Robust Regulation Theory for Transfer Functions With a Coprime Factorization

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***Abstract*—Classical frequency domain results of robust regulation are extended by requiring only a right or a left coprime factorization of a plant, but not both. The famous internal model principle is generalized first, which leads to a necessary and sufficient solvability condition of the robust regulation problem and to a parametrization of all robustly regulating controllers. In addition, a procedure for constructing robustly regulating controllers is proposed.**

***Index Terms*—Distributed parameter systems, linear systems, parametrization, robust control.**

I. Introduction

Consider the control scheme depicted in Fig. 1, where P is a given plant and C is a controller to be designed. The problem addressed in this article is the robust regulation problem in the frequency domain, which, roughly speaking, aims at finding a controller such that the error e between the reference signal y_r and the output y remains stable for all stable y_0 and d_0 despite small perturbations in the plant.

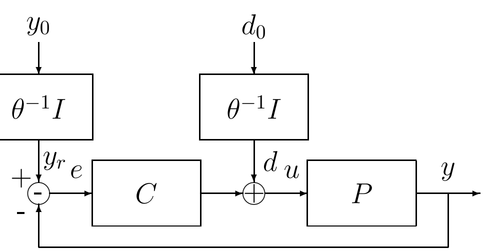

Fig. 1. The control scheme.

The robust regulation problem is a basic problem in control theory, and it has been studied by several authors in the time and frequency domains, see for example [3], [5], [4], [8], [9], [10], [14], [16], [19], [21], and [24]. The frequency domain formulation of the robust regulation problem is thoroughly studied for rational transfer functions in the existing literature. Powerful results allow parametrization of all robustly regulating controllers and state a solvability condition for the problem in terms of the plant and the signal generator [4], [21]. The internal model principle presented by Francis and Wonham [5] has a simple form in the frequency domain [21]. These results have a rather straightforward extension to the abstract algebraic setting, provided that the plant has left and right coprime factorizations, and that the ring of stable transfer functions is topological [14]. The results by Nett in [14] are based on the stability results of [22] that require both coprime factorizations. This is problematic because it is known that there exist algebraic structures where there are stabilizable plants with no coprime factorizations [18]. In addition, finding a coprime factorization even if its existence is known, is far from trivial.

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The purpose of this article is to take a step towards a more general algebraic approach to the regulator problem by requiring the plant to have a right or a left coprime factorization, but not both. It is known that the existence of a right coprime factorization implies the existence of a left coprime factorization, and vice versa, if and only if the ring of stable transfer functions is Hermite [21]. Thus, the results of this article generalize those of [14] to the field of fraction over non-Hermite rings.

The theory contains the case of Hermite rings—including the classical regulator theory dealing with rational functions —as a special case, and is based on the stability results of [11] by Mori. These results as well as the theory developed in this article are presented in terms of left and right coprime factorizations of an extended plant. Mori provided an elementary method to construct the extended plant's both coprime factorizations starting from a left or a right coprime factorization of a given plant. This reveals the practical value of the approach adopted in this article, since even if one is able to show that a ring is Hermite, the construction of both coprime factorizations starting from a given one may rely on theoretical results that are difficult to use in practice. In addition, proving that a ring is Hermitian is not a trivial task in general, e.g. it was shown only quite recently that the Wiener-Laplace algebra $W^+(\mathbb{C}_+)$ is Hermitian [13] and it is not known whether the ring $\mathbf{P}$ of [10] and [15] is Hermite.

In [14] and [21], the robustness is defined with respect to the graph topology of [22]. No topological aspects of the robust regulation problem are needed in this article. This further extends the algebraic structure covered by the theory. There is no need for a topology, because the robustness of regulation is defined as in [1], i.e., the closed loop stability should imply regulation. A robustly regulating controller in the sense of the above definition would be robustly regulating in the sense of a topology provided it is robustly stabilizing. This justifies the definition of robustness of regulation adopted here since the robustness of stability is well-understood in many cases [2], [6], [22].

In addition to the theory of robust regulation, the controller design for unstable plants is addressed. Since any robustly regulating controller contains an internal model by the internal model principle, it is not a surprise that including an internal model to the plant and then stabilizing the resulting system yields a robustly regulating controller. This is the idea of servocompensators developed by Davison [3] which have been generalized to infinite-dimensional systems by [7] and [19]. Having complicated reference signal dynamics may make the stabilization of the closed loop difficult because the dynamics become a part of the closed loop by the internal model principle. While it is not possible to give a detailed controller design because of the general algebraic setting, it is possible to split the design of a robustly regulating controller into two parts: the first part is to find a stabilizing controller for the original plant and the second part is to design a robustly regulating controller for the stable numerator matrix of the plant transfer function. Combining the two controllers appropriately yields a robustly regulating controller. Such an approach makes sense because the original system can be stabilized by any means available and because there are extremely simple controllers available for example for stable transfer functions in the CD-algebra and in H^∞ [7], [19]. This approach was introduced for the first time in [10] for plants with both coprime factorizations over two specified rings. This work further generalizes the results to a general algebraic setting and for plants with a right or a left coprime factorization, but not necessarily both.

