Simultaneous Robust Regulation in the Frequency Domain*

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***Abstract*— This paper aims to find conditions that guarantee the existence of a controller that achieves asymptotic tracking of a given output behavior for two or more plants simultaneously, while being fault-tolerant to small perturbations in the given plants. The problem of finding such a controller is called the simultaneous robust regulation problem. This paper studies the problem with finite-dimensional systems and presents solvability conditions for it using a stable factorization approach based on coprime factorizations of the plant transfer functions. The theoretical results are illustrated with an example.**

I. INTRODUCTION

Robust control deals with control designs that maintain their functionality despite small changes in the controlled system. Such small changes are inevitable in mathematical modeling. In asymptotic tracking, one wants to find a controller for a given plant such that the measured output approaches the desired behavior over time. Finding a controller that achieves asymptotic tracking for all plants close to the given nominal plant is called the robust regulation problem.

Robustness guarantees tolerance to small changes in the plant, but not all changes are necessarily small. For example, a component breakdown can cause the properties of a system to change drastically. Finding a controller that works for systems with broken components in addition to the original system can be formulated as a problem where a single controller is to be found that robustly regulates a finite number of given systems. This problem is called the simultaneous robust regulation problem.

The main feature of a robustly regulating controller is that it should contain an internal model of the dynamics to be tracked [1]. Since the internal model is required even for one plant, it is clear that it is also necessary when solving the simultaneous robust regulation problem. On the other hand, if a controller with an internal model stabilizes a system, then asymptotic tracking is guaranteed. Thus, solving the simultaneous robust regulation problem involves two equally important parts: the construction of an internal model and the simultaneous stabilization of the given plants, both of which are well-studied problems. The robust regulation problem has been studied for various system classes, and there exist several characterizations of internal models and controller design paradigms, see [1]–[6] and the references therein. On the other hand, the simultaneous stabilization problem is a difficult one for three or more plants [7], [8]. Fortunately, relatively easy conditions characterizing solvability exist for two plants [9], [10]. Problems similar to the simultaneous robust regulation problem, where a controller should achieve some design goals in addition to simultaneous stabilization, have been studied in the literature [11], [12]. In particular, simultaneous stabilization with step-input reference tracking was considered in [13], which is a special case of the robust regulation problem.

This article studies the simultaneous robust regulation problem for linear finite-dimensional time-invariant systems and controllers. The problem is examined using stable factorizations of transfer functions [10]. The main results are two solvability conditions for the simultaneous robust regulation problem, which are found by combining the internal model principle with the existing solvability conditions of the simultaneous stabilization problem. The results are similar to those for simultaneous stabilization but take the internal model into account. The first condition works for any number of systems but is hard to use in practice. The second condition is for two plants, providing an easier criterion and showing that solvability is related to the positive real axis behavior of the systems. In more detail, the two given plants are combined into a new system, and the sign of this system's determinant should remain invariant at the blocking zeros of the combined system and the poles of the reference signals' Laplace transforms. The theoretical results are illustrated with an example involving two mass-spring-damper systems. The solvability of the problem is checked, and a simultaneously robustly regulating controller is designed.

For ease of reference, the crucial results on stability are presented in Section II together with the precise problem statement. The theoretical results are given in Section III and demonstrated by an example in Section IV. Finally, concluding remarks are made in Section V.

II. PRELIMINARIES AND THE PROBLEM FORMULATION

The extended set of non-negative real numbers including infinity is denoted by $\mathbb{R}_{+\infty}$. The set of proper rational functions with real coefficients is denoted by $\mathbb{R}(s)$ and the set of all proper real rational functions having only poles with negative real parts is denoted by $\mathbb{R}H_\infty$. The matrices over a set S are denoted by $\mathcal{M}(S)$. Additionally, $n \times m$-matrices and n-vectors with elements in S are denoted by $S^{n\times m}$ and S^n, respectively. The matrices in $\mathcal{M}(\mathbb{R}H_\infty)$ are said to be *stable*. A stable square matrix is *unimodular* if its inverse is also stable. The set of all unimodular matrices is denoted by $\mathcal{U}(\mathbb{R}H_\infty)$. For any stable matrix $M \in \mathcal{M}(\mathbb{R}H_\infty)$ there exist

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unimodular matrices U and V such that $M = U\Lambda V$ where

$$\Lambda = \begin{bmatrix} \mathrm{diag}(h_1, \dots, h_q) & 0 \\ 0 & 0 \end{bmatrix}$$

and h_{i-1} divides h_i for all $i = 2, \dots, q$ [14]. The element h_q is called *the largest invariant factor of* M.

