Inferring the adjoint turnpike property from the primal turnpike property

Timm Faulwasser, Lars Grüne, Jukka-Pekka Humaloja, and Manuel Schaller

Abstract— **This paper investigates an interval turnpike result for the adjoints/costates of finite- and infinite-dimensional nonlinear optimal control problems under the assumption of an interval turnpike on states and controls. We consider stabilizable dynamics governed by a generator of a semigroup with finite-dimensional unstable part satisfying a spectral decomposition condition and show the desired turnpike property under continuity assumptions on the first-order optimality conditions. We further provide a numerical example with a semilinear heat equation to illustrate the results.**

I. Introduction

Turnpike properties are particular features of optimal solutions to optimal control problems (OCPs). The phenomenon can be understood as a property of parametric OCPs, whereby for varying initial conditions and horizon length optimal solutions spend an increasing amount of time close a specific steady state, the so-called turnpike, which in turn corresponds to the solution of the underlying stationary optimization problem. First observations of the phenomenon have been made by von Neumann in the middle of the 20th century [23] and even earlier by Ramsey [27]. The problem has since received vast interest, cf. the recent works [16], [18], [19], [20], [25], [29], [30], [34]. For nonlinear OCPs, turnpike properties can be shown via the linearization of the optimality system, analysis of the linearization, and a smallness assumption, cf. [3], [17], [31], [32]. Alternatively, one may assume a particular notion of dissipativity, cf. [6], [10], [14], [15], which has the advantage to allow for global turnpike properties on state and control, i.e., without a smallness condition on, e.g., the initial distance to the turnpike; see [8] for a recent overview. In that context, however, it remains difficult to characterize the behavior of the corresponding adjoints/costates.

To overcome the above difficulty, the main contribution of this paper is to show that the turnpike behavior of state and control induces turnpike behavior of the adjoints without smallness assumptions. To this end, we analyze the first-order necessary optimality conditions and, loosely speaking, show for problems governed by general evolution equations that continuity of the nonlinearities and convergence of state and control imply convergence of the adjoints. While our results are formulated in an infinite-dimensional setting, the results are new also for finite-dimensional systems, which form a special case of our setting. Besides being an important structural property of the optimal triplet, turnpike properties can be leveraged in design of numerical methods. For example [32] suggests to exploit them in indirect shooting methods, in [16], [17] it is used for tailored discretization of infinite dimensional OCPs in a receding-horizon setting and [11] hinges on them in mixed-integer OCPs.

After introducing the OCP at hand, the first-order optmality conditions and the functional analytic setting in Section II, we prove a turnpike property of the adjoints in Section III for exponentially stable and exactly controllable systems. Assuming that the underlying operator satisfies a spectrum decomposition assumption, we extend the results in Section IV to stabilizable systems with finite-dimensional unstable part. Finally, Section V illustrates the findings with a simulation of a semilinear heat equation on a 2D domain.

This work was supported by the DFG Grants GR 1569/17-1 and SCHI 1379/5-1 and was conducted while the third author was visiting University of Bayreuth under Academy of Finland Grant number 310489 held by Lassi Paunonen and supported by a travel grant from the Magnus Ehrnrooth Foundation. An extended preprint version is available, see [9].

T. Faulwasser is with TU Dortmund University, Institute of Energy Systems, Energy Efficiency and Energy Economics, Germany timm.faulwasser@ieee.org

L. Grüne is with University of Bayreuth, Department of Mathematics, Germany lars.gruene@uni-bayreuth.de

J.-P. Humaloja is with University of Alberta, Department of Chemical and Materials Engineering, AB, Canada jukkapek@ualberta.ca. He is funded by a grant from the Jenny and Antti Wihuri Foundation.

M. Schaller is with Technische Universität Ilmemau, Institute for Mathematics, Germany manuel.schaller@tu-ilmenau.de

II. Setting and preliminaries

Let $(X,\|\cdot\|_X)$ be a Banach space and $(U,\langle\cdot,\cdot\rangle_U)$ be a Hilbert space with corresponding norm $\|\cdot\|_U$. Consider the optimal control problem

$$\begin{aligned} &\min_{u\in L_2(0,T;U)} \int_0^T J(x(t),u(t))\,dt \\ &\text{s.t.} \quad \dot{x}(t) = \mathscr{A}x(t)+\mathscr{B}u(t)+f(x(t),u(t)) \\ &\qquad\; x(0)=x_0, \end{aligned} \tag{1}$$

where $J : X\times U \to \mathbb{R}$ is sufficiently smooth, $x_0 \in X$, $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ generates a strongly continuous semigroup on X, $\mathscr{B} \in L(U,X)$, and $f : X\times U \to X$ is a sufficiently smooth, locally Lipschitz nonlinearity. For simplicity of exposition, we do not involve a terminal cost, i.e., a Mayer term, in the OCP. This would only change the terminal value of the adjoint state and our techniques would still apply. We will assume that the above problem has at least one optimal solution $(x,u) \in C(0,T;X)\times L_2(0,T;U)$, cf. [21, Chap. 3]. Additionally, we consider $(\bar{x},\bar{u}) \in X\times U$ to be an optimal solution of the corresponding steady state system, i.e., $(\bar{x},\bar{u})$ solves

$$\begin{aligned} &\min_{u\in U} J(x,u) \\ &\text{s.t.} \quad 0 = \mathscr{A}x+\mathscr{B}u+f(x,u), \end{aligned} \tag{2}$$

Our goal in this paper is to find conditions under which interval turnpike behavior of the states and control inputs implies interval turnpike behavior of the adjoints. Our basic assumption on the behavior of the optimal solutions is thus the following.

