Approximate Local Output Regulation for a Class of Nonlinear Fluid Flows*

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***Abstract*— We consider output tracking for a class of viscous nonlinear fluid flows including the incompressible 2D Navier–Stokes equations. The fluid is subject to in-domain inputs and disturbances. We construct an error feedback controller which guarantees approximate local velocity output tracking for a class of reference outputs. The control solution covers point velocity observations and assumes a smooth enough state space. Efficacy of the control solution is illustrated through a numerical example.**

I. Introduction

In this work, we consider an output tracking problem for viscous nonlinear fluid flows in the neighborhood of a (locally) exponentially stable steady state solution. We formulate our results for the incompressible Navier–Stokes equations on a sufficiently regular domain $\Omega \subset \mathbb{R}^2$ with boundary Γ. More precisely, we consider controlling an output y of the equations

$$\frac{\partial w}{\partial t} = \nu \Delta w - (w \cdot \nabla) w - \nabla q + f_w + f_u + f_d, \qquad \text{(1a)}$$

$$0 = \nabla \cdot w, \qquad w|_\Gamma = 0, \qquad \text{(1b)}$$

where $w(\xi, t)$ is the fluid velocity, $q(\xi, t)$ is the fluid pressure, ν is the kinematic viscosity, $f_w(\xi)$ is a body force, $f_u(\xi, t)$ is the control action and $f_d(\xi, t)$ is the disturbance action. Our goal is to design a controller such that a chosen velocity output of (1) converges to a desired reference output approximately for initial states which are, in a certain sense, "close enough" to a steady state solution of (1).

Theory of output regulation for nonlinear systems is still under development, especially for infinite-dimensional systems. The finite-dimensional results of [11] have been extended to a class of infinite-dimensional systems in [5] and for co-located inputs and outputs in [6], based on which several example cases are presented in [1]. In this work, we focus on output regulation in an approximate sense utilizing the results of [13]. In [13], the authors use an error feedback controller designed for *robust output regulation* of *exponentially stable regular linear systems* and show that the same controller achieves *approximate local output regulation* for a class of nonlinear systems which they call *regular nonlinear systems*. Similar approach of using linear control solutions for nonlinear systems has been utilized for local stabilization of nonlinear fluid flows in different setups, see e.g. [3] for in-domain inputs, [14] for boundary inputs and [10] for observer design.

As the main contribution of this paper, we show that the equations (1) can be formulated as a regular nonlinear system (in the sense of [13]) for a wide range of velocity observations including the point observation. To achieve this, we consider the equations (1) on a "lifted" state space, i.e. we demand more smoothness from the velocity and the pressure than would typically be required to e.g. solve similar control problems for linear systems. To formulate (1) as a regular nonlinear system, we rely on the fluid being viscous and assume that the domain Ω together with the boundary conditions are such that the system can be formulated on the "lifted" state space. These properties, together with the type $(x \cdot \nabla)x$ of the nonlinearity typical for fluid flows, characterize the fluid flows for which the results can be applied.

Using the results of [13], we show that in the neighborhood of a steady state solution, velocity observations on (1) approximately converge to any desired "small enough" periodic reference signal of the type

$$y_r(t) = a_0 + \sum_{i=1}^{q_s} a_i \cos(\omega_i t) + b_i \sin(\omega_i t)$$

in the sense that for small enough initial data, a finite number of chosen *harmonics* of the system output and the reference output are the same. Here the coefficient vectors $a_i, b_i \in \mathbb{R}^{p_y}$ may be unknown but we expect to know the frequencies ω_i. The controller introduced in [13] also rejects periodic in-domain disturbance signals of the type

$$u_d(t) = c_0 + \sum_{i=1}^{q_s} c_i \cos(\omega_i t) + d_i \sin(\omega_i t)$$

with small enough amplitude, where again $c_i, d_i \in \mathbb{R}^d$ are allowed to be unknown. Note that several controllers have been designed for robust output tracking of similar signal classes in the case of linear systems, see e.g. [16], [15].

Rest of the paper is organized as follows. In Section II, we recall the concepts of regular nonlinear systems and approximate local output regulation. In Section III, we show that the Navier–Stokes equations with in-domain control and point observation fit into the framework of regular nonlinear systems on a suitable state space. In Section IV, we construct, based on [13], a controller for approximate local output regulation for the Navier–Stokes equations and then illustrate the controller's performance through a simulation

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example in Section V. Finally, the paper is concluded in Section VI.

Throughout the paper we denote by $\mathcal{L}(X,Y)$ the set of bounded linear operators from a Hilbert space X to a Hilbert space Y. For a linear operator $A : D(A) \to X$, $D(A)$ is the domain of A, $\rho(A)$ is the resolvent set of A and $\mathbb{T}_A$ is the strongly continuous semigroup generated by A on X. For a fixed $s \in \rho(A)$, we denote by X_{-1} the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(sI - A)^{-1}x\|_X$ and define $X_1 = D(A)$, equipped with the norm $\|x\|_{X_1} = \|(sI - A)x\|_X$. Finally, the L^2-inner product over a domain Ω is denoted by $\langle(\cdot),(\cdot)\rangle_{L^2(\Omega)}$.

