OPTIMAL DECAY FOR A WAVE-HEAT SYSTEM WITH COLEMAN–GURTIN THERMAL LAW

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ABSTRACT. We study the long-term behaviour of solutions to a one-dimensional coupled wave-heat system with Coleman–Gurtin thermal law. Our approach is based on the asymptotic theory of C_0-semigroups and recent results developed for coupled control systems. As our main results, we represent the system as a feedback interconnection between the wave part and the Coleman–Gurtin part and we show that the associated semigroup in the history framework of Dafermos is polynomially stable with optimal decay rate t^{-2} as $t \to \infty$. In particular, we obtain a sharp estimate for the rate of energy decay of classical solutions to the problem.

1. INTRODUCTION

The study of the asymptotic behaviour of solutions to coupled PDE systems has attracted a considerable amount of attention in the recent literature. In this article, we focus on a one-dimensional coupled wave-heat system consisting of a wave equation and a Coleman–Gurtin equation. More specifically, we use the asymptotic theory of strongly continuous semigroups combined with recent results on coupled abstract control systems to derive an optimal rational decay rate for classical solutions to the system

$$(1.1) \qquad \begin{cases} u_{tt}(x,t) = u_{xx}(x,t), & x \in (-1,0), \ t > 0, \\ w_t(x,t) = w_{xx}(x,t) + \displaystyle\int_0^\infty g(s) w_{xx}(x, t-s) \, \mathrm{d}s, & x \in (0,1), \ t > 0. \end{cases}$$

The equations are coupled, for $t > 0$, through the transmission conditions

$$(1.2) \qquad u_t(0,t) = w(0,t), \qquad u_x(0,t) = w_x(0,t) + \int_0^\infty g(s) w_x(0, t-s) \, \mathrm{d}s$$

at the interface $x = 0$, and in addition we impose the Dirichlet boundary conditions

$$(1.3) \qquad u(-1,t) = w(1,t) = 0.$$

The convolution kernel $g : [0, \infty) \to [0, \infty)$ is a convex integrable function (thus non-increasing and vanishing at infinity) of unit total mass, taking the explicit form

$$g(s) = \int_s^\infty \mu(r) \, \mathrm{d}r, \qquad s \geq 0,$$

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where $\mu : (0, \infty) \to [0, \infty)$ is a non-increasing absolutely continuous integrable function (possibly unbounded near zero). In particular, μ is differentiable almost everywhere with $\mu'(s) \leq 0$ for almost every $s > 0$. Finally, we impose initial conditions of the form

$$\begin{cases} u(x,0) = u_0(x), & x \in (-1,0), \\ u_t(x,0) = v_0(x), & x \in (-1,0), \\ w(x,0) = w_0(x), \quad w(x,-s) = \varphi_0(x,s), & x \in (0,1), \ s > 0, \end{cases}$$

where u_0, v_0, w_0, φ_0 are assigned data. In particular, φ_0 accounts for the so-called initial past history of w.

The stability analysis of coupled wave-heat systems has been the subject of intensive investigations over the past few decades. Their intrinsic mathematical interest apart, the main motivation for studying such systems stems from the fact that they can be viewed as linearisations of more complex fluid-structure models arising in fluid mechanics; see for instance [2, 25, 33]. In the absence of the integral term, (1.1) reduces to the classical wave-heat system, whose asymptotic properties have been extensively analysed in the literature; see for instance [1, 3, 5, 15, 23, 32, 33] and the references therein. In particular, it is known that in this case the associated solution semigroup is *semi-uniformly stable* in the sense that all classical solutions converge to zero at a uniform rate, and more specifically the semigroup is *polynomially stable* with optimal decay rate t^{-2} as $t \to \infty$. In particular, the semigroup fails to be exponentially stable. To the best of the authors' knowledge, the system in (1.1) with a non-trivial kernel g was first studied in [31]. In fact, the analysis in [31] deals with a more general system in which the Laplacian w_{xx} appearing in the second equation is replaced by βw_{xx} for some $\beta \geq 0$. The cases $\beta > 0$ and $\beta = 0$ correspond to the so-called Coleman–Gurtin [11] and Gurtin–Pipkin [20] models, respectively. Thus our system (1.1) corresponds to the Coleman–Gurtin case with $\beta = 1$, a choice which entails no essential loss of generality. One of the main results of [31] is that if $\beta = 0$ and if the so-called Dafermos condition