The rest of this article is organized as follows. In Section II, notations and some preliminary results are introduced. Section III contains the main results of the article: The internal model principle and a necessary and sufficient solvability condition. Controller design is addressed in Section IV. Section V contains an example that illustrates the theory.

II. Notations, Preliminary Results and the Problem Formulation

The set of all matrices and the set of all $n\times m$- matrices over a set S are denoted by $\mathcal{M}(S)$ and $\mathcal{M}^{n\times m}(S)$, respectively. The determinant and the transpose of a matrix M are denoted by $\det(M)$ and M^T, respectively. The $n\times n$ identity matrix is denoted by I_n. The $n\times m$ zero matrix is denoted by $0_{n\times m}$. The subscripts of I_n and $0_{n\times m}$ are omitted if the dimensions are clear from the context. The block diagonal matrix with possibly non-square blocks $M_1,\ldots,M_n$ on its diagonal is denoted by $\mathrm{bdiag}(M_1,\ldots,M_n)$.

The set of stable elements is a commutative ring $\mathbf{R}$ that has a unit element and no zero divisors. The quotient field of $\mathbf{R}$ is denoted by $\mathbf{F}$. Plants and controllers are matrices in $\mathcal{M}(\mathbf{F})$. A matrix is *stable* if it is in $\mathcal{M}(\mathbf{R})$. It is said that a controller $C\in\mathcal{M}^{m\times n}(\mathbf{F})$ *stabilizes* a plant $P\in\mathcal{M}^{n\times m}(\mathbf{F})$ if the closed loop system from (y_r,d) to (e,u) given by

$$H(P,C) := \begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ C(I+PC)^{-1} & I-C(I+PC)^{-1}P \end{bmatrix}$$

is well-defined, i.e. $\det(I+PC)\neq 0$, and stable.

A pair (N,D) of stable matrices is said to be *right [left] coprime* if there exist stable matrices X and Y such that $XN+YD=I$ $[NX+DY=I]$. A pair (N_p,D_p) $[(\widetilde{N}_p,\widetilde{D}_p)]$ of matrices is called *a right [left] coprime factorization* of $P\in\mathcal{M}(\mathbf{F})$ if it is right [left] coprime, $\det(D_p)\neq 0$ $[\det(\widetilde{D}_p)\neq 0]$ and $P=N_pD_p^{-1}$ $[P=\widetilde{D}_p^{-1}\widetilde{N}_p]$.

The following two lemmas are well-known result from [21] and [22] and are used extensively in this article.

Lemma II.1. *Let (N_p,D_p) $[(\widetilde{N}_p,\widetilde{D}_p)]$ be a right [left] coprime factorization of a plant P. A controller C stabilizes P if and only if there exists a left [right] coprime factorization $(\widetilde{N}_c,\widetilde{D}_c)$ $[(N_c,D_c)]$ of C such that $\widetilde{N}_cN_p+\widetilde{D}_cD_p=I$ $[\widetilde{N}_pN_c+\widetilde{D}_pD_c=I]$.*

Lemma II.2. *Let (N_p,D_p) and $(\widetilde{N}_p,\widetilde{D}_p)$ be a right and a left coprime factorization of P, respectively. Let $Y,X,\widetilde{Y},\widetilde{X}\in\mathcal{M}(\mathbf{R})$ be such that $\widetilde{N}_pY+\widetilde{D}_pX=I$ and $\widetilde{Y}N_p+\widetilde{X}D_p=I$. Now C stabilizes P if and only if it has a right coprime factorization $(Y+D_pR,X-N_pR)$ for some $R\in\mathcal{M}(\mathbf{R})$ or equivalently a left coprime factorization $(\widetilde{Y}+\widetilde{R}\widetilde{D}_p,\widetilde{X}-\widetilde{R}\widetilde{N}_p)$ for some $\widetilde{R}\in\mathcal{M}(\mathbf{R})$.*

The results of this article are based on the stability results of [11] by Mori, and they use *the extended plant* $P_e\in\mathcal{M}^{(n+m)\times(n+m)}(\mathbf{F})$ of $P\in\mathcal{M}^{n\times m}(\mathbf{F})$ and *the extended controller* $C_e\in\mathcal{M}^{(n+m)\times(n+m)}(\mathbf{F})$ of $C\in\mathcal{M}^{m\times n}(\mathbf{F})$ defined by

$$P_e := \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \text{ and } C_e := \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix},$$

respectively. The first lemma below follows by Theorem 1 of [11]. The second one is Theorem 2 of [11].

