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Polynomial Stability of Coupled Waves with Weak Indirect Damping

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Abstract: We study the stability properties of coupled one-dimensional wave equations with indirect damping. We employ methods based on observability estimates for the undamped system to prove polynomial stability and rational energy decay for the classical solutions of the coupled systems. We present our results for two different kinds of indirect damping — viscous damping and weak damping.

Keywords: Distributed parameter systems, coupled wave equations, polynomial stability, observability estimates, output feedback.
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1. INTRODUCTION

In this paper we investigate the stability properties of a system consisting of two one-dimensional wave equations. The wave equations share the same spatial domain, and they interact via a coupling inside the domain. Only one of the wave equations has damping, and this leads to the system losing uniform exponential stability. However, it has been demonstrated in (Santos et al., 2007; Alabau-Boussouira et al., 2011) that such systems instead exhibit *polynomial stability* (Borichev and Tomilov, 2010). We take a systems theoretic approach and prove stability properties of the coupled wave equations using observability estimates.

We first study the coupled wave equation system

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t) - \kappa v(\xi,t) - d(\xi)u_t(\xi,t) \\ v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t) - \kappa u(\xi,t) \\ u(0,t) = u(1,t) = 0, \qquad v(0,t) = v(1,t) = 0 \end{cases} \tag{1.1}$$

for $\xi \in (0,1)$ and $t > 0$ with initial conditions

$$\begin{cases} u(\xi,0) = u_0(\xi), \quad v(\xi,0) = v_0(\xi), \\ u_t(\xi,0) = u_1(\xi), \quad v_t(\xi,0) = v_1(\xi). \end{cases}$$

In the equations the constant $\kappa > 0$ is the *coupling parameter* which describes the interaction between the two waves, and the *damping profile* $d \in L^\infty(0,1)$ is assumed to satisfy $d \geq 0$ and

$$\operatorname*{ess\,inf}_{\xi\in[a,b]} d(\xi) \geq d_0 > 0 \tag{1.2}$$

for some $0 \leq a < b \leq 1$ and $d_0 > 0$. Note that without the coupling parameter the two wave equations in (1.1) would evolve independently from one another. We also emphasise that the damping only affects one of the wave equations, and therefore the other wave profile experiences only *indirect damping* (Russell, 1993). We assume throughout the paper that $0 < \kappa < \pi^2$.

The system (1.1) is not exponentially stable, and as our main result we show that its classical solutions converge to zero at a specific polynomial rate as $t \to \infty$. More precisely we show that there exist constants $M > 0$ and $t_0 > 0$ such that all solutions of (1.1) corresponding to initial states $u_0, v_0 \in H^2(0,1) \cap H_0^1(0,1)$ and $u_1, v_1 \in H_0^1(0,1)$ satisfy

$$\begin{aligned} &\|u_\xi(\cdot,t)\| + \|u_t(\cdot,t)\| + \|v_\xi(\cdot,t)\| + \|v_t(\cdot,t)\| \\ &\qquad \leq \frac{M}{t^{1/2}} \Big(\|u_0''\| + \|u_1'\| + \|v_0''\| + \|v_1'\| \Big) \end{aligned}$$

for $t \geq t_0$, where all norms are $L^2(0,1)$-norms.

In the case where $d(\xi)$ is a positive constant, the stability result for (1.1) can be deduced from (Santos et al., 2007; Alabau-Boussouira et al., 2011) (which also consider the equation on multi-dimensional spatial domains). These and other earlier articles often utilise energy methods or direct resolvent estimates for the coupled wave system. In this paper we complete the stability analysis by reinterpreting the damping in the system as a *negative output feedback* for an open-loop wave system without damping, and using the results in (Anantharaman and Léautaud, 2014) and (Chill et al., 2023) to prove polynomial stability based on suitable generalised observability estimates. The advantage is that the open-loop wave system without damping is relatively simple and easy to analyse. Because of this, the proof of the polynomial stability becomes relatively short compared to earlier proofs.

Our approach also adapts easily to different types of indirect damping in the coupled wave system. In Section 4 we consider a wave system with *weak indirect damping*, where the damping term $-d(\xi)u_t(\xi,t)$ in (1.1) is replaced with a *averaged* damping of the form

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$$-d_w(\xi)\int_0^1 d_w(r)u_t(r,t)dr$$

for some function $d_w \in L^2(0,1)$. We prove that under mild assumption on d_w, the coupled wave system is polynomially stable and establish a convergence rate for the classical solutions of the system based on the properties of d_w. To the authors' knowledge the stability properties of coupled wave equations with weak indirect damping have not been previously considered in the literature.

