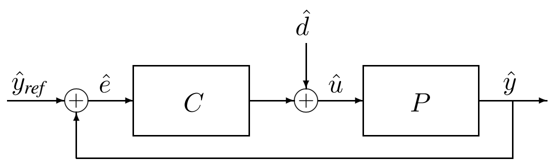

Fig. 1. The closed loop.

The Robust Output Regulation Problem. Given a class $\mathcal{O}$ of admissible perturbations, choose the parameters $C_0(\cdot)$ and $C_1, \dots, C_q$ of the controller (5) in such a way that

(a) The controller $C(\cdot)$ stabilizes the plant $P(\cdot)$, i.e. $H(P,C) \in \mathcal{M}(H_\infty)$.

(b) If $\tilde{P}(\cdot) \in \mathcal{O}$ is such that $C(\cdot)$ stabilizes $\tilde{P}(\cdot)$, then

$$(I - P(\cdot)C(\cdot))^{-1}\hat{y}_{ref}(\cdot) \in \mathcal{M}(H_\infty) \tag{6}$$

If condition (6) is satisfied, we say that $C(\cdot)$ *regulates* $\tilde{P}(\cdot) \in \mathcal{O}$.

III. Characterization of Robustness for a Class of Perturbations

In this section we present a characterization for controllers that are robust with respect to a given class $\mathcal{O}$ of perturbations. The following theorem is the main result of this section.

Theorem 3.1: Assume the controller is of the form (5).

- If $\tilde{P}(\cdot) \in \mathcal{O}$ is such that $C(\cdot)$ stabilizes $\tilde{P}(\cdot)$, then $C(\cdot)$ regulates $\tilde{P}(\cdot)$ if and only if
$$a_k \in \mathcal{R}(\tilde{P}(i\omega_k)C_k) \tag{7}$$
for all $k \in \{1, \dots, q\}$.
- If $C(\cdot)$ stabilizes $P(\cdot)$, then it solves the robust output regulation problem for the class $\mathcal{O}$ of perturbations if and only if (7) is satisfied for all $\tilde{P}(\cdot) \in \mathcal{O}$ that are stabilzed by the controller $C(\cdot)$.

The proof of Theorem 3.1 is based on the following two lemmata.

Lemma 3.2: Assume that the controller $C(\cdot)$ stabilizes the plant $\tilde{P}(\cdot)$. Then $C(\cdot)$ also regulates $\tilde{P}(\cdot)$ if and only if

$$\lim_{s\to i\omega_k} (I - \tilde{P}(s)C(s))^{-1} a_k = 0. \tag{8}$$

Proof: Assume $\tilde{P}(\cdot) \in \mathcal{O}$ is stabilized by $C(\cdot)$. Suppose first that $C(\cdot)$ regulates $\tilde{P}(\cdot)$. For all $s \in \mathbb{C} \setminus \{i\omega_k\}_{k=1}^q$ we have

$$(I - \tilde{P}(s)C(s))^{-1}\hat{y}_{ref}(s) = \sum_{k=1}^{q} \frac{(I - \tilde{P}(s)C(s))^{-1}a_k}{s - i\omega_k}.$$

From this it is clear that the condition (6) can only be satisfied if (8) holds.

On the other hand, since the only poles of the function $\hat{y}_{ref}(\cdot)$ are $\{i\omega_k\}_{k=1}^q$, and since $(I - \tilde{P}(\cdot)C(\cdot))^{-1} \in \mathcal{M}(H_\infty)$, we have that the condition (8) implies (6), and thus $C(\cdot)$ regulates $\tilde{P}(\cdot)$. ■

Lemma 3.3: If H is a $n \times n$-matrix and $F(s)$ is an analytic $n \times n$-matrix valued function approaching zero as $s \to 0$. If $s = 0$ is a pole of order one of $(H + F(s))^{-1}$, then $(H + F(s))^{-1}x$ is bounded near 0 if and only if $x \in \mathcal{R}(H)$.