II. Regular Nonlinear Systems and Output Regulation

Output regulation for fluid flow systems covered by this work is based on the concepts of regular nonlinear systems and approximate local output regulation, which were introduced in [13]. These concepts are presented next, with the definition of regular nonlinear systems formulated in a slightly restricted setting by excluding parts that are not relevant to this work.

Definition 1: Let X, U, Y and V be Hilbert spaces, and let C_Λ defined by $C_\Lambda x = \lim_{s\to+\infty} Cs(sI - A)^{-1}x$ with $D(C_\Lambda) = \{x \in X | \text{the above limit exists}\}$ be the Λ-extension of the observation operator C, see [21]. The system

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t) + B_d u_d(t) + Q\mathcal{F}(x(t)),\\
&x(0) = x_0 \in X,\\
&y(t) = C_\Lambda x(t),
\end{aligned}$$

which we denote by Σ_F, is called a regular nonlinear system if the following hold.

(i) The operator A generates an exponentially stable strongly continuous semigroup $\mathbb{T}_A$ on X.

(ii) It holds that $B \in \mathcal{L}(U, X_{-1})$, $B_d \in \mathcal{L}(U_d, X_{-1})$, $Q \in \mathcal{L}(V, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, and the triples (A, B, C), (A, B_d, C) and (A, Q, C) are regular linear systems in the sense of [21].

(iii) The nonlinear map $\mathcal{F} : X \to V$ satisfies $\mathcal{F}(0) = 0$ and is locally Lipschitz. That is, for every bounded set $O \subset X$, there exists a constant L_O such that for all $x_1, x_2 \in O$

$$\|\mathcal{F}(x_1) - \mathcal{F}(x_2)\|_V \le L_O \|x_1 - x_2\|_X.$$

Furthermore, for each $\gamma > 0$ there exists a $\zeta > 0$ such that if $\sup\left\{\|x\|_X \middle| x \in O\right\} < \zeta$, then $L_O < \gamma$.

To generate the plant input, we use an error feedback controller of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z, \tag{3a}$$

$$u(t) = Kz(t), \tag{3b}$$

where $e(t) = y(t) - y_r(t)$ is the regulation error and Z is a Hilbert space. Coupling the controller with a regular nonlinear system yields the closed-loop system Σ_E defined by

$$\begin{aligned}
&\dot{x}_e(t) = A_e x_e + B_e w_{ext}(t) + Q_e \mathcal{F}(x(t)), \quad x_{e0} \in X_e,\\
&e(t) = C_e x_e(t) + D_e y_r(t)
\end{aligned}$$

on the Hilbert space $X_e = X \times Z$, where $x_e = [x,\ z]^T$, $w_{ext} = [u_d,\ y_r]^T$,

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C_\Lambda & \mathcal{G}_1 \end{bmatrix}, \quad B_e = \begin{bmatrix} B_d & 0 \\ 0 & -\mathcal{G}_2 \end{bmatrix}, \quad Q_e = \begin{bmatrix} Q \\ 0 \end{bmatrix},$$
$$C_e = \begin{bmatrix} C_\Lambda & 0 \end{bmatrix}, \quad D_e = \begin{bmatrix} 0 & -I \end{bmatrix}.$$

Before introducing the output tracking problem, we recall the concept of harmonics of a function. Consider a function $f = f_p + f_e$, where $f_p \in L^2_{loc}([0,\infty);\mathbb{C}^n)$ is T-periodic and $f_e \in L^2([0,\infty);\mathbb{C}^n)$. For a non-negative integer l, the l^{th} harmonic of f is the function

$$f_l(t) = \alpha_l \sin\left(\frac{2\pi l t}{T}\right) + \beta_l \cos\left(\frac{2\pi l t}{T}\right), \qquad t \ge 0,$$

where

$$\begin{aligned}
\alpha_l &= \lim_{k\in\mathbb{N}, k\to\infty} \frac{2}{kT}\int_0^{kT} f(t)\sin\left(\frac{2\pi l t}{T}\right)dt,\\
\beta_l &= \lim_{k\in\mathbb{N}, k\to\infty} \frac{2}{kT}\int_0^{kT} f(t)\cos\left(\frac{2\pi l t}{T}\right)dt,
\end{aligned}$$

thus for frequencies of the harmonics, we have $\omega_i = 2\pi l_i/T$. Now the problem of achieving approximate local output regulation is stated as follows.

Problem 2: Let $T > 0$ be a constant. Assume that y_r and u_d are T-periodic functions and let $\mathbb{V} = \{l_0, l_1, ..., l_{n_v}\}$ be a finite set of non-negative integers. Design an error feedback controller (3) such that:

1) The closed-loop system Σ_E is a regular nonlinear system.
2) There exist positive constants c_y, c_d and c_e such that if $\|y_r\|_{L^2[0,T]} \le c_y$, $\|u_d\|_{L^\infty} \le c_d$ and $\|x_{e0}\|_{X_e} \le c_e$, then x_e converges asymptotically to a T-periodic function and the l^{th} harmonic of $y - y_r$ is 0 for each $l \in \mathbb{V}$. The output satisfies $y = y_p + y_e$, where $y_e \in L^2([0,\infty);Y)$ and $y_p \in L^2_{loc}([0,\infty);Y)$ is a T-periodic function.