Assumption 2.1: Let (x,u) be an optimal solution to (1). We assume there are strictly monotonously increasing functions $t_1,t_2 : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $0 \leq t_1(T) \leq t_2(T) \leq T$ such that $\nu(T) := t_2(T) - t_1(T)$ is strictly monotonously increasing and unbounded and for each $\varepsilon > 0$ there is $T_0 > 0$ such that

$$\|x(t)-\bar{x}\|_X + \|u(t)-\bar{u}\|_U \leq \varepsilon \qquad \forall t \in [t_1(T),t_2(T)],\, T \geq T_0.$$

Note that this bound immediately implies $u \in L_\infty(t_1(T),t_2(T);U)$. We also note that our definition of the turnpike property implies uniform convergence of x to $\bar{x}$ and u to $\bar{u}$ on $[t_1(T),t_2(T)]$. This means that in general the lengths of the intervals $[0,t_1(T)]$ and $[t_2(T),T]$ have to tend to infinity as $T \to \infty$, as these intervals contain the approaching and the leaving arc of the optimal trajectory, respectively, outside an ε-neighborhood of the equilibrium with $\varepsilon \to 0$ as $T \to \infty$, see Remark 2.2 for an example. We emphasize that we do not require any bound on the relation between the lengths of the near-turnpike interval $[t_1(T),t_2(T)]$ and the overall interval $[0,T]$. However, if we have such a bound, then we can take it into account in our analysis, see Remarks 3.4 and 3.9, below.

Remark 2.2: We say that $(x,u) \in C(0,T;X) \times L_2(0,T;U)$ satisfies the exponential turnpike property, cf. e.g., [31], [30], if there is a constant $c > 0$ and a decay parameter $\mu > 0$, both independent of T such that we have

$$\|x(t)-\bar{x}\|_X + \|u(t)-\bar{u}\|_U \leq c\left(e^{-\mu t} + e^{-\mu(T-t)}\right).$$

If this inequality holds, it can be easily seen, cf. [18, Discussion after Rem. 6.3] that we can choose $\delta \in (0,\frac{1}{2})$ such that for each $\varepsilon > 0$ there is a horizon T such that

$$\|x(t)-\bar{x}\|_{L_2(\delta T,(1-\delta)T;X)} + \|u(t)-\bar{u}\|_{L_2(\delta T,(1-\delta)T;U)} \leq \varepsilon$$

and

$$\|x(t)-\bar{x}\|_{C(\delta T,(1-\delta)T;X)} + \|u(t)-\bar{u}\|_{L_\infty(\delta T,(1-\delta)T;U)} \leq \varepsilon,$$

i.e., L_2 and uniform convergence on a fixed part of the time interval $[0,T]$ for $T \to \infty$. Thus, Assumption 2.1 is satisfied with $t_1(T) = \delta T$, $t_2(T) = (1-\delta)T$ and $\nu(T) = (1-2\delta)T$.
By the corresponding necessary optimality conditions of the above problem (1), cf. [21, Chap. 4], there is an adjoint state $\lambda \in C(0,T;X)$ such that

$$\begin{aligned}\dot{\lambda}(t) &= -(\mathscr{A} + f_x(x(t),u(t)))^* \lambda(t) + J_x(x(t),u(t)) \\ 0 &= (\mathscr{B} + f_u(x(t),u(t)))^* \lambda(t) + J_u(x(t),u(t)) \\ \dot{x}(t) &= \mathscr{A}x(t) + \mathscr{B}u(t) + f(x(t),u(t)),\end{aligned} \tag{3}$$

where $x(0) = x_0$ and $\lambda(T) = 0$. Analogously, by the necessary optimality conditions of the steady state problem there is an adjoint state $\bar{\lambda} \in X$ such that

$$\begin{aligned}0 &= -(\mathscr{A} + f_x(\bar{x},\bar{u}))^* \bar{\lambda} + J_x(\bar{x},\bar{u}) \\ 0 &= (\mathscr{B} + f_u(\bar{x},\bar{u}))^* \bar{\lambda} + J_u(\bar{x},\bar{u}) \\ 0 &= \mathscr{A}\bar{x} + \mathscr{B}\bar{u} + f(\bar{x},\bar{u}).\end{aligned}$$

Our goal in this paper is to show the interval turnpike property of the adjoint λ.

Definition 2.3: We say that the adjoint λ satisfies the interval turnpike property at the steady state adjoint $\bar{\lambda}$, if there are strictly monotonously increasing functions $s_1,s_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $0 \leq s_1(T) \leq s_2(T) \leq T$ such that $\theta(T) := s_2(T) - s_1(T)$ is strictly monotonously increasing and unbounded and for each $\varepsilon > 0$ there is $T_0 > 0$ such that

$$\|\lambda(t) - \bar{\lambda}\|_Y \leq \varepsilon \qquad \forall t \in [s_1(T),s_2(T)],\, T \geq T_0.$$

For our analysis, we define the remainder terms

$$r_f(t) := f(x(t),u(t)) - f(\bar{x},\bar{u})$$

and for $\star \in \{x,u\}$

$$\begin{aligned}r_{f_\star}(t) &:= f_\star(x(t),u(t)) - f_\star(\bar{x},\bar{u}), \\ r_{J_\star}(t) &:= J_\star(x(t),u(t)) - J_\star(\bar{x},\bar{u}).\end{aligned}$$