The stability of variants of (1.1) has been studied in the literature, both on one-dimensional and multi-dimensional spatial domains, for example in (Alabau et al., 2002; Alabau-Boussouira, 2002; Santos et al., 2007; Liu and Rao, 2007; Alabau-Boussouira et al., 2011; Abdallah et al., 2012) and (Guglielmi, 2017). The articles (Alabau et al., 2002; Santos et al., 2007) and (Abdallah et al., 2012) study systems that can be expressed as abstract second order equations. The references (Alabau-Boussouira, 2002; Liu and Rao, 2007) also consider boundary damping in one of the wave equations. Our results are most closely related to the ones presented in (Alabau et al., 2002; Santos et al., 2007; Alabau-Boussouira et al., 2011; Abdallah et al., 2012). Our results provide a new and shorter proofs for the wave system in the one-dimensional case and allow a non-constant damping parameter d. Moreover, the stability results for the system with weak indirect damping in Section 4 are completely new.

The paper is organised as follows. In Section 2 we formulate (1.1) as an abstract second order equation on a Hilbert space. Our first main results on stability is presented in Section 3. Section 4 is devoted to the stability analysis of the system with weak indirect damping. Finally, Section 5 contains concluding remarks.

Notation. For two Hilbert spaces X and Y we denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators $A: X \to Y$. If $A : \mathcal{D}(A) \subset X \to Y$ is a linear operator, we denote its domain by $\mathcal{D}(A)$ and kernel by $\mathcal{N}(A)$. If A is a closed operator, we denote by $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ its spectrum, point spectrum, and resolvent set, respectively. Unless otherwise mentioned, the space $\mathcal{D}(A)$ is equipped with the graph norm of A.

2. ABSTRACT FORMULATION

To utilise the results in (Anantharaman and Léautaud, 2014) and (Chill et al., 2023), we recast the coupled wave system (1.1) as an abstract second order equation of the form

$$\begin{cases}\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \\ w(0) = w_0, \ \dot{w}(0) = w_1\end{cases} \tag{2.1}$$

on a Hilbert space H, where $L : \mathcal{D}(L) \subset H \to H$ is a positive operator and $D \in \mathcal{L}(U,H)$ for some other Hilbert space U. If we choose the state $w(t)$ as

$$w(t) = \begin{pmatrix} u(\cdot,t) \\ v(\cdot,t)\end{pmatrix},$$

then we can choose the state space as $H = L^2(0,1) \times L^2(0,1)$. We define the operator L with domain $\mathcal{D}(L) = (H^2(0,1)\cap H_0^1(0,1)) \times (H^2(0,1)\cap H_0^1(0,1))$ by

$$Lf = \begin{pmatrix} -f_1'' + \kappa f_2 \\ \kappa f_1 - f_2''\end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2\end{pmatrix} \in \mathcal{D}(L). \tag{2.2}$$

Moreover, the damping operator $D \in \mathcal{L}(U,H)$ is defined by choosing $U = L^2(0,1)$ and defining $Df = (\sqrt{d}f, 0) \in H$ for all $f \in U$. For convenience, we also define $L_0 : \mathcal{D}(L_0) \subset L^2(0,1) \to L^2(0,1)$ to be the positive Dirichlet Laplacian, i.e., $L_0 f_0 = -f_0''$ for $f_0 \in \mathcal{D}(L_0) = H^2(0,1)\cap H_0^1(0,1)$.

Since L_0 is a positive operator and $\kappa > 0$, it is easy to check that L is a self-adjoint operator. Moreover, L has compact resolvents as a consequence of the Rellich–Kondrachov Theorem. The next lemma shows that L is a positive operator given that the coupling parameter κ is sufficiently small. In our case the smallest eigenvalue of L_0 is exactly $\lambda_1 = \pi^2$.

Lemma 2.1. Suppose λ_1 is the smallest eigenvalue of L_0. If $\kappa < \lambda_1$, then the operator L is a positive operator and $0 \in \rho(L)$.

Proof. Let $f = (f_1, f_2)^T \in \mathcal{D}(L)$ be arbitrary. Since $f_1, f_2 \in H_0^1(0,1)$, the Poincaré inequality implies that $\lambda_1\|f_k\|^2 \le \|f_k'\|^2$ for $k = 1,2$. Using the structure of L and integration by parts we obtain

$$\begin{aligned}\langle Lf, f\rangle_H &= \langle -f_1'' + \kappa f_2, f_1\rangle + \langle \kappa f_1 - f_2'', f_2\rangle \\ &= \langle f_1', f_1'\rangle + \langle f_2', f_2'\rangle + 2\kappa \operatorname{Re}\langle f_1, f_2\rangle \\ &\ge \|f_1'\|^2 + \|f_2'\|^2 - 2\kappa\|f_1\|\|f_2\| \\ &\ge \lambda_1\|f_1\|^2 + \lambda_1\|f_2\|^2 - 2\kappa\|f_1\|\|f_2\| \\ &\ge (\lambda_1 - \kappa)(\|f_1\|^2 + \|f_2\|^2) = (\lambda_1-\kappa)\|f\|^2.\end{aligned}$$

Since L is self-adjoint, this estimate implies that $L > 0$ and $0 \in \rho(L)$. □

3. STABILITY ANALYSIS

Since (1.1) can be expressed in the form (2.1), the coupled wave system has well-defined solutions determined by the associated strongly continuous semigroup (Engel and Nagel, 2000, Sec. VI.3). We can then use (Anantharaman and Léautaud, 2014, Thm. 2.3) to analyse the asymptotic behaviour of the solutions of (1.1). The following theorem is the first main result of the paper.