A point s_0 is said to be a *zero* of $M(s) \in \mathbb{R}(s)^{n\times m}$ if $\mathrm{rank} M(s_0) < \max_{s\in\mathbb{C}} \mathrm{rank} M(s)$ and a *blocking zero* if $M(s_0) = 0$. A point s_0 is a *pole* of $M(s)$ if an element of $M(s)$ has a pole there. The matrix $M(s)$ has a *blocking zero at infinity* if it is strictly proper, i.e., $\lim_{|s|\to\infty} M(s) = 0$.

A. The control configuration

The control configuration considered in this article is given in Fig. 1, where $P \in \mathbb{R}(s)^{n\times n}$ is a given plant and $C \in \mathbb{R}(s)^{n\times n}$ is an error-feedback controller. In addition, $d \in \mathbb{R}(s)^n$, $u \in \mathbb{R}(s)^n$, $y \in \mathbb{R}(s)^n$, $y_r \in \mathbb{R}(s)^n$, and $e \in \mathbb{R}(s)^n$ are the external disturbance, the plant input, the plant output, the reference signal, and the error between the reference signal and the plant output, respectively.

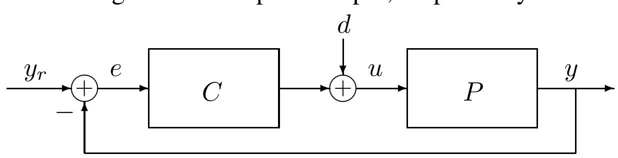

Fig. 1. The error feedback control configuration.

Note that only square plants are considered and it is assumed that $n = \max_{s\in\mathbb{C}} \mathrm{rank} P(s)$. The reference and disturbance signals are assumed to be generated by a signal generator $\Phi \in \mathbb{R}(s)^{n\times p}$, which means that they are of the form $y_r = \Phi y_0$ and $d = \Phi d_0$, where $y_0, d_0 \in (\mathbb{R}H_\infty)^p$. It is assumed that the signal generator is not stable, i.e., it has at least one pole with a non-negative real part. These poles are zeros of the largest invariant factor of Φ.

B. Coprime factorizations

Any matrix in $P \in \mathbb{R}(s)^{n\times m}$ can be written as a *right (left) factorization* $P = ND^{-1}$ ($P = \widetilde{D}^{-1}\widetilde{N}$) of two stable matrices $N, D \in \mathcal{M}(\mathbb{R}H_\infty)$ ($\widetilde{N}, \widetilde{D} \in \mathcal{M}(\mathbb{R}H_\infty)$) with $\det(D) \neq 0$ ($\det(\widetilde{D}) \neq 0$). Such a factorization is *coprime* if there exist $X, Y \in \mathcal{M}(\mathbb{R}H_\infty)$ ($\widetilde{X}, \widetilde{Y} \in \mathcal{M}(\mathbb{R}H_\infty)$) such that

$$XN + YD \in \mathcal{U}(\mathbb{R}H_\infty) \quad (\widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} \in \mathcal{U}(\mathbb{R}H_\infty)). \quad (1)$$

The pair of a left and right coprime factorizations $P = \widetilde{D}^{-1}\widetilde{N} = ND^{-1}$ is called a *doubly coprime factorization*. Theorem 4.1.16 of [10] implies that for the doubly coprime factorization there exist $X, Y, \widetilde{X}, \widetilde{Y} \in \mathcal{M}(\mathbb{R}H_\infty)$ satisfying

$$\begin{bmatrix} Y & X \\ -\widetilde{N} & \widetilde{D} \end{bmatrix} \begin{bmatrix} D & -\widetilde{X} \\ N & \widetilde{Y} \end{bmatrix} = I. \quad (2)$$

C. Stabilizability

We say that a controller C *stabilizes* the plant P, if the closed-loop transfer function

$$H(P, C) = \begin{bmatrix} (I + PC)^{-1} & (I + PC)^{-1}P \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix}$$

from (y_r, d) to (e, u) is stable. A controller C stabilizes P if and only if it has a left (right) coprime factorization $C = Y^{-1}X$ ($C = \widetilde{X}\widetilde{Y}^{-1}$) satisfying (1) [10, Corollary 5.1.8]. Stability is a robust property, i.e., if a plant is stable, then it has a neighborhood with respect to the graph metric where all plants are stable [14, Corollary 7.4.14]. It follows that if C stabilizes P, then it stabilizes all the plants in some neighborhood of P.

A plant P is *strongly stabilizable* if there exists a stable stabilizing controller $C \in \mathcal{M}(\mathbb{R}H_\infty)$. Since a left coprime factorization of a stable controller is $C = I^{-1}C$, then a plant having a right coprime factorization $P = ND^{-1}$ is strongly stabilizable exactly when there exists a stable matrix C such that $D + CN \in \mathcal{U}(\mathbb{R}H_\infty)$. Corollary 5.3.4 of [10] gives a necessary and sufficient condition for strong stabilizability, and it is repeated next for ease of reference.