Accuracy of output tracking by solving the above problem clearly depends on how dominant the harmonics included in $\mathbb{V}$ are. In many cases, ensuring that the first few harmonics of y match those of y_r results in a small tracking error, since higher harmonics of the output are typically small.

III. The Incompressible Navier–Stokes Equations as an Abstract Control System

The goal of this section is to formulate the Navier–Stokes equations (1) as a regular nonlinear system. We start by finding a suitable state space for the formulation and then verify that the requirements of Definition 1 are fulfilled.

A. Choosing the State Space

Translating the equations (1) to the vicinity of a steady state solution (v_e, p_e) using the change of variables $v(\xi,t) = w(\xi,t) - v_e(\xi)$, $p(\xi,t) = q(\xi,t) - p_e(\xi)$ yields

$$\frac{\partial v}{\partial t} = \nu\Delta v - (v_e\cdot\nabla)v - (v\cdot\nabla)v_e \tag{4a}$$

$$- (v\cdot\nabla)v - \nabla p + f_u + f_d, \tag{4b}$$

$$0 = \nabla\cdot v, \qquad v|_\Gamma = 0 \tag{4c}$$

with the initial state $v(\xi,0) = v_0(\xi)$. We assume that the control and the disturbance are defined by

$$f_u(\xi,t) = \begin{bmatrix} g_1(\xi) & g_2(\xi) & \cdots & g_m(\xi)\end{bmatrix} u(t), \tag{5a}$$

$$f_d(\xi,t) = \begin{bmatrix} g_1'(\xi) & g_2'(\xi) & \cdots & g_d'(\xi)\end{bmatrix} u_d(t), \tag{5b}$$

where each $g_1,...,g_m$ and $g_1',...,g_d'$ is an $\mathbb{R}^2$-valued function on Ω, $u(t)\in U \coloneqq \mathbb{C}^m$ is the finite-dimensional control input and $u_d(t)\in U_d \coloneqq \mathbb{C}^d$ is the finite-dimensional disturbance input. Additionally, we assume that the output space Y is also finite-dimensional and $Y = \mathbb{C}^{p_y}$ with $p_y \le m$.

For simpler notation, we define the spaces

$$\tilde{X} = \left\{ v\in (L^2(\Omega))^2 \middle| \nabla\cdot v = 0,\ (v\cdot n)|_\Gamma = 0\right\},$$
$$\tilde{H} = \left\{ v\in (H^1(\Omega))^2 \middle| \nabla\cdot v = 0,\ v|_\Gamma = 0\right\}$$

and the bilinear and trilinear forms

$$a(v,\psi) = 2\nu\langle \epsilon(v),\epsilon(\psi)\rangle_{L^2(\Omega)} \qquad \forall v,\psi\in\tilde{H},$$
$$b(v,\phi,\psi) = \langle (v\cdot\nabla)\phi,\psi\rangle_{L^2(\Omega)} \qquad \forall v,\phi,\psi\in\tilde{H},$$

where $\epsilon(v) = 1/2(\nabla v + (\nabla v)^T)$.

Assumption 3: We assume the following.

(i) The boundary Γ is of class C^3 and $f_w\in (H^1(\Omega))^2$.

(ii) The linearization of (4) is exponentially stable.

The first part of the assumption guarantees sufficient regularity of the solutions to the Navier–Stokes equations (1), while the second part is required for Definition 1.(i) to hold and is satisfied for large enough ν, c.f. [3].

As the first step towards choosing the state space X, we consider semigroup generation for the linearized version of (4) on $\tilde{X}$. A weak formulation for the stationary, linearized version of (4) subject to zero control and disturbance inputs is given by

$$0 = -a(v,\psi) - b(v,v_e,\psi) - b(v_e,v,\psi) \qquad \forall \psi\in\tilde{H}.$$

Lemma 4: The operator $\tilde{A}$ defined by

$$\tilde{A} = \tilde{A}_2 + \tilde{A}_1,$$
$$\langle \tilde{A}_2 x,\psi\rangle_{L^2(\Omega)} = -a(x,\psi),$$
$$\langle \tilde{A}_1 x,\psi\rangle_{L^2(\Omega)} = -b(x,v_e,\psi) - b(v_e,x,\psi),$$
$$D(\tilde{A}) = D(\tilde{A}_2) = \left\{ x\in\tilde{H} \middle| \forall\psi\in\tilde{H}, \psi\to a(x,\psi) \text{ is } \tilde{X}\text{-continuous}\right\}$$

generates an exponentially stable analytic semigroup on $\tilde{X}$.

Proof: We start by showing that $a(\cdot,\cdot)$ is $\tilde{H}$-bounded and $\tilde{H}$-coercive, i.e. $\tilde{H}$ can be continuously and densely embedded in $\tilde{X}$ and there exist $c_1, c_2, \lambda > 0$ such that for every $\phi,\psi\in\tilde{H}$

$$|a(\phi,\psi)| \le c_1\|\phi\|_{\tilde{H}}\|\psi\|_{\tilde{H}}, \tag{6a}$$

$$a(\phi,\phi) \ge c_2\|\phi\|_{\tilde{H}}^2 - \lambda\|\phi\|_{\tilde{X}}^2. \tag{6b}$$