Theorem 3.1. Assume that $0 < \kappa < \pi^2$ and that $d \in L^\infty(0,1)$, $d \ge 0$, satisfies (1.2) for some $0 \le a < b \le 1$ and $d_0 > 0$. The system (1.1) is asymptotically stable, so that all mild solutions corresponding to the initial states $u_0, v_0 \in H_0^1(0,1)$ and $u_1, v_1 \in L^2(0,1)$ satisfy

$$\begin{aligned}\|u(\cdot,t)\| + \|u_\xi(\cdot,t)\| + \|u_t(\cdot,t)\| \to 0 \\ \|v(\cdot,t)\| + \|v_\xi(\cdot,t)\| + \|v_t(\cdot,t)\| \to 0\end{aligned}$$

as $t \to \infty$. In addition, there exist constants $M > 0$ and $t_0 > 0$ such that all solutions of (1.1) corresponding to initial states $u_0, v_0 \in H^2(0,1)\cap H_0^1(0,1)$ and $u_1, v_1 \in H_0^1(0,1)$ satisfy

$$\begin{aligned}&\|u_\xi(\cdot,t)\| + \|u_t(\cdot,t)\| + \|v_\xi(\cdot,t)\| + \|v_t(\cdot,t)\| \\ &\qquad \le \frac{M}{\sqrt{t}}\Big(\|u_0''\| + \|u_1'\| + \|v_0''\| + \|v_1'\|\Big)\end{aligned}$$

for $t \ge t_0$, where all norms are $L^2(0,1)$-norms.

We note that the initial conditions $u_0, v_0 \in H^2(0,1)\cap H_0^1(0,1)$ and $u_1, v_1 \in H_0^1(0,1)$, correspond to classical solutions of (1.1), and thus $u(0,t) = u(1,t) = 0$ and $v(0,t) =$

$v(1,t)=0$ for all $t\geq 0$. Since the Poincaré inequality implies $\|u(\cdot,t)\|\leq \pi^{-1}\|u_\xi(\cdot,t)\|$ and $\|u(\cdot,t)\|\leq \pi^{-1}\|u_\xi(\cdot,t)\|$, also the L^2-norms of $u(\cdot,t)$ and $v(\cdot,t)$ converge to zero at the rate $1/\sqrt{t}$ as $t\to\infty$. To prove Theorem 3.1, we first analyse the spectrum of L.

Proposition 3.2. Suppose the operator L is defined as in (2.2) with $0<\kappa<\pi^2$. Then

$$\sigma(L)=\sigma_p(L)=\{\pi^2n^2\pm\kappa : n\in\mathbb{N}\}$$

and $\mathcal{N}(\pi^2n^2\pm\kappa-L)=\operatorname{span}\{(\sin(\pi n\cdot),\pm\sin(\pi n\cdot))^T\}$ for all $n\in\mathbb{N}$.

Proof. It is straightforward to check that if $n\in\mathbb{N}$, then $\varphi^\pm=(\sin(\pi n\cdot),\pm\sin(\pi n\cdot))^T\in\mathcal{D}(L)$ are eigenvectors of L corresponding to the eigenvalues $\lambda^\pm=\pi^2n^2\pm\kappa$. Deducing that no other eigenvalues or eigenvectors can exist can be done by deriving the forms $\lambda^\pm=\pi^2n^2\pm\kappa$ and $\varphi^\pm=(\sin(\pi n\cdot),\pm\sin(\pi n\cdot))^T$ starting from the eigenvalue equation $(\lambda^\pm-L)\varphi^\pm=0$. This process is somewhat tedious but straightforward, see (Kosonen, 2023, Prop. 5.5) for details. Since L has compact resolvents, we finally have $\sigma(L)=\sigma_p(L)$. □

Proof of Theorem 3.1. If we define $x(t)=(w(t),\dot{w}(t))^T$, then equation (2.1) can be reformulated as a first-order-in-time system of the form

$$\dot{x}(t)=(A-BB^*)x(t),\qquad x(0)=(w_0,w_1)^T$$

on the space $X=\mathcal{D}(L^{1/2})\times H$ with operators $A:\mathcal{D}(A)\subset X\to X$ and $B\in\mathcal{L}(U,X)$ defined so that $\mathcal{D}(A)=\mathcal{D}(L)\times\mathcal{D}(L^{1/2})$,

$$A=\begin{pmatrix}0 & I\\ -L & 0\end{pmatrix},\qquad B=\begin{pmatrix}0\\ D\end{pmatrix},\qquad B^*=(0,D^*). \quad (3.1)$$

Since $L>0$, it follows from standard results that A generates a contraction semigroup on X (Engel and Nagel, 2000, Sec. VI.3), and since $-BB^*\leq 0$, also $A-BB^*$ generates a contraction semigroup $T(t)$ on X.

