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Robust Regulation for First-Order Port-Hamiltonian Systems

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***Abstract*— We present a method for obtaining robust control over a first-order port-Hamiltonian system. The presented method is especially designed for controlling impedance energy-preserving port-Hamiltonian systems. By combining the stabilization results of port-Hamiltonian systems and the theory of robust output regulation for exponentially stable systems, we design a simple finite-dimensional controller for an unstable system that together with output feedback achieves robust output regulation. The method is demonstrated on an example where we implement a robust regulating controller for the one-dimensional wave equation with boundary control and observation.**

I. Introduction

The class of port-Hamiltonian systems provides means for considering Hamiltonian differential equations that interact with their environment via the boundaries of the spatial domain. Such a concept is essential for considering boundary control and boundary observation for these systems. Hamiltonian differential equations include linear and non-linear differential equations, and they occur in many physical models. The class of port-Hamiltonian systems includes models of flexible structures, traveling waves, heat exchangers, bioreactors and, in general, lossless and dissipative hyperbolic systems in one-dimensional spatial domain. [6], [14]

Stability and stabilization of port-Hamiltonian systems has been considered by Villegas et. al. ([13],[14]) and Ramirez et. al. ([10],[11]), where both static and dynamic feedback has been studied. The specific structure of impedance energy-preserving systems has been described in [3] and [13], in latter of which stabilization of such systems has been considered as well as the conditions under which exponential stability is achieved.

The Internal Model Principle (IMS) is the key to understanding how control systems can be robust, i.e., tolerate perturbations in the parameters of the system. The type of robust controller (low-gain controller) proposed by Davison [2] for stable systems has many practical advantages. The structure of the controller is simple and it can be tuned with input-output experiments from the open loop system. The controller was generalized to infinite-dimensional systems and its tuning process was simplified in [4], [5].

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In this paper, the control objective is to design for port-Hamiltonian systems a low-gain controller that stabilizes the closed loop system and achieves robust regulation in the sense of [5], that is, the controller exponentially asymptotically tracks the reference signal y_{ref}, exponentially asymptotically rejects the boundary disturbance signal w and tolerates some perturbations in the plant.

The main contribution of this paper is that we design a simple finite-dimensional robust regulating controller for an unstable, impedance energy-preserving port-Hamiltonian system. By using the stabilization and stability results of [6] and [13] for port-Hamiltonian systems, we can utilize the controller introduced by the authors of [4], [5] for exponentially stable systems, even though the system to be controlled in this paper is initially unstable. As far as the authors know, this is the first time robust regulation is considered for port-Hamiltonian systems.

The structure of this paper is as follows. In Section III we give some background to port-Hamiltonian systems and describe impedance energy-preserving systems. In Section IV we describe the control system including the plant, the exosystem and the controller. In Section V we formulate the robust output regulation problem and define the internal model principle. In Section VI we construct the robust controller and prove that it solves the robust output regulation problem. In Section VII, we implement a controller for a vibrating string (the one-dimensional wave equation) as an example to illustrate the theory. Finally, in section VIII we conclude this paper.

II. Notation

Here $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, null space and resolvent of a linear operator A are denoted by $\mathcal{D}(A), \mathcal{R}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively. A strongly continuous (C_0-) semigroup $T_A(t)$ generated by A is exponentially stable if there are positive constants M and α such that $||T_A(t)|| \leq Me^{-\alpha t}$. We denote k times continuously differentiable functions from the interval $[a,b]$ to $\mathbb{C}^n$ by $C^k([a,b];\mathbb{C}^n)$. Furthermore, the set $H^1([a,b];\mathbb{C}^n)$ is defined as $H^1([a,b];\mathbb{C}^n) = \{f \in L^2([a,b];\mathbb{C}^n) \,|\, f$ is absolutely continuous and $\partial_\zeta f \in L^2([a,b];\mathbb{C}^n)\}$, where $L^2([a,b];\mathbb{C}^n)$ is the set of square integrable functions from the interval $[a,b]$ to $\mathbb{C}^n$.