Proof: Denote

$$h(s) := (H + F(s))^{-1}x$$

and assume $h(\cdot)$ is bounded near 0. Due to our assumptions, this is equivalent to $\lim_{s\to 0} h(s) = z$ for some $z \in \mathbb{C}^n$. Now $x = (H + F(s))h(s) = Hh(s) + F(s)h(s) \to Hz$ as $s \to 0$, which shows that $x \in \mathcal{R}(H)$.

Assume now that $x \in \mathcal{R}(H)$. Due to our assumptions, the Laurent series of $(H + F(s))^{-1}$ at $s = 0$ is of the form

$$(H + F(s))^{-1} = s^{-1}X_0 + X_1 + sX_2 + \cdots$$

The identity $(H + F(s))^{-1}(H + F(s)) = I$ implies $X_0 H = 0$, which further shows that $(H + F(s))^{-1}x$ is bounded near $s = 0$. ■

Proof of Theorem 3.1. The second part of the Theorem 3.1 is a direct consequence of the first part and the statement of the robust output regulation problem. To prove the first part, let $\tilde{P}(\cdot) \in \mathcal{O}$ be such that $C(\cdot)$ stabilizes $\tilde{P}(\cdot)$ and let $k \in \{1, \dots, q\}$ be arbitrary. Since $\tilde{P}(\cdot)$ is analytic at $i\omega_k$, we can write

$$\tilde{P}(s) = \tilde{P}(i\omega_k) + H_0(s)$$

where $H_0(s)$ is an analytic function such that $H_0(s) \to 0$ as $s \to i\omega_k$. Define

$$C_{0k}(s) := C_0(s) + \sum_{\substack{l\neq k \\ 1\leq l\leq q}} \frac{C_l}{s - i\omega_k}.$$

Using the structure (5) of $C(\cdot)$ we can write

$$\begin{aligned}&(I-\tilde{P}(s)C(s))^{-1}\\&\quad=(I-(s-i\omega_k)^{-1}(\tilde{P}(i\omega_k)+H_0(s))\\&\qquad\times(C_k+(s-i\omega_k)C_{0k}(s)))^{-1}\\&\quad=(i\omega_k-s)(\tilde{P}(i\omega_k)C_k+F(s))^{-1}\end{aligned}$$

where $F(s)$ is an analytic function approaching zero as $s\to i\omega_k$. Since $C(\cdot)$ stabilizes $\tilde{P}(\cdot)$ we have $(I-\tilde{P}(s)C(s))^{-1}\in\mathcal{M}(H_\infty)$. Because of this, the above equation implies that $(\tilde{P}(i\omega_k)C_k+F(s))^{-1}$ can have a pole of order at most one at $i\omega_k$. Since

$$(I-\tilde{P}(s)C(s))^{-1}a_k=(i\omega_k-s)(\tilde{P}(i\omega_k)C_k+F(s))^{-1}a_k,$$

we have that $\lim_{s\to i\omega_k}(I-\tilde{P}(s)C(s))^{-1}a_k=0$ if and only if $(\tilde{P}(i\omega_k)C_k+F(s))^{-1}a_k$ is bounded near $s=i\omega_k$. By Lemma 3.3 this is in turn equivalent to $a_k\in\mathcal{R}(\tilde{P}(i\omega_k)C_k)$. Since $k\in\{1,\dots,q\}$ was arbitrary, we have that (8) is equivalent to (7). This completes the proof. ∎

If the controller $C(\cdot)$ solves the robust output regulation problem for a class $\mathcal{O}$ of perturbations, then Theorem 3.1 gives us the following lower bounds for the ranks of the matrices C_k.

Theorem 3.4: Let σ be the minimum number of elements over the sets $\mathcal{K}$ of linearly independent vectors such that $\tilde{P}^{-1}(i\omega_k)a_k\cap\operatorname{span}\mathcal{K}\neq\emptyset$ – here $\tilde{P}^{-1}(i\omega_k)a_k$ is the preimage of a_k – for all $\tilde{P}(\cdot)\in\mathcal{O}$. If $C(\cdot)$ of the form (5) solves the robust output regulation problem for a class $\mathcal{O}$ of perturbations, then the rank of C_k is greater than or equal to σ.

