Robust Output Regulation of Counter-Flow Heat Exchangers ⋆

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Abstract: We consider a partial differential equation model widely used for counter-flow heat exchangers and the related robust output regulation problem with boundary control and boundary observation. We show that the control system is an exponentially stable regular linear system, which enables us to use a specific known controller design to robustly regulate the system. The results are illustrated with numerical simulations.

Keywords: Heat exchangers, linear control systems, output regulation, partial differential equations, robust control

1. INTRODUCTION

The goal of the *output regulation problem* is to assure asymptotical convergence of the system output to some desired reference output signal with the help of an appropriate controller. Adding the *robustness* requirement means that the controller should solve the tracking problem despite some known disturbances, and uncertainties or perturbations in the system parameters. The *internal model principle* gives conditions which are both necessary and sufficient for the controller to be able to solve the robust output regulation problem.

Dating back to 1970s, the internal model principle was first introduced in Francis and Wonham (1975), Francis and Wonham (1976) and Davison (1976). Initially developed for finite-dimensional systems, the theory has since been expanded to infinite-dimensional systems as well, see for example Rebarber and Weiss (2003), Hämäläinen and Pohjolainen (2010) and Paunonen and Pohjolainen (2010). Recent advances have been made in developing the theory for *regular systems* in Paunonen and Pohjolainen (2014) and Paunonen (2016), as well as for *boundary control systems* in Humaloja et al. (2018).

Heat exchangers have a wide range of applications. They are an essential part of controlling the temperature of devices of different scales of size, both for heating and cooling, and from power plants to household electric devices. For that reason, it is natural that the properties of different types of heat exchanger and the possibilities to model them have been studied extensively.

For heat exchangers, it is common to consider 1D models and assume advection being dominant over diffusion to the extent of completely ignoring the latter property in many of the heat exchanger models. Doing so leads to considering systems of coupled transport equations, which will also be the model considered in this paper. There are also "full flux" models which take diffusion into account and have their benefits when it comes to for example numerics. This has been studied in Aulisa et al. (2015) and Burns and Kramer (2015). Counter-flow heat exchangers have been studied in articles Burns and Cliff (2014), Chen (2014), Maidi et al. (2009), Heo et al. (2011) and Xu and Dubljevic (2016), to name a few. First two of the preceeding papers, similarly to this work of ours, focus on temperature based control, while the latter three also leave the door open for fluid velocity based control.

As the main result of this paper, we design a controller to solve the robust output regulation problem for the considered counter-flow heat exchanger model. To get there, we first show that the considered system of partial differential equations (PDE) forms a regular linear system. Together with the result that the system operator of the PDE system in question generates an exponentially stable strongly continuous semigroup, this enables us to use the *minimal order robust controller* previously introduced in Hämäläinen and Pohjolainen (2000), Rebarber and Weiss (2003) and Paunonen (2016) to solve the robust output regulation problem for reference signals consisting of linear combinations of sinusoidal functions. The system model considered is of a form commonly used to model the behaviour of counter-flow heat exchangers.

The paper is organized as follows. In Section 2, we first present the considered model for counter-flow heat exchanger and then familiarize the reader with the robust output regulation problem and the abstract representation of the *plant*, the *controller* and the *exosystem*. In Section 3, we first verify all the necessary properties for our system and then present a controller capable of solving the robust output regulation problem for systems with the verified properties. In Section 4, we demonstrate the controller in action with a numerical example using finite elements

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and piecewise polynomial interpolation. The numerical example is followed by conclusion in Section 5.

We use the following notation. $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from a normed space X to a normed space Y and $\mathcal{L}(X) := \mathcal{L}(X,X)$. For $q \in L^\infty(\Omega)$, $M_q \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ denotes the multiplication operator with $(M_q x)(\xi) = q(\xi) \cdot x(\xi)$ for all $\xi \in \Omega$. The resolvent operator of a linear operator A is denoted by $R(s,A) := (s-A)^{-1}$ for $s \in \rho(A)$, the resolvent set of A.

2. PROBLEM FORMULATION

We start with the presentation of the system model considered in the paper. Afterwards we present the robust output regulation problem followed by the abstract representations of the plant, the controller and the exosystem.