Theorem 1: A plant P with a right coprime factorization $P = ND^{-1}$ is strongly stabilizable if and only if $\det(D)$ has the same sign at every blocking zero of P in $\mathbb{R}_{+\infty}$.

A finite number of plants $P_0, P_1, \dots, P_n$ are said to be *simultaneously stabilizable* if there exists a controller C that stabilizes all of them. Theorem 5.4.6 of [10] gives a necessary and sufficient condition for simultaneous stabilizability, and it is given here for later use.

Theorem 2: Given doubly coprime factorizations $P_i = \widetilde{D}_i^{-1}\widetilde{N}_i = N_i D_i^{-1}$, $i = 0, 1, \dots, n$, and the related matrices $X_0, Y_0, \widetilde{X}_0, \widetilde{Y}_0$ satisfying (2) when $i = 0$, then the plants P_i are simultaneously stabilizable if and only if there exists a stable matrix $M \in \mathcal{M}(\mathbb{R}H_\infty)$ such that

$$U_i = A_i + MB_i \in \mathcal{U}(\mathbb{R}H_\infty), \quad \forall i = 1, 2, \dots, n, \quad (3)$$

where

$$A_i = Y_0 D_i + X_0 N_i \quad \text{and} \quad B_i = \widetilde{D}_0 N_i - \widetilde{N}_0 D_i.$$

Since $\widehat{P}_i = B_i A_i^{-1}$ turns out to be a right coprime factorization of $\widehat{P}_i$, (3) is equivalent to saying that the plants $\widehat{P}_i$ are simultaneously strongly stabilizable.

D. The simultaneous robust regulation problem

It is said that a controller C *regulates* P if for all stable y_0 and zero disturbance, the error is stable, i.e.,

$$e = (I + PC)^{-1}y_r = (I + PC)^{-1}\Phi y_0 \in \mathcal{M}(\mathbb{R}\mathcal{H}_\infty).$$

The controller is said to be *robustly regulating* for a given plant P if it is regulating and stabilizing for all plants in a neighborhood of P. A controller is known to be robustly regulating exactly when it is stabilizing and contains an internal model of the signal generator Φ. This is called the internal model principle and was formulated in Theorem 7.5.1 and Lemma 7.5.5 of [14] within the framework adopted here and is repeated in the next theorem. Given a left coprime factorization $\Phi = D_r^{-1}N_r$, the internal model is characterized by the largest invariant factor ϕ of D_r. In this paper, $\phi \in \mathbb{R}H_\infty$ is called *the internal model function* related to the signal generator Φ.

Theorem 3: Let ϕ be the internal model function related to the signal generator Φ. Then C is robustly regulating for P if and only if $C = \phi^{-1}C_0$ where C_0 stabilizes the plant $\phi^{-1}P$. Furthermore, a stabilizing controller that has a right coprime factorization $C = N_cD_c^{-1}$ solves the robust regulation problem if and only if $D_c = \phi D_{c0}$ where $D_{c0} \in \mathcal{M}(\mathbb{R}H_\infty)$.

The simultaneous robust regulation problem for a given finite number of plants P_i, where $i = 0, 1, \dots, n$, and a signal generator Φ, is to find a controller C that is robustly regulating for all the plants P_i. The controllers solving the problem, are said to simultaneously robustly regulate the plants P_i.

The controller is *disturbance rejecting* if for all stable y_0 and zero reference signal, the error is stable, i.e.,

$$e = (I + PC)^{-1}Pd = (I + PC)^{-1}P\Phi d_0 \in \mathcal{M}(\mathbb{R}\mathcal{H}_\infty).$$

Controllers having an internal model of the signal generator are disturbance rejecting whenever the controller stabilizes a plant [6]. Thus, it is possible to consider disturbance rejection as a part of the simultaneous robust regulation problem without affecting its solvability or the related results.

III. SOLVABILITY CONDITIONS

The main theoretical results of this article are presented in this section. First, a necessary and sufficient solvability condition for the simultaneous robust regulation problem is found by combining Theorems 2 and 3.

Theorem 4: Let ϕ be the internal model function related to the signal generator Φ and assume that there exists a robustly regulating controller for all $P_i \in \mathbb{R}(s)^{n\times n}$ individually. Let $P_i = \widetilde{D}_i^{-1}\widetilde{N}_i = N_iD_i^{-1}$, $i = 0, 1, \dots, n$, be doubly coprime factorizations, and let the matrices $X_{0e}, Y_{0e} \in \mathcal{M}(\mathbb{R}H_\infty)$ satisfy the equation

$$X_{0e}N_0 + \phi Y_{0e}D_0 = I. \tag{4}$$

The simultaneous robust regulation problem is solvable if and only if there exists $M \in \mathcal{M}(\mathbb{R}H_\infty)$ such that

$$U_{ie} = A_{ie} + MB_{ie} \in \mathcal{U}(\mathbb{R}H_\infty), \quad \forall i = 1, 2, \dots, n, \tag{5}$$

where

$$A_{ie} = \phi Y_{0e}D_i + X_{0e}N_i \quad \text{and} \quad B_{ie} = \phi(\widetilde{D}_0N_i - \widetilde{N}_0D_i). \tag{6}$$