Our plan is to apply the stability result (Anantharaman and Léautaud, 2014, Thm. 2.3) which is based on the observability of the so-called *Schrödinger group* (D^*,iL). This result states that if the pair (D^*,iL) is exactly observable in the sense of (Tucsnak and Weiss, 2009, Def. 6.1.1), then there exist constants $M'>0$ and $t_0>0$ such that the solutions of (2.1) corresponding to the initial states $w_0\in\mathcal{D}(L)$ and $w_1\in\mathcal{D}(L^{1/2})$ satisfy

$$\|L^{1/2}w(t)\|_H+\|\dot{w}(t)\|_H\leq\frac{M'}{t^{1/2}}\Big(\|Lw_0\|_H+\|L^{1/2}w_1\|_H\Big)$$

for $t\geq t_0$ [2]. Since L has the form

$$L=\begin{pmatrix}L_0 & 0\\ 0 & L_0\end{pmatrix}+\kappa\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

where the second term is a bounded operator and the first term is boundedly invertible, it is easy to check that the norms defined by $f\to\|Lf\|$ and $f\to\|\operatorname{diag}(L_0,L_0)f\|$ are equivalent. The result (Kato, 1961, Corollary) therefore further shows that $\mathcal{D}(L^{1/2})=\mathcal{D}(L_0^{1/2})\times\mathcal{D}(L_0^{1/2})$ and also the norms defined by $f\to\|L^{1/2}f\|$ and $f\to\|\operatorname{diag}(L_0^{1/2},L_0^{1/2})f\|$ are equivalent. Since $L_0f=-f''$ for $f\in\mathcal{D}(L_0)=H^2(0,1)\cap H_0^1(0,1)$, we have $\mathcal{D}(L_0^{1/2})=H_0^1(0,1)$ and $\|L_0^{1/2}f\|_{L^2}=\|f'\|_{L^2}$. Therefore the above equivalences of the norms imply that there exist constants $c_k>0$ for $k\in\{1,2,3,4\}$ such that

$$c_1\big(\|f_1'\|+\|f_2'\|\big)\leq\|L^{1/2}f\|\leq c_2\big(\|f_1'\|+\|f_2'\|\big)$$
$$c_3\big(\|g_1''\|+\|g_2''\|\big)\leq\|Lg\|\leq c_4\big(\|g_1''\|+\|g_2''\|\big)$$

for all $f=(f_1,f_2)\in\mathcal{D}(L^{1/2})=H_0^1(0,1)\times H_0^1(0,1)$ and $g=(g_1,g_2)\in\mathcal{D}(L)$. Because of this, there exist $c_5,c_6>0$ such that for all solutions $w(t)=(u(\cdot,t),v(\cdot,t))^T$ corresponding to initial states $u_0,v_0\in H^2(0,1)\cap H_0^1(0,1)$ and $u_1,v_1\in H_0^1(0,1)$ we have

$$\|u_\xi(\cdot,t)\|+\|u_t(\cdot,t)\|+\|v_\xi(\cdot,t)\|+\|v_t(\cdot,t)\|\leq c_5\Big(\|L^{1/2}w(t)\|_H+\|\dot{w}(t)\|_H\Big)$$

$$\|Lw_0\|_H+\|L^{1/2}w_1\|_H\leq c_6\big(\|u_0''\|+\|u_1'\|+\|v_0''\|+\|v_1'\|\big).$$

This implies that the convergence rate in the claim follows from (Anantharaman and Léautaud, 2014, Thm. 2.3) if (D^*,iL) is exactly observable. We note that since the semigroup $T(t)$ generated by $A-BB^*$ is uniformly bounded (more precisely contractive), this estimate in (Anantharaman and Léautaud, 2014, Thm. 2.3) together with (Batty and Duyckaerts, 2008, Thm. 1.1) and (Engel and Nagel, 2000, Thm. V.2.21) also implies that $T(t)$ is strongly stable, i.e., $\|L^{1/2}w(t)\|+\|\dot{w}(t)\|\to 0$ as $t\to\infty$ for all $w_0\in\mathcal{D}(L)$ and $w_1\in H$, implying the first claim of the theorem.