Since the norm $\|\epsilon(\cdot)\|_{\tilde{X}}$ is equivalent to the $\|\cdot\|_{\tilde{H}}$ norm through Korn's and Poincare's inequalities, we immediately have for a constant $c_1 > 0$ and for any $v\in\tilde{H}$

$$a(v,v) = 2\nu\|\epsilon(v)\|_{\tilde{X}}^2 \ge c_1\|v\|_{\tilde{H}}^2.$$

Similarly, for a constant $c_2 > 0$ and any $v,\phi\in\tilde{H}$,

$$|a(v,\phi)| \le 2\nu\|\epsilon(v)\|_{\tilde{X}}\|\epsilon(\phi)\|_{\tilde{X}} \le c_2\|v\|_{\tilde{H}}\|\phi\|_{\tilde{H}}.$$

As such, $a(\cdot,\cdot)$ is $\tilde{H}$-bounded and $\tilde{H}$-coercive, which implies that $\tilde{A}_2$ generates an analytic semigroup $\mathbb{T}_{\tilde{A}_2}$ on $\tilde{X}$ [2, Sec. 2].

Regarding the trilinear form $b(\cdot,\cdot,\cdot)$, Assumption 3.(i) guarantees that $v_e\in\tilde{H}$, c.f. [12, Ch. 5]. We have for constants $c_3, c_4 > 0$ using integration by parts and Sobolev embeddings

$$\begin{aligned}|b(v_1,v_2,\psi)| &\le |\langle v_1,(v_2\cdot\nabla)\psi\rangle_\Omega| + |\langle v_1\cdot n, v_2\cdot\psi\rangle_\Gamma| \\ &\le c_3\|v_1\|_{L^4(\Omega)}\|v_2\|_{L^4(\Omega)}\|\psi\|_{\tilde{H}} \\ &\le c_4\|v_1\|_{\tilde{H}}\|v_2\|_{\tilde{H}}\|\psi\|_{\tilde{H}} \qquad \forall v_1,v_2,\psi\in\tilde{H}.\end{aligned}$$

Now $\tilde{A}_1\in\mathcal{L}(\tilde{H},\tilde{X})$, thus perturbation theory of semigroups, see e.g. [7, Ch. III], implies that $\tilde{A}$ generates an analytic semigroup $\mathbb{T}_{\tilde{A}}$ on $\tilde{X}$. By Assumption 3.(ii), $\mathbb{T}_{\tilde{A}}$ is exponentially stable. ■

The fact that we may choose $\lambda = 0$ in (6b) implies that the semigroup $\mathbb{T}_{\tilde{A}_2}$ is exponentially stable for any $\nu > 0$. Furthermore, $\tilde{A}_2$ is self-adjoint and the fractional powers $(-\tilde{A}_2)^\delta$ are well defined. Domains of the fractional powers are defined by, c.f. [18, Ch. 2], [14],

$$D((-\tilde{A}_2)^\delta) = \left\{ v\in (H^{2\delta}(\Omega))^2 \middle| \nabla\cdot v = 0,\ (v\cdot n)|_\Gamma = 0\right\}, \quad 0\le\delta<\frac{1}{4},$$
$$D((-\tilde{A}_2)^\delta) = \left\{ v\in (H^{2\delta}(\Omega))^2 \middle| \nabla\cdot v = 0,\ v|_\Gamma = 0\right\}, \quad \frac{1}{4}<\delta\le 1.$$

The norms corresponding to domains of the fractional powers for the full range $\delta\in\mathbb{R}$ are given by

$$\|x\|_{D((-\tilde{A}_2)^\delta)} = \|(-\tilde{A}_2)^\delta x\|_{\tilde{X}}.$$

We next utilize domains of the fractional powers to find a "lifted" state space X such that in particular Definition 1.(iii) is satisfied by (4).

For the translated Navier–Stokes equations (4), nonlinearity in the abstract framework Σ_F is described by

$$\mathcal{F}(v) = -\mathbb{P}\big((v\cdot\nabla)v\big), \qquad Q = I, \tag{7}$$

where $\mathbb{P}$ is the Leray projector, see e.g. [8], [14]. The domains of definition for $\mathcal{F}$ and Q are dictated by the following Lemma.

Lemma 5: For a (small) $\delta > 0$, choose $X = D((-\tilde{A}_2)^{1/2+\delta})$ and $V = D((-\tilde{A}_2)^{\delta})$. Then Definition 1.(iii) holds for $\mathcal{F}$.

Proof: The proof is based on the "properties of multipliers", see [17, Ch. 4.6.1, Thm. 1], [4, Lemma 5.4], which state that if

$$s_2 > s_1, \qquad s_2 > \frac{d_\Omega}{2}, \tag{8}$$

where d_Ω is the spatial dimension, then

$$H^{s_1} \cdot H^{s_2} \to H^{s_1}$$

is continuous, where

$$H^{s_1} \cdot H^{s_2} := \big\{ fg \;\big|\; f \in H^{s_1},\ g \in H^{s_2} \big\}.$$

