Linear Model Predictive Control for Time Delay Systems*

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***Abstract*— This paper studies linear model predictive control of real matrix-valued single delay systems. The delay system is written as an abstract infinite-dimensional control system which is then mapped into an infinite-dimensional discrete-time control system using Cayley-Tustin discretization. A constrained model predictive control (MPC) problem is formulated for the discrete-time system where a terminal penalty function is utilized to cast the infinite-horizon optimization problem into a finite-horizon one. The proposed MPC design is demonstrated on an example of constrained stabilization of a 2×2 system. We will demonstrate that the proposed discrete-time MPC law not only stabilizes the discrete-time system but can be utilized in stabilizing the original continuous-time system as well, which is due to several favorable properties of the Cayley-Tustin discretization.**

I. Introduction

Time delay systems belong to the class of infinite-dimensional functional differential equations and can be used to model aftereffect phenomena in processes with examples in biology, chemistry, economics etc. (see the survey [12] and the references therein for the full list of examples and further motivation). A delay system can be equivalently written as a finite-dimensional distributed parameter system as described in [2, Sect. 2.4 and various examples], which allows us to analyze time delay systems by the means of infinite-dimensional systems theory, as well as utilize existing control strategies developed for distributed parameter systems.

Model predictive control (MPC) has been studied for nonlinear delay systems without terminal constraints in [11] and for discrete-time delay systems with robustness with respect to linear parameter uncertainties in [8], in both of which the considered delay systems are of the similar single delay type as what will be considered in this paper. Additionally, there are delay MPC papers that address solely systems under input delays, which are outside the scope of this paper (see (1) for the type of systems considered).

The model predictive control approach that will be utilized in designing a control law for the single delay systems is based on earlier work by the authors [13], [5], where the proposed design has been applied to transport-reaction processes and Schrödinger equation, respectively. In addition to the preceding works, the manuscript [4] by the authors addresses the theoretical basis for applying the proposed design to distributed parameter systems in general, especially to the class of regular linear systems. As opposed to [8], [11], the our design accounts for output constraints but is on the other hand limited to linear delay systems and presently without parameter uncertainties.

The proposed model predictive control design is based on Cayley-Tustin discretization that maps a continuous-time system into a discrete-time one. The discretization preserves several important properties between the continuous- and discrete-time systems such as asymptotic stability, controllability and observability [1]. Furthermore, the discretization is convergent with respect to the inputs and outputs of the continuous- and discrete-time systems [3], which essentially allows us to design a control law for the continuous-time system based on the Cayley-Tustin discretized system. Eventually, the discrete-time MPC design reduces to the classical MPC design for which optimality and convergence have been proved [9], [10].

As the proposed MPC design is essentially a direct application of authors' previous work, the contribution of the paper comprises several novel remarks on the design:

1) In the previous work, the applicability of the discrete-time control law to the continuous-time model has been justified by the properties of Cayley-Tustin discretization without a proper demonstration. In this paper, we will show that the discrete-time controls can be directly utilized to stabilize the original continuous-time system as well.
2) In their previous work, the authors have applied the control strategy only for single-input-single-output systems. In this paper, the example delay system has two-dimensional input and output spaces.
3) The proposed MPC design is applied to time delay systems and accounts for simultaneous satisfaction of input and output constraints, which to our knowledge is a novelty with respect to existing MPC designs for delay systems.

The paper is organized as follows. In Section II, we present the considered class of single delay systems and reformulate them to the abstract differential equation framework. In Section III, Cayley-Tustin time discretization is invoked and utilized in computing the discretized operators and their adjoints for the class of single delay systems. In Section IV, the model predictive control problem is formulated and solved, which provides the optimal controls for the discrete-time system. In Section

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V, the control design is demonstrated on an example, where we demonstrate that the proposed control design stabilizes not only the Cayley-Tustin discretized system but also the original continuous-time system. Finally, the paper is concluded in Section VI.