2.1 The Counter-Flow Heat Exchanger

Counter-flow heat exchanger on interval $0 \le \xi \le l$, the interval denoted from now on by Ω, can be modeled by the system of PDEs

$$\frac{\partial \Theta_h}{\partial t} = -v_h \frac{\partial \Theta_h}{\partial \xi} - k_h(\xi)(\Theta_h - \Theta_c), \tag{1a}$$

$$\frac{\partial \Theta_c}{\partial t} = v_c \frac{\partial \Theta_c}{\partial \xi} - k_c(\xi)(\Theta_c - \Theta_h), \tag{1b}$$

where $\Theta_h = \Theta_h(\xi,t)$ and $\Theta_c = \Theta_c(\xi,t)$ are the temperatures of the fluid inside the heat exchanger. We refer to the channel with the state Θ_h as the "hot" channel and the channel with the state Θ_c as the "cold" channel. Constants $v_h, v_c > 0$ are the flow velocities and $k_h(\xi), k_c(\xi)$ are the source terms related to conduction of heat between the two channels. From now on it is assumed that $k_h, k_c \in C^\infty(\Omega)$ with $k_h(\xi), k_c(\xi) \ge r > 0$ for some $r \in \mathbb{R}_+$ for all $\xi \in \Omega$. Physically this means that the heat conducting properties of the wall separating the two channels may be ξ dependant on Ω as long as the properties change smoothly. At the same time, the wall must not be insulating at any point within the heat exchanger.

Boundary and initial conditions of the system are given by

$$\Theta_h(\xi,0) = \Theta_{h0}(\xi), \qquad \Theta_c(\xi,0) = \Theta_{c0}(\xi), \tag{1c}$$

$$\Theta_h(0,t) = u(t), \qquad \Theta_c(l,t) = 0, \tag{1d}$$

where $u(t)$ is the *control input* of the system. In addition, we are observing the temperature of the fluid in the left end of the cold channel, and we express this observation as

$$y(t) = \Theta_c(0,t). \tag{1e}$$

The whole setup of the counter-flow heat exchanger with boundary control and observation is illustrated in Figure 1.

2.2 The Robust Output Regulation Problem

The goal of the robust output regulation problem is to design a controller such that the *output signal* $y(t)$ of the plant converges to some desired *reference signal* $y_{ref}(t)$. For a more complete description of the following system, see for example Paunonen and Pohjolainen (2014) and Paunonen (2016).

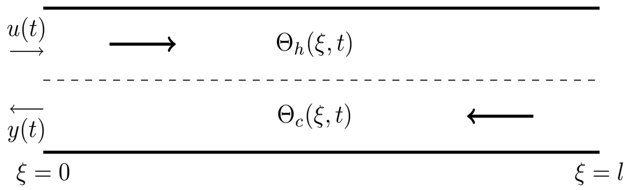

Fig. 1. Counter-flow heat exchanger with boundary control $u(t)$ and boundary observation $y(t)$

The plant is taken to be of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{2a}$$

$$y(t) = Cx(t) \tag{2b}$$

with linear operators A, B and C, where $x(t)$ is the state of the plant with x_0 being the initial state, and $u(t)$ is the control input. We assume $A : X \supset D(A) \longrightarrow X$, where X is a Hilbert space, $B \in \mathcal{L}(U, X_{-1})$, where $U = \mathbb{C}$ and X_{-1} is the extension of X with repect to $||R(s,A)x||$-norm, and $C \in \mathcal{L}(X_1, Y)$, where $Y = \mathbb{C}$ and $X_1 = (D(A), ||\cdot||_{gr})$ with $||\cdot||_{gr}$ denoting the graph norm. Going forward, we will have to replace operator C in (2) with its Λ-*extension*

$$C_\Lambda x = \lim_{s\to\infty} sCR(s,A)x,$$

$$D(C_\Lambda) = \left\{ x \in X \,\middle|\, \lim_{s\to\infty} sCR(s,A)x \text{ exists} \right\}.$$

The *dynamic error feedback controller* with state $z(t) \in Z = Y^q$, where q is defined later in (5), and initial state z_0 is then of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0, \tag{3a}$$

$$u(t) = Kz(t), \tag{3b}$$

where $\mathcal{G}_1 \in \mathcal{L}(Z)$, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$, is designed such that the *regulation error* $e(t) = y(t) - y_{ref}(t)$ converges to zero. Finally, the reference signal is generated by the exosystem