Since $d_\Omega = 2$, we choose $s_1 = 2\delta$ and $s_2 = 1+2\delta$, and apply the above result. Now for a constant $c_1 > 0$ and $\phi, \psi \in X$

$$\|\phi_i \partial_j \psi_k\|_{H^{s_1}(\Omega)} \leq c_1 \|\phi_i\|_{H^{s_2}(\Omega)} \|\partial_j \psi_k\|_{H^{s_1}(\Omega)} \tag{9a}$$

for $i, j, k \in \{\xi_1, \xi_2\}$, thus for some constants $c_2, c_3 > 0$ also

$$\begin{aligned} \|(\phi \cdot \nabla)\psi\|_V &\leq c_2 \|\phi\|_X \|\nabla \psi\|_V \\ &\leq c_3 \|\phi\|_X \|(-\tilde{A}_2)^{1/2} \psi\|_V \\ &= c_3 \|\phi\|_X \|\psi\|_X. \end{aligned} \tag{9b}$$

Utilizing (9), for $v_1, v_2 \in X$ and some constants $c_4, c_5 > 0$ we have

$$\begin{aligned} &\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_V \\ &= \| -\mathbb{P}\big((v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2\big)\|_V \\ &= \|\mathbb{P}\big(((v_1 - v_2) \cdot \nabla)v_1 + (v_2 \cdot \nabla)(v_1 - v_2)\big)\|_V \\ &\leq c_4 \big(\|(v_1 - v_2)\|_X \|\nabla v_1\|_V + \|v_2\|_X \|\nabla(v_1 - v_2)\|_V\big) \\ &\leq c_5 (\|v_1\|_X + \|v_2\|_X) \|v_1 - v_2\|_X, \end{aligned}$$

thus $\mathcal{F}$ is locally Lipschitz. Clearly $\mathcal{F}(0) = 0$, and if $\|v_1\|_X, \|v_2\|_X < \frac{\gamma}{c}$ for a large enough constant $c > 0$, then $(\|v_1\|_X + \|v_2\|_X) < \gamma$, which completes the proof. ∎

Due to Lemma 5, we choose for a fixed (small) $\delta > 0$

$$X = D((-\tilde{A}_2)^{1/2+\delta})$$

as the state space for our abstract system presentation and denote

$$X_s = D((-\tilde{A}_2)^{1/2+\delta+s}) \qquad \forall s \in \mathbb{R},$$

with the corresponding norms defined accordingly by

$$\|x\|_{X_s} = \|(-\tilde{A}_2)^s x\|_X = \|(-\tilde{A}_2)^{1/2+\delta+s} x\|_{\tilde{X}}.$$

Now $V = X_{-1/2}$, $\mathcal{F} : X \to V$ and $Q = I_V \in \mathcal{L}(V)$.

B. The Abstract System Formulation

We define the operators

$$A = A_2 + A_1 : D(A) \to X, \tag{10a}$$

$$A_2 = \nu \mathbb{P} \Delta, \quad A_1 v = -\mathbb{P}\big((v_e \cdot \nabla)v + (v \cdot \nabla)v_e\big), \tag{10b}$$

$$D(A) = D(A_2) = D((-\tilde{A}_2)^{3/2+\delta}). \tag{10c}$$

Now $A_2 v = \tilde{A}_2 v$ and $A_1 v = \tilde{A}_1 v$ for $v \in D(A)$. To verify that Definition 1.(i) holds on the state space X, we note that A generates a strongly continuous semigroup $\mathbb{T}_A$ on X, c.f. [7, Ch. 5]. The semigroup $\mathbb{T}_A$ is exponentially stable, since for $x \in X$

$$\begin{aligned} \|\mathbb{T}_A x\|_X &= \|(-\tilde{A}_2)^{1/2+\delta} \mathbb{T}_A x\|_{\tilde{X}} \\ &= \|\mathbb{T}_{\tilde{A}} (-\tilde{A}_2)^{1/2+\delta} x\|_{\tilde{X}} \\ &\leq \|\mathbb{T}_{\tilde{A}}\|_{\mathcal{L}(\tilde{X})} \|x\|_X. \end{aligned}$$

We still need to verify Definition 1.(ii). We do so for controls and disturbances of the form (5) and observations up to the "level of unboundedness" of a point observation. Using integration by parts, we have for the X-adjoint of A_1

$$A_1^* \phi = \mathbb{P}\big((v_e \cdot \nabla)\phi - (\nabla v_e)^T \phi\big).$$

Properties of multipliers with the choices $s_1 = 1 + 2\delta$, $s_2 = 2 + 2\delta$ to satisfy (8) imply, similarly to (9), for any $\phi, \psi \in X_{1/2}$ and a constant $c > 0$

$$\begin{aligned} \|\mathbb{P}\big((\phi \cdot \nabla)\psi\big)\|_X &\leq c \|\phi_2\|_{X_{1/2}} \|\psi_2\|_{X_{1/2}}, \\ \|\mathbb{P}\big((\nabla \phi)^T \psi\big)\|_X &\leq c \|\phi_2\|_{X_{1/2}} \|\psi_2\|_{X_{1/2}}. \end{aligned}$$

Since Assumption 3.(i) implies $v_e \in X_{1/2} \subset (H^{2+2\delta}(\Omega))^2$ [12, Ch. 5], we have $A_1, A_1^* \in \mathcal{L}(X_{1/2}, X)$. As such, theory of admissible control and observation operators, see [20, Ch. 4-5], now states that

- Let $\tilde{Y}$ be a Hilbert space. If $\tilde{C} \in \mathcal{L}(X_{1/2}, \tilde{Y})$, then $\tilde{C}$ is an *admissible observation operator* for $\mathbb{T}_{A_2}$ and its adjoint $\tilde{C}^* \in \mathcal{L}(\tilde{Y}, X_{-1/2})$ is an *admissible control operator* for $\mathbb{T}_{A_2}$.
- The sets of admissible control (observation) operators for $\mathbb{T}_A$ and $\mathbb{T}_{A_2}$ are the same.