II. Delay Systems

Consider single time delay systems of the form

$$\dot{z}(t) = A_0 z(t) + A_1 z(t-\tau) + B_0 u(t), \quad z(0) = r \tag{1a}$$
$$z(\theta) = f(\theta), \quad -\tau \le \theta < 0 \tag{1b}$$
$$y(t) = C_0 z(t) \tag{1c}$$

where $A_0, A_1 \in \mathbb{R}^{n\times n}, B_0 \in \mathbb{R}^{n\times m}, C_0 \in \mathbb{R}^{p\times n}$ for some $n, m, p \in \mathbb{N}$, $\tau > 0$ is the time delay and $f \in L^2(-\tau, 0; \mathbb{R}^n)$ is the past data for $\theta \in [-\tau, 0]$. By [2, Sect. 2.4], the system (1) can be formulated as an abstract differential equation of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{2a}$$
$$y(t) = Cx(t) \tag{2b}$$

on the Hilbert space $M^2(-\tau, 0; \mathbb{R}^n) := \mathbb{R}^n \oplus L^2(-\tau, 0; \mathbb{R}^n)$ equipped with the corresponding inner product which we denote by $\langle\cdot,\cdot\rangle$. In (2), A is the generator of a C_0-semigroup, given by [2, Thm 2.4.6]

$$A \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} A_0 r + A_1 f(-\tau) \\ \dfrac{df}{d\theta}(\cdot) \end{bmatrix}, \tag{3}$$

with domain

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} \in M^2(-\tau, 0; \mathbb{R}^n) \mid f \text{ abs. cont.}, \frac{df}{d\theta} \in L^2(-\tau, 0; \mathbb{R}^n),\ f(0) = r \right\} \tag{4}$$

and $B = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$, $C = [C_0 \quad 0]$ are bounded operators. For simplicity, we will make a standing assumption that the delay system is exponentially stable.

III. Cayley-Tustin Time Discretization

Consider a system of the form given in (2). Given a discretization parameter $h > 0$, a Crank-Nicolson type time discretization of (2) is given by

$$\frac{x(ih) - x((i-1)h)}{h} \approx A\frac{x(ih) + x((i-1)h)}{2} + Bu(ih)$$
$$y(ih) \approx C\frac{x(ih) + x((i-1)h)}{2}$$

for $i \ge 1$. Let $u_k^{(h)}/\sqrt{h}$ be an approximation of $u(t)$ on the interval $t \in ((k-1)h, kh)$, e.g., by the mean value sampling as in [3]:

$$\frac{u_k^{(h)}}{\sqrt{h}} = \frac{1}{h}\int_{(k-1)h}^{kh} u(t)dt.$$

It has been shown in [3] that Cayley-Tustin discretization is a convergent time discretization scheme for input-output stable well-posed systems with finite dimensional input and output spaces, that is, $y_k^{(h)}/\sqrt{h} \to y(t)$ as $h \to 0$. Thus, denoting $u_k^h/\sqrt{h}$ and $y_k^h/\sqrt{h}$ by $u(k)$ and $y(k)$, respectively, the Cayley-Tustin transform of the system (2) to a discrete-time system (A_d, B_d, C_d, D_d) is given by

$$x(k) = A_d x(k-1) + B_d u(k), \quad x(0) = x_0 \tag{5a}$$
$$y(k) = C_d x(k-1) + D_d u(k) \tag{5b}$$

with the operators defined as

$$\begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} := \begin{bmatrix} (\delta + A)(\delta - A)^{-1} & \sqrt{2\delta}(\delta - A)^{-1}B \\ \sqrt{2\delta}C(\delta - A)^{-1} & G(\delta) \end{bmatrix},$$

where $G(\delta) := C(\delta - A)^{-1}B$ denotes the transfer function of the original system (2) and $\delta = 2/h$ which needs to be in the resolvent set of A, i.e., $\delta \in \rho(A)$. It is easy to see that the operator A_d can be equivalently expressed as $A_d = -I + 2\delta(\delta - A)^{-1}$.