Proof: Let σ_0 be the rank of C_k, and let $x_1,\dots,x_{\sigma_0}$ be linearly independent columns of C_k. We set $\mathcal{K}'=\{x_1,\dots,x_{\sigma_0}\}$. Since $C(\cdot)$ is robustly regulating, Theorem 3.1 implies that for all $\tilde{P}(\cdot)\in\mathcal{O}$ there exists a vector h such that

$$a_k=\tilde{P}(i\omega_k)C_kh=\tilde{P}(i\omega_k)\sum_{j=1}^{\sigma_0}\alpha_jx_j.$$

Thus, $\tilde{P}^{-1}(i\omega_k)a_k\cap\operatorname{span}\mathcal{K}'\neq\emptyset$. By the assumption $\sigma_0\geq\sigma$. ∎

Corollary 3.5: If $C(\cdot)$ of (5) robustly regulates and $\tilde{P}(i\omega_k)$ is invertible for all $\tilde{P}(\cdot)\in\mathcal{O}$, then the rank of C_k is greater than or equal to

$$p_k:=\dim\left(\operatorname{span}\{\,\tilde{P}^{-1}(i\omega_k)a_k\mid\tilde{P}(\cdot)\in\mathcal{O}\,\}\right).$$

Proof: The corollary follows by the previous theorem because $P^{-1}(i\omega_k)$ is a unique element. ∎

IV. Controller Design for Stable $P(\cdot)$

In this section we construct a controller of the form

$$C(s)=\epsilon\sum_{k=1}^{q}\frac{C_k}{s-i\omega_k}\tag{9}$$

to solve the robust output regulation problem for a class $\mathcal{O}$ of perturbations. We in particular present appropriate choices for the matrices C_k, and show that for all sufficiently small $\epsilon>0$ the closed loop system $H(P,C)$ is stable.

The simple controller (9) suffices since we have a stable plant. Choosing the matrices C_k appropriately guarantees minimal structure of the controller, which is our aim here. Additional structure is needed for example if the plant is unstable or there are some other design goals, e.g. optimization, which together with the minimal internal model leads to the controller (5).

A. Assumptions on the perturbation class and the nominal plant

Before proceeding we need to do some standing assumptions on the given nominal plant $P(\cdot)$ and the perturbation class $\mathcal{O}$.

Assumption 4.1: For all $k=1,\dots,q$, we denote

$$\mathcal{V}_k=\operatorname{span}\{\,\tilde{P}^{+}(i\omega_k)a_k\mid\tilde{P}\in\mathcal{O}\,\}$$

and $p_k=\dim(\mathcal{V}_k)$. We assume that

(i) $a_k\in\mathcal{R}(\tilde{P}(i\omega_k))$ for all $\tilde{P}\in\mathcal{O}$,
(ii) $P\in\mathcal{M}(H_\infty)$, and
(iii) $\dim(P(i\omega_k)\mathcal{V}_k)=p_k$

for all $k=1,\dots,q$.

It is an immediate consequence of Theorem 3.1 that the robust output regulation problem can only be solvable if condition (i) is satisfied. The second assumption is a simplifying condition and the third one is needed for stabilization of the closed loop.

In what follows, we construct a controller (9) with $\operatorname{rank}(C_k)=p_k$. In the light of Theorem 3.4 this is not optimal if $\sigma<p_k$. Thus, the last assumption above may be weakened in that case. On the other hand, if $\tilde{P}(i\omega_k)$ are invertible for all $\tilde{P}(\cdot)\in\mathcal{O}$, then by Corollary 3.5 condition (iii) is also necessary for robustness.

B. The design parameters C_k

We show how to construct a suitably reduplicated internal model into the controller. This is done by choosing the design parameters C_k appropriately.

Theorem 4.2: Define the design parameters C_k in the following way:

- Choose a basis $\{h_1,\dots,h_{p_k}\}$ of $\mathcal{V}_k$.
- Define $H_k:=[h_1,\dots,h_{p_k},0,\dots,0]$.
- Choose an invertible matrix D_k so that the eigenvalues of $P(i\omega_k)H_kD_k$ are zero or have negative real parts.
- Choose $C_k:=H_kD_k$.