$$(1.4) \qquad\qquad\qquad \mu'(s) + \delta\mu(s) \leq 0$$

holds for some $\delta > 0$ and almost every $s > 0$, then the semigroup associated with the wave-Gurtin–Pipkin system in the history space framework of Dafermos [13] is exponentially stable. Since the Gurtin–Pipkin dissipation given solely by the convolution term $\int_0^\infty g(s) w_{xx}(t-s) \, ds$ is weaker than the dissipation provided by the Laplacian w_{xx}, this result serves to illustrate that the classical wave-heat system fails to be exponentially stable on account of *overdamping*. It is a reasonable guess, therefore, that the wave-Coleman–Gurtin system (1.1), too, fails to be exponentially stable, and this has been confirmed in [31], at least in the special case where g is an exponential function.

In the present paper, we complete the analysis begun in [31] by finding the optimal (semi-uniform) decay rate of the semigroup $(S(t))_{t\geq 0}$ associated with (1.1) in the history space framework of Dafermos. More precisely, assuming the condition

$$(1.5) \qquad\qquad\qquad \mu(t+s) \leq C\mathrm{e}^{-\delta t}\mu(s)$$

for some $C \geq 1$ and $\delta > 0$ and for every $t \geq 0$ and $s > 0$, we show in Theorem 4.2 and Proposition 4.3 that $(S(t))_{t\geq 0}$ is polynomially stable with optimal decay rate t^{-2} as $t \to \infty$. Observe that this decay rate coincides with that of the classical wave-heat

system. Note also that (1.5) is weaker than (1.4). For instance, condition (1.4), in contrast to (1.5), does not allow flat zones or horizontal inflection points; see for instance [8, 9]. Our approach consists in writing the system in (1.1)–(1.2) as a *feedback interconnection* between the wave part and the Coleman–Gurtin part. Based on this decomposition we show that the infinitesimal generator $\mathbb{A}$ of $(S(t))_{t\geq 0}$ has a special block operator structure which can be used to derive a sharp resolvent estimate; see Theorems 3.1 and 3.4. This resolvent estimate, combined with the asymptotic theory of C_0-semigroups, finally leads to the desired decay estimates. The general decomposition approach used in this paper extends to the case where the wave part in (1.1) has spatially varying parameters, and also to more complex systems, such as chains consisting of several coupled wave and Coleman–Gurtin-type equations. In the latter case the decoupling approach reduces the study of the more complicated system to the analysis of its simpler constituent parts, and in this way facilitates efficient treatment of chains of coupled equations. The same approach can potentially also be employed in the stability analysis of coupled PDEs on networks.

Notation. We introduce the (complex) Hilbert spaces

$$H_l^1(-1,0) = \{\varphi \in H^1(-1,0) : \varphi(-1) = 0\}, \quad H_r^1(0,1) = \{\varphi \in H^1(0,1) : \varphi(1) = 0\},$$

with the inner products $\langle \varphi, \psi \rangle_{H_l^1(-1,0)} = \langle \varphi', \psi' \rangle_{L^2(-1,0)}$ and $\langle \varphi, \psi \rangle_{H_r^1(0,1)} = \langle \varphi', \psi' \rangle_{L^2(0,1)}$. We also introduce the so-called memory space $\mathcal{M} = L_\mu^2(0,\infty; H_r^1(0,1))$ of $H_r^1(0,1)$-valued functions on $(0,\infty)$ which are square-integrable with respect to the measure $\mu(s)\mathrm{d}s$, endowed with the natural inner product

$$\langle \eta, \xi \rangle_{\mathcal{M}} = \int_0^\infty \mu(s) \langle \eta(s), \xi(s) \rangle_{H_r^1(0,1)} \, \mathrm{d}s.$$