Lemma II.3. *If $P\in\mathcal{M}^{n\times m}(\mathbf{F})$ has a right or a left coprime factorization, then P_e has right and left coprime factorizations.*

Proof: Assume that P has a right coprime factorization (N_p,D_p). Theorem 1 of [11] shows that $\begin{bmatrix} P^T & 0_{m\times m}\end{bmatrix}^T$ has a left coprime factorization (N,D). It is easy to verify that $(\mathrm{bdiag}(N_p,0_{m\times n}),\mathrm{bdiag}(D_p,I_n))$ and $\left(\begin{bmatrix} N & 0_{(n+m)\times n}\end{bmatrix},D\right)$ are right and left coprime factorizations of P_e, respectively. If P has a left coprime factorization, then one can construct the left and right coprime factorizations of P_e similarly by using a left coprime factorization of $\begin{bmatrix} P & 0_{n\times n}\end{bmatrix}$. ■

Lemma II.4. *The controller $C\in\mathcal{M}^{m\times n}(\mathbf{F})$ stabilizes $P\in\mathcal{M}^{n\times m}(\mathbf{F})$ if and only if $C=\begin{bmatrix} I_m & 0\end{bmatrix}C_0\begin{bmatrix} I_n & 0\end{bmatrix}^T$ where C_0 stabilizes P_e.*

Remark II.5. *One can change the roles of the plant and the controller in the above Lemmas II.1 and II.2 to yield an existence result for a coprime factorization of a plant and a parametrization of all the plants stabilized by C, respectively.*

Problem 1. *Let the plant $P\in\mathcal{M}(\mathbf{F})$ and* the generating element $\theta\in\mathbf{R}$ *be given in the control configuration of Fig. 1.* The robust regulation problem *aims at finding a controller $C\in\mathcal{M}(\mathbf{F})$ such that it*

i) stabilizes P, and
ii) regulates any plant P' that it stabilizes, i.e.

$$\theta^{-1}\begin{bmatrix}(I+P'C)^{-1} & -(I+P'C)^{-1}P'\end{bmatrix}\in\mathcal{M}(\mathbf{R}). \quad (1)$$

The assumption that the reference signals are generated by a generator of the form $\theta^{-1}I$ is not as restrictive as it may first appear. For example, for rational matrices—suitable to handle finite-dimensional systems and sinusoidal reference signals—or more generally if $\mathbf{R}$ is any principal ideal domain, this is not restrictive at all, since if the reference signals are of the form Θy_0, then there exists a generating element θ such that the robust regulation problem with Θ is solvable if and only if it is solvable with $\theta^{-1}I$ [21]. A similar result holds also with the $\mathbf{P}$-stability that is suitable to handle infinite-dimensional systems and exosystems [10].

The following lemma gives a formulation of (1) in terms of the extended plant and controller, and their coprime factorizations.

Lemma II.6. *Let $(\widetilde{N}_{pe}, \widetilde{D}_{pe})$ be a left coprime factorization of the extended plant P_e of $P \in \mathcal{M}^{n\times m}(\mathbf{F})$. If $C = \begin{bmatrix} I_m & 0 \end{bmatrix} C_0 \begin{bmatrix} I_n & 0 \end{bmatrix}^T$ where C_0 is a controller that stabilizes P_e and (N_0, D_0) is the right coprime factorization of C_0 such that $\widetilde{N}_{pe}N_0 + \widetilde{D}_{pe}D_0 = I$, then*

$$(I+PC)^{-1} = \begin{bmatrix} I_n & 0 \end{bmatrix} D_0 \widetilde{D}_{pe} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \tag{2}$$

and

$$(I+PC)^{-1}P = \begin{bmatrix} I_n & 0 \end{bmatrix} D_0 \widetilde{N}_{pe} \begin{bmatrix} I_m \\ 0 \end{bmatrix}. \tag{3}$$

Proof: Factorize

$$C_0 = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix},$$

where the dimensions of the blocks are compatible with the blocks of P_e. A direct calculation shows that

$$\begin{aligned}(I+PC)^{-1} &= (I+PC_1)^{-1} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix} (I+P_eC_0)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix} D_0 \widetilde{D}_{pe} \begin{bmatrix} I_n \\ 0 \end{bmatrix},\end{aligned}$$

and (3) follows easily by (2). ■

III. Solvability of the Robust Regulation Problem

The next theorem is a frequency domain formulation of the internal model principle of robust regulation due to Francis and Wonham [5]. It states a necessary and sufficient condition for a stabilizing controller to be robustly regulating.