Furthermore, if (5) holds for some $M \in \mathcal{M}(\mathbb{R}H_\infty)$, then a controller that simultaneously robustly regulates the plants P_i is given by

$$C = \phi^{-1}(Y_{0e} - M\widetilde{N}_0)^{-1}(X_{0e} + \phi M\widetilde{D}_0) \tag{7}$$

provided that $\det(Y_{0e} - M\widetilde{N}_0) \neq 0$.

Proof: By Theorem 3, the existence of a simultaneously robustly regulating controller for the plants P_i is equivalent to the existence of a simultaneously stabilizing controller for the plants $P_{ie} = \phi^{-1}P_i$. This in turn is equivalent to the existence of M satisfying (5) by Theorem 3 provided that $P_{ie} = (\phi\widetilde{D}_i)^{-1}\widetilde{N}_i = N_i(\phi D_i)^{-1}$ are doubly coprime factorizations.

The proof of the first part is completed by showing that $(\phi\widetilde{D}_i)^{-1}\widetilde{N}_i$ and $N_i(\phi D_i)^{-1}$ are coprime. By assumption, there exists a robustly regulating controller C for a fixed P_i. Let $C = N_cD_c^{-1} = \widetilde{D}_c^{-1}\widetilde{N}_c$ be a doubly coprime factorization. Since C is stabilizing for P_i, the matrices $\widetilde{N}_iN_c + \widetilde{D}_iD_c$ and $\widetilde{N}_cN_i + \widetilde{D}_cD_i$ are unimodular. In addition, $D_c = \phi D_{c0}$ for some stable D_{c0} by Theorem 3. Since the plants and controllers have the same number of inputs and outputs, D_c can be divisible by ϕ if and only if $\widetilde{D}_c$ is, i.e., $\widetilde{D}_c = \phi\widetilde{D}_{c0}$ for some stable $\widetilde{D}_{c0}$. This means that

$$\widetilde{N}_iN_c + (\phi\widetilde{D}_i)D_{c0},\ \widetilde{N}_cN_i + \widetilde{D}_{c0}(\phi D_i) \in \mathcal{U}(\mathbb{R}H_\infty)$$

and consequently $(\phi\widetilde{D}_i)^{-1}\widetilde{N}_i$ and $N_i(\phi D_i)^{-1}$ are coprime.

It remains to show that (7) is simultaneously robustly regulating. By the above discussion, it suffices to show that

$$C_0 = (Y_{0e} - M\widetilde{N}_0)^{-1}(X_{0e} + \phi M\widetilde{D}_0)$$

simultaneously stabilizes the plants P_{ie}, $i = 0, 1, \dots n$. Since $\det(Y_{0e} - M\widetilde{N}_0) \neq 0$, C_0 is well-defined. Since $P_{ie} = N_i(\phi D_i)^{-1}$ is a right coprime factorization, stability of the closed-loop transfer matrix $H(P_i, C_0)$ follows if (1) holds for $N = N_i$, $D = \phi D_i$, $X = X_{0e} + \phi M\widetilde{D}_0$, and $Y = \phi Y_{0e} - M\widetilde{N}_0$. Substitution yields

$$\begin{aligned}
&(X_{0e} + \phi M\widetilde{D}_0)N_i + (Y_{0e} - M\widetilde{N}_0)\phi D_i \\
&\quad = \phi Y_{0e}D_i + X_{0e}N_i + M\phi(\widetilde{D}_0N_i - \widetilde{N}_0D_i) \\
&\quad = A_{ie} + MB_{ie}
\end{aligned}$$

which is unimodular by assumption. Thus, C stabilizes all P_i, $i = 1, 2, \dots, n$. In addition, C_0 stabilizes P_0 since

$$\begin{aligned}
&(X_{0e} + \phi M\widetilde{D}_0)N_0 + (Y_{0e} - M\widetilde{N}_0)\phi D_0 \\
&\quad = \phi Y_{0e}D_0 + X_{0e}N_0 + M\phi(\widetilde{D}_0N_0 - \widetilde{N}_0D_0) \\
&\quad = I + \phi M\widetilde{D}_0(P_0 - P_0)D_0 \\
&\quad = I.
\end{aligned}$$

■

As mentioned earlier, simultaneous stabilization of plants P_i, $i = 0, 1, \dots, n$, is equivalent to simultaneous strong stabilization of plants $B_iA_i^{-1}$ where $i = 1, \dots, n$ and the matrices A_i and B_i are those in (3). Strong stabilization of $B_{ie}A_{ie}^{-1}$ is in turn related to (5). This leads to the following solvability condition.