III. Background on Port-Hamiltonian Systems

Consider a linear first order port-Hamiltonian system on the spatial interval $\zeta \in [a,b]$, given by

$$\frac{\partial}{\partial t}x(\zeta,t) = \mathcal{A}x(\zeta,t), \quad x(0) = x_0, \tag{1a}$$

$$u(t) = \mathcal{B}x(\cdot,t), \tag{1b}$$

$$y(t) = \mathcal{C}x(\cdot,t), \tag{1c}$$

where the operators $\mathcal{B}$ and $\mathcal{C}$ will be defined shortly, and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}x(\zeta,t) := P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t), \tag{2}$$

where $\mathcal{H}, P_1$ and P_0 satisfy the following assumptions [6]

Assumption 1: $P_1 \in \mathbb{C}^{n\times n}$ is invertible and self-adjoint $P_0 \in \mathbb{C}^{n\times n}$ is skew-adjoint, and $\mathcal{H}(\zeta) \in C^1([a,b];\mathbb{C}^{n\times n})$ such that $\mathcal{H}(\zeta)$ is self-adjoint for all $\zeta \in [a,b]$, and there exists $M,m > 0$ such that $mI \le \mathcal{H}(\zeta) \le MI$ for all $\zeta \in [a,b]$.

The state-space is defined as $X = L^2([a,b];\mathbb{C}^n)$ with the inner product

$$\langle f,g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^*\mathcal{H}(\zeta)f(\zeta)d\zeta, \tag{3}$$

and hence, X is a Hilbert space. The Hamiltonian associated with the port-Hamiltonian system (1) satisfying Assumption 1 is given by $E(t) = \langle x(\cdot,t), x(\cdot,t)\rangle_X = ||x(\cdot,t)||_X^2$.

By using the concepts of boundary effort e_∂ and a boundary flow f_∂, defined by

$$\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix}, \tag{4}$$

the operators $\mathcal{B} : \mathcal{H}^{-1}(H^1([a,b],\mathbb{C}^n)) \to \mathbb{C}^n$ and $\mathcal{C} : \mathcal{H}^{-1}(H^1([a,b],\mathbb{C}^n)) \to \mathbb{C}^n$ of equations (1b)–(1c) are given in the the form

$$\mathcal{B}x(\cdot,t) := W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{5a}$$

$$\mathcal{C}x(\cdot,t) := W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{5b}$$

where $W_B, W_C \in \mathbb{C}^{n\times 2n}$. Let us define the matrix $\Sigma \in \mathbb{R}^{2n\times 2n}$ by

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{6}$$

in order to make the following assumption on the matrices W_B and W_C:

Assumption 2: The matrices W_B and W_C have full rank, W_B satisfies $W_B\Sigma W_B^* \ge 0$, and the matrix $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible.

Before continuing, we need to define the concept of boundary control systems [1]

Definition 1: The control system (1a)–(1b) is a boundary control system if the following hold:

1) The operator $A : \mathcal{D}(A) \to X$ with $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$ and $Ax = \mathcal{A}x$ for $x \in \mathcal{D}(A)$ is the infinitesimal generator of a C_0-semigroup on X.
2) There exists an operator $B \in \mathcal{L}(U,X)$ such that for all $u \in U$, $Bu \in \mathcal{D}(\mathcal{A})$, the operator $\mathcal{A}B$ is an element of $\mathcal{L}(U,X)$ and $\mathcal{B}Bu = u$ for $u \in U$.

Note that even though the above definition does not mention the output $y(t) = \mathcal{C}x(t)$ in any way, we will consider it being a part of the boundary control system.