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0, \tag{4a}$$

$$y_{ref}(t) = -Fv(t), \tag{4b}$$

with state $v(t)$, initial state $v_0 \in \mathbb{C}^q$ and yet another set of linear operators $S \in \mathbb{C}^{q\times q}$ and $F \in \mathbb{C}^{1\times q}$. We aim to track reference signals that are linear combinations of sinusoidal signals, which can be generated by an exosystem with

$$S = \operatorname{diag}(i\omega_1, -i\omega_1 ..., -i\omega_q), \tag{5}$$

where ω_i are the frequencies of the reference signal. More specific properties of the operators in (2)-(4) will be stated in Section 3. The *closed-loop system* can now be written as

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0},$$

$$e(t) = C_{e\Lambda} x_e(t) + D_e v(t),$$

where $x_e := [x, z]^T$, $D_e := F$, $C_{e\Lambda} := [C_\Lambda, 0]$,

$$A_e := \begin{bmatrix} A_{-1} & BK \\ \mathcal{G}_2 C_\Lambda & \mathcal{G}_1 \end{bmatrix} \text{ with}$$

$$D(A_e) = \left\{ x_e \in D(C_\Lambda) \times Z \,\middle|\, A_{-1}x + BKz \in X \right\}$$

$$\text{and } B_e := \begin{bmatrix} 0 \\ \mathcal{G}_2 F \end{bmatrix}$$

Finally, A_{-1} is the extension of A from $D(A)$ to X.

The Robust Output Regulation Problem. Design a controller such that

(1) The closed-loop semigroup $T_e(t)$ generated by A_e is exponentially stable.
(2) For every x_{e0} and v_0,
$$||e(t)|| \leq M_e e^{-\omega_e t}(||x_{e0}||^2 + ||v_0||^2) \tag{6}$$
for some $M_e, \omega_e > 0$.
(3) If the operators A, B, C, F are perturbed in a way that the closed-loop system remains exponentially stable, then (6) is still satisfied for all x_{e0}, v_0 for some $M_{ep}, \omega_{ep} > 0$.

In this paper the three requirements are fulfilled for a regular linear system by first including a suitable internal model of the dynamics of the exosystem into the operator $\mathcal{G}_1$ and then choosing $\mathcal{G}_2$ and K such that the closed-loop system is exponentially stable. For a regular linear system of the form (2) and an exosystem of the form (4) with S being a diagonal matrix, the controller design implemented will be the minimal order robust controller from Rebarber and Weiss (2003) and Paunonen (2016).

3. ROBUST OUTPUT REGULATION OF THE COUNTER-FLOW HEAT EXCHANGER

In order to design a controller to solve the robust ouput regulation problem, we need to confirm certain properties of the heat exchanger model. These properties will be checked one by one starting with stability and then moving on towards showing regularity of the system. Once all of the required properties have been verified, we present the controller structure for solving the robust output regulation problem.

3.1 Stability of the Plant

Stability of a system much like (1) but with constant coefficients k_h and k_c has been considered in Burns and Cliff (2014), and we will be following the same steps in verifying the stability properties of our plant. We start our analysis of the system with the change of variables
$$\theta_h = \sqrt{k_c}\Theta_h, \qquad \theta_c = \sqrt{k_h}\Theta_c$$
for symmetry reasons. Now an alternative representation for system (1) is given by
$$\frac{\partial \theta_h}{\partial t} = -v_h \frac{\partial \theta_h}{\partial \xi} - k_h\theta_h + \sqrt{k_h k_c}\theta_c, \tag{7a}$$
$$\frac{\partial \theta_c}{\partial t} = v_c \frac{\partial \theta_c}{\partial \xi} - k_c\theta_c + \sqrt{k_h k_c}\theta_h, \tag{7b}$$
$$\theta_h(0,t) = u(t), \qquad \theta_c(l,t) = 0, \tag{7c}$$
$$\theta_h(\xi,0) = \theta_{h0}(\xi), \qquad \theta_c(\xi,0) = \theta_{c0}(\xi), \tag{7d}$$
$$y(t) = \theta_c(0,t), \tag{7e}$$
which we will be working with from now on.

We take as our state space $X := L^2(\Omega) \times L^2(\Omega) =: X_h \times X_c$. To construct the abstract linear system representation (2) for our system, we first define operator $A_b : X \longrightarrow X$,
$$A_b = \begin{bmatrix} -M_{k_h} & M_{\sqrt{k_h k_c}} \\ M_{\sqrt{k_h k_c}} & -M_{k_c} \end{bmatrix}.$$
Clearly A_b is now a bounded operator, and it can also be verified that it is self-adjoint and dissipative.