In addition to the input-output-convergence, Cayley-Tustin discretization has several other favorable properties for the purposes of stabilization and control as shown by the results in [1]. For example, by [1, Lem. 2.2] the discrete-time system is asymptotically stable if and only the continuous-time system is. Moreover, the continuous- and discrete-time systems have equivalent controllability and observability Gramians and the solutions of the continuous- and discrete-time Lyapunov and Riccati equations are equivalent [1, Thm. 2.4]. We will utilize some of these properties when deriving the model predictive control law for the delay system (1).

A. Cayley-Tustin Discretization for the Delay System

In order to compute the Cayley-Tustin discretization for the delay system, we first need to find the resolvent operator of the system operator A. By [2, Lem 2.4.5], the resolvent is given for any $\delta \in \rho(A)$ by

$$(\delta - A)^{-1} \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} g(0) \\ g(\cdot) \end{bmatrix},$$

where

$$g(\theta) = e^{\delta\theta}g(0) - \int_0^\theta e^{\delta(\theta - s)}f(s)ds, \quad \theta \in [-\tau, 0]$$

and

$$g(0) = \Delta(\delta)^{-1}\left(r + \int_{-\tau}^0 e^{-\delta(\theta+\tau)}A_1 f(\theta)d\theta \right),$$

where

$$\Delta(\delta) = \delta - A_0 - A_1 e^{-\delta\tau},$$

which directly yields the expression for $A_d = -I + 2\delta(\delta - A)^{-1}$. Based on the preceding, it can be easily seen that

$$B_d = \sqrt{2\delta}(\delta - A)^{-1}B = \sqrt{2\delta}\begin{bmatrix} \Delta(\delta)^{-1}B_0 \\ e^{\delta\cdot}\Delta(\delta)^{-1}B_0 \end{bmatrix} \tag{6}$$

and

$$C_d \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \sqrt{2\delta} C(\delta - A)^{-1} \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \sqrt{2\delta} C_0 g(0) \quad (7)$$

and finally,

$$D_d = C(\delta - A)^{-1} B = C_0 \Delta(\delta)^{-1} B_0. \quad (8)$$

B. Adjoint Operators

In order to compute the adjoints for the discretized operators (A_d, B_d, C_d, D_d), we utilize [2, Lem. 2.4.8] where it is given that the adjoint of $(\delta - A)^{-1}$ is given by

$$(\delta - A)^{-*} \begin{bmatrix} q \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} \Delta^T(\delta)^{-1} \left(q + \int_{-\tau}^{0} e^{\delta\theta} f(\theta) d\theta \right) \\ \int_{-\tau}^{s} e^{-\delta(s-\theta)} f(\theta) d\theta + e^{-\delta(s+\tau)} A_1^T q \end{bmatrix}$$

where the second component is defined for $s \in [-\tau, 0]$. Using the preceding, we directly obtain

$$A_d^* = -I + 2\delta(\delta - A)^{-*}$$

and it is easy to compute

$$\begin{aligned} B_d^* \begin{bmatrix} q \\ f(\cdot) \end{bmatrix} &= \sqrt{2\delta} B^* (\delta - A)^{-*} \begin{bmatrix} q \\ f(\cdot) \end{bmatrix} \\ &= \sqrt{2\delta} B_0^T \Delta^T(\delta)^{-1} \left(q + \int_{-\tau}^{0} e^{\delta\theta} f(\theta) d\theta \right) \end{aligned}$$

and

$$C_d^* = \sqrt{2\delta} (\delta - A)^{-*} C^* = \sqrt{2\delta} \begin{bmatrix} \Delta^T(\delta)^{-1} C_0^T \\ e^{-\delta(\cdot+\tau)} A_1^T C_0^T, \end{bmatrix},$$