Thus it only remains to show that the pair (D^*,iL) is exactly observable. Since iL is a skew-adjoint operator with compact resolvents and simple uniformly separated eigenvalues $\sigma_p(iL)=\{i\pi^2n^2\pm i\kappa : n\in\mathbb{N}\}$, we have from (Tucsnak and Weiss, 2009, Thm. 6.9.3) that the pair (D^*,iL) is exactly observable if there exists $c_0>0$ such that

$$\|D^*\phi_n\|_U\geq c_0\|\phi_n\|_H$$

for every eigenvector ϕ_n of L. We have $D^*f=\sqrt{d}f_1$ for $f=(f_1,f_2)\in H$ due to the definition of $D\in\mathcal{L}(L^2(0,1),H)$. By assumption, the function $d\in L^\infty(0,1)$ satisfies (1.2) for some $0\leq a<b\leq 1$ and $d_0>0$. We have from Proposition 3.2 that the eigenvectors of iL corresponding to $i\lambda_n^\pm=i\pi^2n^2\pm i\kappa$ have the forms $\varphi_n^\pm=(\sin(\pi n\cdot),\pm\sin(\pi n\cdot))^T$. The properties of trigonometric functions imply that there exists $q>0$ such that $\|\sin(\pi n\cdot)\|_{L^2(a,b)}\geq q\|\sin(\pi n\cdot)\|_{L^2(0,1)}$ for all $n\in\mathbb{N}$. Thus we can estimate

$$\|D^*\varphi_n^\pm\|_U^2=\|\sqrt{d}\sin(\pi n\cdot)\|_{L^2(0,1)}^2\geq d_0\int_a^b\sin^2(\pi n\xi)d\xi$$
$$\geq d_0q^2\|\sin(\pi n\cdot)\|_{L^2(0,1)}^2=\frac{d_0q^2}{2}\|\varphi_n^\pm\|_H^2.$$

Thus (D^*,iL) is exactly observable by (Tucsnak and Weiss, 2009, Thm. 6.9.3) and the proof is complete. □

4. THE WAVE SYSTEM WITH INDIRECT WEAK DAMPING

In this section we consider a version of the coupled wave system (1.1) where the viscous damping has been replaced with *weak damping*. More precisely, we study the system

[2] In (Anantharaman and Léautaud, 2014, Thm. 2.3) the right-hand side of the estimate has an additional term $\|DD^*w_1\|$, but since $0\in\rho(L^{1/2})$, this norm is bounded by a constant times $\|L^{1/2}w_1\|$.

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t) - \kappa v(\xi,t) - d_w(\xi)\int_0^1 d_w(r)u_t(r,t)dr \\ v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t) - \kappa u(\xi,t) \\ u(0,t) = u(1,t) = 0, \qquad v(0,t) = v(1,t) = 0 \end{cases} \tag{4.1}$$

with initial conditions

$$\begin{cases} u(\xi,0) = u_0(\xi), \quad v(\xi,0) = v_0(\xi), \\ u_t(\xi,0) = u_1(\xi), \quad v_t(\xi,0) = v_1(\xi). \end{cases}$$

In this model the *damping parameter* is a function $d_w \in L^2(0,1)$. The damping in (4.1) acts in an averaged manner, and this generally causes the damping to be weaker than viscous damping. This feature is understood well in the case of a single wave equation, where the viscous damping achieves uniform exponential stability, whereas weak damping results in polynomial stability (Russell, 1969, Thm. 1). Moreover, polynomial decay rate depends on the properties of the function d_w (Chill et al., 2023, Sec. 6B2).

Also the weakly damped system (4.1) can be expressed as an abstract second order equation (2.1) on $H = L^2(0,1) \times L^2(0,1)$ with the same operator $L : \mathcal{D}(L) \subset H \to H$ as before, and with the operator $D \in \mathcal{L}(\mathbb{C}, H)$ defined by

$$Du = \begin{pmatrix} d_w u \\ 0 \end{pmatrix}, \qquad u \in \mathbb{C}.$$

Because of this, the coupled wave system (4.1) has well-defined solutions which are determined by a strongly continuous semigroup (Engel and Nagel, 2000, Sec. VI.3).

The following theorem is the second main result of the paper. The result gives conditions for the polynomial stability of the coupled wave system (4.1) with weak indirect damping in terms of the Fourier sine coefficients of the damping function d_w.

Theorem 4.1. Assume that $0 < \kappa < \pi^2$. Denote the (scaled) Fourier sine coefficients of $d_w \in L^2(0,1)$ by

$$d_n = \int_0^1 d_w(\xi)\sin(\pi n\xi)d\xi, \qquad n \in \mathbb{N}.$$

Assume that there exist $\beta, K > 0$ such that $|d_n| \ge K/n^\beta$ for all $n \in \mathbb{N}$. Then system (4.1) is asymptotically stable, so that for all solutions corresponding to the initial states $u_0, v_0 \in H_0^1(0,1)$ and $u_1, v_1 \in L^2(0,1)$ satisfy

$$\|u(\cdot,t)\| + \|u_\xi(\cdot,t)\| + \|u_t(\cdot,t)\| \to 0$$
$$\|v(\cdot,t)\| + \|v_\xi(\cdot,t)\| + \|v_t(\cdot,t)\| \to 0$$

as $t \to \infty$. In addition, there exist constants $M > 0$ and $t_0 > 0$ such that all solutions of (4.1) corresponding to initial states $u_0, v_0 \in H^2(0,1) \cap H_0^1(0,1)$ and $u_1, v_1 \in H_0^1(0,1)$ satisfy

$$\begin{aligned} &\|u(\cdot,t)\| + \|u_t(\cdot,t)\| + \|v(\cdot,t)\| + \|v_t(\cdot,t)\| \\ &\qquad \le \frac{M}{t^{1/(2+2\beta)}}\left(\|u_0''\| + \|u_1'\| + \|v_0''\| + \|v_1'\|\right) \end{aligned}$$

for $t \ge t_0$, where all norms are $L^2(0,1)$-norms.

