A Finite-Dimensional Controller for Robust Output Tracking of an Euler–Bernoulli Beam*

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***Abstract*— In this paper, we consider robust output tracking problem of an undamped Euler-Bernoulli beam with boundary control and boundary observation. In particular, we study a cantilever beam which has control and observation at the free end. As our main result, we construct a finite-dimensional, internal model based controller for the output tracking of the beam system. In addition, we consider a case where the controller achieves the robust output tracking for the cantilever beam with distributed control and observation. Numerical simulations demonstrating the effectiveness of the controller are presented.**

I. Introduction

In this paper, we consider output tracking of an Euler Bernoulli beam with conservative clamped boundary conditions at one end and control at the other end. The beam system we study is given by

$$\begin{aligned} \rho(\xi)w_{tt}(\xi,t)+(EI(\xi)w_{\xi\xi})_{\xi\xi}(\xi,t)=0,\ 0<\xi<1,t>0,\\ w(0,t)=0,\ w_\xi(0,t)=0,\\ (EI(\xi)w_{\xi\xi})(1,t)=0,\\ -(EI(\xi)w_{\xi\xi})_\xi(1,t)=u(t),\\ y(t)=w_t(1,t),\\ w(\xi,0)=w_0(\xi),\ w_t(\xi,0)=w_1(\xi),\ 0<\xi<1, \end{aligned} \tag{I.1}$$

where $w(\xi,t)$ is the transverse displacement of the beam at position ξ and time t, $w_t(\xi,t)$ and $w_\xi(\xi,t)$ denote time and spatial derivatives of $w(\xi,t)$, respectively, $\rho(\xi)$ and $EI(\xi)$ are linear density and flexural rigidity of the beam, respectively, $u(t)$ is an external boundary input and $y(t)$ is a boundary observation. The parameters $\rho(\xi)$ and $EI(\xi)$ satisfy the conditions

$$\rho(\cdot),\ EI(\cdot)\in C^4([0,1]),\quad \rho(\xi),\ EI(\xi)>0\quad \forall\,\xi\in[0,1]. \tag{I.2}$$

Our goal is to design a controller in such a way that the output $y(t)$ tracks a given reference signal $y_{ref}(t)$ asymptotically despite uncertainties and perturbations in the system. In other words, the objective is to find a controller that produces the input $u(t)$ such that

$$\int_t^{t+1}\|y(s)-y_{ref}(s)\|ds\to 0\quad\text{as}\quad t\to\infty.$$

The reference signal to be considered is of the form

$$y_{ref}(t)=\sum_{k=1}^{q}a_k\cos(\omega_k t)+b_k\sin(\omega_k t) \tag{I.3}$$

where $(\omega_k)_{k=1}^q$ are known frequencies and $(a_k)_{k=1}^q$ and $(b_k)_{k=1}^q$ are possibly unknown constant coefficients.

This so-called *Robust Output Regulation Problem* has been studied widely in the literature for distributed parameter systems ([1], [2], [3], [4], [5], [6]), for regular and well-posed linear systems ([7], [8], [9]) and for boundary control systems ([10], [11]). The main key in the construction of robust regulating controllers is the *Internal model principle* which states that a controller can solve the robust output regulation problem if the dynamics of the controller contains copies of the frequencies from the reference signal. The internal model principle was introduced by Francis and Wonham in [12], [13] for finite-dimensional systems and since then it has been developed for infinite-dimensional systems by many authors, see for example, [5], [8], [11].

Robust output tracking of Euler-Bernoulli beam models has been studied recently in [14], [15] using infinite-dimensional controllers. In this paper, we solve the output tracking problem for the considered beam system (I.1) using a finite-dimensional dynamic error feedback controller.

As the main contribution, we construct a finite-dimensional, internal model based controller which achieves output tracking of given combination of sinusoidal signals as in (I.3). We formulate the beam system as an impedance passive well-posed linear system ([16], [17], [18]) and show that it can be stabilized exponentially using negative output feedback. The controller construction is based on the results for abstract well-posed linear systems [7]. As the main novelty compared to the recent articles [14] and [15] on output regulation of Euler-Bernoulli beam models, we consider spatially varying parameters in the beam system and solve the output tracking problem using a finite-dimensional controller.