Thus, denoting $A := \mathscr{A} + f_x(\bar{x},\bar{u})$, $B := \mathscr{B} + f_u(\bar{x},\bar{u})$, and $(\delta x, \delta u, \delta\lambda) := (x-\bar{x}, u-\bar{u}, \lambda-\bar{\lambda})$, we have that

$$\dot{\delta\lambda}(t) = -A^*\delta\lambda(t) - r_{f_x}(t)^*\lambda(t) + r_{J_x}(t) \tag{4}$$

$$0 = B^*\delta\lambda(t) + r_{f_u}(t)^*\lambda(t) + r_{J_u}(t) \tag{5}$$

$$\dot{\delta x}(t) = \mathscr{A}\delta x + \mathscr{B}\delta u + r_f(t) \tag{6}$$

with $\delta x(0) = x_0 - \bar{x}$ and $\delta\lambda(T) = -\bar{\lambda}$. It is clear that the solutions x, u, and λ of eq. (3) depend on T and hence also δx, δu, and $\delta\lambda$ do. However, for the sake of readability, we do not explicitly indicate this dependence. We note that, using the definition of δx, δu, and $\delta\lambda$, the inequalities from Assumption 2.1 and Definition 2.3 can be written as $\|\delta x(t)\|_X + \|\delta u(t)\|_U \leq \varepsilon$ and $\|\delta\lambda(t)\|_Y \leq \varepsilon$, respectively.

We note that box constraints can be incorporated into the problem if one assumes that the turnpike lies in the interior of the constraints. For details see [9, Remark 2.4].

In our subsequent analysis, we will exploit that, due to Assumption 2.1 and continuity, the remainder terms r_f, $r_{f_\star}$ and $r_{J_\star}$ defined above approach zero for $t \in [t_1(T),t_2(T)]$. In order to make this property rigorous in the appropriate function spaces, we introduce superposition operators. Intuitively, a superposition operator is a nonlinear map between function spaces induced by a given nonlinear function defined on, e.g., finite-dimensional spaces by superposition. We refer the interested reader to [33, Sec. 4.3.3] for a short introduction and [2], [12] for an in-depth treatment of these topics in Sobolev and Lebesgue spaces of abstract functions. In order to not hide the main steps behind technical details, we only consider the case of scalar nonlinearities here.

Definition 2.4: Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Consider a mapping $\varphi : \mathbb{R} \to \mathbb{R}$. Then the mapping Φ defined by

$$\Phi(x)(\omega) = \varphi(x(\omega)) \qquad \text{for } \omega \in \Omega$$

assigns to a function $x : \Omega \to \mathbb{R}$ a new function $z : S \to \mathbb{R}$ via the relation $z(\omega) = \varphi(x(\omega))$ for $\omega \in \Omega$ and is called a Nemytskij operator or superposition operator.
An immediate question that arises is the following: Given a function $x \in L_p(\Omega)$, which integrability does the image

$\Phi(x)$ have? It turns out that in case $p<\infty$, this is coupled to growth assumptions on the underlying nonlinearity. It is to be expected as, e.g., for $\varphi(x)=x^3$, the corresponding superposition operator maps $L_{3p}(\Omega)$ to $L_p(\Omega)$.

Proposition 2.5: Let $\varphi:\mathbb{R}\to\mathbb{R}$ be continuous. For $1\le p,q<\infty$ let

$$|\varphi(s)|\le c_1+c_2|s|^{\frac{p}{q}} \qquad \forall s\in\mathbb{R} \tag{7}$$

for constants $c_1\in\mathbb{R}$ and $c_2\ge 0$. Then the corresponding superposition operator Φ maps $L_p(\Omega)$ into $L_q(\Omega)$. Additionally, it is continuous as a nonlinear map from $L_p(\Omega)$ to $L_q(\Omega)$, i.e., if $\|x-z\|_{L_p(\Omega)}\to 0$, we have that

$$\|\Phi(x)-\Phi(z)\|_{L_q(\Omega)}\to 0.$$

Proof: See [12, Thm. 1] and [12, Thm. 4]. ∎

It can also be shown that the assumptions of Proposition 2.5 are not only sufficient for continuity, but also necessary, cf. [12, Thm. 3.1]. Thus, e.g., for a cubic nonlinearity $f(x,u)=-x^3$ and assuming that the state and control approach the turnpike in some L_p-norm, the remainder term r_f will vanish in the $L_{p/3}$-norm. We now formulate the main assumption considering the continuity of the remainder terms.

Assumption 2.6: We assume that there is a real Hilbert space $(Y,\langle\cdot,\rangle_Y)$ with corresponding norm $\|\cdot\|_Y$ such that the superposition operators induced by the remainder terms $r^*_{f_x}(t)$ and $r_{J_x}(t)$ for any $t\in[0,T]$ are continuous from X to $L(X,Y)$ and X to Y respectively. Additionally, we assume that the superposition operators corresponding to the remainder terms $r^*_{f_u}(t)$ and $r_{J_u}(t)$ for any $t\in[0,T]$ are continuous from U to $L(U,Y)$ and U to $U^*\simeq U$ respectively.

We note that the continuity required in this assumption allows to deduce vanishing right-hand sides of the adjoint system by continuity and the primal turnpike property. Without vanishing source terms in the governing equation of $\delta\lambda$, the turnpike, i.e., the convergence of $\delta\lambda(t)$ to zero, can not be expected (even in the finite dimensional case).

Remark 2.7: In the finite-dimensional setting with $X=Y=\mathbb{R}^n$ and $U=\mathbb{R}^m$, $n,m\in\mathbb{N}$, the superposition operator concept is not needed and the subsequent results will hold for all Lipschitz nonlinearities. In the infinite-dimensional setting the assumption on continuity of the superposition operators corresponding to $r_{f_u}(t)$ and $r_{J_u}(t)$ allows, e.g., for $Y=L_2(\Omega)$ if the cost functional is quadratic in the control and the dynamics include a polynomial nonlinearity in the control, if U is embedded into an L_p space with large p. The continuity of the superposition operators corresponding to $r_{f_x}(t)$ and $r_{f_u}(t)$ can be verified if the state space X is sufficiently regular and embedded into an L_p-space with p large and the nonlinearity is, e.g., polynomial in x. Additionally if the superposition operator corresponding to $f_x(\bar{x},\bar{u})^*$ can be extended to a compact operator from the domain of $\mathscr{A}^*$ in Y to Y and if the semigroup generated by $\mathscr{A}^*$ is exponentially stable, the perturbed operator $A^*=(\mathscr{A}+f_x(\bar{x},\bar{u}))^*$ generates a semigroup on Y, cf. [7, Sec. III.2] and Section V.