Consider now the system described by equations (1a)–(1c) satisfying Assumption 1 with $\mathcal{B}$ and $\mathcal{C}$ defined in equations (5a)–(5b) with W_B and W_C satisfying Assumption 2. Due to [3, Thm. 4.2] we have that the system described by equations (1a)–(1c) is a boundary control system on X, and the operator $\mathcal{A}$ with domain

$$\mathcal{D}(\mathcal{A}) = \left\{\mathcal{H}x \in H^1([a,b],\mathbb{C}^n) \,\middle|\, W_B\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0\right\} \tag{7}$$

generates a contraction semigroup on X. Furthermore, a matrix P_{W_B,W_C}, given by

$$P_{W_B,W_C} = \begin{bmatrix} W_B\Sigma W_B^* & W_B\Sigma W_C^* \\ W_C\Sigma W_B^* & W_C\Sigma W_C^* \end{bmatrix}^{-1}, \tag{8}$$

is well defined, and for $u \in C^2([0,\infty);\mathbb{C}^k)$, $\mathcal{H}x(0) \in H^1([a,b];\mathbb{C}^n)$ and $u(0) = W_B\begin{bmatrix} f_\partial(0) \\ e_\partial(0) \end{bmatrix}$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}||x(t)||_X^2 = \frac{1}{2}\begin{bmatrix} u^*(t) & y^*(t) \end{bmatrix} P_{W_B,W_C}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}. \tag{9}$$

In this paper we assume that the boundary control system satisfies the relation

$$\frac{1}{2}\frac{d}{dt}||x(t)||_X^2 = u^*(t)y(t), \tag{10}$$

i.e., it is an impedance energy-preserving system [13]. It follows from the balance equation (9) that the relation (10) is satisfied exactly when W_B and W_C are such that $P_{W_B,W_C} = \Sigma$, which is equivalent to [3, Thm. 4.4]

$$W_B = Q_B\,[\,I + V_B, \quad I - V_B\,], \tag{11a}$$

$$W_C = Q_C\,[\,I + V_C, \quad I - V_C\,], \tag{11b}$$

$$2Q_C(I - V_CV_B^*)Q_B^* = I, \tag{11c}$$

where Q_B and Q_C are invertible, and V_B and V_C are unitary. Clearly W_B and W_C satisfying equations (11a)–(11c) also satisfy Assumption 2.

Impedance energy-preserving port-Hamiltonian systems are of particular interest due to the property that they can be exponentially stabilized by using output feedback. Let the system (1) satisfying Assumption 1 be impedance energy-preserving, and let us apply feedback $u(t) = r(t) - \kappa y(t)$, $\kappa \in \mathbb{R}^+$, to the system. It has been shown in [13] that the resulting closed-loop system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) \\ (W_B + \kappa W_C)\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= (\mathcal{B} + \kappa\mathcal{C})x(t) = r(t) \\ \mathcal{C}x(t) &= y(t) \end{aligned} \tag{12}$$

is a boundary control system, and furthermore, the operator $\mathcal{A}x = P_1\partial_\zeta(\mathcal{H}x) + P_0\mathcal{H}x$ with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \mathcal{H}x \in H^1([a,b],\mathbb{C}^n) \,\middle|\, W_\kappa \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \quad (13)$$

where $W_\kappa = W_B + \kappa W_C$, generates a contraction semigroup on X. In fact, since we have $W_\kappa \Sigma W_\kappa^* = 2\kappa I > 0$, due to [6, Lem. 9.1.4] the semigroup generated by $\mathcal{A}$ with domain (13) is exponentially stable.

IV. The Plant, Exosystem and Controller

In this section, we will introduce the control system consisting of the plant, the exosystem and the controller. Using the results presented in the previous section, we can define the plant of the control system to be an impedance energy-preserving port-Hamiltonian system. The plant is given by

$$\dot{x}(t) = \mathcal{A}x(t), \qquad x(0) = x_0, \quad (14a)$$
$$\mathcal{B}x(t) = u(t) + w(t), \quad (14b)$$
$$\mathcal{C}x(t) = y(t), \quad (14c)$$

where $\mathcal{A}x(t) = P_1\partial_\zeta(\mathcal{H}x(t)) + P_0\mathcal{H}x(t)$ with P_0, P_1 and $\mathcal{H}$ satisfying Assumption 1, $\mathcal{B}$ and $\mathcal{C}$ are defined in equations (5a)–(5b) with W_B and W_C satisfying equations (11a)–(11c), and $w(t)$ is bounded and differentiable disturbance signal.