Theorem 5: Assume that a robust controller exists for the plants P_0 and P_1 individually and that the plants have no common poles in $\mathbb{R}_{+\infty}$. Let ϕ be the internal model function related to the signal generator Φ, $P_i = N_iD_i^{-1} = \widetilde{D}_i^{-1}\widetilde{N}_i$, $i = 0, 1$, be doubly coprime factorizations, and $Y_{0e}, X_{0e} \in \mathcal{M}(\mathbb{R}H_\infty)$ such that (4) holds.The simultaneous robust regulation problem is solvable if and only if

1) $\det(X_{0e}N_1)$ at the $\mathbb{R}_{+\infty}$-zeros of ϕ and
2) $\det(D_0D_1)$ at the $\mathbb{R}_{+\infty}$-blocking zeros of $P_0 - P_1$ that are not zeros of ϕ

all have the same sign.

Proof: According to Theorem 4, the simultaneous robust regulation problem is solvable for the given two plants if and only if the matrix in (5) is unimodular for some stable matrix M. This is equivalent to saying that the associated plant $\widehat{P} = B_{1e}A_{1e}^{-1}$ is strongly stabilizable. Theorem 1 implies that this is possible exactly when the sign of $\det(A_{1e})$ at the blocking zeros of $\widehat{P}$ is the same. The blocking zeros of $\widehat{P}$ are the blocking zeros of $B_{1e} = \phi(-\widetilde{N}_0 D_1 + \widetilde{D}_0 N_1)$. Obviously, the zeros of ϕ are blocking zeros of B_{1e}. The remaining blocking zeros are the blocking zeros of $-\widetilde{N}_0 D_1 + \widetilde{D}_0 N_1$. These are exactly the blocking zeros of $P_0 - P_1$ by [10, Lemma 5.4.4]. If $\phi(s) = 0$, one has $A_{1e}(s) = (X_{0e}N_1)(s)$. Thus, the claim follows if the signs of $\det(A_{1e})$ and $\det(D_0 D_1)$ are the same at the blocking zeros of $P_0 - P_1$ that are not zeros of ϕ. At the blocking zeros $(P_0 - P_1)(s) = 0$, so $P_0(s) = P_1(s)$ and consequently,

$$\begin{aligned} A_{1e}(s) &= (\phi Y_{0e}D_1 + X_{0e}N_1)(s) \\ &= ((Y_{0e} + X_{0e}N_1(\phi D_1)^{-1})(\phi D_1))(s) \\ &= ((Y_{0e} + X_{0e}N_0(\phi D_0)^{-1})(\phi D_1))(s) \\ &= ((X_{0e}N_0 + \phi Y_{0e}D_0)(\phi D_0)^{-1}(\phi D_1))(s) \\ &= (D_0^{-1}D_1)(s). \end{aligned}$$

This shows the final step of the proof since the signs of $\det(D_0(s))$ and $\det(D_0^{-1}(s))$ are the same. ■

Condition 1 in Theorem 5 relates the solvability of the simultaneous robust regulation problem to the internal model. Condition 2 is a necessary and sufficient condition for simultaneous stabilization of the two plants as shown by the proof of Lemma 5.4.5 in [10]. This is natural since a simultaneously robustly regulating controller is also simultaneously stabilizing. The observations lead to the following corollary.

Corollary 1: Assume that a robust controller exists for the plants P_0 and P_1 individually and that the plants have no common poles in $\mathbb{R}_{+\infty}$. If the internal model function ϕ related to the signal generator Φ does not have $\mathbb{R}_{+\infty}$-zeros, then the simultaneous robust regulation problem is solvable if and only if P_0 and P_1 are simultaneously stabilizable.

IV. EXAMPLE

The theoretical results of the previous section are illustrated using a mass-spring-damper system depicted in Fig. 2. The calculations are done using MATLAB. The system has two carts with masses $m_1 = 1 = m_2$. The spring constants are $k_1 = 2$ and $k_2 = 1$ and the damping coefficients are $d_1 = 3$ and $d_2 = \frac{15}{16}$. The deviations of the carts from their equilibrium positions are denoted by x_1 and x_2. The outputs of the system are the deviations, i.e., $y_1 = x_1$ and $y_2 = x_2$. The inputs of the system are two forces, u_1 and u_2, each acting on the cart with the corresponding index.