We assume that $A^*=(\mathscr{A}+f_x(\bar{x},\bar{u}))^*$ generates a strongly continuous semigroup $(\mathscr{T}^*(t))_{t\ge 0}$ on Y, $B\in L(U,Y)$ and that $\bar{\lambda}\in Y$ and whenever we refer to a solution of (4), we mean it in the mild sense, i.e., for the adjoint, we have the variation of constants formula, cf. [24, Sec. 4.2],

$$\begin{aligned}\delta\lambda(t)=&\,\mathscr{T}^*(T-t)\delta\lambda(T)\\&+\int_t^T \mathscr{T}^*(s-t)\left(r_{f_x}(s)^*\lambda(s)+r_{J_x}(s)\right)ds.\end{aligned} \tag{8}$$

III. Stable or exactly controllable systems

We first analyze two particular cases, to which we will reduce more general systems in Section IV: On the one hand the case where A^* generates an exponentially stable semigroup on Y and on the other hand the case of (A,B) being exactly controllable.

Theorem 3.1: Let Assumption 2.6 hold. Let (x,u) satisfy the interval turnpike property of Assumption 2.1 with the intervals $[t_1(T),t_2(T)]$ and assume that the adjoints satisfy $\rho:=\sup_{T\ge 0}\|\lambda\|_{C(t_1(T),t_2(T);X)}<\infty$. Assume that A^* generates an exponentially stable semigroup $(\mathscr{T}^*(t))_{t\ge 0}$ on Y. Then λ satisfies the interval turnpike property from Definition 2.3.

Proof: First, we write the adjoint equation, i.e., the first equation of (4), by means of the variation of constants formula (8) for $t\in[t_1(T),t_2(T)]$ on $[t,t_2(T)]$

$$\begin{aligned}\delta\lambda(t)=&\,\mathscr{T}^*(t_2(T)-t)\delta\lambda(t_2(T))\\&+\int_t^{t_2(T)} \mathscr{T}^*(s-t)\left(r_{f_x}(s)^*\lambda(s)+r_{J_x}(s)\right)ds.\end{aligned}$$

By exponential stability of the semigroup there is $M\ge 1$ and $\mu>0$ such that $\|\mathscr{T}^*(t)\|_{L(Y,Y)}\le Me^{-\mu t}$ for all $t\ge 0$. This implies the existence of $c>0$ such that the estimate

$$\begin{aligned}\|\delta\lambda(t)\|_Y\le&\, Me^{-\mu(t_2(T)-t)}\|\delta\lambda(t_2(T))\|_Y\\&+c\left(\|r_{f_x}\|_{C(t,t_2(T);L(X,Y))}\rho+\|r_{J_x}\|_{C(t,t_2(T);Y)}\right)\end{aligned}$$

holds for all $t\in[t_1(T),t_2(T)]$. Setting $s_1(T):=t_1(T)$, $s_2(T):=t_1(T)+(t_2(T)-t_1(T))/2$, and recalling that $t_2(T)-t_1(T)\to\infty$ as $T\to\infty$, we obtain for sufficiently large T that $Me^{-\mu(t_2(T)-t)}\le 1/2$ for all $t\in[s_1(T),s_2(T)]$. This implies

$$\|\delta\lambda(t)\|_Y\le 2c\left(\|r_{f_x}\|_{C(t,t_2(T);L(X,Y)}\rho+\|r_{J_x}\|_{C(t,t_2(T);Y)}\right)$$

for all $t\in[s_1(T),s_2(T)]$. The assertion follows since $\|r_{f_x}\|_{C(t,t_2(T);L(X,Y)}\to 0$ and $\|r_{J_x}\|_{C(t,t_2(T);Y)}\to 0$ as $T\to\infty$ due to Assumptions 2.1 and 2.6. ∎

Remark 3.2: The above choice of $s_2(T)$ is not unique, as we can also choose $s_2(T):=t_1(T)+(t_2(T)-t_1(T))/n$ for any $n\in\mathbb{N}$ with $n\ge 2$.

Remark 3.3: If we add a term $\sigma_T(t)$ with $\|\sigma_T\|_{C(t,t_2(T);Y)}<\infty$ on the right hand side of (4), then a straightforward extension of the proof shows that for all sufficiently large T we obtain

$$\|\delta\lambda(t)\|_Y\le\varepsilon+2c\|\sigma_T\|_{C(t,t_2(T);Y)}$$

for all $t\in[s_1(T),s_2(T)]$.

Remark 3.4: If the times $t_1(T)$ and $t_2(T)$ in the interval turnpike property for state and control satisfy $(t_2(T)-t_1(T))/T\ge C(T)$ for a function $C:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$, then the times $s_1(T)$ and $s_2(T)$ of the adjoint turnpike property can be chosen to satisfy $(s_2(T)-s_1(T))/T\ge C(T)/2$.

Next, we discuss the case of (A,B) being exactly controllable.

Definition 3.5: [5, Def. 4.1.3] For any $\tau \in [0,T]$, we define the controllability map $\phi_\tau : L_2(0,\tau;U) \to Y$ by

$$\phi_\tau u := \int_0^\tau \mathscr{T}(\tau - s) B u(s)\, ds.$$

We call (A,B) exactly controllable in time $t_c > 0$ if $\operatorname{ran}\phi_{t_c} = Y$. Similarly, we call (A,B) approximately controllable in time t_c if $\overline{\operatorname{ran}\phi_{t_c}} = Y$.