and finally

$$D_d^* = D_d^T = B_0^T \Delta^T(\delta)^{-1} C_0^T.$$

IV. The Model Predictive Control Problem

The moving horizon regulator is based on a similar formulation emerging from the finite-dimensional framework [9], [10] and has been formulated for general input and output spaces in the distributed parameter system setting in [4]. For delay systems with real vector valued input and output spaces, the objective function with constraints at a given sampling time k is given by

$$\begin{aligned} \min_u \sum_{i=k}^{\infty} & y_i^T Q y_i + u_i^T R u_i \\ s.t. \quad & x_i = A_d x_{i-1} + B_d u_i \\ & y_i = C_d x_{i-1} + D_d u_i \\ & u_{\min} \le u_i \le u_{\max} \\ & y_{\min} \le y_i \le y_{\max} \qquad \forall i \ge k \end{aligned} \quad (9)$$

where Q and R are positive definite matrices. In the upper and lower bounds, the inequalities are interpreted elementwise which allows different components of the inputs and outputs to have independent upper and lower bounds.

The aforementioned infinite-horizon open-loop objective function can be cast into a finite-horizon open-loop objective function under the assumption that the input u is zero beyond the control horizon N, i.e., $u_{k+N+i} = 0$ for all $i \in \mathbb{N}_0$. Additionally, a penalty term needs to be included to account for the cost of the outputs beyond the control horizon. As we have assumed the considered delay system to be exponentially stable, the extended output operator C is infinite-time admissible for the strongly continuous semigroup generated by A in (2), and thus, the penalty term can be written according to [4, Sect. 3.1] as a state penalty term $< x_{k+N-1}, \bar{Q} x_{k+N-1} >$, where $\bar{Q}$ is the solution of the Lyapunov equation

$$A^* \bar{Q} + \bar{Q} A = -C^* Q C \quad (10)$$

or equivalently its discrete-time counterpart

$$A_d^* \bar{Q} A_d - \bar{Q} = -C_d^* Q C_d \quad (11)$$

which have unique and equivalent solutions by [1, Thm. 2.4]. Thus, the finite horizon objective function is given by

$$\min_u \sum_{i=k}^{k+N-1} y_i^T Q y_i + u_i^T R u_i + < x_{k+N-1}, \bar{Q} x_{k+N-1} > \quad (12)$$

with the same constraints as in (9).

Similar to [4], we use the notation $U_k := (u_{k+i})_{i=0}^{N-1}$ and $Y_k := (y_{k+i})_{i=0}^{N-1}$, and write (12) as a quadratic optimization problem for U_k:

$$\min_{U_k} \quad U_k^T H U_k + 2 U_k^T P x_{k-1} \quad (13)$$

where $H \in \mathbb{R}^{mN \times mN}$ is self-adjoint given by

$$h_{i,j} = \begin{cases} D_d^* Q D_d + B_d^* \bar{Q} B_d + R & \text{for } i = j \\ D_d^* Q C_d A_d^{i-j-1} B_d + B_d^* \bar{Q} A_d^{i-j} B_d & \text{for } i > j \\ h_{j,i}^* & \text{for } i < j \end{cases}$$

and $P : M^2(-\tau, 0; \mathbb{R}^n) \to \mathbb{R}^{mN}$ is a bounded linear operator given by

$$P = \begin{bmatrix} D_d^* Q C_d + B_d^* \bar{Q} A_d \\ D_d^* Q C_d A_d + B_d^* \bar{Q} A_d^2 \\ \vdots \\ D_d^* Q C_d A_d^{N-2} + B_d^* \bar{Q} A_d^{N-1} \end{bmatrix}.$$