Theorem III.1. *Assume that $C \in \mathcal{M}^{m\times n}(\mathbf{F})$ stabilizes $P \in \mathcal{M}^{n\times m}(\mathbf{F})$ and that P has a coprime factorization. If (N_{ce}, D_{ce}) is a right coprime factorization of C_e, then C solves the robust regulation problem if and only if $\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce} \in \mathcal{M}(\mathbf{R})$.*

Proof: Since C is stabilizing it is sufficient to show that $\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce} \in \mathcal{M}(\mathbf{R})$ holds if and only if (1) holds for any P' that C stabilizes. To this end, let P' be a plant that is stabilized by C. Let (N_{pe}, D_{pe}) and $(\widetilde{N}_{pe}, \widetilde{D}_{pe})$ be left and right coprime factorizations of the extended plant P_e of P that exists by Lemma II.3. Since C_e stabilizes P'_e—the extended plant of P'—Remark II.5 and Lemma II.2 show that P'_e has a left coprime factorization $\left(\widetilde{N}_{pe} + \widetilde{R}\widetilde{D}_{ce}, \widetilde{D}_{pe} - \widetilde{R}\widetilde{N}_{ce}\right)$, where $(\widetilde{N}_{ce}, \widetilde{D}_{ce})$ is a left coprime factorization of C_e satisfying $\widetilde{N}_{ce}N_{pe} + \widetilde{D}_{ce}D_{pe} = I$.

The sufficiency follows by the assumption and Lemma II.6. In order to show necessity, assume that C is robustly regulating. This implies by (2) of Lemma II.6 that

$$\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce}\left(\widetilde{D}_{pe} - \widetilde{R}\widetilde{N}_{ce}\right)\begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathcal{M}(\mathbf{R})$$

for all matrices $\widetilde{R}$ such that $\det\left(\widetilde{D}_{pe} - \widetilde{R}\widetilde{N}_{ce}\right) \neq 0$. Since C is regulating, Lemma II.6 shows that $\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce}\widetilde{D}_{pe}\begin{bmatrix} I_n & 0 \end{bmatrix}^T \in \mathcal{M}(\mathbf{R})$. Thus, the above equation shows that $\theta^{-1}\begin{bmatrix} I & 0 \end{bmatrix} D_{ce}\widetilde{R}\widetilde{N}_{ce}\begin{bmatrix} I & 0 \end{bmatrix}^T \in \mathcal{M}(\mathbf{R})$. Choosing $\widetilde{R} = -R_0N_{pe}$, where $R_0 \in \mathcal{M}(\mathbf{R})$ is chosen so that $\det\left(\widetilde{D}_{pe} + R_0N_{pe}\widetilde{N}_{ce}\right) \neq 0$, gives the stable matrix

$$\begin{aligned}&\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce}\left(R_0 - (R_0 - R_0N_{pe}\widetilde{N}_{ce})\right)\begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &\quad= \theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce}R_0\left(I - (I+P_eC_e)^{-1})\right)\begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &\quad= \begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce}R_0\left(\theta^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix} - \begin{bmatrix} \theta^{-1}(I+PC)^{-1} \\ 0 \end{bmatrix}\right).\end{aligned}$$

Since C regulates P, $\theta^{-1}(I+PC)^{-1} \in \mathcal{M}(\mathbf{R})$. The matrix in the above equation is stable, so $\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix} D_{ce}R_0\begin{bmatrix} I_n & 0 \end{bmatrix}^T \in \mathcal{M}(\mathbf{R})$ for any stable R_0 matrix such that $\det\left(\widetilde{D}_{pe} + R_0N_{pe}\widetilde{N}_{ce}\right) \neq 0$.

Denote the ith natural basis vector of $\mathbf{F}^{n+m}$ by e_i, and choose $R_1 = \begin{bmatrix} e_i & 0 & \cdots 0 \end{bmatrix}$ and $R_2 = (1+\theta)R_1$. Since $\det(\widetilde{D}_{pe}) \neq 0$, its rows are linearly independent. The rows of $\widetilde{D}_{pe} + R_jN_{pe}\widetilde{N}_{ce}$ must be linearly independent for $j = 1$ or $j = 2$. Consequently, by choosing $R_0 = R_1$ or $R_0 = R_2$ one has $\det(\widetilde{D}_{pe} + R_0N_{pe}\widetilde{N}_{ce}) \neq 0$. Since $\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix}\widetilde{D}_{pe}R_0\begin{bmatrix} I_n & 0 \end{bmatrix}^T \in \mathcal{M}(\mathbf{R})$ it follows that the ith column of $\theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix}\widetilde{D}_{pe}$ is stable. Varying i from one to $n+m$ completes the proof. ■

The next theorem is an extension of the solvability condition of Theorem 7.5.2 in [21]. The invariant factors of rational matrices are used in the proof of Theorem 7.5.2. They are not defined for general $\mathbf{R}$, so the theorem and its proof given here generalizes the result to a wider class of plants even if both coprime factorizations are assumed to exist.