Since the plant (14) is a boundary control system, there are operators $A : \mathcal{D}(A) \to X$ with $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$ and $Ax = \mathcal{A}x$ for $x \in \mathcal{D}(A)$ and $B \in \mathcal{L}(U,X)$ such that $\mathcal{R}(B) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{B}Bu = u$. Using the presented operators the transfer function from u to y is given by [6]

$$P(s) = \mathcal{C}(sI - A)^{-1}(\mathcal{A}B - sB) + \mathcal{C}B. \quad (15)$$

Let us now describe the exosystem that generates the boundary disturbance signal $w(t)$ and the reference signal $y_{ref}(t)$. The exosystem is defined by the following equations

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \quad (16a)$$
$$w(t) = Ev(t), \quad (16b)$$
$$y_{ref}(t) = -Fv(t) \quad (16c)$$

on a finite-dimensional space $W = \mathbb{C}^q$. Here $S = \mathrm{diag}(i\omega_1, i\omega_2, \ldots, i\omega_q)$ with $\{\omega_i\}_{i=1}^q \in \mathbb{R}$ and $\omega_i \neq \omega_j$ for $i \neq j$, $E \in \mathcal{L}(W,U)$ and $F \in \mathcal{L}(W,Y)$. We make the following assumption that is crucial for the solvability of the robust output regulation problem: [9]

Assumption 3: For every $k \in \{1,2,\ldots,q\}$ the transfer function $P(i\omega_k) \in \mathcal{L}(U,Y)$ is surjective.

The dynamic error feedback controller to be designed is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0, \quad (17a)$$
$$r(t) = Kz(t), \quad (17b)$$

where $e = y - y_{ref}$ is the error signal and the parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are to be chosen such that robust output regulation is achieved for the system (14). The controller (17) is an abstract linear system on Banach space Z. The operator $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$. [7]

In order to discuss the robust regulation problem (RORP) in the next section, we give the state-space presentation for the closed-loop control system. In order to do this, we define a new variable $\xi = x - Bu - Gv$, where the operator $G \in \mathcal{L}(W,X)$ is such that $\mathcal{R}(G) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{B}Gv = Ev$. Such an operator exists as the plant (14) is a boundary control system [1]. Now, if $x(t) \in \mathcal{D}(\mathcal{A})$, then $\xi(t) \in \mathcal{D}(\mathcal{A}$ and $\mathcal{B}\xi(t) = 0$. Hence, $\xi(t) \in \mathcal{D}(A)$ and, assuming that u and v are differentiable, we get the new state equation [5]

$$\dot{\xi} = Ax + \mathcal{A}Bu - B\dot{u} + \mathcal{A}Gv - G\dot{v}. \quad (18)$$

When the control loop is closed, i.e., the plant and the controller are connected, we have $u = r = Kz$, and hence, equation (18) is well defined. Let us now define the extended state-space as $X_e = X \times \mathbb{C}^q$, and let $\xi_e(t) = (\xi(t), z(t))$ be the extended state. Following [5], the closed-loop control system can be written as

$$\dot{\xi}_e = A_e\xi_e + Hv + G_e\dot{v} + Dy_{ref}, \quad (19)$$

where $D(A_e) = D(A) \times \mathbb{C}^q$ and

$$A_e = \begin{bmatrix} A - BK\mathcal{G}_2\mathcal{C} & \mathcal{A}BK - BK(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}BK) \\ \mathcal{G}_2\mathcal{C} & \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}BK \end{bmatrix},$$
$$H = \begin{bmatrix} \mathcal{A}G - BK\mathcal{G}_2\mathcal{C}G \\ \mathcal{G}_2\mathcal{C}G \end{bmatrix}, \quad G_e = \begin{bmatrix} -G \\ 0 \end{bmatrix},$$
$$D = \begin{bmatrix} BK\mathcal{G}_2 \\ -\mathcal{G}_2 \end{bmatrix}. \quad (20)$$