The objective function in (13) is subjected to constraints

$$U_{\min} \le U_k \le U_{\max} \quad (14a)$$

$$Y_{\min} \le SU + Tx(\zeta, k) \le Y_{\max} \quad (14b)$$

where, e.g., $U_{\min}$ contains N copies of $u_{\min}$. The constraints can be written in the form

$$\begin{bmatrix} I_{mN \times mN} \\ -I_{mN \times mN} \\ S \\ -S \end{bmatrix} U_k = \begin{bmatrix} U_{\max} \\ -U_{\min} \\ Y_{\max} - T x_k \\ -Y_{\min} + T x_k \end{bmatrix}$$

where $S \in \mathbb{R}^{pN \times mN}$ is block lower triangular given by

$$s_{i,j} = \begin{cases} D_d & \text{for } i = j \\ C_d A_d^{i-j-1} B_d & \text{for } i > j \\ 0 & \text{for } i < j \end{cases}$$

and $T : M^2(-\tau, 0; \mathbb{R}^n) \to \mathbb{R}^{pN}$ is a bounded linear operator given by $T = (C_d A_d^{k-1})_{k=1}^N$.

The model predictive control law is obtained by solving the optimization problem (13) for each step $k = 1, 2, \ldots$. When the set of optimal control moves U_k^* is obtained for the kth step, the first m components of U_k^* corresponding to u_k^* is given as an input to the system. Then, the optimization problem is solved again for the next step to obtain a new set of optimal inputs U_{k+1}^*, of which the first m components provide the next input for the system. This procedure is then repeated in the subsequent time instances.

As Cayley-Tustin discretization preservers asymptotic stability and observability Gramians between continuous and discrete time, stability results from classical discrete-time MPC, e.g., [10, Thm. 3] can be extended to the infinite-dimensional discrete-time system (5). This observation was already made in our previous work and we merely repeat the result in the following:

Theorem 1: [4, Thm. 3.1] Provided that the input and output constraints are feasible, under the controls u_k^* from (12) with N sufficiently large the output $y(k)$ of the discrete-time system (5) converges asymptotically to zero and satisfies the output constraints.

By the input-output convergence of Cayley-Tustin discretization, the above result yields the following approximate result for the continuous-time system under discrete-time controls.

Theorem 1: With $y(k)$ and u_k^* from Theorem 1, under controls $u(t) = u_k^*/\sqrt{h}$ for $t \in [h(k-1), hk]$ the output $y(t)$ of the continuous-time system (2) converges asymptotically to zero and satisfies the output constraints approximately in the sense that $y(t) \to y(k)/\sqrt{h}$ as $h \to 0$.

Note that due to the assumed exponential stability, the states of the discrete- and continuous-time systems naturally go to zero as well. Further note that there is no uniform convergence rate for $y(t) \to y(k)/\sqrt{h}$ [3, Sect. 5], due to which the result of Corollary 1 cannot be improved with a convergence estimate. Finally note that the proposed design is note limited to stable systems as certain instabilities can be dealt with as in [4, Sect. 3.2], or in [13] as a delay system can only have finitely many unstable eigenvalues by [2, Thm. 2.4.6], but for simplicity we restrict here to the stable case.

A. On solving the Lyapunov equation

Solving the Lyapunov equation (10) for the delay system is different from how it was solved in the previous works by the authors [4], [5], [13], where in most of the examples the system operator A was a Riesz spectral operator which was utilized in solving the Lyapunov equation. However, the operator A associated with the delay system (1) is not a Riesz spectral operator, and hence, other strategies need to be incorporated in order to solve the Lyapunov equation.

It is known by standard infinite-dimensional systems theory that the unique positive solution of the Lyapunov equation (10) is in general given by the extended observability Gramian [2, Sect. 4.1]

$$\bar{Q}x = \int_0^\infty T^*(t) C^* C T(t) x dt$$

for $x \in \mathcal{D}(A)$, where $T(t)$ is the strongly continuous semigroup generated by A. However, even having the expression for the semigroup $T(\cdot)$ by [2, Sect. 2.4] does not make the extended observability Gramian a particularly reasonable way to solve the Lyapunov equation.