Theorem III.2. *Let $P \in \mathcal{M}^{n\times m}(\mathbf{F})$ be a stabilizable plant having a coprime factorization, and let (N_{pe}, D_{pe}) be a right coprime factorization of P_e. The robust regulation problem is solvable if and only if $(\theta I_n, \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe})$ is left coprime.*

Proof: In order to show the necessity, assume that there exists a robustly regulating controller C. Lemma II.1 implies that there exists a left coprime factorization $(\widetilde{N}_{ce}, \widetilde{D}_{ce})$ of C_e satisfying $\widetilde{N}_{ce}N_{pe} + \widetilde{D}_{ce}D_{pe} = I$. Since C is robustly regulating,

$$\begin{aligned}V &:= \theta^{-1}(I+PC)^{-1} \\ &= \theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix}(I+P_eC_e)^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &= \theta^{-1}\begin{bmatrix} I_n & 0 \end{bmatrix}\left(I + N_{pe}\widetilde{N}_{ce}\right)\begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathcal{M}(\mathbf{R}),\end{aligned}$$

so

$$\theta V + \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} \widetilde{N}_{ce} \begin{bmatrix} I_n \\ 0 \end{bmatrix} = I,$$

which shows the coprimeness.

The sufficiency is shown by constructing a robustly regulating controller. To this end, assume that $\left(\theta I, \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe}\right)$ is left coprime, i.e. there exist $V, W \in \mathcal{M}(\mathbf{R})$ such that

$$\theta V + \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} W = I_n. \tag{4}$$

Let (N_{pe}, D_{pe}) and $(\widetilde{N}_{pe}, \widetilde{D}_{pe})$ be right and left coprime factorizations of P_e, respectively, and let $X, Y \in \mathcal{M}(\mathbf{R})$ be such that $\widetilde{N}_{pe} Y + \widetilde{D}_{pe} X = I$. One can assume that $\det(X) \neq 0$ since P and consequently P_e are stabilizable. By Lemma II.2, all stabilizing controllers of P_e have a right coprime factorization of the form $(X + D_{pe} R, X - N_{pe} R)$. By Lemma II.6, a robustly regulating controller exists if one is able to find $R \in \mathcal{M}(\mathbf{R})$ such that $\det(X - N_{pe} R) \neq 0$ and $\theta^{-1} \begin{bmatrix} I_n & 0 \end{bmatrix} (X - N_{pe} R) \in \mathcal{M}(\mathbf{R})$. Choose $R = \begin{bmatrix} W + \theta R_0 & 0 \end{bmatrix} X$, where $R_0 \in \mathcal{M}(\mathbf{R})$ is to be chosen appropriately. By (4), this choice yields

$$\begin{aligned} &\theta^{-1} \begin{bmatrix} I_n & 0 \end{bmatrix} (X - N_{pe} R) \\ &\quad = \theta^{-1} \begin{bmatrix} I_n - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} W - \theta \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} R_0 & 0 \end{bmatrix} X \\ &\quad = \begin{bmatrix} V - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} R_0 & 0 \end{bmatrix} X \in \mathcal{M}(\mathbf{R}). \end{aligned}$$

It remains to choose $R_0 \in \mathcal{M}(\mathbf{R})$ so that the determinant of $\left(I - N_{pe} \begin{bmatrix} W + \theta R_0 & 0 \end{bmatrix}\right) X$ is not zero which is equivalent to $\det\left(I - N_{pe} \begin{bmatrix} W + \theta R_0 & 0 \end{bmatrix}\right) \neq 0$ since $\det(X) \neq 0$. By using (4) one gets

$$\begin{aligned} &I - N_{pe} \begin{bmatrix} W + \theta R_0 & 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} I_n - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe}(W + \theta R_0) & 0 \\ -\begin{bmatrix} 0 & I_m \end{bmatrix} N_{pe}(W + \theta R_0) & I_m \end{bmatrix} \\ &\quad = \begin{bmatrix} \theta(V - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} R_0) & 0 \\ -\begin{bmatrix} 0 & I_m \end{bmatrix} N_{pe}(W + \theta R_0) & I_m \end{bmatrix}. \end{aligned}$$

Thus, R_0 must be chosen so that $\det(V - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} R_0) \neq 0$. Equation (4) shows that $\begin{bmatrix} V & \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} \end{bmatrix}$ is right invertible, so its columns span $\mathbf{F}^n$. Consequently, it is possible to choose R_0 so that $V - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} R_0$ has linearly independent columns, which completes the proof. ■

IV. Controller Design

Some ideas on design of robustly regulating controllers are discussed in this section. It is shown first that including the internal model in the plant and then stabilizing the combined system gives a robustly regulating controller. This is done in the next theorem. Actually, this leads to a parametrization of all robustly regulating controllers. Next a controller design where the problem is split into two main parts is presented. The main parts are: stabilization of the original plant and design of a robustly regulating controller for a numerator of the plant. An idea of how to do this can be found in the proof of Theorem III.2.