V. The Robust Output Regulation Problem and the Internal Model Principle

In this section we formulate the robust output regulation problem and define the internal model principle. We consider perturbations $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ of the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ where the operators in the class $\mathcal{O}$ of admissible perturbations are such that (i) the perturbed plant $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is a boundary control system and (ii) $i\omega_k \in \rho(\tilde{\mathcal{A}})$ for $k \in \{1,2,\ldots,q\}$. It is easy to see that there conditions are satisfied for all bounded and sufficiently small perturbations to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and for arbitrary bounded perturbations to the operators E and F. [9]

The following formulation of the robust output regulation problem is given in [9]:

The Robust Output Regulation Problem. Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha\cdot}e(\cdot) \in L^2([0,\infty);Y)$ for some $\alpha > 0$.

3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\tilde{\alpha}\cdot} e(\cdot) \in L^2([0,\infty); Y)$ for some $\tilde{\alpha} > 0$.

In the following two definitions for an internal model are given. [7], [8]

Definition 2: Assume $\dim(Y) < \infty$. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to incorporate a p-copy of the internal model of the exosystem S if for all $k \in \{1, 2, \ldots, q\}$ we have

$$\dim(\mathcal{N}(i\omega_k - \mathcal{G}_1)) \geq \dim(Y)$$

and $\mathcal{G}_1$ has at least $\dim(Y)$ independent Jordan chains of length greater than or equal to n_k associated to the eigenvalue $i\omega_k$.

Definition 3: A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to satisfy the $\mathcal{G}$-conditions if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \tag{21a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{21b}$$

$$\mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k - 1} \subset \mathcal{R}(i\omega_k - \mathcal{G}_1) \tag{21c}$$

for all $k \in \{1, 2, \ldots, q\}$.

The following theorem from [9] presents the internal model principle for regular linear systems with finite-dimensional exosystems and exponential closed-loop stability. The theorem is applicable for boundary control systems as well.

Theorem 1: Assume that the controller stabilizes the closed-loop system exponentially. Then the controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions. Moreover, if $\dim(Y) < \infty$, then the controller solves the robust output regulation problem if and only if it incorporates a p-copy of the internal model of the exosystem.

□

VI. Construction of the Robust Controller

In this section, we will show that robust control over the plant (14) is achieved with control input of the form $u(t) = r(t) - \kappa \mathcal{C} x(t)$, where $\kappa \in \mathbb{R}^+$ and $r(t)$ is the output of the controller of the form (17). That is, we will use negative output feedback $-\kappa \mathcal{C} x(t)$ to exponentially stabilize the plant (14), and the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ with output $r(t)$ solves the robust output regulation problem for the stabilized plant.

Let us first consider the stabilization of the plant (14). Following Section III we can write the plant (14) with input $u(t) = r(t) - \kappa \mathcal{C} x(t)$ as

$$\begin{aligned} \dot{x}(t) &= \mathcal{A} x(t), \\ r(t) + w(t) &= (\mathcal{B} + \kappa \mathcal{C}) x(t), \\ y(t) &= \mathcal{C} x(t), \end{aligned} \tag{22}$$

where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are the same as in the system (14). Furthermore, as shown in Section III, the system (22) is a boundary control system, and thus, there is an exponentially stable operator $A : \mathcal{D}(A) \to X$ such that $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B} + \kappa \mathcal{C})$ and $Ax = \mathcal{A}x$ for $x \in \mathcal{D}(A)$.