As the second contribution, we consider a case where the cantilever beam (I.1) has distributed control and observation instead of boundary control and observation. We formulate the beam system as an impedance passive abstract linear system which can be stabilized strongly using negative output feedback. We show that the same finite-dimensional

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controller structure achieves robust output tracking of the given sinusoidal reference signals.

The paper is organized as follows. In Section II, we formulate the robust output regulation problem for the beam system. In Section III, we construct the controller for the robust output tracking of the reference signals. In addition, we present results related to stabilizability and well-posedness of the beam system. In Section IV, we consider the robust output tracking problem for the beam system with distributed control and observation. Section V is devoted to numerical simulations which demonstrate the performance of the controller for the robust output tracking of the beam system (I.1). In Section VI, we conclude our results.

A. Notation

For normed linear spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X to Y. For a linear operator A, $D(A), \mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the domain, range and the kernel of A, respectively. The resolvent and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$, respectively. The resolvent operator is denoted by $R(\lambda, A) = (\lambda - A)^{-1}, \lambda \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|,\ x \in X, \beta \in \rho(A)$ and by $A_{-1} \in \mathcal{L}(X, X_{-1})$ the extension of A to X_{-1}. For any $a \in \mathbb{R}$, $\mathbb{C}_a = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > a\}$.

II. Problem Formulation

In this section, we formulate the robust output regulation problem for the considered beam system (I.1). The dynamic error feedback controller to be constructed is of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t),\ z(0) = z_0, \\ u(t) &= K z(t) - k_1 e(t), \end{aligned} \tag{II.1}$$

where $z \in Z,\ Z = \mathbb{R}^{2q}$, $\mathcal{G}_1 \in \mathbb{R}^{2q\times 2q}$, $\mathcal{G}_2 \in \mathbb{R}^{2q\times 1}$, $K \in \mathbb{R}^{1\times 2q}$, $k_1 > 0$ and $e(t) = y(t) - y_{ref}(t)$ is the regulation error. Here q is the number of frequencies in the reference signal.

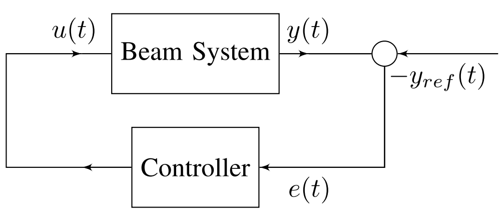

Fig. 1. The closed-loop system interconnecting the beam system and the controller

Robust Output Regulation Problem. Choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, k_1)$ in such a way that

(a) The closed-loop system in Figure 1 is exponentially stable in the sense that the closed-loop semigroup decays to zero exponentially.
(b) There exists $\alpha > 0$ such that for all reference signals of the form (I.3) and for all initial conditions $w_0(\xi), w_1(\xi)$ of the beam system and $z_0 \in Z$, the regulation error satisfies $e^{\alpha\cdot}e(\cdot) \in L^2([0,\infty), \mathbb{C})$.
(c) If (a) holds despite uncertainties, perturbations and disturbances in the system, then (b) is still satisfied for all initial conditions and some $\tilde{\alpha} > 0$.

III. Robust Output Regulation of the Cantilever Beam

In this section, we construct the controller for the robust output tracking of the sinusoidal reference signal y_{ref}. We start with presenting the controller. Based on [7], we choose the controller parameters as

$$\begin{aligned} \mathcal{G}_1 &= \operatorname{diag}(G_1, G_2, \cdots, G_q), \\ G_k &= \begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix}, \quad k = 1, 2, \cdots, q, \\ \mathcal{G}_2 &= -[1, 0, \cdots, 1, 0]^T, \\ K &= [2, 0, \cdots, 2, 0], \\ k_1 &> \frac{1}{2}. \end{aligned} \tag{III.1}$$

We note that the above choice of controller parameters does not depend on the coefficients a_k and b_k, $k = 1, 2, \cdots, q$ in the reference signal (I.3), a_k and b_k can possibly be unknown. The controller with the above choices of parameters solves the robust output regulation problem if the beam system is impedance passive, exponentially stabilizable using negative output feedback and well-posed linear system [16, Def. 1.1]. Therefore, in order to solve the output tracking problem, we need to verify the stabilizability of the beam system and formulate the beam system (I.1) as an impedance passive abstract well-posed linear system.