Next consider operator $A_d : D(A_d) \longrightarrow X$,
$$A_d = \begin{bmatrix} -v_h \frac{\partial}{\partial \xi} & 0 \\ 0 & v_c \frac{\partial}{\partial \xi} \end{bmatrix},$$
$$D(A_d) = \left\{ x \in H^1(\Omega) \times H^1(\Omega) \,\middle|\, \theta_h(0) = \theta_c(l) = 0 \right\},$$
where $x = [\theta_h \ \theta_c]^T$. It has been shown in (Burns and Cliff, 2014, Thm. 2) that A_d generates a dissipative strongly continuous semigroup on X. We note that representing (7) as an abstract linear system of the form (2), we have as the system operator $A : D(A) = D(A_d) \longrightarrow X$,
$$A = A_d + A_b = \begin{bmatrix} -v_h \frac{\partial}{\partial \xi} - M_{k_h} & M_{\sqrt{k_h k_c}} \\ M_{\sqrt{k_h k_c}} & v_c \frac{\partial}{\partial \xi} - M_{k_c} \end{bmatrix}.$$
We now get the following result regarding the stability of our system.

Theorem 1. The operator A generates an exponentially stable strongly continuous semigroup $T(t)$ on X.

Proof. Recall that $v_h, v_c > 0$, $k_h, k_c \in C^\infty(\Omega)$ and A is dissipative. Thus the requirements of (Besson et al., 2006, Thm. 1.1) are fulfilled, which yields the result. □

Finally, we formulate (7) as an abstract linear system
$$\dot{x}(t) = Ax(t) + Bu(t), \tag{8a}$$
$$y(t) = C_\Lambda x(t), \tag{8b}$$
where $B \in \mathcal{L}(U, X_{-1})$, $B = [\delta_0, 0]^T$ with Dirac delta δ_0 and C_Λ is the Λ-extension of $C \in \mathcal{L}(X_1, Y)$, $Cx(\xi,t) = \theta_c(0,t)$.

3.2 Control and Observation

We now turn our attention to the admissibility of B and C_Λ in (8). We start with the definitions of an *admissible control operator* and an *admissible observation operator* as defined in Tucsnak and Weiss (2009).

Definition 2. Let $B \in \mathcal{L}(U, X_{-1})$, $\tau > 0$ and define $\Phi_\tau \in \mathcal{L}(L^2([0,\infty); U), X_{-1})$ by
$$\Phi_\tau u = \int_0^\tau T(\tau - \sigma)Bu(\sigma)d\sigma,$$
where $T(t)$ is the semigroup generated by the system operator A. Then operator B is an admissible control operator for the semigroup T if Ran $(\Phi_\tau) \subset X$ for some $\tau > 0$.

Definition 3. Let $C \in \mathcal{L}(X_1, Y)$. Then C is an admissible observation operator for the semigroup $T(t)$ if there exists $\tau > 0$ such that for some $K_\tau \geq 0$
$$\int_0^\tau ||CT(t)x_0||_Y^2 dt \leq K_\tau^2 ||x_0||_X^2 \text{ for all } x_0 \in D(A).$$

For later use, we now present (7) as a combination of two subsystems, one for each channel of the heat exchanger. For the hot channel we get the subsystem
$$\dot{x}_h = A_h x_h + B_s u + B_h u_h, \tag{9a}$$
$$y_h = C_h x_h, \tag{9b}$$
where $x_h = \theta_h, A_h = v_h A_{rs} - A_{hb}$, and

$$A_{rs} = -\frac{\partial}{\partial \xi}, \tag{10a}$$

$$D(A_{rs}) = D(A_h) = \{x \in H^1(\Omega) \big| x(0) = 0\} \tag{10b}$$

is the generator of the right shift semigroup T_{rs}. The operators $A_{hb}, B_h, C_h \in \mathcal{L}(X_h)$, $A_{hb} = M_{k_h}, B_h = C_h = M_{\sqrt{k_h k_c}}$, $u_h = \theta_c$ and finally $B_s \in \mathcal{L}(U, (X_h)_{-1})$, $B_s = \delta_0$. On the other hand, for the cold channel the subsystem reads