We will now make the choices on the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$. Since we have $Y = U = \mathbb{C}^k$, according to Theorem 1 the system operator $\mathcal{G}_1$ has to contain the internal model of the exosystem (16). Following [9] we define $Z = Y^q$ and

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y) \in \mathcal{L}(Z), \tag{23a}$$

$$K = \epsilon K_0 = \epsilon \left[K_0^1, K_0^2, \ldots, K_0^q\right] \in \mathcal{L}(Z, U), \tag{23b}$$

where $\epsilon > 0$ and $K_0 \in \mathcal{L}(Z, U)$. We choose the components K_0^k of K_0 such that the operators $P(i\omega_k) K_0^k$ are invertible, which is possible due to Assumption 3. Note that even though we assumed $P(i\omega_k)$ of the original system (14) to be surjective, we know from [12] that the transfer functions $P_1(s)$ and $P_2(s)$ of the systems (14) and (22), respectively, are related by $P_2(s) = P_1(s)(I + \kappa P_1(s))^{-1}$, from which it follows that $P_2(s)$ is surjective exactly when $P_1(s)$ is surjective. Thus, Assumption 3 holds, and the components K_0^k can be chosen such that the operators $P(i\omega_k) K_0^k$ are invertible, e.g., by choosing $K_0^k = P(i\omega_k)^\dagger$ (the Moore-Penrose pseudoinverse of $P(i\omega_k)$) for all $k \in \{1, 2, \ldots, q\}$. Finally, we choose

$$\begin{aligned} \mathcal{G}_2 &= (\mathcal{G}_2^k)_{k=1}^q = (-(P(i\omega_k) K_0^k)^*)_{k=1}^q \\ &= \begin{bmatrix} -(P(i\omega_1) K_0^1)^* \\ \vdots \\ -(P(i\omega_q) K_0^q)^* \end{bmatrix} \in \mathcal{L}(Y, Z). \end{aligned} \tag{23c}$$

As noted in [9], If we choose $K_0^k = P(i\omega_k)^\dagger$, then $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1, 2, \ldots, q\}$. It is also proved in [9] that the proposed controller satisfies the $\mathcal{G}$-conditions (21).

Now that we have chosen the parameters of the controller, let us show that the resulting closed-loop control system is exponentially stable. Since the plant (22) is a boundary control system, there are operators $B \in \mathcal{L}(U, X)$ and $G \in \mathcal{L}(W, X)$ such that $(\mathcal{B} + \kappa \mathcal{C}) B u = u$ and $(\mathcal{B} + \kappa \mathcal{C}) G v = E v$, and we can define a new variable $\xi = x - Br - Gv$. Since in this case r and v are continuously differentiable, we get the state equation

$$\dot{\xi} = A\xi + \mathcal{A}Br - B\dot{r} + \mathcal{A}Gv - G\dot{v}. \tag{24}$$

Define the extended state-space $X_e = X \times \mathbb{C}^q$ and $\xi_e(t) = (\xi(t), z(t))$ as the extended state. When the control system is closed, we have $r = Kz$, and hence, similar to [5], we get the closed-loop system

$$\dot{\xi}_e = A_e \xi_e + Hv + G_e \dot{v} + D y_{ref}, \tag{25}$$

where $D(A_e) = D(A) \times \mathbb{C}^q$ and

$$\begin{aligned} A_e &= \begin{bmatrix} A - \epsilon B K_0 \mathcal{G}_2 \mathcal{C} & \epsilon \mathcal{A} B K_0 - \epsilon B K_0 (\mathcal{G}_1 + \epsilon \mathcal{G}_2 \mathcal{C} B K_0) \\ \mathcal{G}_2 \mathcal{C} & \mathcal{G}_1 + \epsilon \mathcal{G}_2 \mathcal{C} B K_0 \end{bmatrix}, \\ H &= \begin{bmatrix} \mathcal{A} G - \epsilon B K_0 \mathcal{G}_2 \mathcal{C} G \\ \mathcal{G}_2 \mathcal{C} G \end{bmatrix}, \quad G_e = \begin{bmatrix} -G \\ 0 \end{bmatrix}, \\ D &= \begin{bmatrix} \epsilon B K_0 \mathcal{G}_2 \\ -\mathcal{G}_2 \end{bmatrix}. \end{aligned} \tag{26}$$