In the following, we present the abstract representation and stabilizability of the beam system followed by well-posedness results for the beam system. Afterward, we show that the controller presented in (II.1) and (III.1) solves the robust output tracking problem. Here we emphasize that the construction of the controller does not require the beam system as an abstract well-posed linear system. We will verify the above properties to prove that the controller in (II.1) and (III.1) solves the robust output regulation problem for the system (I.1).

A. Abstract Formulation of the Beam System

We formulate (I.1) in the state space $X = H_E^2(0,1) \times L^2(0,1)$ where $H_E^2(0,1) = \{f \in H^2(0,1) \mid f(0) = f'(0) = 0\}$. The norm on X is defined as

$$\|(f,g)^T\|_X^2 = \int_0^1 [\rho(\xi)|g(\xi)|^2 + EI(\xi)|f''(\xi)|^2] d\xi, \qquad \forall (f,g)^T \in X.$$

The total energy of the beam system is given by

$$E(t) = \frac{1}{2}\int_0^1 [\rho(\xi) w_t^2(\xi,t) + EI(\xi) w_{\xi\xi}^2(\xi,t)] d\xi. \tag{III.2}$$

We define

$$x(t) = \begin{bmatrix} x_1(\cdot,t) \\ x_2(\cdot,t) \end{bmatrix} = \begin{bmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{bmatrix}.$$

Now (I.1) on X has the form

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathcal{A}x(t), \quad x(0) = x_0, \\ \mathcal{B}x(t) &= u(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{III.3}$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$,

$$\mathcal{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-1}{\rho(\xi)}(EI(\xi)x_1''(\xi))'' \end{bmatrix},$$

$$D(\mathcal{A}) = \{(x_1, x_2)^T \in [H^4(0,1) \cap H_E^2(0,1)] \times H_E^2(0,1) \mid x_1''(1) = 0\},$$

the operators $\mathcal{B} : D(\mathcal{A}) \to U$ and $\mathcal{C} : D(\mathcal{A}) \to Y$ with $U = \mathbb{C}$ and $Y = \mathbb{C}$ are given by

$$\mathcal{B}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -(EI(\xi)x_1''(\xi))'(1,t),$$

$$\mathcal{C}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2(1,t).$$

Let us introduce the operator $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ with

$$D(A) = \{(f,g)^T \in [H^4(0,1) \cap H_E^2(0,1)] \times H_E^2(0,1) \mid f''(1) = f'''(1) = 0\}.$$

We have that A is a skew-adjoint operator with compact resolvent [19, Sec. 3]. This implies that A generates a unitary group on X. Moreover, we have that $\mathcal{N}(\mathcal{B}) = D(A)$. Therefore, $\mathcal{N}(\mathcal{B})$ is dense in X. Thus $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a boundary control system in the sense of [20, Def. 10.1.1]. Next, we show that the boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is impedance passive which is defined as follows.

Definition III.1. (*Impedance Passive System*). A boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is an impedance passive system on (X, U, Y) if $U = Y$ and

$$\operatorname{Re}\langle \mathcal{A}x, x\rangle_X \leq \operatorname{Re}\langle \mathcal{B}x, \mathcal{C}x\rangle_U, \quad x \in D(\mathcal{A}).$$

Lemma III.2. *The boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in (III.3) is an impedance passive system.*

Proof. We have that for $x \in D(\mathcal{A})$,

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}x, x\rangle_X &= \operatorname{Re}\left\langle \begin{bmatrix} x_2 \\ \frac{-1}{\rho(\xi)}(EI(\xi)x_1''(\xi))'' \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle_X, \\ &= \operatorname{Re}\int_0^1 \rho(\xi)\frac{-1}{\rho(\xi)}(EI(\xi)x_1''(\xi))''\overline{x_2(\xi)}d\xi \\ &\quad + \operatorname{Re}\int_0^1 EI(\xi)\overline{x_1''(\xi)}x_2''(\xi)d\xi. \end{aligned}$$