Lyapunov matrices for matrix-valued single delay systems have been considered, e.g., in [6], [7]. However, in those references the past data $f(\cdot)$ is not accounted for and the solution obtained for the delay Lyapunov equation is a matrix-valued function of time. Thus, the Lyapunov matrix approach does not seem to be compatible with the semigroup approach that we are using, so we cannot utilize the results provided in the preceding references.

Another approach that can be utilized also in the semigroup setting comes from the controllability and observability considerations of delay systems in [14] using the matrix Lambert W function. The preceding article provides an expression for the observability Gramian of a matrix-valued single delay system, which can then be utilized in computing the extended observability Gramian to solve the Lyapunov equation. The expression for the observability Gramian is given in [14, Thm. 2] which involves computing coefficients based on the matrices A_0, A_1, the delay τ and the past data $f(\cdot)$ along with the matrix Lambert W function. As we are not aware of better ways to solve the Lyapunov equation for the delay system, the approximate solution using the method developed in [14] is utilized in the ensuing section.

V. Numerical Example

Based on the expressions derived for the discretized operators A_d, B_d, C_d, D_d and their adjoints in Section III, the discretized operators can be easily computed for any single delay system by substituting the actual parameter values to the corresponding equations. For brevity, we consider as a numerical example a simple 2×2 system of the form (1) with parameters

$$A_0 = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix},$$

$C_0 = I_{2\times 2}$ and $\tau = 1$. Moreover, let the initial data be given by $f(\theta) = [\sin(\theta) - 1/4,\ \cos(\theta) - 1/2]^T$ for $\theta \in [-1,\ 0]$ and $z(0) = f(0) = [-1/4, 1/2]^T$.

For Cayley-Tusting discretization, let us at this point choose the time discretization parameter as $h = 2^{-6}$ so that $\delta = 128$. For numerical integration, we approximate $d\theta \approx 2^{-9}$. For the MPC formulation, let us choose the weights for the objective function as $Q = R = I_{2\times 2}$ and the prediction horizon as $N = 10$. The intended upper and lower bounds for the continuous-time inputs and outputs are given by $-0.3 \leq u(t) \leq 0.9$ and $-0.25 \leq y(t) \leq 0.6$ so that in the discrete-time formulation we will be using bounds $-0.3\sqrt{h} \leq u(k) \leq 0.9\sqrt{h}$ and $-0.25\sqrt{h} \leq y(k) \leq 0.6\sqrt{h}$. Naturally we could impose the bounds for both of the input and output components independently but opted not to do so for the sake of brevity.

A. Simulation with the discrete-time system

The simulation results for the Cayley-Tustin discretized version of the delay system under the model predictive control law obtained by solving (13) repeatedly are shown in Figure 1. The simulation is run for $k = 1, 2, \ldots 256$ steps which is equivalent to $t \in [0,\ 4]$ with the time discretization parameter $h = 2^{-6}$. The discrete-time inputs and outputs are normalized by $h^{-1/2}$ in order to eliminate the scaling effect the discretization parameter has on them (recall that $y(k)/\sqrt{h}$ approximates $y(t)$).

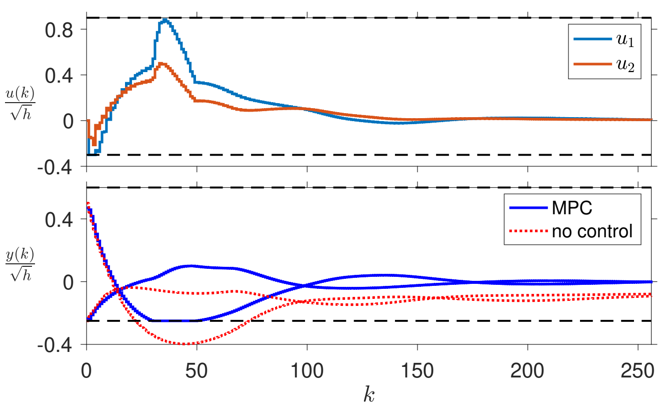

Fig. 1. Normalized controls and outputs for the discrete-time model for time discretization $h = 2^{-6}$ and comparison to non-controlled outputs. The dashed lines denote the input and output constraints.