The system (4.1) is not exponentially stable. Moreover, if $d_n = 0$ for some $n \in \mathbb{N}$, then (1.1) is not asymptotically stable.

Proof. If we define $x(t) = (w(t), \dot{w}(t))^T$, we can again express equation (2.1) as a first-order-in-time system of the form

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = (w_0, w_1)^T$$

on the space $X = \mathcal{D}(L^{1/2}) \times H$ with operators $A : \mathcal{D}(A) \subset X \to X$ and $B \in \mathcal{L}(U, X)$ defined by (3.1) and $\mathcal{D}(A) = \mathcal{D}(L) \times \mathcal{D}(L^{1/2})$. We begin by noting that since A generates a unitary group on the infinite-dimensional space X and since the operator BB^* is of rank one, the semigroup $T(t)$ generated by $A - BB^*$ cannot be exponentially stable (Engel and Nagel, 2000, Prop. IV.2.12).

Moreover, if $d_n = 0$ for some fixed $n \in \mathbb{N}$, then the definition of D implies that $D^*\varphi^\pm = \langle \sin(\pi n \cdot), d_w\rangle = \overline{d_n} = 0$. If we define $\psi_n = (\varphi_n^+, i\sqrt{\pi^2 n^2 + \kappa}\varphi_n^+)^T$, then a direct computation shows that $A\psi_n = i\sqrt{\pi^2 n^2 + \kappa}\psi_n$ and

$$\begin{aligned} (i\sqrt{\pi^2 n^2 + \kappa} - A + BB^*)\psi_n &= BB^*\psi_n \\ &= \begin{pmatrix} 0 \\ i\sqrt{\pi^2 n^2 + \kappa} DD^*\varphi_n^+ \end{pmatrix} = 0. \end{aligned}$$

Thus $\sigma(A - BB^*) \cap i\mathbb{R} \neq \varnothing$ and the semigroup generated by $A - BB^*$ is not asymptotically stable.

Assume now that $|d_n| \ge K/n^\beta$ for all $n \in \mathbb{N}$. We will use the stability result (Chill et al., 2023, Thm. 3.9) which utilises so-called "wavepackets" associated to the operator $L^{1/2}$. This result will imply that there exist constants $M' > 0$ and $t_0 > 0$ such that the solutions of (2.1) corresponding to the initial states $w_0 \in \mathcal{D}(L)$ and $w_1 \in \mathcal{D}(L^{1/2})$ satisfy

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \le \frac{M'}{t^{1/(2+2\beta)}}\left(\|Lw_0\| + \|L^{1/2}w_1\|\right) \tag{4.2}$$

for $t \ge t_0$. Equivalences of the graph norms associated to the operators L and $L^{1/2}$ will allow us to deduce the main decay estimate in the claim exactly as in the proof of Theorem 3.1. Moreover, the above estimate together with the uniform boundedness of $T(t)$ again implies that $T(t)$ is asymptotically stable, proving the first claim of the theorem.

To apply (Chill et al., 2023, Thm. 3.9), we will define the *wavepackets* $w \in \mathrm{WP}_{s,\delta_0(s)}(L^{1/2})$ of the operator $L^{1/2}$. According to (Chill et al., 2023, Def. 3.4), for $s \in \mathbb{R}$ and $\delta_0(s) > 0$ the set $\mathrm{WP}_{s,\delta_0(s)}(L^{1/2})$ is defined as the spectral subspace of the positive operator $L^{1/2}$ associated to the interval $(s - \delta_0(s), s + \delta_0(s)) \subset \mathbb{R}$. Proposition 3.2 implies that we have $\sigma(L^{1/2}) = \{\sqrt{\lambda} : \lambda \in \sigma_p(L)\} = \{\sqrt{\pi^2 n^2 \pm \kappa} : n \in \mathbb{N}\}$, and thus

$$\mathrm{WP}_{s,\delta_0(s)}(L^{1/2}) = \mathrm{span}\{\varphi_n^\pm : |s - \sqrt{\pi^2 n^2 \pm \kappa}| < \delta_0(s)\}$$