Theorem IV.1. *Assume that a plant $P \in \mathcal{M}^{n\times m}(\mathbf{F})$ has a coprime factorization and the robust regulation problem is solvable. A controller C solves the robust regulation problem if and only if it is of the form*

$$C = \theta^{-1} \begin{bmatrix} I_m & 0 \end{bmatrix} C_0 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{5}$$

where C_0 is a stabilizing controller of the plant $P_0 = \theta^{-1} P_e$.

Proof: In order to show necessity, assume that C solves the robust regulation problem. It follows that C_e stabilizes P_e. By Lemmas II.3 and II.1 there exists a left coprime factorization $(\widetilde{N}_{pe}, \widetilde{D}_{pe})$ of P_e and a right coprime factorization (N_{ce}, D_{ce}) of C_e such that

$$I = \widetilde{N}_{pe} N_{ce} + \widetilde{D}_{pe} \begin{bmatrix} \theta I_n & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \theta^{-1} I_n & 0 \\ 0 & I_m \end{bmatrix} D_{ce}. \tag{6}$$

Theorem III.1 implies that

$$D_0 := \begin{bmatrix} \theta^{-1} I_n & 0 \\ 0 & I_m \end{bmatrix} D_{ce} \in \mathcal{M}(\mathbf{R}).$$

Equation (6) and Lemma II.1 show that $C_0 := N_{ce} D_0^{-1}$ stabilizes P_0. A direct calculation shows that (5) holds with this choice.

The sufficiency is showed next. Assume that C is of the form (5). A robustly regulating controller C' exists by the assumptions. If $(\widetilde{N}_{pe}, \widetilde{D}_{pe})$ is a left coprime factorization of P_e then one can choose a right coprime factorization (N_{ce}, D_{ce}) of the extended controller C'_e of C' such that (6) is valid. This shows that P_0 has the coprime factorization

$$\left(\widetilde{N}_{pe}, \widetilde{D}_{pe} \begin{bmatrix} \theta I_n & 0 \\ 0 & I_m \end{bmatrix}\right).$$

Since C_0 stabilizes P_0, Lemma II.1 shows that it has a right coprime factorization (N_0, D_0) such that

$$\widetilde{N}_{pe} N_0 + \widetilde{D}_{pe} \begin{bmatrix} \theta I_n & 0 \\ 0 & I_m \end{bmatrix} D_0 = I.$$

This implies that

$$C'_0 := C_0 \begin{bmatrix} \theta^{-1} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

stabilizes P_e, which in turn implies by Lemma II.4 that $C = \begin{bmatrix} I_m & 0 \end{bmatrix} C'_0 \begin{bmatrix} I_n & 0 \end{bmatrix}^T$ stabilizes P. Since

$$\left(N_0, \begin{bmatrix} \theta I_n & 0 \\ 0 & I_m \end{bmatrix} D_0\right)$$

is a right coprime factorization of C'_0, Lemma II.6 implies that C regulates any plant it stabilizes. ■

Remark IV.2. *If one knows a coprime factorization of P then left and right coprime factorizations of P_e can be constructed algorithmically by using the methods of [11]. Thus, by* (6) *of the above proof one finds a left coprime factorization of P_0 and a similar construction gives a right coprime factorization. Consequently, Lemma II.2 gives a parameterization of all stabilizing controllers of P_0. This together with Theorem IV.1 provides a parametrization of all robustly regulating controllers.*

Theorem IV.3. *Let $P \in \mathcal{M}^{n\times m}(\mathbf{F})$ have a coprime factorization. Provided that the robust regulation problem is solvable* *a robustly regulating controller can be found by using the following procedure:*

1) *Find a stabilizing controller C_s for P.*
2) *Construct a right coprime factorization (N_{pe}, D_{pe}) of P_e.*
3) *Find a robustly regulating controller C_i for $\begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe}$, and define the stable matrices*

$$D_i := \left(I + \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} C_i\right)^{-1} \text{ and } N_i := C_i D_i.$$

4) *A robustly regulating controller for P is given by*

$$C := C_s D_i^{-1} + \begin{bmatrix} I_m & 0 \end{bmatrix} D_{pe} C_i. \tag{7}$$

Proof: One needs to show that (7) is robustly regulating. To this end, let $(\widetilde{N}_{pe}, \widetilde{D}_{pe})$ be a left coprime factorization of P_e and define the coprime factorization (N_{se}, D_{se}) of C_{se}—the extended matrix of C_s—by setting

$$D_{se} := \left(\widetilde{D}_{pe} + \widetilde{N}_{pe} C_{se}\right)^{-1} \quad \text{and} \quad N_{se} := C_{se} D_{se}.$$