It can be seen from Figure 1 that in the beginning of the simulation some control effort is applied to bring the output faster towards zero. Then a higher-gain control is applied to keep the second output component within the allowed limits. At this point, it can be seen that the corresponding output of the non-controlled system (the dotted red lines) violates the lower output constraint. Finally, the MPC controls and outputs decay close to zero whereas the non-controlled outputs decay towards zero much slower. Naturally the non-controlled outputs will eventually decay to zero as well due to the stability of the considered system.

B. Discrete-time MPC for the continuous-time system

In this section, we utilize the discrete-time model predictive control law to control the continuous-time system and see how the continuous-time output behaves. In order to do this, we define a continuous-time control law as $u(t) = u(k)/\sqrt{h}$ for $t \in [(k-1)h,\ kh]$, under which the continuous-time output $y(t)$ will behave approximately as $y(k)/\sqrt{h}$ by the input-output convergence of Cayley-Tustin discretization. In order to simulate the continuous time system, we solve the considered delay system using the `dde23`-solver in Matlab.

In Figure 2, the output $y(t)$ of the continuous-time system is displayed for $t \in [0,\ 4]$ under the controls derived from the discrete-time controls displayed in Figure 1. The continuous-time output under no controls is displayed there as well for reference. It can be seen by comparing Figures 1 and 2 that while the discrete-time output $y(k)/\sqrt{h}$ seems to approximate the continuous output $y(t)$ rather accurately, the second component of $y(t)$ does however violate the lower output constraint while close to its minimum. This behavior is expected as the values of $y(t)$ do deviate from the ones of $y(k)/\sqrt{h}$, so we cannot guarantee that the continuous-time output would satisfy the output constraints even if the discrete-time output does. Regardless, the discrete-MPC-based control law still stabilizes the continuous-time system much faster than having no controls. The approximation error $y(t) - y(k)/\sqrt{h}$ is shown in Figure 3.

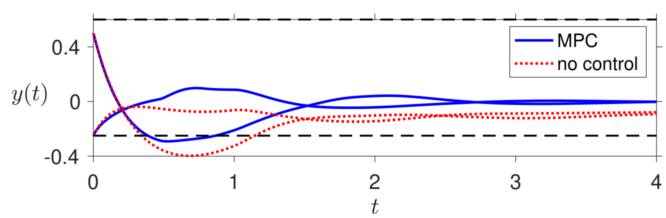

Fig. 2. Outputs of the continuous-time model under the MPC control law obtained with time discretization $h = 2^{-6}$ along with output constraints drawn with dashed lines.

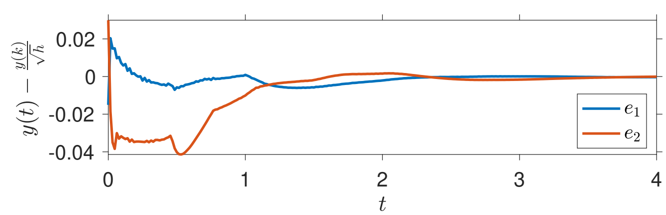

Fig. 3. Approximation error $y(t) - y(k)/\sqrt{h}$ for $h = 2^{-6}$.

Even though there is no way to guarantee that the continuous output $y(t)$ would satisfy the output constraints in the proposed setting, we can improve the convergence of the discrete-time outputs $y(k)/\sqrt{h}$ to the continuous-time outputs $y(t)$ by using a smaller discretization parameter h. In order to demonstrate this, we repeat the MPC computations of the previous section with a denser time discretization $h = 2^{-8}$. The resulting control law and the discrete-time outputs can be seen in Figure 4, where the outputs of the non-controlled discrete-time system are displayed as well (the same dotted red lines as in Figure

1). Due to the more accurate time discretization, the obtained control law is somewhat different from the one shown in Figure 1, but there are no notable differences between the discrete-time outputs.