If Assumption 4.1 holds, then $\operatorname{rank}(P(i\omega_k)C_k)=p_k$ and the regulation condition (7) holds for every $\tilde{P}\in\mathcal{O}$.

Proof: The rank of $P(i\omega_k)C_k$ is p_k by (iv) of Assumption 4.1 and the fact that $\mathcal{V}_k=\mathcal{R}C_k$. Thus, it remains to show that (7) holds for an arbitrary $\tilde{P}(\cdot)\in\mathcal{O}$.

By the choice of C_k, we see that for every $\tilde{P}(\cdot)\in\mathcal{O}$ there exists y such that

$$\tilde{P}^{+}(i\omega)a_k=C_ky.$$

Left multiplying by $\tilde{P}(i\omega)$, we get

$$\tilde{P}(i\omega)\tilde{P}^{+}(i\omega)a_k=\tilde{P}(i\omega)C_ky.$$

Since $a_k \in \mathcal{R}(\tilde{P}(i\omega_k))$ by (i) of Assumption 4.1 and $\tilde{P}(i\omega_k)\tilde{P}^+(i\omega_k)$ is an orthogonal projector on $\mathcal{R}(\tilde{P}(i\omega_k))$, the above equation shows that (7) holds. ■

The above theorem together with Theorem 3.1 show that the controller is robustly regulating if it stabilizes $P(\cdot)$. The fact that $P(i\omega_k)C_k$ has no positive eigenvalues was not needed to prove the regulation condition (7). We need it to prove the existence of an $\epsilon > 0$ implying stability.

C. Stability and the choice of ϵ

Theorem 4.3: If Assumption 4.1 holds and C_k are chosen as in Theorem 4.2, then there exists $\epsilon^* > 0$ such that $C(\cdot)$ of (9) stabilizes $P(\cdot)$ for every $\epsilon \in (0, \epsilon^*]$.

Lemma 4.4: If Assumption 4.1 holds and C_k are chosen as in Theorem 4.2, then $(I - \frac{\epsilon}{s-i\omega_k}P(i\omega_k)C_k)^{-1}$ is bounded in $\mathbb{C}_+$ by a bound independent of $\epsilon > 0$.

Proof: By Theorem 4.2, $\operatorname{rank}(P(i\omega_k)C_k) = \operatorname{rank}(C_k)$, so $\mathcal{N}(P(i\omega_k)C_k) = \mathcal{N}(C_k)$. Because of the structure of C_k, we know that the Jordan blocks of $P(i\omega_k)C_k$ related to the eigenvalue 0 are trivial. The non-zero eigenvalues of $P(i\omega_k)C_k$ have negative real parts. This means that there exist a matrix S and a negative-definite matrix M such that

$$P(i\omega_k)C_k = S\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} S^{-1},$$

so

$$\left(I - \frac{1}{z}P(i\omega_k)C_k\right)^{-1} = S\begin{bmatrix} z\,(zI-M)^{-1} & 0 \\ 0 & I \end{bmatrix} S^{-1}.$$

Since $-M$ is positive definite, $H(z) = z\,(zI-M)^{-1}$ is analytic in $\mathbb{C}_+$. In addition, it approaches I as $z \to \infty$. Thus, it is bounded in $\mathbb{C}_+$. We see that $(I - \frac{\epsilon}{s-i\omega_k}M)^{-1}$ is bounded in $\mathbb{C}_+$ by a bound independent of $\epsilon > 0$ since $\{\, \frac{1}{z} \mid 0 \neq z \in \mathbb{C}_+ \,\} = \{\, \frac{\epsilon}{s-i\omega_k} \mid i\omega_k \neq s \in \mathbb{C}_+ \,\}$. Thus, $(I - \frac{\epsilon}{s-i\omega_k}P(i\omega_k)C_k)^{-1}$ is bounded in $\mathbb{C}_+$ by a bound independent of $\epsilon > 0$. ■