The state space of our problem will be

$$\mathcal{H} = H_l^1(-1,0) \times L^2(-1,0) \times L^2(0,1) \times \mathcal{M},$$

with the natural inner product

$$\langle (u,v,w,\eta), (\tilde{u},\tilde{v},\tilde{w},\tilde{\eta}) \rangle_{\mathcal{H}} = \langle u, \tilde{u} \rangle_{H_l^1(-1,0)} + \langle v, \tilde{v} \rangle_{L^2(-1,0)} + \langle w, \tilde{w} \rangle_{L^2(0,1)} + \langle \eta, \tilde{\eta} \rangle_{\mathcal{M}}.$$

Throughout the paper, the Young, Hölder and Poincaré inequalities will be used without explicit mention. Square roots of complex numbers are defined with a branch cut along $(-\infty, 0]$. In particular, $\mathrm{Re}\,\sqrt{\lambda} \geq 0$ for all $\lambda \in \mathbb{C}$, with strict inequality for $\lambda \notin (-\infty, 0]$. We denote the open right and left half-planes in the complex plane by $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \gtrless 0\}$. Given (complex) Banach spaces X and Y we write $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X to Y, and we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$. If A is a closed linear operator acting on a Banach or Hilbert space, we denote its spectrum by $\sigma(A)$ and its resolvent set by $\rho(A)$. We frequently consider the domain $D(A)$ of A as being endowed with the graph norm $\|x\|_A = (\|x\|^2 + \|Ax\|^2)^{1/2}$. In particular, $D(A)$ is a Hilbert space whenever X is. Moreover, for $\lambda \in \rho(A)$ we write $R(\lambda, A)$ for the resolvent operator $(\lambda - A)^{-1}$. Finally, we use conventional asymptotic notation, including 'big O' and 'little o', and we occasionally write $p \lesssim q$ to indicate that $p \leq Cq$ for some (implicit) constant $C > 0$.

2. The System Operator and Wellposedness

We begin by introducing the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, that is, the linear operator

$$T\eta = -\eta_s, \qquad D(T) = \big\{\eta \in \mathcal{M} : \eta_s \in \mathcal{M}, \ \lim_{s \to 0} \|\eta(s)\|_{H^1_r(0,1)} = 0\big\},$$

where η_s denotes the (weak) derivative of η with respect to the variable $s > 0$. Integration by parts with respect to s together with a limiting argument can be used to show (as in [18]) that

$$(2.1) \qquad \operatorname{Re}\langle T\eta, \eta\rangle_{\mathcal{M}} = \frac{1}{2} \int_0^\infty \mu'(s)\|\eta(s)\|^2_{H^1_r(0,1)} \, \mathrm{d}s \leq 0, \qquad \eta \in D(T).$$

With a view to rewriting (1.1)–(1.2) in the *history space framework* of Dafermos [13], we consider for each $t > 0$ the auxiliary function

$$\eta^t(x, s) = \int_0^s w(x, t - \sigma) \, \mathrm{d}\sigma, \qquad x \in (0, 1), \ s > 0,$$

accounting for the integrated past history of w. We further introduce, still in the spirit of [31], the function

$$(2.2) \qquad \phi(x, t) = w(x, t) + \int_0^\infty \mu(s)\eta^t(x, s) \, \mathrm{d}s, \qquad x \in (0, 1), \ t > 0.$$

Integrating by parts (formally) we obtain the identity

$$w(x, t) + \int_0^\infty g(s)w(x, t - s) \, \mathrm{d}s = \phi(x, t), \qquad x \in (0, 1), \ t > 0.$$

The system (1.1)–(1.2) can now be rewritten as

$$(2.3) \qquad \begin{cases} u_{tt}(x, t) = u_{xx}(x, t), & x \in (-1, 0), \ t > 0, \\ w_t(x, t) = \phi_{xx}(x, t), & x \in (0, 1), \ t > 0, \\ \eta^t_t(x, s) = T\eta^t(x, s) + w(x, t), & x \in (0, 1), \ s, t > 0, \end{cases}$$

with the boundary conditions (1.3) and the coupling conditions

$$(2.4) \qquad u_t(0, t) = w(0, t), \qquad u_x(0, t) = \phi_x(0, t)$$

for $t > 0$. By introducing the state vector $z(t) = (u(\cdot, t), v(\cdot, t), w(\cdot, t), \eta^t(\cdot, \cdot))^T$, we may convert the above problem into an abstract Cauchy problem in the space $\mathcal{H}$, namely