It is clear that exact and approximate controllability coincide in finite-dimensions. An important characterization of controllability is the following observability inequality, which was proven first in the seminal paper [22] with the Hilbert Uniqueness Method.

Theorem 3.6: [5, Thm. 4.1.7] (A,B) is exactly controllable in time $t_c > 0$ if and only if there is $\alpha_{t_c} > 0$ such that

$$\int_0^{t_c} \|B^* \mathscr{T}^*(s) x_0\|_U^2\, ds \ge \alpha_{t_c} \|x_0\|_Y^2 \quad \forall x_0 \in Y.$$

Using substitution in the previous estimate we immediately obtain that

$$\int_{t-t_c}^{t} \|B^* \mathscr{T}^*(t-s) \delta\lambda(t)\|_U^2\, ds \ge \alpha_{t_c} \|\delta\lambda(t)\|_Y^2 \qquad (9)$$

for all $\delta\lambda(t) \in Y$ and $t \in [t_c, T]$.

This enables us to derive the following bound on $\|\delta\lambda(t)\|_Y$.

Theorem 3.7: Let (A,B) be exactly controllable in time $t_c > 0$. Then there is $c > 0$ independent of T, such that

$$\|\delta\lambda(t)\|_Y^2 \le c \int_{t-t_c}^{t} \|r_{f_u}(s)^* \lambda(s) + r_{J_u}(s)\|_U^2 + \|-r_{f_x}(s)^* \lambda(s) + r_{J_x}(s)\|_Y^2\, ds.$$

Proof: The proof of this estimate is inspired by [26, Proof of Rem. 2.1], where the finite-dimensional case is considered. We decompose $\delta\lambda = \delta\lambda_1 + \delta\lambda_2$, where for any $s < t$

$$\delta\dot\lambda_1(s) = -A^* \delta\lambda_1(s), \qquad \delta\lambda_1(t) = \delta\lambda(t),$$
$$\delta\dot\lambda_2(s) = -A^* \delta\lambda_2(s) - r_{f_x}(s)^* \lambda(s) + r_{J_x}(s), \quad \delta\lambda_2(t) = 0,$$

and apply the observability estimate (9) to $\delta\lambda_1(s) = \mathscr{T}^*(t-s)\delta\lambda(t)$. This yields

$$\alpha_{t_c} \|\delta\lambda(t)\|_Y^2 \le \int_{t-t_c}^{t} \|B^* \delta\lambda_1(s)\|_U^2\, ds \le \int_{t-t_c}^{t} \|B^* \delta\lambda(s)\|_U^2 + \|B^* \delta\lambda_2(s)\|_U^2\, ds.$$

Further, we estimate

$$\int_{t-t_c}^{t} \|B^* \delta\lambda_2(s)\|_U^2\, ds \le \int_{t-t_c}^{t} \|B^* \int_s^t \mathscr{T}^*(\tau - s)(-r_{f_x}(\tau)^* \lambda(\tau) + r_{J_x}(\tau))\, d\tau\|_U^2\, ds \le c(t_c) \int_{t-t_c}^{t} \|-r_{f_x}(s)^* \lambda(s) + r_{J_x}(s)\|_Y^2\, ds.$$

Finally, by (5), we have that

$$\int_{t-t_c}^{t} \|B^* \delta\lambda(s)\|_U^2 = \int_{t-t_c}^{t} \|r_{f_u}(s)^* \lambda(s) + r_{J_u}(s)\|_U^2\, ds,$$

which concludes the proof. ■

Since the right hand side of the inequality from Theorem 3.7 obviously tends to zero if the integrands tend to zero, we immediately obtain the following corollary.

Corollary 3.8: Let Assumption 2.6 hold and let (A,B) be exactly controllable in time $t_c > 0$. Let (x,u) satisfy the turnpike property of Assumption 2.1 with the intervals $[t_1(T), t_2(T)]$ and assume that the adjoints satisfy $\rho := \sup_{T\ge 0} \|\lambda\|_{C(t_1(T),t_2(T);Y)} < \infty$. Then λ satisfies the interval turnpike property from Definition 2.3 with $s_1(T) = t_1(T) + t_c$ and $s_2(T) = t_2(T)$.

Proof: Follows immediately from Theorem 3.7. ■

Remark 3.9: Under the assumptions of Remark 3.4, for the times from Corollary 3.8 the estimate $(s_2(T) - s_1(T))/T \ge C(T) - t_c/T$ holds.

Remark 3.10: Similar to Remark 3.3, it is easily seen from the proof of Theorem 3.7 that if we add a term $\sigma_T(t)$ with $\|\sigma_T\|_{C(t-t_c,t;Y)} \le \bar\sigma_T < \infty$ and $\rho_T(t)$ on the right hand sides of (4) and (5), respectively, then the result of Theorem 3.7 changes to

$$\|\delta\lambda(t)\|_Y^2 \le c \int_{t-t_c}^{t} \|r_{f_u}(s)^* \lambda(s) + r_{J_u}(s) + \sigma_T(s)\|_U^2 + \|-r_{f_x}(s)^* \lambda(s) + r_{J_x}(s) + \rho_T(s)\|_Y^2\, ds. \qquad (10)$$

We then obtain as a counterpart for the inequality in Definition 2.3 the bound

$$\|\delta\lambda(t)\|_Y \le \varepsilon + c(\bar\sigma_T + \bar\rho_T) \qquad \forall t \in [s_1(T), s_2(T)],\ T \ge T_0.$$

IV. Stabilizable systems with finite-dimensional unstable part

In this section we extend our results to exponentially detectable (A^*, B^*), where the unstable part of A^* is finite-dimensional and B^* has finite rank. We note that this includes all finite-dimensional systems with stabilizable linearization. In order to define the correct setting for infinite-dimensional systems, we present the spectrum decomposition assumption as follows.