As noted in [5], the operator A_e generates a C_0-semigroup, and hence, it follows from the smoothness of y_{ref} and v that equation (25) has a unique, continuously differentiable solution. It has also been shown in [5] that there exists an $\epsilon^* > 0$ such that the operator A_e is exponentially stable for every $0 < \epsilon \leq \epsilon^*$, and hence, the closed-loop system is exponentially stable. Thus, we have shown that the input $u(t) = r(t) - \kappa\mathcal{C}x(t)$ with $\kappa \in \mathbb{R}^+$ and $r(t)$ being the output of the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ achieves robust control over the system (14).

VII. An Example

As an example we consider a vibrating string on the spatial interval $[0, 1]$. The vibrating string satisfies the one-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}p(\zeta,t) = c^2\frac{\partial^2}{\partial \zeta^2}p(\zeta,t), \quad \zeta \in [0,1], \tag{27a}$$

where $c^2 = T/\rho$ with the Young's modulus T and the mass density ρ being positive constants. We decide to apply force to one end of the string and control the velocity of the other end. Thus, the boundary controls with the boundary disturbances are given by

$$\begin{aligned} u_1(t) + w_1(t) &= T\frac{\partial}{\partial \zeta}p(1,t) \quad \text{and} \\ u_2(t) + w_2(t) &= \frac{\partial}{\partial t}p(0,t). \end{aligned} \tag{27b}$$

Furthermore, we choose to observe the velocity at the end where the force is applied and the negative force at the end where the velocity is controlled. Thus, our outputs are given by

$$y_1(t) = \frac{\partial}{\partial t}p(1,t) \quad \text{and} \quad y_2(t) = -T\frac{\partial}{\partial \zeta}p(0,t). \tag{27c}$$

Let us start writing the system described by the equations (27a)–(27c) as a port-Hamiltonian system. In order to do this, we define the new state variables $x_1 = \rho\partial_t p$ (momentum) and $x_2 = \partial_\zeta p$ (strain) [6]. Now, equation (27a) can equivalently be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} = P_1\frac{\partial}{\partial \zeta}\left(\mathcal{H}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix}\right), \tag{28}$$

where

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{H} = \begin{bmatrix} 1/\rho & 0 \\ 0 & T \end{bmatrix}, \tag{29}$$

which is a port-Hamiltonian system satisfying Assumption 1 with $P_0 = 0$.

Using the new state variables we can now define the operators $\mathcal{B}$ and $\mathcal{C}$ as

$$\begin{aligned} u(t) &= \begin{bmatrix} T\partial_\zeta p(1,t) \\ \partial_t p(0,t) \end{bmatrix} = \begin{bmatrix} Tx_2(1,t) \\ \rho^{-1}x_1(0,t) \end{bmatrix} = \mathcal{B}x(t), \\ y(t) &= \begin{bmatrix} \partial_t p(1,t) \\ T\partial_\zeta p(0,t) \end{bmatrix} = \begin{bmatrix} \rho^{-1}x_1(1,t) \\ Tx_2(0,t) \end{bmatrix} = \mathcal{C}x(t). \end{aligned} \tag{30}$$

Equivalently, we can express the inputs and outputs with respect to the boundary effort e_∂ and boundary flow f_∂ (4), which results in

$$\begin{aligned} u(t) &= W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \\ y(t) &= W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}. \end{aligned} \tag{31}$$

We see from (31) that the matrices W_B and W_C are such that $P_{W_B,W_C} = \Sigma$, and hence, the system described by equations (27a)–(27c) is an impedance energy-preserving port-Hamiltonian system. Thus, the controller structure proposed in Section VI with $u(t) = r(t) - \kappa\mathcal{C}x(t), \kappa \in \mathbb{R}^+$ achieves robust regulation on the system.