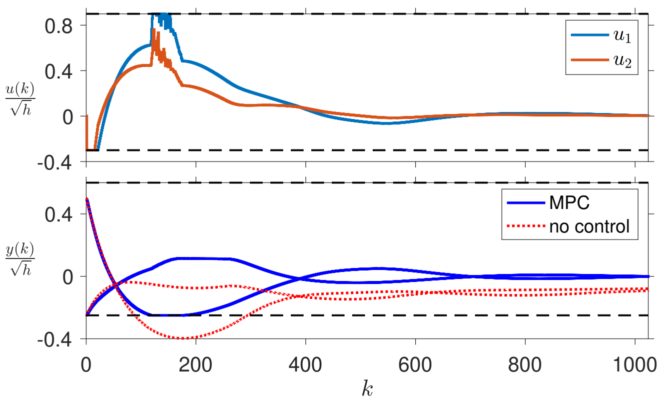

Fig. 4. Normalized controls and outputs for the discrete-time model for time discretization $h = 2^{-8}$ and comparison to non-controlled outputs. The dashed lines denote the input and output constraints.

Finally, the output of the continuous-time system under the discrete-MPC-based control law is displayed in Figure 5. As expected, the continuous-time output behaves more closely as the discrete-time output in Figure 4 (or rather vice versa but regardless, they are closer to one another). The continuous-time output does still violates the lower output constraint around its minimum value, but the violation is much smaller than with the coarser time discretization $h = 2^{-6}$. The approximation error $y(t) - y(k)/\sqrt{h}$ for $h = 2^{-8}$ is shown in Figure 6 which shows that the errors have diminished roughly by a factor of four.

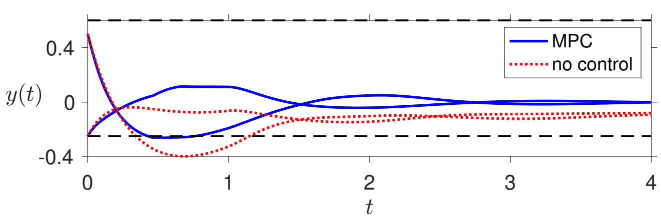

Fig. 5. Outputs of the continuous-time model under the MPC control law obtained with time discretization $h = 2^{-8}$ along with output constraints drawn with dashed lines.

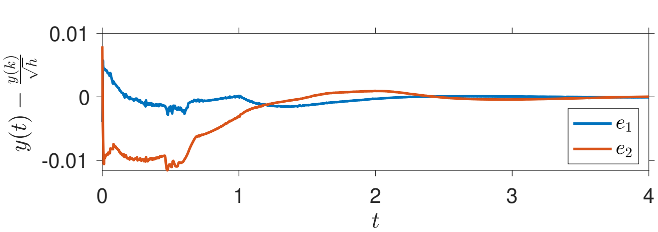

Fig. 6. Approximation error $y(t) - y(k)/\sqrt{h}$ for $h = 2^{-8}$.

VI. Conclusions

Cayley-Tustin time discretization was applied to single delay systems in order to design a discrete-time model predictive control law for the class of systems based on the earlier work on the subject by the authors. Once the delay system was written as an abstract differential equation by the means described in [2, Sect. 2.4], the control design was a direct application of the design originally proposed in [13]. The performance of the control design was demonstrated by a numerical example.

The numerical tests also demonstrated the result of Corollary 1, i.e., that the discrete-MPC-based control law stabilizes the continuous-time system, but there is no guarantee that the continuous output would strictly satisfy the output constraints. The constraint violation could possibly be avoided by adding a certainty factor to the discrete-time constraints, so that the discrete-time output should actually satisfy stricter constraints than the continuous output, where the level of certainty should depend on the discretization parameter h. Investigation of this idea will be a topic of future research.

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