A direct calculation shows that

$$\widetilde{N}_{pe} N_{se} + \widetilde{D}_{pe} D_{se} = I \tag{8}$$

Since $\left(\begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe}, I\right)$ is a right coprime factorization of $\begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe}$, Theorem III.2 shows that the third step is possible if and only if the robust regulation problem is solvable. The matrices defined in the third step satisfy

$$\begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} N_i + D_i = I_n. \tag{9}$$

Define

$$N_0 := N_{se} + D_{pe} \begin{bmatrix} N_i & 0 \end{bmatrix} D_{se} \tag{10}$$

and

$$\begin{aligned} D_0 &:= \begin{bmatrix} \theta^{-1} I_n & 0 \\ 0 & I_m \end{bmatrix} \left(I - N_{pe} \begin{bmatrix} N_i & 0 \end{bmatrix}\right) D_{se} \\ &= \begin{bmatrix} \theta^{-1} I_n & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n - \begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe} N_i & 0 \\ -\begin{bmatrix} 0 & I_m \end{bmatrix} N_{pe} N_i & I_m \end{bmatrix} D_{se} \\ &= \begin{bmatrix} \theta^{-1} D_i & 0 \\ -\begin{bmatrix} 0 & I_m \end{bmatrix} N_{pe} N_i & I_m \end{bmatrix} D_{se}, \end{aligned} \tag{11}$$

where the last equality follows by (9). Since C_i robustly regulates $\begin{bmatrix} I_n & 0 \end{bmatrix} N_{pe}$ and $(\text{bdiag}(N_i, 0), \text{bdiag}(D_i, I))$ is a right coprime factorization of its extended controller, Theorem III.1 shows that $\theta^{-1} D_i \in \mathcal{M}(\mathbf{R})$. Thus, $D_0 \in \mathcal{M}(\mathbf{R})$. By (8) – (11),

$$\begin{aligned} &\widetilde{N}_{pe} N_0 + \widetilde{D}_{pe} \begin{bmatrix} \theta I_n & 0 \\ 0 & I_m \end{bmatrix} D_0 \\ &\quad = I + \left(\widetilde{N}_{pe} D_{pe} - \widetilde{D}_{pe} N_{pe}\right) \begin{bmatrix} N_i & 0 \end{bmatrix} D_{se} \\ &\quad = I + \left(\widetilde{D}_{pe} P_e D_{pe} - \widetilde{D}_{pe} P_e D_{pe}\right) \begin{bmatrix} N_i & 0 \end{bmatrix} D_{se} = I \end{aligned}$$

This and Lemma II.1 show that

$$\begin{aligned} C_0 &:= N_0 D_0^{-1} \\ &= \left(C_{se} + D_{pe} \begin{bmatrix} N_i & 0 \end{bmatrix}\right) \begin{bmatrix} \theta D_i^{-1} & 0 \\ \begin{bmatrix} 0 & I_m \end{bmatrix} N_{pe} N_i D_i^{-1} & I_m \end{bmatrix} \\ &= \begin{bmatrix} \theta C_s D_i^{-1} & 0 \\ 0 & 0 \end{bmatrix} + D_{pe} \begin{bmatrix} \theta C_i & 0 \end{bmatrix}. \end{aligned} \tag{12}$$

stabilizes

$$P_0 := \left(\widetilde{D}_{pe} \begin{bmatrix} \theta I_n & 0 \\ 0 & I_m \end{bmatrix}\right)^{-1} \widetilde{N}_{pe} = \theta^{-1} P_e.$$

Theorem IV.1 implies that C defined by (7) is robustly regulating since $C = \theta^{-1} \begin{bmatrix} I_m & 0 \end{bmatrix} C_0 \begin{bmatrix} I_n & 0 \end{bmatrix}^T$ by (12). ■

V. An Example

In this section the robust regulation problem is solved for the plant

$$P(s) := (e^{-s/4} - 1)^{-1} \begin{bmatrix} s\left(e^{-s/2} - 1\right)/(s+1)^3 & 1/s \end{bmatrix}$$

and the generating element $\theta(s) = \frac{s^2+1}{(s+1)^2}$ by using the controller design procedure of Theorem IV.3. The plant P is the result of adding the internal model of 1/4-periodic functions to a retarded system with the transfer function

$$P_0(s) := \begin{bmatrix} s\left(e^{-s/2} - 1\right)/(s+1)^3 & 1/s \end{bmatrix}$$

when designing a repetitive controller [23]. The generating element is chosen so that θ^{-1} can generate $1/(s^2+1)$ which corresponds to the time domain reference signal $\sin(t)$.

Before applying Theorem III.2 note that it is not possible to stabilize P in the H^∞-framework [24], which implies it does not have a coprime factorization over H^∞. Thus, the ring $\mathbf{R} = \mathbf{P}$ from [10] is used. Denote $\mathbb{C}_\beta := \{s \in \mathbb{C} \mid \text{Re}(s) > \beta\}$, where $\beta \in \mathbb{R}$. By the definition, $\mathbf{P}$ is the set of all functions f that satisfy

1) f is analytic in the closure of $\mathbb{C}_0$,
2) f is uniformly bounded in $\mathbb{C}_\beta$ for all $\beta > 0$, and
3) there exist constants $M, \alpha > 0$ such that $|f(\mathrm{i}\omega)| \leq M(1+|\omega|)^\alpha$ for all $\omega \in \mathbb{R}$.