Using integration by parts twice for the first term and applying boundary conditions, we obtain

$$\begin{aligned} &\operatorname{Re}\langle \mathcal{A}x, x\rangle_X \\ &= \operatorname{Re}\Bigg[-\overline{x_2}(1)(EI(\xi)x_1'')'(1) + \overline{x_2}(0)(EI(\xi)x_1'')'(0) \\ &\quad + \overline{x_2}'(1)(EI(\xi)x_1'')(1) - \overline{x_2}'(0)(EI(\xi)x_1'')(0) \\ &\quad - \int_0^1 EI(\xi)x_1''(\xi)\overline{x_2''(\xi)}d\xi + \int_0^1 EI(\xi)x_2''(\xi)\overline{x_1''(\xi)}d\xi \Bigg] \\ &= \operatorname{Re}[-\overline{x_2(1)}(EI(\xi)x_1'')'(1)] \\ &= \operatorname{Re}\mathcal{B}x\overline{\mathcal{C}x} \\ &= \operatorname{Re}\langle \mathcal{B}x, \mathcal{C}x\rangle_{\mathbb{C}} \end{aligned}$$

which implies that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in (III.3) is impedance passive. □

B. Stabilization of the Beam

In [19, Thm. 2.5] it is shown that the beam (I.1) with output feedback $u(t) = -\kappa w_t(1,t),\ \kappa > 0$ is exponentially stable in the sense that the energy $E(t)$ of the solutions decays to zero exponentially. Here we note that $E(t) = \frac{1}{2}\|x(t)\|_X^2$. Therefore we have the following lemma.

Lemma III.3 ([19, Thm. 2.5])**.** *The beam (I.1) with new input $u(t) = \tilde{u}(t) - \kappa y(t),\ \kappa > 0$ is exponentially stable in the sense that for the semigroup $T(t)$ generated by $A_{cl} = \mathcal{A}|_{\mathcal{N}(\mathcal{B}+\kappa\mathcal{C})}$, there exist $\omega > 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{-\omega t}, \quad t \geq 0.$$

C. Well-posedness of the Beam system

In this section, we present results related to the well-posedness ([18, Def. 3.1]) of the beam system.

Lemma III.4 ([19, Lem. 3.4])**.** *The eigenvalues $\{i\lambda_n, \overline{i\lambda_n}\}$ and the corresponding eigenfunctions $((i\lambda_n)^{-1}\phi_n, \phi_n)$ of A have the following asymptotic expressions*

$$\begin{aligned} i\lambda_n &= \frac{\mu_n^2}{h^2}, \quad h = \int_0^1 \left(\frac{\rho(s)}{EI(s)}\right)^{\frac{1}{4}} ds, \\ \mu_n &= \frac{1}{\sqrt{2}}(n + \frac{1}{2})\pi(1+i) + \mathcal{O}(\frac{1}{n}), \end{aligned} \tag{III.4}$$

as $n \to \infty$, n is a large positive integer and

$$\begin{aligned} \phi_n(\xi) &= e^{-\frac{1}{4}\int_0^z a(s)ds}\sqrt{2}(i-1)[\sin((n+\frac{\pi}{2})z) \\ &\quad - \cos((n+\frac{\pi}{2})z) + e^{-(n+\frac{1}{2})\pi z} \\ &\quad + (-1)^n e^{-(n+\frac{1}{2})\pi(1-z)} + \mathcal{O}(\frac{1}{n})] \end{aligned} \tag{III.5}$$

where

$$z = z(\xi) = \frac{1}{h}\int_0^\xi \left(\frac{\rho(s)}{EI(s)}\right)^{\frac{1}{4}} ds$$

$$\begin{aligned} a(z) &= \frac{3h}{2}\left(\frac{\rho(\xi)}{EI(\xi)}\right)^{-\frac{5}{4}}\frac{d}{d\xi}\left(\frac{\rho(\xi)}{EI(\xi)}\right) \\ &\quad + h\frac{2\frac{d}{d\xi}EI(\xi)}{EI(\xi)}\left(\frac{\rho(\xi)}{EI(\xi)}\right)^{-\frac{1}{4}}. \end{aligned}$$

Next, we show that the boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in (III.3) defines a well-posed system node on (X, U, Y), where system node is defined in the sense of [17, Def. 2.1] or [21, Def. 2.1] and well-posed system node is defined in the sense of [17, Def. 2.6], [18].