The state-space model and the transfer matrix are

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \end{cases}$$

and $P(s) = C(sI - A)^{-1}B$, where the state, the input and the output vectors are $\mathbf{x} = [x_1\ \dot{x}_1\ x_2\ \dot{x}_2]^T$, $\mathbf{u} = [u_1\ u_2]^T$ and $\mathbf{y} = [y_1\ y_2]^T$, and the related matrices are

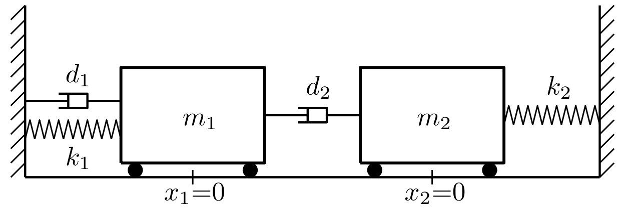

Fig. 2. The error feedback control configuration.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1} & \frac{-d_1-d_2}{m_1} & 0 & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_2}{m_2} & \frac{-k_2}{m_2} & \frac{-d_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is assumed that the system is initially at rest in its equilibrium state, i.e., $\mathbf{x}(0) = 0$. The aim is to find a controller for the given system P_0 so that the deviations $x_1(t)$ and $x_2(t)$ asymptotically follow the reference signals $\hat{y}_1(t) = \sin(2t)$ and $\hat{y}_2(t) = 1 + \sin(2t)$, respectively. That is, the first cart oscillates around its equilibrium position and the second cart one unit to the right of its equilibrium position. The controller should work despite possible small variations in the parameters m_i, k_i and d_i due to inaccurate parameter estimation and also if the damper between the carts is removed or breaks down, i.e., $d_2 = 0$. Removing the damper counts as a large perturbation and leads to a new system P_1. Thus, the problem considered is the simultaneous robust regulation problem with two plants P_0 and P_1.

The transfer functions of the plant with and without the second damper are

$$P_0(s) = \begin{bmatrix} \frac{16s^2+15s+16}{\alpha(s)} & \frac{15s}{\alpha(s)} \\ \frac{15s}{\alpha(s)} & \frac{16s^2+63s+32}{\alpha(s)} \end{bmatrix}$$

and

$$P_1(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{1}{s^2+1} \end{bmatrix} = \begin{bmatrix} p_1(s) & 0 \\ 0 & p_2(s) \end{bmatrix}$$

where $\alpha(s) = 16s^4 + 78s^3 + 93s^2 + 93s + 32$. The Laplace transforms of the reference signals are $\hat{y}_1(s) = \frac{1}{s^2+4}$ and $\hat{y}_2(s) = \frac{1}{s^2+4} + \frac{1}{s}$, so the signal generator can be chosen to be $\Phi(s) = \left[\frac{1}{s^2+4} \quad \frac{1}{s^2+4} + \frac{1}{s}\right]^T$. The internal model function related to Φ is $\phi(s) = \frac{s(s^2+4)}{(s+1)^3}$.

Solvability of the simultaneous robust regulation problem is verified using Theorem 5. To this end, one needs to verify the assumptions of the theorem and find coprime factorizations for the two plants as well as the related matrices X_{0e} and Y_{0e} satisfying (4). First, one verifies that robustly regulating controllers exist for both plants individually. By Lemma 7.5.2 of [14], this follows if neither of the plants has common zeros with ϕ. As the zeros of ϕ are 0 and $\pm 2i$ and the plant transfer functions have full rank at these points, robust controllers exist for both plants.

The next step is to find a doubly coprime factorization $P_0 = \widetilde{D}_0^{-1}\widetilde{N}_0 = N_0 D_0^{-1}$ and matrices X_{0e} and Y_{0e} solving (4). This means that one needs to find a robustly regulating controller for P_0 since the controller $\phi^{-1}Y_{0e}^{-1}X_{0e}$ is stabilizing by (4) and contains an internal model due to the term ϕ^{-1}. The damper connected to the wall makes the whole system stable, which can be verified by numerically calculating the roots of $\alpha(s)$. The roots are approximately -0.5, -3.7, and $-0.3 \pm i$ all having negative real parts as expected. It follows that $I^{-1} \cdot P_0 = P_0 \cdot I^{-1}$ is a doubly coprime factorization.

A simple robustly regulating controller for the stable plant P_0 is given by a controller of the form

$$C_i(s) = \varepsilon_i \left(\frac{M_0}{s} + \frac{M_{-2i}}{s+2i} + \frac{M_{2i}}{s-2i} \right) \tag{8}$$

where the constant 2×2-matrices M_0, M_{-2i}, and M_{2i} are chosen so that all the eigenvalues of $P_0(0)M_0$, $P_0(-2i)M_{-2i}$, and $P_0(2i)M_{2i}$ have strictly positive real parts, and $\varepsilon_i > 0$ is chosen to be sufficiently small [3, Corollary 7]. This is easily achieved by choosing the matrices as inverses of P_0 at the related points. However, in order to keep the structure of the controller simple, diagonal matrices are used. Numerical calculation shows that $P_0(0)$ has positive eigenvalues, whereas all the eigenvalues of $P_0(2i)$ and $P_0(-2i)$ have negative real parts. Thus, one can choose $M_0 = I$ and $M_{-2i} = -I = M_{2i}$. Setting $\varepsilon_i = 0.1$ leads to