The first step of the design procedure is to find a stabilizing controller for P. To this end, it is shown that $\left(\widetilde{N}_p, \widetilde{D}_p\right)$ is a left coprime factorization of P over $\mathbf{P}$ if

$$\widetilde{N}_p(s) := \frac{1}{f(s)} \begin{bmatrix} s^2(e^{-s/2} - 1)/(s+1)^4 & 1/(s+1) \end{bmatrix}$$

and

$$\widetilde{D}_p(s) := \frac{1}{f(s)} \frac{s(e^{-s/4} - 1)}{s+1},$$

where $f(s) := s e^{-s/4}/(s+1) - 1 \in \mathbf{P}$.

In order to see that this actually is a left coprime factorization, note that $P = \widetilde{D}_p^{-1} \widetilde{N}_p$ and

$$\widetilde{N}_p(s) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \widetilde{D}_p(s) = 1. \tag{13}$$

Thus, $(\widetilde{N}_p, \widetilde{D}_p)$ is coprime if $\widetilde{N}_p$ and $\widetilde{D}_p$ are stable. This is clear if $1/f(s) \in \mathbf{P}$ which is shown next. Note that $|f(s)| \geq 1 - \left|e^{-s/4}\right| |s|/|s+1|$ if $\text{Re}(s) \geq 0$. This shows that

$$|f(s)| \geq 1 - \left|e^{-s/4}\right| > 1 - e^{-\beta/4} > 0$$

whenever $\text{Re}(s) > \beta > 0$, and

$$|f(s)| \geq 1 - |s|/|s+1| > 0$$

if $s \in \mathrm{i}\mathbb{R}$. Thus, $1/f(s) \in \mathbf{P}$. Equation (13) shows that $C_s := \begin{bmatrix} 0 & -1 \end{bmatrix}^T$ stabilizes P.

The second step of the procedure is to find a right coprime factorization for P_e. By following the proof of Lemma II.3 and [11], one finds the right coprime factorizations (N_{pe}, D_{pe}) of P_e where

$$N_{pe}(s) := \begin{bmatrix} \frac{s^2(e^{-s/2}-1)}{f(s)(s+1)^4} & \frac{1}{f(s)(s+1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$D_{pe}(s) := \begin{bmatrix} 1 & 0 & 0 \\ \frac{s^2(e^{-s/2}-1)}{f(s)(s+1)^4} & \frac{s(e^{-s/4}-1)}{f(s)(s+1)} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The third step is to construct a robustly regulating controller for $\begin{bmatrix}1 & 0 & 0\end{bmatrix} N_{pe}(s)$. The robust regulation problem is solvable by Theorem III.2 since

$$\theta + \begin{bmatrix}1 & 0 & 0\end{bmatrix} N_{pe}(s) \cdot \frac{2s}{s+1} f(s) \begin{bmatrix}0\\1\\0\end{bmatrix} = 1.$$

By Theorem III.1 and the above equation,

$$C_i(s) := \frac{2sf(s)}{(s+1)\theta(s)} \begin{bmatrix}0\\1\\0\end{bmatrix}$$

robustly regulates $\begin{bmatrix}1 & 0 & 0\end{bmatrix} N_{pe}(s)$. Define

$$D_i(s) := \left(1 + \begin{bmatrix}1 & 0 & 0\end{bmatrix} N_{pe}(s) C_i(s)\right)^{-1} = \frac{s^2+1}{(s+1)^2}.$$

Finally, the robustly regulating controller (7) is

$$C(s) = \frac{2s^2(e^{-s/4}-1) + 2s + 1 - (s+1)^2}{s^2+1} \begin{bmatrix}0\\1\end{bmatrix}.$$

One can use Theorem IV.1 to show that multiplying $C(s)$ by $(e^{-s/4}-1)^{-1}$ results in a controller of P_0 that robustly regulates all 1/4-periodic signals in addition to the signal $\sin(t)$. Technical details are omitted.

VI. Concluding Remarks

In this article, some classical results of frequency domain robust regulation were presented in an abstract algebraic framework. The results of this article require a left or a right coprime factorization of the plant but not both. While all stabilizable plants have a coprime factorization in many common algebraic structures in control theory [17], [20], that is not always the case, see [18] and the references therein. Even if there exist coprime factorizations they may be hard to find. This is why there is a need to develop the theory presented in this article towards a theory of robust regulation that is independent of coprime factorizations. A logical starting point would be the theory of stabilization based on general factorizations [12], [18].

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