$$(2.5) \qquad \begin{cases} \dot{z}(t) = \mathbb{A}z(t), & t \geq 0, \\ z(0) = z_0, \end{cases}$$

where $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator

$$
\mathbb{A} \begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ u'' \\ \phi'' \\ T\eta + w \end{pmatrix}, \qquad D(\mathbb{A}) = \left\{ \begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} \in \mathcal{H} \left| \begin{array}{c} u \in H^2(-1,0) \\ v \in H^1_l(-1,0) \\ w \in H^1_r(0,1) \\ \eta \in D(T) \\ \phi \in H^2(0,1) \\ u'(0) = \phi'(0) \\ v(0) = w(0) \end{array} \right. \right\},
$$

and $z_0 = (u_0, v_0, w_0, \eta^0)^T \in \mathcal{H}$ with $\eta^0(x, s) = \int_0^s \varphi_0(x, \sigma) \, d\sigma$ for $x \in (0, 1)$ and $s > 0$. Theorem 2.2 below shows that $\mathbb{A}$ generates a contraction semigroup on $\mathcal{H}$. The proof is based on the special block operator structure of $\mathbb{A}$ introduced in the same result. This structure of $\mathbb{A}$ also plays a central role later in Section 3, where we use it together with the results in [24] in order to derive an optimal resolvent estimate for $\mathbb{A}$ on $i\mathbb{R}$.

To state the theorem, we first define some notation related to extrapolation spaces for semigroup generators. If $A : D(A) \subset Z \to Z$ generates a C_0-semigroup $(T(t))_{t \geq 0}$ on a Hilbert space Z, then $D(A)$ is a Hilbert space with respect to the graph norm of A. We define Z_{-1} to be the completion of the space Z with respect to the norm $\|z\|_{Z_{-1}} = \|(\lambda_0 - A)^{-1}z\|_Z$ with $\lambda_0 \in \rho(A)$ (the space Z_{-1} is independent of the choice of $\lambda_0 \in \rho(A)$). The operator $A : D(A) \subset Z \to Z$ extends to $A_{-1} : D(A_{-1}) \subset Z_{-1} \to Z_{-1}$ with domain $D(A_{-1}) = Z$; see for instance [16, Sec. II.5]. The operator A_{-1} generates a C_0-semigroup $(T_{-1}(t))_{t \geq 0}$ on the Banach space Z_{-1} such that for every $t \geq 0$ the operator $T_{-1}(t) \in \mathcal{L}(Z_{-1})$ is an extension of $T(t) \in \mathcal{L}(Z)$. Finally, for an operator $B \in \mathcal{L}(\mathbb{C}^m, Z_{-1})$ we let

$$
Z^B = D(A) + \text{Ran}(R(\lambda_0, A_{-1})B)
$$

for $\lambda_0 \in \rho(A)$ (the space $Z^B \subset Z$ is again is independent of the choice of $\lambda_0 \in \rho(A)$).

Definition 2.1 ([29, Def. 5.1]). Assume that $A : D(A) \subset Z \to Z$ generates a C_0-semigroup on Z and that $C \in \mathcal{L}(D(A), \mathbb{C}^m)$. The Λ-*extension* of C is defined as the operator

$$
C_\Lambda z = \lim_{\substack{\lambda \to \infty \\ \lambda > 0}} \lambda C R(\lambda, A) z
$$

and the domain $D(C_\Lambda)$ consists of those $z \in Z$ for which the limit exists.