Theorem III.5. *The boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in (III.3) defines a well-posed system node on (X, U, Y).*

Proof. We have shown that the system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is an impedance passive boundary control system. In addition, since A generates a unitary group, the boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is internally well-posed in the sense of [21, Def. 1.1]. Therefore, by [21, Thm. 2.3], $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ defines a system node $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : D(S) \subset X \times \mathbb{C} \to X \times \mathbb{C}$ and the system node is impedance passive [17, Thm. 4.2]. The system node S is defined by

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} A_{-1} & B \\ \mathcal{C} & 0 \end{bmatrix} \Bigg|_{D(S)},$$

$$D(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \;\middle|\; A_{-1}x + Bu \in X \right\}$$

where $B \in \mathcal{L}(U, X_{-1})$ is uniquely determined by the relation $\mathcal{A} = A_{-1} + B\mathcal{B}$ on $D(\mathcal{A})$ [20, Prop. 10.1.2]. Next, we show that the transfer function of the system node S is bounded on some vertical line in the complex right half plane.

Using [20, Rem. 10.1.6], we obtain $B^*x = x_2(1) = Cx,\ x = (x_1, x_2)^T \in D(A^*)$, where $C = \mathcal{C}|_{\mathcal{N}(\mathcal{B})}$. The operator $B^* \in \mathcal{L}(D(A^*), U)$ is the adjoint of $B \in \mathcal{L}(U, X_{-1})$ in the sense that

$$\langle x, Bu \rangle_{D(A^*), X_{-1}} = \langle B^*x, u \rangle_{\mathbb{C}},\ x \in D(A^*), u \in U.$$

Therefore, (III.3) can be equivalently written as a second order system

$$\begin{aligned} w_{tt}(\cdot, t) + A_0 w(\cdot, t) &= B_0 u(t) \\ y(t) &= B_0^* w_t(\cdot, t) \end{aligned} \tag{III.6}$$

where $A_0 f = \frac{1}{\rho(\xi)}(EI(\xi) f'')''$ is a positive self-adjoint operator with $D(A_0) = \{f \in H^4(0,1) \cap H_E^2(0,1) \mid f''(1) = (EIf'')'(1) = 0\}$ and $B_0 = \delta(\cdot - 1)$, $\delta(\cdot)$ is the Dirac delta distribution. Then λ_n^2 and ϕ_n from Lemma III.4 are the eigenvalues and the corresponding eigenfunctions of A_0.

From the expression (III.4), we have that $(\lambda_n)_{n\geq 1}$ are increasing. In addition, $|B_0^*\phi_n| = |\phi_n(1)|$ which from (III.5) is bounded for $n \geq 1$. This implies that B^* is admissible [20, Sec. 5.3], [22, Prop. 2]. By duality ([20, Sec. 4.4]), we have that B is admissible. Moreover, using [22, Rem. 4], we have that the eigenvalues of A_0 satisfies the spectral condition

$$\lambda_{n+1} - \lambda_n \geq \beta \lambda_{n+1}^{\gamma}, \quad \forall\ n \text{ large},$$

for some $\beta, \gamma > 0$. Therefore, using [22, Thm. 4], we conclude that the transfer function $s \mapsto G(s) = sB_0^*(s^2 + A_0)^{-1}B_0 \in \mathcal{L}(U)$ of (III.6) is bounded on some vertical line in the complex right half plane. Since