Proof of Theorem 4.3. First we show the stability of $(I - P(\cdot)C(\cdot))^{-1}$. To this end, we choose

$$\gamma < \min\{\, |i\omega_k - i\omega_l| \mid 1 \le k < l \le q \,\}$$

and define the half discs $D_k := \mathbb{C}_+ \cap \{\, s \in \mathbb{C} \mid |s - i\omega_k| < \gamma \,\}$. Our aim is to show the existence of $\epsilon' > 0$ such that $(I - P(\cdot)C(\cdot))^{-1}$ is bounded in $\mathbb{C}_+ \setminus \bigcup_{k=1}^q D_k$ whenever $0 < \epsilon \le \epsilon'$, and of $\epsilon_k > 0$ such that $(I - P(\cdot)C(\cdot))^{-1}$ is bounded in D_k whenever $0 < \epsilon \le \epsilon_k$. Then $(I - P(\cdot)C(\cdot))^{-1}$ is stable for all $\epsilon \in (0, \epsilon^*]$ where $\epsilon^* = \min\{\epsilon', \epsilon_1 \ldots, \epsilon_q\}$.

By the stability $P(\cdot)$ and the definition of $C(\cdot)$, $P(\cdot)C(\cdot)$ is bounded in $\mathbb{C}_+ \setminus \bigcup_{k=1}^q D_k$. Thus, there exists small enough $\epsilon' > 0$ such that $(I - P(\cdot)C(\cdot))^{-1}$ is bounded in $\mathbb{C}_+ \setminus \bigcup_{k=1}^q D_k$ whenever $0 < \epsilon < \epsilon'$.

Next we show the existence of suitable $\epsilon_k > 0$. We decompose

$$(I - P(s)C(s))^{-1} = Q_{1k}(s)(I - \epsilon Q_{2k}(s)Q_{1k}(s))^{-1} \quad (10)$$

where

$$Q_{1k}(s) = \left(I - \frac{\epsilon P(i\omega_k)C_k}{s - i\omega_k}\right)^{-1}, \text{and}$$
$$Q_{2k}(s) = \frac{P(s) - P(i\omega_k)}{(s - i\omega_k)} - P(s)\sum_{\substack{l \neq k \\ 1 \le l \le q}} \frac{C_l}{s - i\omega_l}.$$

By Lemma 4.4, $Q_{1k}(s)$ is bounded in D_k by a bound independent of ϵ. In addition, $Q_{2k}(s)$ is bounded in D_k since $P(\cdot)$ and $\sum_{\substack{l \neq k \\ 1 \le l \le q}} \frac{C_l}{s-i\omega_l}$ are analytic in D_k. The decomposition (10) implies that we can choose $\epsilon_k > 0$ such that that $(I - P(s)C(s))^{-1}$ is bounded in D_k for all $\epsilon \in (0, \epsilon_k]$. This completes the proof of the stability of $(I - P(\cdot)C(\cdot))^{-1}$.

Since $P(\cdot)$ is stable, it remains to show that $C(\cdot)(I - P(\cdot)C(\cdot))^{-1}$ is stable. By the stability of $(I - P(\cdot)C(\cdot))^{-1}$ and the decomposition (10), we only need to show that

$$\begin{aligned} H(s) &:= \frac{s - i\omega_k}{\epsilon} C_k Q_{1k}(s) \\ &= C_k\left(\frac{s - i\omega_k}{\epsilon} I - P(i\omega_k)C_k\right)^{-1} \end{aligned}$$

is stable. By the above discussion $H(s)$ can only have poles of order one. Thus, it has the representation

$$H(s) = \frac{\epsilon}{s - i\omega_k} E + F_1(s), \quad (11)$$

where E is the projection to $\mathcal{N}(P(i\omega_k)C_k)$ along $\mathcal{R}(P(i\omega_k)C_k)$ and $F_1(s)$ is an analytic function [12]. As was mentioned in the proof of Lemma 4.4, $\mathcal{N}(P(i\omega_k)C_k) = \mathcal{N}(C_k)$, which implies that $C_k E = 0$. Thus, $H(s) = F_1(s)$ is analytic and the proof is completed. ■