Definition 4.1: [5, Def. 5.2.5] Denoting $\sigma^+(A) := \sigma(A) \cap \{s \in \mathbb{C} : \operatorname{Re} s \ge 0\}$ and $\sigma^-(A) := \sigma(A) \cap \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$, an operator A satisfies the spectral decomposition assumption if $\sigma^+(A)$ is bounded and separated from $\sigma^-(A)$ in such a way that a rectifiable, simple, closed curve Γ can be drawn so as to enclose an open set containing $\sigma^+(A)$ in its interior and $\sigma^-(A)$ in its exterior.

If A^* satisfies the spectrum decomposition assumption, there exists a decomposition of Y given by $Y = Y_u \oplus Y_s$, where $Y_u = PY$, $Y_s = (I-P)Y$, and P is the projection according to [5, Lem. 2.5.7]. Moreover, the spectral projection yields a linear coordinate transform such that the pair (A^*, B^*) can be transformed into the form

$$\widetilde{A^*} = \begin{bmatrix} A_u^* & 0 \\ 0 & A_s^* \end{bmatrix}, \qquad \widetilde{B^*} = \begin{bmatrix} B_u^* & B_s^* \end{bmatrix} \qquad (11)$$

where $A_u^*, B_u^*, A_s^*, B_s^*$ are restrictions of A^* and B^* to Y_u and Y_s, respectively. Note that A_u^* and B_u^* are bounded operators. We impose the following assumption on A^*.

Assumption 4.2: A^* satisfies the spectrum decomposition assumption such that it has the decomposition according to (11), where A_u^* is finite-dimensional and A_s^* is exponentially stable.

If we split up the transformed adjoint accordingly via

$$\widetilde{\delta\lambda} = \begin{bmatrix}\delta\lambda_u\\ \delta\lambda_s\end{bmatrix}, \qquad \widetilde{\lambda} = \begin{bmatrix}\lambda_u\\ \lambda_s\end{bmatrix}, \tag{12}$$

then the equations (4) and (5) attain the form

$$\delta\dot{\lambda}_u = -A_u^*\delta\lambda_u - \tilde{r}_1^*\lambda_u - \tilde{r}_2^*\lambda_s + \tilde{r}_3 \tag{13}$$

$$\delta\dot{\lambda}_s = -A_s^*\delta\lambda_s - \tilde{r}_7^*\lambda_u - \tilde{r}_8^*\lambda_s + \tilde{r}_9 \tag{14}$$

$$0 = B_u^*\delta\lambda_u + B_s^*\delta\lambda_s + \tilde{r}_4\lambda_u + \tilde{r}_5\lambda_s + \tilde{r}_6. \tag{15}$$

Here, the terms $\tilde{r}_j$ are derived via coordinate transformation and splitting from the remainder terms in (4)–(6) and—up to multiplication by appropriate constants—satisfy the same estimates as these remainder terms. Using this decomposition, we can prove the following theorem.

Theorem 4.3: Let Assumption 2.6 hold. Let (x,u) satisfy the turnpike property of Assumption 2.1 on $[t_1(T),t_2(T)]$ and assume that $\rho := \sup_{T\geq 0}\|\lambda\|_{C(t_1(T),t_2(T);Y)} < \infty$. Let Assumption 4.2 hold and further assume that B^* has finite rank and (A^*,B^*) is exponentially detectable. Then λ satisfies the interval turnpike property from Definition 2.3.

Proof: First note that the claimed property holds for $\delta\lambda$ if and only if it holds for the transformed adjoint $\widetilde{\delta\lambda}$ from (12). The property for $\widetilde{\delta\lambda}$, in turn, holds if and only if it holds for the two components $\delta\lambda_u$ and $\delta\lambda_s$. Moreover, note that the assumed exponential detectability of (A^*,B^*) implies that the finite-dimensional pair (A_u^*,B_u^*) is (exponentially) detectable, and hence, (exactly) observable by Hautus rank condition [5, Def. 1.2.6].

We start by applying the extension of Theorem 3.1 described in Remark 3.3 to $\delta\lambda_s$, with $\sigma_T = -\tilde{r}_7^*\lambda_u$. We note that the fact that equation (15) contains additional terms compared to equation (5) does not affect the applicability of Theorem 3.1 and Remark 3.3, because equation (5) is not used in its proof. Due to the uniform boundedness of λ which implies uniform boundedness of λ_u, σ_T tends to zero as $T\to\infty$ on $[t_1(T),t_2(T)]$. Hence, we obtain the desired property for $\delta\lambda_s$ on an interval $[\tilde{s}_1(T),\tilde{s}_2(T)]$. We note that by the construction in the proof of Theorem 3.1 we obtain $[\tilde{s}_1(T),\tilde{s}_2(T)] \subset [t_1(T),t_2(T)]$.

Now for $\delta\lambda_u$ we use the extension of Theorem 3.7 described in Remark 3.10 with $\sigma_T = -\tilde{r}_2^*\lambda_s$ and $\rho_T = B_s^*\delta\lambda_s - \tilde{r}_5^*\lambda_s$, on $[\tilde{s}_1(T),\tilde{s}_2(T)]$. Since all terms become arbitrarily small on $[\tilde{s}_1(T),\tilde{s}_2(T)]$ as $T\to\infty$, we obtain the desired estimate for $\delta\lambda_u$ on $[s_1(T),s_2(T)]$ with $s_1(T) = \tilde{s}_1(T)+t_c$ and $s_2(T) = \tilde{s}_2(T)$. ∎

So far we do not know general conditions for the boundedness of the adjoints. However, for concrete examples this property can be checked. For instance, it holds if we set $X = H_0^1(\Omega)$, $Y = L_2(\Omega)$, choose $\mathscr{A} = \Delta$ with Dirichlet boundary conditions and consider $f(x,u) = f(x)$ monotonously non-increasing, and running cost $J(x,u) = \frac{1}{2}\|x-x_d\|^2_{L_2(\Omega_o)} + \frac{1}{2}\|u-u_d\|^2_{L_2(\Omega_c)}$. For details see [9, Example 6.3].