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & B_0^* \end{bmatrix},$$

we have that the transfer function G_S of the system node S which is given by [17, Def. 2.1], [18, Sec. 6]

$$\begin{aligned} G_S(s)u &= C\&D \begin{bmatrix} R(s, A_{-1})Bu \\ u \end{bmatrix} = \mathcal{C}R(s, A_{-1})Bu \\ &= sB_0^*(s^2 + A_0)^{-1}B_0 u \\ &= G(s)u \end{aligned}$$

is bounded on $\mathbb{C}_0$. Therefore, by [17, Thm. 5.1], we conclude that the system node S is well-posed. ☐

Remark III.6. Since B is an admissible control operator, using [23, Thm. 2.7], we can deduce that

$$\lim_{s\to+\infty} G(s) = 0, \quad s \in \mathbb{R}.$$

Since the above limit exists, we have that the beam system is a regular linear system [24].

D. Robust Regulating Controller for the Beam System

In this section, we show that the controller (II.1), (III.1) presented in Section II solves the robust output tracking problem.

We note that the transfer function $G(s)$ in Section III-C can also be written in terms of the solution of the elliptic problem corresponding to I.1 ([25, Sec. 12.1], [26])

$$\begin{aligned} \frac{1}{\rho(\xi)}(EI(\xi)\hat{w}_{\xi\xi})_{\xi\xi} &= -s^2\hat{w}, \quad \xi \in [0,1], \\ (EI(\xi)\hat{w}_{\xi\xi})_{\xi}(1) &= \hat{u}, \\ G(s)\hat{u} = \hat{y} &= s\hat{w}(1), \end{aligned}$$

for $(\hat{w}, s\hat{w}) \in D(\mathcal{A}),\ \hat{u}, \hat{y} \in \mathbb{C}$ and $s \in \rho(A)$.

Theorem III.7. *Let $\omega_j \in \mathbb{R},\ j = 1, 2, \cdots, q$ be the frequencies from the reference signal. Assume that $\operatorname{Re} G(i\omega_j)\hat{u} = \operatorname{Re} i\omega_j \hat{w}(1) \neq 0$ for all j. Then the controller (II.1), (III.1) solves the robust output regulation problem for (I.1).*

Proof. We consider the input $u(t) = Kz(t) - k_1 e(t)$. Let us write $k_1 = C_0 + \kappa$, where $C_0 \geq \frac{1}{2}$ and $\kappa > 0$. Then we have $u(t) = Kz(t) - C_0 e(t) - \kappa y(t) + \kappa y_{ref}(t) = u_1(t) - \kappa y(t) + \kappa y_{ref}(t)$ where $u_1(t) = Kz(t) - C_0 e(t)$.

With this input, (III.3) can be written as

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathcal{A}x(t), \quad x(0) = x_0, \\ (\mathcal{B} + \kappa\mathcal{C})x(t) &= u_1(t) + \kappa y_{ref}(t), \\ \mathcal{C}x(t) &= y(t). \end{aligned} \tag{III.7}$$

From Lemma III.3, we have that the system $(\mathcal{A}, \mathcal{B} + \kappa\mathcal{C}, \mathcal{C})$ is exponentially stable and from Theorem III.5, we have that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a well-posed linear system since every well-posed system node defines a well-posed linear system ([18]). Moreover, due to Remark III.6, we have that κ is an admissible output feedback operator. This implies that the system $(\mathcal{A}, \mathcal{B} + \kappa\mathcal{C}, \mathcal{C})$ is a well-posed linear system

[24, Thm. 4.7]. Therefore, by considering $\kappa y_{ref}(t)$ as an external disturbance to the system (I.1), then we have that (III.7) is an exponentially stable well-posed linear system with input $u_1(t)$. In addition, the impedance passivity of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ implies that the transfer function $G(s)$ is positive, i.e., $\operatorname{Re} G(s) = \frac{1}{2}[G(s) + G(s^*)] \geq 0, \ \forall \ s \in \mathbb{C}_0$ ([17], [18]). This further implies that the transfer function $G_\kappa(s)$ of the system $(\mathcal{A}, \mathcal{B} + \kappa\mathcal{C}, \mathcal{C})$ is positive and the assumption $\operatorname{Re} G(i\omega_j) \neq 0, \ j = 1, 2, \cdots, q$ implies that $\operatorname{Re} G_\kappa(i\omega_j) \neq 0$ for all $j = 1, 2, \cdots, q$. Therefore, using [7, Thm. 3.4], a minimal realization of

$$C(s) = -C_0 - \sum_{j \in \mathcal{J}} \frac{1}{s - i\omega_j}, \tag{III.8}$$

where $C_0 \geq \frac{1}{2}$, $\mathcal{J} = \{-q, \cdots, -1, 1, \cdots, q\}$ and $\omega_{-j} = -\omega_j$, solves the robust output tracking problem and rejects the disturbance $\kappa y_{ref}(t)$.