V. EXAMPLE

Let the given stable nominal plant be

$$P(s) = \begin{bmatrix} -\frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & \frac{s}{s+1} \end{bmatrix},$$

and assume that the plant is subject to an upper triangular additive perturbation Δ. This leads to the perturbation class

$$\mathcal{O} = \{\, P(\cdot) + \Delta(\cdot) \mid \Delta \text{ is upper triangular} \,\}.$$

We want to find a controller solving the robust regulation problem for the reference signal

$$\begin{aligned} y_r &= \frac{\alpha}{s-i}\begin{bmatrix} \frac{-i}{2} \\ 0 \\ 0 \end{bmatrix} + \frac{\alpha}{s+i}\begin{bmatrix} \frac{i}{2} \\ 0 \\ 0 \end{bmatrix} + \frac{\beta}{s}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{\alpha}{s-i}a_1 + \frac{\alpha}{s+i}a_{-1} + \frac{\beta}{s}a_0, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. This is the Laplace transform of the time domain reference signal $(\alpha\sin(t), \beta, 0)$.

In order to find a robustly regulating controller, we define a basis of

$$\mathcal{V}_k = \operatorname{span}\{\, (P(i\omega_k) + \Delta(i\omega_k))^+ a_k \mid \Delta \text{ upper triangular} \,\}$$

for $\omega_k = k$ and $k = -1, 0, 1$.

Let e_l be the lth natural basis vector of $\mathbb{C}^3$. Since the only non zero element in the first column of $P(i\omega_k) + \Delta(i\omega_k)$ is the first one, it is easy to verify that

$$\mathcal{V}_{-1} = \mathcal{V}_1 = \operatorname{span}\{e_1\}.$$

Similarly, because of the upper triangular structure we have

$$\mathcal{V}_0 = \operatorname{span}\{e_1, e_2\}.$$

We choose $C_{-1} = \operatorname{diag}(1,0,0) = C_1$ and $C_0 = \operatorname{diag}(1,-1,0)$. The controller (9) satisfies the regulation property (7) according to Theorem 4.2.

It remains to choose a small enough scaling factor $\epsilon > 0$. By the proof of Theorem 4.3 it is sufficient to choose ϵ so that $(I - P(\cdot)C(\cdot))^{-1}$ remains bounded in $\mathbb{C}_+$. To this end, note that

$$(I - P(s)C(s))^{-1} = \operatorname{diag}\left(f_1(s), f_2(s), 1\right)$$

where

$$f_1(s) = \left(1 + \frac{\epsilon}{s+1}\left(\frac{1}{s+i} + \frac{1}{s-i} + \frac{1}{s}\right)\right)^{-1}$$

and

$$f_2(s) = \left(1 + \frac{\epsilon}{s+1}\frac{1}{s}\right)^{-1}.$$

Choose $\gamma = \frac{1}{2}$ and $\epsilon^* = \frac{1}{7}$. Using similar arguments as in the proof of Theorem 4.3 it is straightforward to verify that $f_1(s)$ and $f_2(s)$ are bounded in $\mathbb{C}_+$.

We note that the internal model is minimal in the sense that the ranks of the matrices C_k for $k = -1, 0, 1$ are minimal. The same structure of the controller is required for notably smaller perturbation classes, e.g. if

$$\Delta(s) = \delta \begin{bmatrix} 0 & \frac{s-1}{s+5} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\delta \in \mathbb{R}$. On the other hand, allowing perturbations in the lower triangular parts of Δ or, for example, adding a sine wave to the last element of the reference signal would force us to increase the size of the internal model because of Theorem 3.1.

The nominal plant $P(\cdot)$ has a transmission zero at 0 which is also a pole of the reference signal. It is well-known that in such a situation it is impossible to achieve robustness with respect to arbitrary perturbations, since if C_0 is of full rank and $P(0)$ is not, the closed-loop system cannot be stabilized [5]. In particular, it is not possible to allow perturbations in the last element of the second column. This situation illustrates that a controller with a reduced order internal model may provide us the desired robustness properties even if robustness with respect to general perturbations is not achievable.

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