V. Numerical Example

We present an example with a semilinear heat equation on a domain $\Omega = [0,1]^2$. The control acts on a subdomain $\Omega_c = \{(\omega_1,\omega_2)\in\Omega \,|\, \omega_2 \geq 0.5\}$. We consider homogeneous Dirichlet boundary conditions and zero initial value, i.e.,

$$\begin{aligned} \dot{x} - (0.1\Delta - 1)x + x^3 &= \chi_{\Omega_c}u && \text{in } [0,T]\times\Omega,\\ x &= 0 && \text{in } [0,T]\times\partial\Omega,\\ x(0) &= 0 && \text{in } \Omega. \end{aligned}$$

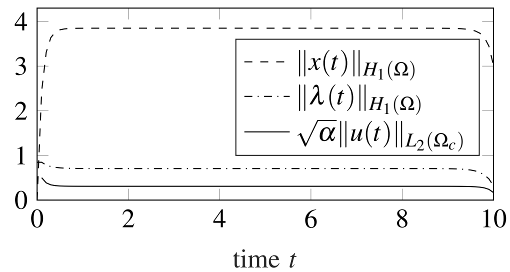

Fig. 1. Turnpike property of state, adjoint and control.

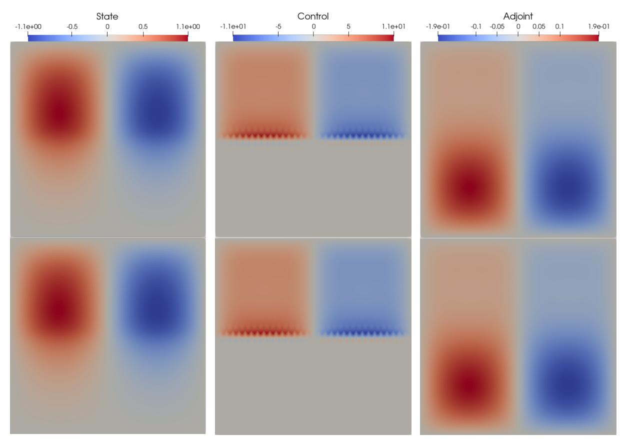

Fig. 2. Comparison of dynamic variables (x,u,λ) at time $t=5$ (above) vs steady state variables $(\bar{x},\bar{u},\bar{\lambda})$ (below)

As a cost function, we consider

$$\int_0^T J(x(t),u(t))dt = \frac{1}{2}\int_0^T \|x(t)-x_d\|^2_{L_2(\Omega)} + \alpha\|u(t)\|^2_{L_2(\Omega_c)}\,dt,$$

and we set $\alpha = 10^{-2}$. We choose the time horizon $T = 10$. The reference trajectory is defined by $x_d(\omega_1,\omega_2) := 1$ for $\omega_1 < 0.5$ and $x_d(\omega_1,\omega_2) := -1$ otherwise.

We implement and solve the optimal control problem with the C++-library for vector space algorithms *Spacy*[1] using the finite element library *Kaskade7* [13]. For in-depth analysis of semilinear parabolic optimal control problems we refer the reader to [28] or [33, Chap. 5]. Considering the regularity of the static adjoint, for sufficiently smooth data we obtain that $\bar{\lambda}\in C(\bar{\Omega})$, cf. [4]. We set $\mathscr{A} = 0.1\Delta + 1$ and $\varphi(x) = x^3$ and denote the superposition operator corresponding to

[1] https://spacy-dev.github.io/Spacy/

$\varphi'(x) = 3x^2$ by Φ. Note that the semigroup of $\mathscr{A}$ is not exponentially stable due to the addition of 1 and the fact that the smallest eigenvalue of -0.1Δ with Dirichlet boundary conditions is given by $0.1 \cdot 2\pi < 1$ on the unit square. We numerically verify that the turnpike property of Assumption 2.1 for the optimal state and control holds in $X = H^1(\Omega)$ and $U = L_2(\Omega_c)$, cf. Figure 1. There, we also depict the resulting turnpike for the adjoint. In Figure 2, we depict a snapshot of the dynamic solution and compare it to the steady state solution for state, control and adjoint.

While the turnpike property of the state and adjoints is verified numerically, the remaining assumptions can be checked analytically as follows. By the classical embeddings $H^1(\Omega) \hookrightarrow L_p(\Omega)$ for any $1 \le p < \infty$ for $\Omega \subset \mathbb{R}^2$, cf. [33, Sec. 7.1] or [1, Chap. V] and as the nonlinearity is cubic, the occurring superposition operators satisfy Assumption 2.6 for $Y = L_2(\Omega)$. Moreover, as $D(\mathscr{A}^*) = \{v \in H^2(\Omega) \,|\, v = 0 \text{ on } \partial\Omega\} \hookrightarrow H^1(\Omega) = X$ compactly, the perturbation $\Phi(\bar{x}) \in L(X,Y)$ is $\mathscr{A}^*$-compact and $A^* = \mathscr{A}^* + \Phi(\bar{x})^*$ generates an analytic semigroup on $L_2(\Omega)$, and $D(A^*) = D(\mathscr{A}^*)$, cf. [7, Chap. III, Thm. 2.10]. Thus, assuming additionally a boundedness condition of the adjoint, Theorem 4.3 applies and we obtain the turnpike property for the adjoint in $Y = L_2(\Omega)$. In Figure 1 we observe that the turnpike property for the adjoint even holds in the stronger $H^1(\Omega)$-norm.