It can be verified from (III.1) that $(\mathcal{G}_1, \mathcal{G}_2)$ is controllable, $(\mathcal{G}_1, K)$ is observable and the transfer function of $(\mathcal{G}_1, \mathcal{G}_2, K, -C_0)$ is given by (III.8). Therefore, the controller given in (II.1) and (III.1) is a minimal realization of (III.8). Combining the above arguments and using [7, Thm. 3.4], we have that the controller (II.1), (III.1) solves the robust tracking problem for (I.1). □

IV. A Robust Regulating Controller for an Euler-Bernoulli Beam with Distributed Control and Observation

In this section, we consider robust output tracking of a cantilever beam which has distributed control and observation. The beam system that we study is described by

$$\begin{aligned} \rho(\xi) w_{tt}(\xi, t) &= -(EI(\xi) w_{\xi\xi})_{\xi\xi}(\xi, t) + b(\xi) u_2(t) \\ w(0, t) &= 0, \ w_\xi(0, t) = 0, \\ (EI(\xi) w_{\xi\xi})(1, t) &= 0, \ -(EI(\xi) w_{\xi\xi})_\xi(1, t) = 0, \\ w(\xi, 0) &= w_0(\xi), \ w_t(\xi, 0) = w_1(\xi), \\ y_2(t) &= \int_0^1 b(\xi) w_t(\xi, t) d\xi \end{aligned} \tag{IV.1}$$

where $0 < \xi < 1$, $t > 0$, $u_2(t)$ and $y_2(t)$ are the external control input and observation respectively and $b(\cdot) \in L^2(0, 1)$ is a real-valued function. The parameters $\rho(\xi)$ and $EI(\xi)$ satisfy (I.2). The beam system (IV.1) cannot be stabilized exponentially [27, Cor. 3.58], [28, Sec. 8.4].

Assumption IV.1. Under negative output feedback $u_2(t) = -\kappa y_2(t)$, $\kappa > 0$, the solutions of the beam system (IV.1) satisfy

$$\|w(\cdot, t)\|_{L^2} + \|w_t(\cdot, t)\|_{L^2} \quad \to 0 \quad \text{as} \quad t \to \infty \tag{IV.2}$$

for any initial conditions.

Assumption IV.1 implies that the system (IV.1) can be stabilized strongly by negative output feedback.

Robust Output Regulation Problem (Strongly Stable Version). Choose $(\mathcal{G}_1, \mathcal{G}_2, K, k_1)$ in (II.1) such that

(a) The closed-loop system comprising the controller and the beam system (IV.1) is strongly stable.
(b) The regulation error $\tilde{e}(t) = y_2(t) - y_{ref}(t)$ satisfies
$$\int_t^{t+1} \|\tilde{e}(s)\| ds \to 0 \quad \text{as} \quad t \to \infty$$
for all initial conditions $w_0(\xi), w_1(\xi)$ and $z_0 \in Z$.
(c) If (a) holds despite uncertainties, perturbations and disturbances in the system, then (b) is still satisfied for all initial conditions.