Theorem 2.2. *Let* $Z_1 = H^1_l(-1, 0) \times L^2(-1, 0)$ *and* $Z_2 = L^2(0, 1) \times \mathcal{M}$. *There exist semigroup generators* $A_k : D(A_k) \subset Z_k \to Z_k$ *and operators* $B_k \in \mathcal{L}(\mathbb{C}, Z_{k,-1})$, $C_k \in \mathcal{L}(D(A_k), \mathbb{C})$ *for* $k = 1, 2$, *and a constant* $D_1 > 0$ *such that*

$$
(2.6a) \qquad \mathbb{A} = \begin{pmatrix} A_{1,-1} & B_1 C_{2\Lambda} \\ -B_2 C_{1\Lambda} & A_{2,-1} - B_2 D_1 C_{2\Lambda} \end{pmatrix},
$$

$$
(2.6b) \qquad D(\mathbb{A}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z_1^{B_1} \times Z_2^{B_2} \left| \begin{array}{c} A_{1,-1} z_1 + B_1 C_{2\Lambda} z_2 \in Z_1 \\ A_{2,-1} z_2 - B_2(C_{1\Lambda} z_1 + D_1 C_{2\Lambda} z_2) \in Z_2 \end{array} \right. \right\}.
$$

Moreover, the operator $\mathbb{A}$ *generates a contraction semigroup* $(S(t))_{t \geq 0}$ *on* $\mathcal{H} = Z_1 \times Z_2$.

The proof of Theorem 2.2 is a direct consequence of Proposition 2.14 at the end of this section. Our approach in the proof of this result and subsequent ones does not require us to derive explicit expressions for the operators B_1, B_2 or the Λ-extensions $C_{1\Lambda}$ and $C_{2\Lambda}$ of the operators C_1 and C_2. In particular, explicit knowledge of these operators is not required for the purposes of proving well-posedness or deriving resolvent estimates for $\mathbb{A}$ by means of the results in [24].

Remark 2.3. The fact that $\mathbb{A}$ generates a contraction semigroup was already proved in [31] under slightly stronger assumptions on the memory kernel; cf. hypotheses (H1)-(H2) in [31]. We also stress that (1.5) is not needed in the semigroup generation part, but only in the resolvent estimates carried out in the next section.

Before proceeding to prove Theorem 2.2, we shall motivate the block operator structure of $\mathbb{A}$ based on the properties of the coupled PDE system (2.3) with the boundary conditions (1.3) and the coupling conditions (2.4). The block structure in (2.6) arises from the decomposition of the full coupled PDE system into two natural subparts: a wave equation and a Coleman–Gurtin-type diffusion equation. Indeed, if we introduce two auxiliary functions U_1 and Y_1, the 'wave part' of the coupled PDE system is given by

$$(2.7) \quad \begin{cases} u_{tt}(x,t) = u_{xx}(x,t), & x \in (-1,0), \ t > 0, \\ u(-1,t) = 0, \quad u_t(0,t) = U_1(t), \quad Y_1(t) = u_x(0,t), & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), & x \in (-1,0). \end{cases}$$

Thus, for $t > 0$, the value for $u_t(\cdot, t)$ at $x = 0$ is given by $U_1(t)$, while $Y_1(t)$ is determined by the value of $u_x(\cdot, t)$ at $x = 0$.

Introducing two further auxiliary functions U_2 and Y_2, the remaining 'Coleman–Gurtin part' is given by

$$(2.8) \quad \begin{cases} w_t(x,t) = \phi_{xx}(x,t), & x \in (0,1), \ t > 0, \\ \eta_t^t(x,s) = T\eta^t(x,s) + w(x,t), & x \in (0,1), \ s, t > 0, \\ -\phi_x(0,t) = U_2(t), \quad Y_2(t) = w(0,t), \quad w(1,t) = 0, & t > 0, \\ w(x,0) = w_0(x), \quad \eta^0(x,s) = \int_0^s \varphi_0(x,\sigma)\,\mathrm{d}\sigma, & x \in (0,1), \ s > 0. \end{cases}$$

For $t > 0$, the value of $\phi_x(\cdot, t)$ at $x = 0$ is determined by $U_2(t)$, and $Y_2(t)$ is determined by the value of $w(\cdot, t)$ at $x = 0$.