(where we interpret $\mathrm{span}\,\varnothing = \{0\}$). The results (Chill et al., 2023, Thm. 3.9 & Lem. 2.5(b)) imply that if there exist exponents $\gamma, \kappa > 0$ and constants $c_0, d_0 > 0$ such that

$$\|D^*w\| \ge \frac{d_0}{1 + s^\gamma}\|w\|, \quad s \ge 0, w \in \mathrm{WP}_{s,\delta_0(s)}(L^{1/2}) \tag{4.3}$$

where $\delta_0(s) = c_0/(1 + s^\kappa)$, then $i\mathbb{R} \subset \rho(A - BB^*)$ and there exists $M_R > 0$ such that

$$\|(is - A + BB^*)^{-1}\| \le M_R(1 + |s|^{\gamma+\kappa}), \qquad s \in \mathbb{R}.$$

If $2(\gamma + \kappa) \le 2 + 2\beta$, then we finally have by (Borichev and Tomilov, 2010, Thm. 2.4) that

$$\|T(t)x\| \leq \frac{M''}{t^{1/(2\beta+2)}}\|(A-BB^*)x\| \\ \leq \frac{M''}{t^{1/(2\beta+2)}}\|(I-BB^*A^{-1})\|\|Ax\|$$

for all $x \in \mathcal{D}(A)$, and this immediately implies (4.2).

Our aim is to choose the parameters $\kappa, c_0 > 0$ of $\delta_0(s) = c_0/(1+s^\kappa)$ in such a way that for every $s \geq 0$ the interval $(s-\delta_0(s), s+\delta_0(s))$ only contains a single spectral point of $L^{1/2}$. In this situation for each $s \geq 0$ the space $\mathrm{WP}_{s,\delta_0(s)}(L^{1/2})$ is either a one-dimensional space spanned by a single eigenvector $\varphi_n^\pm$ of $L^{1/2}$, or alternatively the trivial subspace $\{0\}$. The eigenvalues of $L^{1/2}$ are exactly $\mu_n^\pm = \sqrt{\pi^2 n^2 \pm \kappa}$. We first note that $\kappa < \pi^2$ implies $|\mu_n^+ - \mu_{n+1}^-| \geq \sqrt{4\pi^2-\kappa} - \sqrt{\pi^2+\kappa} > (\sqrt{3}-\sqrt{2})\pi > 0$. Thus $\mu_n^\pm$ and $\mu_m^\pm$ are uniformly separated for $n \neq m$. On the other hand, for every $n \in \mathbb{N}$ we have

$$\mu_n^+ - \mu_n^- = \frac{\pi^2 n^2 + \kappa - (\pi^2 n^2 - \kappa)}{\sqrt{\pi^2 n^2+\kappa} + \sqrt{\pi^2 n^2 - \kappa}} \geq \frac{\kappa}{\sqrt{\pi^2 n^2+\kappa}} = \frac{\kappa}{\mu_n^+}.$$

This implies that we if we choose $\delta_0(s) = c_0/(1+s^\kappa)$ where $\kappa = 1$ and $c_0 > 0$ is sufficiently small, then for every $s \geq 0$ the space $\mathrm{WP}_{s,\delta_0(s)}(L^{1/2})$ is either one-dimensional (spanned by a single eigenvector φ_n^- or φ_n^+ for some $n \in \mathbb{N}$) or trivial. To verify condition (4.3) it is therefore sufficient to consider elements $w \in \mathrm{WP}_{s,\delta_0(s)}(L^{1/2})$ of the form $w = c\varphi_n^\pm$ for $n \in \mathbb{N}$ and $c \in \mathbb{C}$. Our assumption $|d_n| \geq K/n^\beta$ and $\|\varphi_n^\pm\| = 1$ imply that

$$\|D^*w\|_U = \|D^*(c\varphi_n^\pm)\|_U = |c|\left|\int_0^1 d_w(r)\sin(\pi n r)\,dr\right| \\ = |d_n|\|w\| \geq K n^{-\beta}\|w\|_H.$$

If $s > 0$ and $w = c\varphi_n^\pm \in \mathrm{WP}_{s,\delta_0(s)}(L^{1/2})$, then $n \in \mathbb{N}$ is such that $|s-\mu_n^\pm| < \delta_0(s)$. Since $\mu_n^\pm \sim \pi n$ for $n \in \mathbb{N}$ large, the above lower bound for $\|D^*w\|$ implies that if we choose $\gamma = \beta$, then (4.3) holds for a sufficiently small constant $d_0 > 0$. Since our choices of $\kappa = 1$ and $\gamma = \beta$ satisfy $2(\gamma+\kappa) = 2+2\beta$, the decay estimate (4.2) follows from (Chill et al., 2023, Thm. 3.9) and (Borichev and Tomilov, 2010, Thm. 2.4) in the way described above. □

The damping function $d_w \in L^2(0,1)$ and its Fourier sine coefficients play an important role in the stability of the system (4.1). We can obtain an explicit expression for d_n — and consequently an explicit decay rate — for a large class of damping functions. Table 1 contains this information for a few particular example functions.