Note that above we assumed for $\tilde{Y}$ to be self-dual.

We first search for admissible observations for (4) on the state space X by considering observations such that $C \in \mathcal{L}(X_{1/2}, Y)$. Typically the "most unbounded" observation of interest would be the point observation

$$C_p x(\xi, t) = x(\xi_p, t) \tag{11}$$

for some $\xi_p \in \Omega$. By Sobolev embeddings, when $\Omega \subset \mathbb{R}^2$, $H^s(\Omega) \subset C(\bar{\Omega})$ for $s > 1$, thus $C_p \in \mathcal{L}(X, \mathbb{C})$. That is, all the observations of interest for (4) are bounded operators from X to Y. As such, $C_\Lambda = C$ and if U, U_d and Q are admissible control operators for $\mathbb{T}_{A_2}$, then Definition 1.(ii) holds.

Consider next admissible control operators for $\mathbb{T}_{A_2}$, thus also for $\mathbb{T}_A$. We start with the operator $Q = I_V = I_{X_{-1/2}}$. Note that in this case the "input space" V is not self-dual, but instead the correct dual is the X-dual of $X_{-1/2}$, i.e. $V' = X_{1/2}$. Thus we have $Q^* \in \mathcal{L}(X_{1/2}, V')$ and $Q \in \mathcal{L}(V, X_{-1/2})$, i.e. the triple (A, Q, C) is a regular linear system.

For a single control input of the type (5a), we have $B = g(\xi)$. If $g \in X_s$, then $B \in \mathcal{L}(\mathbb{C}, X_s)$. That is, if

$$g \in X_{-1/2} = D((-\tilde{A}_2)^{\delta}),$$

then B is an admissible control operator for $\mathbb{T}_{A_2}$.

We conclude the section by gathering our findings in the following result.

Theorem 6: Given Assumption 3, assume that the control shape functions g_i and the disturbance shape functions g_j' satisfy $g_i, g_j' \in X_{-1/2} = D((-\tilde{A}_2)^{\delta})$ for each $i = 1, 2, ..., m$, $j = 1, 2, ..., d$ and a small $\delta > 0$. Then the translated Navier–Stokes equations (4) with the dynamics operator (10), the nonlinearity (7), the control (5a), the disturbance (5b) and p_y point observations (11) form a regular nonlinear system on the state space $X = D((-\tilde{A}_2)^{1/2+\delta})$.

IV. The Controller

We use a low-gain -type controller design introduced in [13] to solve Problem 2. The only system information required to construct the controller is the *transfer function* gains

$$G(\pm i\omega_k) = C(\pm i\omega_k I - A)^{-1}B$$

of the linearized system (A, B, C) for the frequencies $\omega_k = 2\pi l_k/T$ for each $l_k \in \mathbb{V}$. A good estimate for these gains of the linearized system can be obtained experimentally from the gains of the nonlinear system Σ_N, see [13], and robustness of the controller means that the approximate gains can be used to achieve the output tracking goal.

The controller consists of two finite-dimensional systems. The first system Σ_F is described by the transfer function

$$G_F(s) = I_Y - \prod_{k=0}^{n_v} \frac{s^2 + \omega_k^2}{s^2 + \varepsilon s + \omega_k^2} I_Y,$$

where $\varepsilon > 0$ is the control tuning parameter. The second system Σ_R is described by the transfer function

$$G_R(s) = \prod_{k=0}^{n_v} \frac{s^2 + \omega_k^2}{s^2 + 2s + \omega_k^2} \times \sum_{k=0}^{n_v} \left(\frac{\rho_k R_k}{s - i\omega_k} + \frac{\rho_{-k} R_{-k}}{s + i\omega_k} \right),$$

where

$$\begin{aligned} R_k &= G^*(i\omega_k)(G(i\omega_k)G^*(i\omega_k))^{-1}, \\ R_{-k} &= G^*(-i\omega_k)(G(-i\omega_k)G^*(-i\omega_k))^{-1}, \\ \rho_k &= \prod_{j\neq k, j=0}^{n_v} \frac{\omega_j^2 - \omega_k^2 + 2i\omega_k}{\omega_j^2 - \omega_k^2}, \\ \rho_{-k} &= \prod_{j\neq k, j=0}^{n_v} \frac{\omega_j^2 - \omega_k^2 - 2i\omega_k}{\omega_j^2 - \omega_k^2}. \end{aligned}$$

We denote a state space realization of G_F on $X_F = \mathbb{C}^{n_F}$ by

$$\begin{aligned} \dot{x}_F(t) &= A_F x_F(t) + B_F u_F(t), \quad x_F(0) = x_{F0} \in X_F, \\ y_F(t) &= C_F x_F(t), \end{aligned}$$

and a state space realization of G_R on $X_R = \mathbb{C}^{n_R}$ by

$$\begin{aligned} \dot{x}_R(t) &= A_R x_R(t) + B_R u_R(t), \quad x_R(0) = x_{R0} \in X_R, \\ y_R(t) &= C_R x_R(t) + D_R u_R(t). \end{aligned}$$