$$\dot{x}_c = A_c x_c + B_c u_c, \tag{11a}$$

$$y_c = C_c x_c, \tag{11b}$$

$$y = C_{s\Lambda} x_c, \tag{11c}$$

where $x_c = \theta_c, A_c = v_c A_{ls} - A_{cb}$, and

$$A_{ls} = \frac{\partial}{\partial \xi}, \tag{12a}$$

$$D(A_{ls}) = D(A_c) = \{x \in H^1(\Omega) \big| x(l) = 0\} \tag{12b}$$

is the generator of the left shift semigroup T_{ls}. The operators $A_{cb}, B_c, C_c \in \mathcal{L}(X_c), A_{cb} = M_{k_c}, B_c = C_c = M_{\sqrt{k_h k_c}}$ and $u_c = \theta_h$. Finally, $C_{s\Lambda}$ is the Λ-extension of $C_s \in \mathcal{L}((X_c)_1, Y)$, $C_s x_c(\xi, t) = \theta_c(0, t)$.

Now that we have defined $u_h = \theta_c$, $u_c = \theta_h$, we see that the dynamics of (8) are shared by system

$$\begin{bmatrix} \dot{\theta}_h \\ \dot{\theta}_c \end{bmatrix} = \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} + \begin{bmatrix} B_s & B_h & 0 \\ 0 & 0 & B_c \end{bmatrix} \begin{bmatrix} u \\ u_h \\ u_c \end{bmatrix}, \tag{13a}$$

$$\begin{bmatrix} y_h \\ y \\ y_c \end{bmatrix} = \begin{bmatrix} C_h & 0 \\ 0 & C_{s\Lambda} \\ 0 & C_c \end{bmatrix} \begin{bmatrix} \theta_h \\ \theta_c \end{bmatrix}, \tag{13b}$$

and the only differences between the two systems are the additional outputs $y_h = C_h \theta_h$, $y_c = C_c \theta_c$ of (13). We aim to show the following.

Theorem 4. B is an admissible control operator and C_Λ is an admissible observation operator for the exponentially stable semigroup T.

Proof. It is straightforward to show that B_s is an admissible control operator for T_{rs} and C_s is an admissible observation operator for T_{ls}, cf. (Tucsnak and Weiss, 2009, Ch. 4) for the unilateral case, and admissibility of $C_{s\Lambda}$ for T_{ls} follows immediately. Additionally, as bounded operators B_h and C_h are admissible for T_{rs} while B_c and C_c are admissible for T_{ls}, which means that the control and observation operators of our subsystems (9)-(12) are admissible for their respective semigroups.

Operator A_h is received by boundedly perturbing A_{rs}, which means that it generates a strongly continuous semigroup T_h. Furthermore, the admissible control operators for boundedly perturbed semigroups are exactly the same as those for the original ones, thus B_s is an admissible control operator for T_h. By similar arguments, $C_{s\Lambda}$ is an admissible observation operator for T_c generated by A_c, and as bounded operators B_h, C_h, B_c and C_c remain admissible control and observation operators for T_h and T_c, respectively. Thus the control and observation operators in (13) are admissible, and we conclude that B is an admissible control operator and C is an admissible observation operator for T. □

3.3 Regularity

With admissibility of the control and observation operators verified in Theorem 4, the last step to prove regularity of (7) is to consider regularity of its *transfer function* $P(s)$. Regularity is defined as follows according to Weiss (1994b).

Definition 5. Transfer function $P(s)$ is regular if there exists $D \in \mathcal{L}(U, Y)$, called the feedthrough operator of the system, such that

$$\lim_{s \to \infty} P(s)u = Du \text{ for all } u \in U. \tag{14}$$

We will start by constructing a suitable expression for the transfer function, after which the regularity can be verified easily. To figure out the transfer function, we use the connection between *open-loop* and closed-loop transfer functions through a feedback structure presented in Weiss (1994a). Looking at (13) together with (9)-(12), we see that the coupling between the hot and cold channels of the heat exchanger can be thought of as an output feedback structure of the form

$$\hat{u} = K\hat{y} + \hat{v}$$

$$\iff \begin{bmatrix} u \\ u_h \\ u_c \end{bmatrix} = \begin{bmatrix} 0 \\ y_c \\ y_h \end{bmatrix} + \hat{v} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_h \\ y \\ y_c \end{bmatrix} + \hat{v}.$$