$$C_i(s) = \frac{4-s^2}{10s(s^2+4)} I.$$

It can be verified numerically that this controller stabilizes P_0. The controller contains an internal model since $\frac{4-s^2}{10s(s^2+4)} \cdot \phi(s) = \frac{4-s^2}{(s+1)^3} \in \mathbb{R}H_\infty$, so it robustly regulates P_0. By stability of the closed-loop, the matrices

$$\widetilde{D}_i = (I + C_i P_0)^{-1} \quad \text{and} \quad \widetilde{N}_i = \widetilde{D}_i C_i$$

are stable. In addition, $\widetilde{D}_i + \widetilde{N}_i P_0 = I$, so $C_i = \widetilde{D}_i^{-1}\widetilde{N}_i$ is a left coprime factorization of C_i. By Theorem 3 and the fact that the plants are square, $X_{0e} = \phi^{-1}\widetilde{D}_i$ is stable. Choosing $Y_{0e} = N_i$ gives the matrices satisfying (4).

Plant P_1 has two independent subsystems p_1 and p_2 as there is no damper connecting them. The subsystem p_1 is stable due to the damper connected to the wall, which is easily verified since the poles of $p_1(s)$ are located at -1 and -2. Thus, only p_2 needs to be stabilized. It has two unstable poles located at $\pm i$. A stabilizing controller is found by introducing a stable pole on the real axis and then choosing a sufficiently small feedback gain so that the two stable poles move to the left of the imaginary axis and the introduced stable pole does not become unstable. This is achieved by a stable controller $c_2(s) = \frac{-\varepsilon_1}{s+1}$, where $\varepsilon_1 = 0.5$. Thus, a strongly stabilizing controller for P_1 is

$$C_1(s) = \begin{bmatrix} 0 & 0 \\ 0 & c_2(s) \end{bmatrix} \in \mathcal{M}(\mathbb{R}H_\infty)$$

and its doubly coprime factorization is $I \cdot C_1 = C_1 \cdot I$. A doubly coprime factorization of P_1 is $N_1 D_1^{-1} = D_1^{-1} N_1$ where $N_1 = (I + P_1 C_1)^{-1} P_1$ and $D_1 = (I + P_1 C_1)^{-1}$.

It remains to find the non-negative real blocking zeros of $P_0 - P_1$. They can be found, for example, by finding the Smith-McMillan form or by finding $s \in \mathbb{R}_{+\infty}$ where zero is a repeated eigenvalue of $(P_0 - P_1)(s)$. Symbolically solving the eigenvalues of $P_0 - P_1$ shows that zero is a repeated eigenvalue only if $15s(2s^4 + 6s^3 + 15s^2 + 12s + 5) = 0$. Thus, 0 is the only blocking zero in $\mathbb{R}_{+\infty}$, which is also a zero of the internal model function. As the plants are strictly proper, there is also a blocking zero at infinity. Calculation reveals that $\det(X_{0e}(0)N_1(0)) = 2$ and $\det(D_0(\infty)D_1(\infty)) = 1$ have the same sign, so Theorem 5 implies that a simultaneously robustly regulating controller exists.

A robustly regulating controller is constructed next. It is known that given a robustly regulating controller of the form (8), one obtains another robustly regulating controller by adding a sufficiently small stable matrix [5, Theorem 1.1]. Recall that C_1 is a stable matrix, so $C = C_1 + C_i$ is robustly regulating for P_0 if ε_1 is sufficiently small. On the other hand, it is known that the sum of a stabilizing controller C_1 of P_1 and a robustly regulating controller of the plant $N_1 = (I + P_1 C_1)^{-1} P_1$ is robustly regulating for P_1 [6, Section 6.2]. The given controller C_i is robustly regulating for N_1 if $M_0 N_1(0) = N_1(0)$ and $M_{\pm 2i} N_1(\pm 2i) = -N_1(\pm 2i)$ have eigenvalues with positive real parts, and the chosen parameter ε_i is sufficiently small. Calculation of the eigenvalues reveals that the real parts are positive, so the controller $C = C_1 + C_i$ can be made robustly regulating if the parameters ε_1 and ε_i are small enough. Numerical calculations show that

$$C(s) = C_1(s) + C_i(s) = \begin{bmatrix} -\frac{s^2-4}{10s(s^2+4)} & 0 \\ 0 & -\frac{6s^3+s^2+16s-4}{10s(s^2+4)(s+1)} \end{bmatrix}$$

is stabilizing for the plants P_0 and P_1. As it contains an internal model of the signal generator, it is simultaneously robustly regulating. This also implies that (7) should hold for some stable M that makes $U_{1e} = A_{1e} + MB_{1e}$ in (5) unimodular. The denominator and numerator matrices of the coprime factorization $C = D_c^{-1} N_c$ with $D_c = (I + CP_0)^{-1}$ and $N_c = D_c C$ should be the same as those in (7), i.e.,

$$N_c = X_{0e} + \phi M \widetilde{D}_0 \quad \text{and} \quad D_c = \phi Y_{0e} - \phi M \widetilde{N}_0.$$