After coupling the two subsystems of the controller as depicted in Fig. 1, i.e. by setting $u_F = y - y_r + y_F$ and $u_R = y_F$, we have the structure of an error feedback controller (3) with $z = [x_F, \ x_R]^T \in X_F \times X_R$,

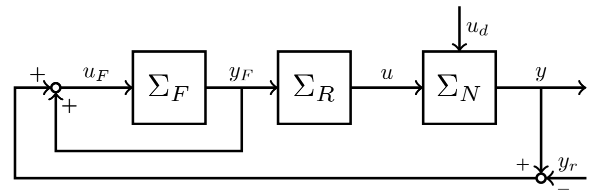

Fig. 1: The closed-loop system

$$\mathcal{G}_1 = \begin{bmatrix} A_F + B_F C_F & 0 \\ B_R C_F & A_R \end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix} -B_F \\ 0 \end{bmatrix}, \quad (12a)$$

$$K = \begin{bmatrix} D_R C_F & C_R \end{bmatrix}. \quad (12b)$$

The following result is obtained in [13] for the class of regular nonlinear systems and we formulate it for the incompressible 2D Navier–Stokes equations.

Theorem 7: Assume that $G(i\omega_k)$ is surjective for each $k = 1, 2, \ldots n_v$ and the assumptions of Thm. 6 hold. There exists $\varepsilon^* > 0$ such that an error feedback controller (3) with the operators chosen as (12) with $0 < \varepsilon \leq \varepsilon^*$ solves Problem 2 for the system (10)-(11).

Proof: The proof follows directly from Theorem 6 and [13, Sec. 5.2]. ■

V. A Numerical Example

Let Ω be the unit disk and consider the Navier–Stokes equations (1) with $\nu = 1/25$ around a steady state solution corresponding to the body force

$$f_w(\xi_1, \xi_2) = \left[\xi_2(1 - \xi_1^2 - \xi_2^2), \quad -\xi_1(1 - \xi_1^2 - \xi_2^2)\right]^T \in \tilde{H}$$

and $f_u = 0$, $f_d = 0$. Our output tracking goal is to have the point observation

$$y(t) = C \begin{bmatrix} v_1(\xi, t) \\ v_2(\xi, t) \end{bmatrix} = v_2\left(\begin{bmatrix} 0.4, & -0.4 \end{bmatrix}, t\right) \in \mathcal{L}(X, \mathbb{R})$$

track the reference signal

$$y_r(t) = 0.5 \sin(2t) \quad (13)$$

despite the disturbance

$$f_d(\xi, t) = B_d u_d(t) = \mathbb{P} \left[0, \quad \chi_{\Omega_d}(\xi)\right]^T (1 + \cos(2t)),$$

where χ_{Ω_d} is the characteristic function and $\Omega_d = [-0.4, -0.1] \times [-0.4, -0.1]$. The output tracking is to be achieved, approximately and locally, by using the control

$$f_u(\xi, t) = Bu(t) = \mathbb{P} \left[\chi_{\Omega_u}(\xi), \quad 0\right]^T u(t)$$

where $\Omega_u = [-0.6, -0.3] \times [0.1, 0.4]$. Now $U = U_d = Y = \mathbb{R}$ and since $\chi_{\Omega_u}, \chi_{\Omega_d} \in H^s(\Omega)$ for any $s < 1/2$ [19], also $B \in \mathcal{L}(U, X_{-1/2})$ and $B_d \in \mathcal{L}(U_d, X_{-1/2})$. As such, if the steady state (v_e, p_e) is locally exponentially stable, then the translated system (4) forms a regular nonlinear system on the state space X.

We use the Taylor-Hood finite element spatial discretization for the Navier–Stokes equations. With the help of functions included in the Matlab PDE toolbox, the unit disk is approximated by 694 triangles with the maximum edge length of ≈ 0.1, which leads to approximation order of 1453 for each of the velocity components and 380 for the pressure. The steady state solution (v_e, p_e), with the steady state velocity depicted in Fig 2, is calculated using the Newton's method, and we assume $p_e(0) = 0$ to obtain a unique steady state pressure.

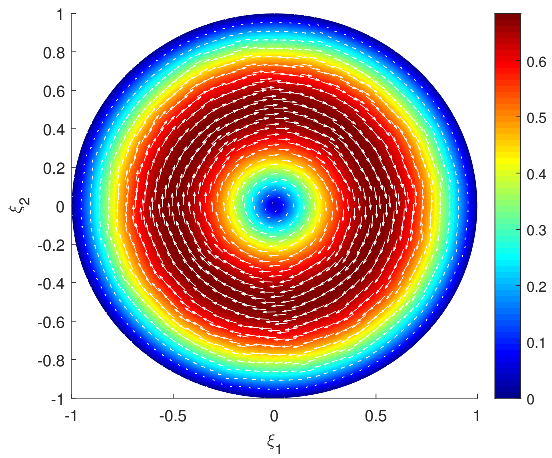

Fig. 2: The steady state velocity field $v_e(\xi)$, where color depicts speed of the fluid