VI. Summary

This paper presented a turnpike result for the adjoints of finite- and infinite-dimensional nonlinear optimal control problems under the assumption of an interval turnpike on states and controls. It focused on the case of stabilizable dynamics governed by a generator of a semigroup with finite dimensional unstable part satisfying a spectral decomposition condition. We have shown the desired turnpike property under continuity assumptions on the first-order optimality conditions. We illustrated our results considering a semilinear heat equation.

References

[1] R. A. Adams. *Sobolev Spaces*. Academic Press, Amsterdam, Boston, 1975.

[2] J. Appell and P. P. Zabrejko. *Nonlinear Superposition Operators*. Cambridge Tracts in Mathematics. Cambridge University Press, 1990.

[3] T. Breiten and L. Pfeiffer. On the turnpike property and the receding-horizon method for linear-quadratic optimal control problems. *SIAM Journal on Control and Optimization*, 58(2):1077–1102, 2020.

[4] E. Casas. Boundary control of semilinear elliptic equations with pointwise state constraints. *SIAM Journal on Control and Optimization*, 31(4):993–1006, 1993.

[5] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 1995.

[6] T. Damm, L. Grüne, M. Stieler, and K. Worthmann. An exponential turnpike theorem for dissipative discrete time optimal control problems. *SIAM Journal on Control and Optimization*, 52(3):1935–1957, 2014.

[7] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer-Verlag New York, 2000.

[8] T. Faulwasser and L. Grüne. *Handbook of Numerical Analysis*, chapter Turnpike Properties in Optimal Control: An Overview of Discrete-Time and Continuous-Time Results. Elsevier, 2021. arxiv: 2011.13670. In press.

[9] T. Faulwasser, L. Grüne, J.-P. Humaloja, and M. Schaller. The interval turnpike property for adjoints. arXiv Preprint 2005.12120, 2020.

[10] T. Faulwasser, M. Korda, C. N. Jones, and D. Bonvin. On turnpike and dissipativity properties of continuous-time optimal control problems. *Automatica*, 81:297–304, 2017.

[11] T. Faulwasser and A. Murray. Turnpike properties in discrete-time mixed integer optimal control. *IEEE Control Systems Letters*, 4:704–709, 2020.

[12] H. Goldberg, W. Kampowsky, and F. Tröltzsch. On Nemytskij operators in Lp-spaces of abstract functions. *Mathematische Nachrichten*, 155(1):127–140, 1992.

[13] S. Götschel, A. Schiela, and M. Weiser. Kaskade 7 – A flexible finite element toolbox. *Computers & Mathematics with Applications*, 2020.

[14] L. Grüne and R. Guglielmi. On the relation between turnpike properties and dissipativity for continuous time linear quadratic optimal control problems. *Math. Control Rel. Fields*, 2020. To appear.

[15] L. Grüne and M. A. Müller. On the relation between strict dissipativity and turnpike properties. *Systems & Control Letters*, 90:45–53, 2016.

[16] L. Grüne, M. Schaller, and A. Schiela. Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control. *SIAM Journal on Control and Optimization*, 57(4):2753–2774, 2019.

[17] L. Grüne, M. Schaller, and A. Schiela. Exponential sensitivity analysis for model predictive control of PDEs. urn:nbn:de:bvb:703-epub-4590-3, 2020.

[18] L. Grüne, M. Schaller, and A. Schiela. Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations. *Journal of Differential Equations*, 268(12):7311–7341, 2020.

[19] M. Gugat and F. Hante. On the turnpike phenomenon for optimal boundary control problems with hyperbolic systems. *SIAM Journal on Control and Optimization*, 57(1):264–289, 2019.

[20] M. Gugat, E. Trélat, and E. Zuazua. Optimal Neumann control for the 1D wave equation: Finite horizon, infinite horizon, boundary tracking terms and the turnpike property. *Systems & Control Letters*, 90:61–70, 2016.

[21] X. Li and J. Yong. *Optimal Control Theory for Infinite Dimensional Systems*. Birkhäuser, 1995.

[22] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM review*, 30(1):1–68, 1988.

[23] J. v. Neumann. A model of general economic equilibrium. *The Review of Economic Studies*, 13(1):1–9, 1945.

[24] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York, NY, 1983.

[25] A. Porretta. On the turnpike property for mean field games. *Minimax Theory and its Applications*, 3(2):285–312, 2018.

[26] A. Porretta and E. Zuazua. Long time versus steady state optimal control. *SIAM Journal on Control and Optimization*, 51(6):4242–4273, 2013.

[27] F. P. Ramsey. A mathematical theory of saving. *The Economic Journal*, 38(152):543–559, 1928.

[28] J. P. Raymond and H. Zidani. Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations. *Applied Mathematics and Optimization*, 39(2):143–177, 1999.

[29] N. Sakamoto, D. Pighin, and E. Zuazua. The turnpike property in nonlinear optimal control — A geometric approach. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 2422–2427, 2019.

[30] E. Trélat and C. Zhang. Integral and measure-turnpike properties for infinite-dimensional optimal control systems. *Mathematics of Control, Signals, and Systems*, 30(1):3, 2018.

[31] E. Trélat, C. Zhang, and E. Zuazua. Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces. *SIAM Journal on Control and Optimization*, 56(2):1222–1252, 2018.

[32] E. Trélat and E. Zuazua. The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations*, 258(1):81–114, 2015.

[33] F. Tröltzsch. *Optimal control of partial differential equations: theory, methods, and applications*, volume 112. American Mathematical Soc., 2010.

[34] A. J. Zaslavski. *Turnpike theory of continuous-time linear optimal control problems*. Springer, 2016.