Now the closed-loop transfer function $P^K(s)$, which includes the connection

$$y = P(s)u,$$

where $P(s)$ is the transfer function for (7) of our interest, is given by

$$P^K(s) = P^O(s)(I - KP^O(s))^{-1}. \tag{15}$$

From (13) we see that the transfer function of our complete open-loop system for a fixed frequency s_* is given by

$$\hat{y} = P^O(s_*)\hat{u} \iff$$

$$P^O = C^O R(s_*, A) B^O = \begin{bmatrix} C_h R_h B_s & C_h R_h B_h & 0 \\ 0 & 0 & C_{s\Lambda} R_c B_c \\ 0 & 0 & C_c R_c B_c \end{bmatrix},$$

where $R_h = R(s_*, A_h)$ and $R_c = R(s_*, A_c)$. The closed-loop transfer function is now to be solved from (15). We start by noting that

$$I - KP^O = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -C_c R_c B_c \\ -C_h R_h B_s & -C_h R_h B_h & I \end{bmatrix}.$$

By using Schur's complement, we first get that

$$M_1 := \begin{bmatrix} I & -E_2 \\ -E_1 & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I - E_2 E_1)^{-1} & (I - E_2 E_1)^{-1} E_2 \\ E_1 (I - E_2 E_1)^{-1} & (I - E_1 E_2)^{-1} \end{bmatrix},$$

where $E_1 := C_h R_h B_h$ and $E_2 := C_c R_c B_c$, and afterwards that

$$\left(I - KP^O\right)^{-1} = \begin{bmatrix} I & 0 \\ M_1 \begin{bmatrix} 0 \\ C_h R_h B_s \end{bmatrix} & M_1 \end{bmatrix}.$$

Finally equation (15) yields

$$P^K = \begin{bmatrix} C_h R_h B_s & C_h R_h B_h & 0 \\ 0 & 0 & C_{s\Lambda} R_c B_c \\ 0 & 0 & C_c R_c B_c \end{bmatrix} \left(I - KP^O\right)^{-1}$$
$$= \begin{bmatrix} * & * & * \\ C_{s\Lambda} R_c B_c \left(I - KP^O\right)^{-1}_{31} & * & * \\ * & * & * \end{bmatrix}.$$

We are interested in solving the transfer function connecting y to u, which is now seen to be given by

$$y = P^K_{21} u = Pu$$
$$= C_{s\Lambda} R_c B_c (I - C_h R_h B_h C_c R_c B_c)^{-1} C_h R_h B_s u. \quad (16)$$

With a suitable transfer function expression constructed, we are ready to prove the following theorem.

Theorem 6. The heat exchanger representation (7) is a regular linear system with $D = 0$.

Proof. With the control and observation operators shown admissible, it suffices to check that (14) of Definition 5 holds. We know that

$$||R(s, A_h)|| \to 0, \qquad ||R(s, A_c)|| \to 0 \quad \text{and}$$
$$||(I - C_h R(s, A_h) B_h C_c R(s, A_c) B_c)^{-1}|| \to I$$

as $s \to \infty$. From (16) we now immediately see that

$$\lim_{s \to \infty} P(s) u = 0$$

for every $u \in \mathbb{C}$. Thus (14) holds and $D = 0$. □

3.4 The Controller

So far we have shown in Theorems 1 and 6 that (7) is indeed an exponentially stable regular linear system. This allows us to make use of controller structures designed particularly for systems of this kind.

Recall that the operator S of our exosystem (4) is assumed to be a diagonal matrix. From (Paunonen, 2016, Thm. 4.1) we have that controller of the form (3) designed with the following parameter choices, the so called minimal order robust controller for stable systems, solves the robust output regulation problem.

$$\mathcal{G}_1 = \mathrm{diag}(i\omega_1 I_Y, ..., i\omega_q I_Y) \in \mathcal{L}(Z), \quad (17a)$$
$$K = \epsilon K_0 = \epsilon(K_0^1, ..., K_0^q) \in \mathcal{L}(Z, U), \quad (17b)$$
$$\mathcal{G}_2 = (-(P(i\omega_k) K_0^k)^*)_{k=1}^q \in \mathcal{L}(Y, Z) \quad (17c)$$

with ω_k received from (5). Components K_0^q are to be chosen such that $P(i\omega_k) K_0^k$ are invertible, and $\epsilon > 0$ is to be chosen such that the closed-loop system is stable.