We check numerically that linearization of the translated system (4) is exponentially stable. Then we design an error feedback controller (12) with $\mathbb{V} = \{0, 1, 2, 3\}$ and choose as the control tuning parameter $\varepsilon = 0.095$ to roughly maximize stability margin of the linearized closed-loop system. For the simulation, we relax the incompressibility condition by using a penalty method with the penalty parameter $\epsilon_p = 10^{-5}$, see e.g. [9, Ch. 5.2], to decouple the fluid pressure from the fluid velocity. As the initial state, we use

$$x_{e0} = \begin{bmatrix} v_e - v_{e1/2}, & 0 \end{bmatrix}^T \in X \times Z,$$

where $v_{e1/2}(\xi)$ is the steady state velocity corresponding to the body force $f_{w1/2} = 0.5 f_w$ and $f_u = 0$, $f_d = 0$. Evolution of the closed-loop system is then solved using Crank–Nicolson method with the time step $\Delta t = 0.01$ together with Newton iteration.

Output tracking performance of the controller is depicted in Fig. 3 and a snapshot of the fluid velocity at the time $t = 120$ is shown in Fig. 4. The controller achieves output tracking of (13) with satisfactory performance for the chosen initial state despite the disturbance. The effect of the disturbance is not clearly visible in Fig. 3, since the frequencies of y_r and u_d coincide. The locations of Ω_u and Ω_d with respect to the observed point also lead to the disturbance not being clearly visible in Fig. 4, although the fluid velocity inside the disturbed region has a dominantly positive ξ_2-component for the most part.

A small tracking error remains after the transient behavior. This could be due to the approximate nature of the output

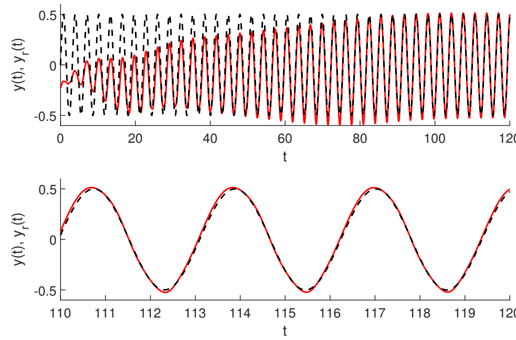

Fig. 3: The point observation $y(t)$ (red) and the reference output $y_r(t)$ (black) for $t \in [0, 120]$ and $t \in [110, 120]$

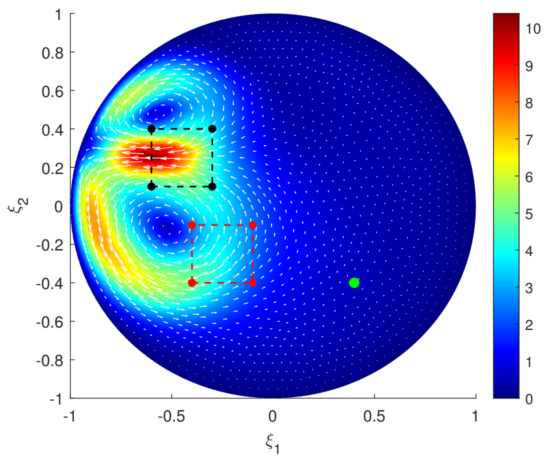

Fig. 4: The velocity field $v(\xi, 120)$, where color depicts speed of the fluid. Boundaries of the control and the disturbance domains are highlighted with black and red, respectively, and the observation point is highlighted with green.

tracking, but also at least partially due to the discretization schemes. A comparison of tracking errors using different time step sizes for the implemented Crank–Nicolson method is presented in Figure 5. The figure indicates that refining step size from 0.1 to 0.025 is beneficial, but further refinement to the implemented 0.01 has little effect. Recall that in

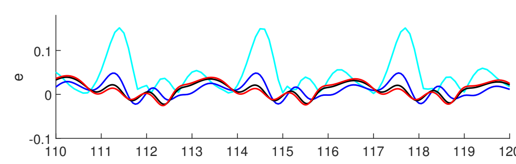

Fig. 5: Tracking error comparison of the implemented time step 0.01 (red) to the time steps 0.025 (black), 0.05 (blue) and 0.1 (cyan) for $t \in [110, 120]$.

practice the controller would be constructed without having to rely on system approximations, since the construction only requires knowledge of the transfer function gains at certain frequencies, which can be experimentally estimated with good accuracy.

VI. Conclusion

We studied a velocity output tracking problem for the incompressible 2D Navier–Stokes equations. As the main result, we showed that the studied equations subject to in-domain control and point observation form a regular nonlinear system, in the sense of [13], on a smooth enough state space. As such, a specific error feedback controller, introduced in [13], achieves approximate local velocity output tracking of periodic sinusoidal reference signals. Achieved output tracking is approximate in the sense that a finite number of harmonics of the system output and the reference output are the same.

The same control approach can be implemented directly for other fluid flow models as well. To do so, the fluid should be viscous for the decomposition similar to (10a) to exist, and with the nonlinearity modeled by a term of the type $(x \cdot \nabla)x$. Additionally, the domain Ω together with the boundary Γ should be such that the estimates used in Sec. III are justified. This means that no loss of regularity of the solutions may occur at least until the regularity level associated to $X_{1/2}$.

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