Now with all the necessary operators defined and the input being of the form $u(t) = r(t) - \kappa\mathcal{C}x(t)$, we can write the system described by equations (27a)–(27c) as the following port-Hamiltonian system:

$$\dot{x}(t) = \mathcal{A}x(t), \tag{32a}$$

$$r(t) - w(t) = (\mathcal{B} + \kappa\mathcal{C})x(t), \tag{32b}$$

$$y(t) = \mathcal{C}x(t), \tag{32c}$$

where $\mathcal{A}x = P_1\partial_\zeta(\mathcal{H}x)$ with P_1 and $\mathcal{H}$ given in (29), and $\mathcal{B}$ and $\mathcal{C}$ given in (30).

Now that we have

$$(\mathcal{B} + \kappa\mathcal{C})x(t) = \begin{bmatrix} Tx_2(1,t) + \kappa\rho^{-1}x_1(1,t) \\ \rho^{-1}x_1(0,t) - \kappa Tx_2(0,t) \end{bmatrix},$$

the operator $B \in \mathcal{L}(U, X)$ satisfying $(\mathcal{B} + \kappa\mathcal{C})Bu = u$ is given by

$$B = \frac{\mathbf{1}_{[0,1]}}{(T\rho)^2 + 1}\begin{bmatrix} T\rho^2 & \rho \\ \rho\kappa^{-1} & -T\rho^2\kappa^{-1} \end{bmatrix} \tag{33}$$

Following [6] we can compute the transfer function from u to y which is given by

$$P(s) = \frac{1}{f_4(s)}\begin{bmatrix} f_1(s) + f_2(s) & 1 \\ -1 & f_1(s) + f_3(s) \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} f_1(s) &= \kappa\cosh(cs), \\ f_2(s) &= c\rho^{-1}\sinh(cs), \\ f_3(s) &= \rho c^{-1}\sinh(cs), \\ f_4(s) &= (\kappa^{-1} + \kappa)f_1(s) + \kappa(f_2(s) + f_3(s)). \end{aligned}$$

It should be noted that $P(i\omega)$ is surjective for all $\omega \in \left(\mathbb{R} \setminus \left\{\frac{(2m+1)\pi}{2c} \mid m \in \mathbb{Z}\right\}\right)$, and thus, provided that the signal generator of the exosystem does not have eigenvalues of the form $i\frac{(2m+1)\pi}{2c}, m \in \mathbb{Z}$, the controller proposed in Section VI achieves robust output regulation for the system (32).

Let $T = \rho = 1$ and choose $\kappa = 1$. Let the exosystem be given by $S = \mathrm{diag}(-i\pi, i\pi)$ and $E = F = I$. Thus, we have $c = 1$,

$$P(-i\pi) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = P(i\pi), \tag{35}$$

and the operator $G \in \mathcal{L}(W, X)$ satisfying $(\mathcal{B} + \kappa\mathcal{C})Gv = Ev$ is given by $G = B$. Furthermore, the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ that solves the robust regulation problem for the stabilized plant (32) is given by

$$\begin{aligned} \mathcal{G}_1 &= \operatorname{diag}(-i\pi, -i\pi, i\pi, i\pi) \in \mathbb{C}^{4\times 4}, \\ \mathcal{G}_2 &= \begin{bmatrix} -I_Y \\ -I_Y \end{bmatrix}, \\ K &= \epsilon \left[P(-i\pi)^{-1}, P(i\pi)^{-1} \right], \end{aligned} \tag{36}$$

where we made the choice $K_0^k = P(i\omega_k)^{-1}$.

VIII. Conclusions

We presented a method for constructing a simple finite-dimensional robust regulating controller for a first-order impedance energy-preserving port-Hamiltonian system. Even though the system is initially unstable, by exponentially stabilizing the system with negative output feedback we were able to utilize the results of robust output regulation for exponentially stable systems when designing the controller. We showed that the proposed controller together with negative output feedback solves the Robust Output Regulation Problem for the considered system, and as an example we implemented such a controller for the one-dimensional wave equation with boundary control and observation.

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