Since $\widetilde{D}_0 = I$, solving for M gives $M = \phi^{-1}(N_c - X_{0e})$. This matrix is stable. The roots of the denominators of U_{1e}^{-1} are plotted in Fig. 3. The only root not in the figure is approximately at -3.7. The roots have negative real parts, so U_{1e}^{-1} is stable and U_{1e} is unimodular.

Finally, the controlled system is simulated by finding a state-space realization for the controller. The cart deviations from their equilibrium positions are illustrated in Fig. 4. It is observed that the deviations of the first cart (black line) and the second cart (gray line) approach the reference signals $\sin(2t)$ and $1 + \sin(2t)$ as expected.

The norm of the error $e(t) = y_r(t) - y(t)$ over time is depicted in Fig. 5 for the controlled plant P_0. It is seen

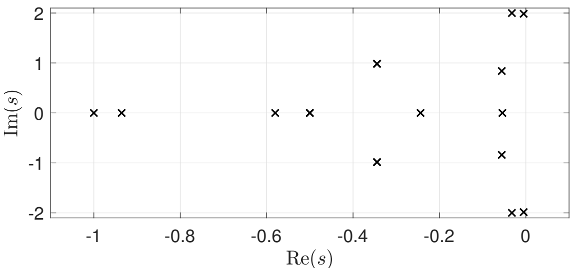

Fig. 3. The poles of $U_{1e}^{-1} = (A_{1e} + MB_{1e})^{-1}$.

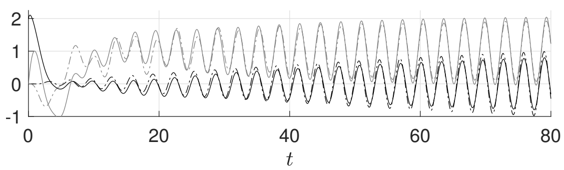

Fig. 4. The cart deviations $x_1(t)$ (gray line) and $x_2(t)$ (black line) in the controlled systems P_0 (solid) and P_1 (dashed).

that the error converges to zero as expected. The situation is similar for the plant P_1. The closed-loop eigenvalues of the state matrices closest to the imaginary axis have real parts -0.005 with P_0 and -0.003 with P_1. Thus, the stability margin is very small and convergence is slow.

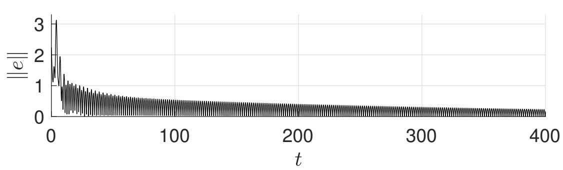

Fig. 5. The norm of the error $\|e(t)\|$ for the controlled system P_0.

V. CONCLUSIONS

The problem of finding a robustly regulating controller working simultaneously for two or more given plants was studied, and two solvability conditions were given. The solvability condition of Theorem 4 applies to any finite number of plants, but requires solving the system (5) over a ring of matrices. This is challenging, so finding systematic ways of solving such systems is needed. Theorem 5 is suitable for two plants and provides an easily checkable condition once the related coprime factorizations are found. Finding the two matrices X_{0e} and Y_{0e} means solving the robust regulation problem for P_0, but for other systems finding any right coprime factorization suffices. Corollary 1 shows that, with two plants, solving the simultaneous robust regulation problem is equivalent to solving the simultaneous stabilization problem, provided the signal generator lacks non-negative real poles and the individual plants have robustly regulating controllers. Unless one wants to track exponentially growing reference signals, the only time domain signals producing $\mathbb{R}_{+\infty}$-poles are non-zero constants. The corollary is unlikely to generalize for more than two plants, since it is known that with more than two plants the simultaneous stabilization is not anymore strictly related to the $\mathbb{R}_{+\infty}$-behavior of the plants [15]. The example in Section IV illustrates the theoretical results, so compromises were made to keep the structure of the controller simple, and its performance was not optimized. A prominent approach for designing simultaneously robustly regulating controllers with optimized performance is to use software packages, see for example [16] and the references therein, suitable for finding simultaneously stabilizing controllers. The idea is simply to add an internal model to all plants and to simultaneously stabilize these extended plants. The results here concern linear time-invariant controllers, but periodic controllers can solve problems unsolvable by time-invariant ones [17], thus extending the possibilities in simultaneous robust regulation.

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