4. NUMERICAL EXAMPLE

We consider robust output regulation of the heat exchanger given by (7) on the interval $0 \leq \xi \leq 2$, and start by choosing the plant parameters $v_h = 1.0$, $v_c = 0.7$, $k_h(\xi) = 1.2 - 0.1\xi$ and $k_c(\xi) = 0.05\xi^2 + 1$. The goal is to track the sinusoidal reference signal $y_{ref}(t) = \sin(t)$, which can be generated by the exosystem with $S = \mathrm{diag}(i, -i), F = [-1, 1]$ and $v_0 = \frac{1}{2i}[1, \ -1]^T$. The initial state of the plant is set to $[\theta_{h0}, \theta_{c0}]^T = [1.5, \ 0.5]^T$.

We use the minimal order robust controller, so from (17) we get that $\mathcal{G}_1 = S$. By choosing

$$K_0 = \left[P(i)^{-1}, \ P(-i)^{-1}\right],$$

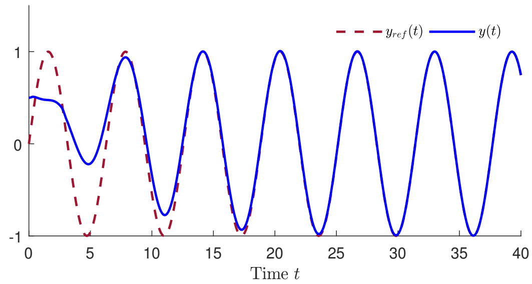

Fig. 2. Output $y(t)$ and reference output $y_{ref}(t)$ of the heat exchanger for $t \in [0, 40]$

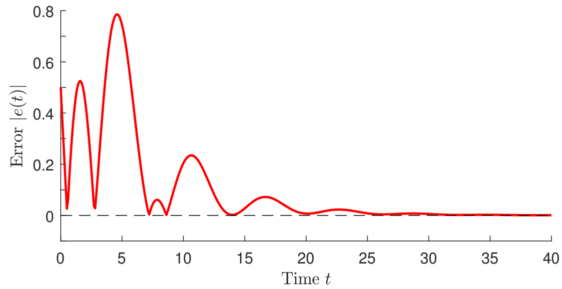

Fig. 3. Absolute value of the regulation error $e(t)$ for $t \in [0, 40]$

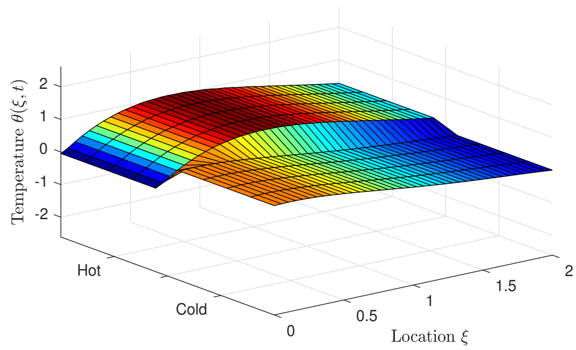

Fig. 4. State of the heat exchanger at time $t = 40$

we have $\mathcal{G}_2 = -I \in \mathbb{R}^{2\times 2}$. Finally we choose $\epsilon = 0.2$ based on trial and error approach with the simulation. Transfer function for the desired frequencies is solved directly from (7) using `Chebfun`-package in MATLAB. Chebfun is based on representing functions by polynomial interpolation in Chebyshev points, see Trefethen (2013). System (7) is discretized in the spatial domain with the hot and cold channels divided to $n = 40$ nodes each. Solution of (7) at each node is then approximated using the finite element method.

Figure 2 illustrates the behaviour of the observed system output compared to the reference signal. Together with the information on the absolute value of the regulation error illustrated in Figure 3, we can say that relatively accurate tracking is achieved after two periods of the reference signal with $|e(t)| < 0.1$ from there on, and the regulation error converges to 0. Additionally, the states of both of the channels of the heat exchangers at the end of the simulation is presented in Figure 4.

5. CONCLUSION

We examined a 1D PDE model commonly used for counter-flow heat exchangers. The considered heat exchanger model with boundary control and observation was shown to form an exponentially stable regular linear system, which enabled us to use controller structures designed for such systems in order to solve the robust output regulation problem for the heat exchanger for a class of reference signals. The controller was demonstrated in action using numerical simulations performed with MATLAB.

Possible avenues for future research include implementation of more complicated controller structures in order to possibly reach faster convergence to the reference signal. The nature of coupling between the channels of the heat exchanger, in this paper governed by $C^\infty(\Omega)$ functions, could also be further examined and possibly extended to a larger class of functions.

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