$d(\xi)$	ξ	ξ^2	$\xi^2(1-\xi)$
d_n	$\frac{(-1)^n}{\pi n}$	$\frac{2((-1)^n-1)-(-1)^n n^2\pi^2}{n^3\pi^3}$	$\frac{2(2(-1)^n-1)}{\pi^3 n^3}$
β	1	1	3
Decay rate	$t^{-1/4}$	$t^{-1/4}$	$t^{-1/8}$

Table 1. The scaled Fourier sine coefficients d_n and decay rates for selected $d_w \in L^2(0,1)$.

The Theorem 4.1 guarantees polynomial stability if the damping function $d_w \in L^2(0,1)$ is such that $|d_n| \geq K/n^\beta$ for all $n \in \mathbb{N}$. Conversely we can ask if it is possible to find a suitable damping function $d_w \in L^2(0,1)$ that achieves a given decay rate. Fortunately, Theorem 4.1 allows us to construct such a damping function under some additional assumptions.

Corollary 4.2. Suppose $0 < \alpha < 1/3$. With the damping function $d_w \in L^2(0,1)$ defined by

$$d_w(\xi) = \sum_{n=1}^\infty n^{1-\frac{1}{2\alpha}}\sin(\pi n\xi), \qquad \xi \in [0,1],$$

the system (4.1) is polynomially stable and there exist constants $M > 0$ and $t_0 > 0$ such that all solutions of (4.1) corresponding to initial states $u_0, v_0 \in H^2(0,1) \cap H_0^1(0,1)$ and $u_1, v_1 \in H_0^1(0,1)$ satisfy

$$\|u_\xi(\cdot,t)\| + \|u_t(\cdot,t)\| + \|v_\xi(\cdot,t)\| + \|v_t(\cdot,t)\| \\ \leq \frac{M}{t^\alpha}\Big(\|u_0''\| + \|u_1'\| + \|v_0''\| + \|v_1'\|\Big)$$

for $t \geq t_0$.

Proof. Let $0 < \alpha < 1/3$. Since $(n^{1-1/(2\alpha)})_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$, we have that d_w defined in the claim satisfies $d_w \in L^2(0,1)$. Moreover, since its Fourier sine coefficients are given by $d_n = 2^{-1}n^{1-1/(2\alpha)}$, the condition $|d_n| \geq K/n^\beta$ holds for a constant $K > 0$ and for all $n \in \mathbb{N}$ if we choose $\beta = -1 + 1/(2\alpha)$. Since $1/(2+2\beta) = \alpha$, the claim follows from Theorem 4.1. □

Theorem 4.1 implies that if we aim to construct a damping function corresponding to fastest decay rate for classical solutions, we should choose a d_w whose Fourier sine coefficients decay as slowly as possible. Since $\sqrt{2}\sin(\pi n\cdot)$ are the orthonormal eigenvectors of the (positive) Dirichlet Laplacian $L_0 : \mathcal{D}(L_0) \subset L^2(0,1) \to L^2(0,1)$, we have for $m \in \mathbb{N}$

$$\mathcal{D}(L_0^m) \\ = \{f \in H^{2m}(0,1) : f^{(2k)} \in H_0^1(0,1), 0 \leq k \leq m-1\} \\ = \{f \in L^2(0,1) : \sum_{n\in\mathbb{N}} n^{2m}|\langle f, \sin(\pi n\cdot)\rangle|^2 < \infty\}.$$

Because of this, a damping function satisfying $d_w \in \mathcal{D}(L_0^m)$ for a large $m \in \mathbb{N}$ will result in slow decay of solutions. If we want to achieve a good degree of stability it is necessary to look for a damping function such that $d_w \in \mathcal{D}(L_0^m)$ for a small $m \in \mathbb{N}$. We in particular note that $d_w \notin \mathcal{D}(L_0)$ if the function violates either the regularity requirement ($d_w \notin H^2(0,1)$) or the boundary conditions ($d_w(0) \neq 0$ or $d_w(1) \neq 0$). These two factors are not by itself a guarantee of a fast decay rate, since for instance the characteristic function $\chi_{[0,1/2]}$ on the interval $[0,1/2]$ satisfies $\chi_{[0,1/2]} \notin H^1(0,1)$ and $\chi_{[0,1/2]}(0) \neq 0$, but $d_w = \chi_{[0,1/2]}$ does not stabilise (4.1) due to the fact that $d_{4k} = 0$ for all $k \in \mathbb{N}$. However, the requirement $d_w \notin \mathcal{D}(L_0)$ can nevertheless be used as a starting point for constructing dampings which result in fast decay.

5. CONCLUSIONS

We have investigated the stability properties of coupled systems of waves with indirect damping. We have considered two different types of damping, namely, viscous damping and weak averaged damping. In both of the cases we have proved that the classical solutions of the system decay at a rational rate as $t \to \infty$. In the proofs we have utilised methods that are based on the observability estimates for the undamped system. Finally, we have discussed the effect of the damping function on the decay rate in the case of the weak damping.

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