Theorem IV.2. *Under the Assumption IV.1, the controller* (II.1) *and* (III.1) *solves the robust output regulation problem (Strongly Stable Version) for the beam system (IV.1).*

Proof. The system (IV.1) can be formulated as an abstract linear system

$$\begin{aligned} \frac{d}{dt} x(t) &= A x(t) + \tilde{B} u_2(t), \quad x(0) = x_0, \\ y_2(t) &= \tilde{C} x(t) \end{aligned}$$

in the state space $X = H_E^2(0, 1) \times L^2(0, 1)$ with state variable $x(t) = (w(\cdot, t), w_t(\cdot, t))^T$. The norm on X is defined as in Section III-A. The operator A corresponds to the skew-adjoint operator in Section III-A and the operators $\tilde{B} \in \mathcal{L}(\mathbb{C}, X)$ and $\tilde{C} \in \mathcal{L}(X, \mathbb{C})$ are given by

$$\tilde{B} u_2 = \begin{bmatrix} 0 \\ \tilde{B}_0 \end{bmatrix} u_2, \quad \tilde{B}_0 = \frac{b(\cdot)}{\rho(\cdot)}, \ u_2 \in \mathbb{C},$$
$$\tilde{C} x = \int_0^1 b(\xi) x_2(\xi) d\xi, \quad (x_1, x_2)^T \in X.$$

Here $\tilde{B}^* = \tilde{C}$.

By direct computation, we obtain

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|_X^2 = \operatorname{Re} \langle u_2(t), y_2(t) \rangle_{\mathbb{C}}.$$

This implies that the system $(A, \tilde{B}, \tilde{B}^*, 0)$ is an impedance passive system.

Now we have that the system (IV.1) is passive and assumed to be strongly stabilizable by negative output feedback. Therefore, by [9, Thm. 5.2], we conclude that the controller (II.1) and (III.1) solves the robust output tracking problem. □

V. Numerical Simulations

Simulations are carried out in Matlab for the beam system (I.1) with the following choices of parameters on the time interval [0,15]. We consider the case where $\rho(\xi) = 1, \ EI(\xi) = 1$. We aim to track the reference signal $y_{ref}(t) = \sin 2t + \cos t$. So, the frequencies are $\{2, 1\}$. We choose the beam initial state $w_0(\xi) = 0.1(\sin(\pi\xi) - \pi\xi), \ w_1(\xi) = (1 + \frac{\pi^3}{60})\xi^2$ and the controller initial state $z_0 = 0$. The beam system is approximated using Legendre spectral Galerkin method [29]. The number of basis functions used for the approximation is 20. The controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are chosen as in (III.1) with $k_1 = 6$. Figure 2 shows that the tracking of the given reference signal is achieved asymptotically. Velocity profile of the controlled beam is shown in Figure 3.

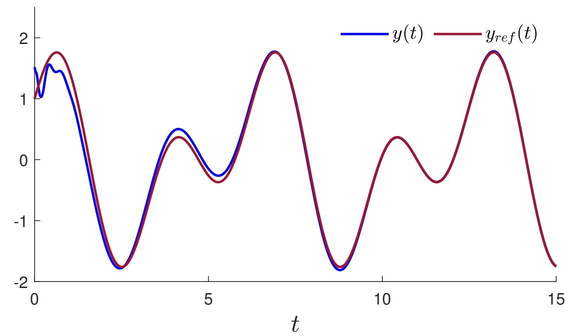

Fig. 2. Output tracking

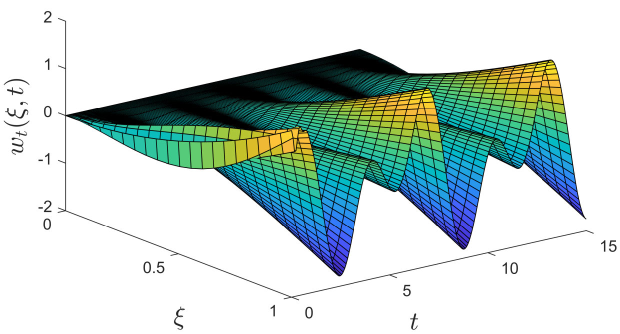

Fig. 3. Velocity profile of the controlled beam

VI. Conclusions

In this paper, we studied the robust output tracking of a cantilever beam. As the main problem, we considered the cantilever beam which has control and observation at the free end. In addition, we considered the case where the beam has distributed control and observation. We solved the output regulation problem using a finite-dimensional, internal model based controller. The advantage of using this controller is that the controller is simple and able to handle the spatially varying parameters in the beam system. Numerical simulations demonstrating the effectiveness of the controller were presented.

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