The PDE models (2.7) and (2.8) become equivalent to the coupled PDE system (2.3) once we require that for all $t > 0$ the auxiliary functions $U_1(t)$, $U_2(t)$, $Y_1(t)$, and $Y_2(t)$, satisfy the identities

$$(2.9) \quad \begin{cases} U_1(t) = Y_2(t) \\ U_2(t) = -Y_1(t) \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} u_t(0,t) = w(0,t) \\ u_x(0,t) = \phi_x(0,t), \end{cases}$$

which are precisely the coupling conditions (2.4). The block operator structure (2.6) follows this decomposition of the coupled PDE into two parts. In particular, the operators (A_1, B_1, C_1, D_1) are related to the wave part (2.7) and (A_2, B_2, C_2) are related to the Coleman–Gurtin part (2.8). This decomposition is moreover closely connected to mathematical systems theory, where U_1 and U_2 would be interpreted as the *inputs* of the PDE models (2.7) and (2.8), respectively, and Y_1 and Y_2 would define the *outputs* of the two

systems [12, 28, 22]. In the terminology of systems theory, the coupling conditions (2.9) on the inputs $U_1(t)$ and $U_2(t)$ and the outputs $Y_1(t)$ and $Y_2(t)$ in (2.9) define a *feedback interconnection* between the wave part and the Coleman–Gurtin part.

In the remaining part of this section, we shall use the results from infinite-dimensional systems theory in [28, 22, 30] to prove the block operator representation (2.6) of $\mathbb{A}$.

2.1. **Background on regular tuples and boundary nodes.** The operators appearing in (2.6) form "regular tuples" in the sense of Definition 2.5 below. Such operators are closely related to the theory *regular linear systems* [30], [29, Sec. 5].

Definition 2.4. Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space Z. An operator $B \in \mathcal{L}(\mathbb{C}^m, Z_{-1})$ is *admissible* [29, Rem. 3.3] with respect to $(T(t))_{t\geq 0}$ if there exists $\tau > 0$ such that

$$\int_0^\tau T_{-1}(t) B u(t)\, \mathrm{d}t \in Z, \qquad u \in L^2(0, \tau; \mathbb{C}^m).$$

Correspondingly, an operator $C \in \mathcal{L}(D(A), \mathbb{C}^m)$ is *admissible* [29, Rem. 3.4] with respect to $(T(t))_{t\geq 0}$ if there exist $\tau, \kappa > 0$ such that

$$\int_0^\tau \|CT(t)z\|_{\mathbb{C}^m}^2\, \mathrm{d}t \leq \kappa \|z\|_Z^2, \qquad z \in D(A).$$

Definition 2.5. Assume that $A : D(A) \subset Z \to Z$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space Z and that $B \in \mathcal{L}(\mathbb{C}^m, Z_{-1})$ and $C \in \mathcal{L}(D(A), \mathbb{C}^m)$ are admissible with respect to $(T(t))_{t\geq 0}$. Then the tuple (A, B, C, D) is said to be *regular* if $D \in \mathbb{C}^{m \times m}$, $\mathrm{Ran}(R(\lambda, A_{-1})B) \subset D(C_\Lambda)$ for some (or, equivalently, all) $\lambda \in \rho(A)$ and

$$\sup_{\mathrm{Re}\,\lambda \geq \sigma} \|C_\Lambda R(\lambda, A_{-1})B\|_{\mathbb{C}^m} < \infty$$

for some $\sigma \geq 0$. The *transfer function* P of the regular tuple (A, B, C, D) is defined by

$$P(\lambda) = C_\Lambda R(\lambda, A_{-1})B + D, \qquad \lambda \in \rho(A).$$

The regular tuple (A, B, C, D) is called *impedance passive* if

(2.10) $$\mathrm{Re}\langle A_{-1}z + BU, z\rangle_Z \leq \mathrm{Re}\langle C_\Lambda z + DU, U\rangle_{\mathbb{C}^m}$$

for all $U \in \mathbb{C}^m$ and $z \in Z^B$ satisfying $A_{-1}z + BU \in Z$.

Choosing $z \in D(A)$ and $U = 0 \in \mathbb{C}^m$ in (2.10) shows that the semigroup generated by A in an impedance passive regular tuple (A, B, C, D) is contractive.

Our aim is to relate the wave part (2.7) and the Coleman–Gurtin part (2.8) of our coupled PDE system to regular tuples (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), respectively. We shall do this by first formulating both of these PDEs as *abstract boundary control systems* [7, 22, 27] of the form

(2.11) $$\begin{cases} \dot{z}(t) = Lz(t), & t \geq 0, \\ Gz(t) = U(t), & t \geq 0, \\ Y(t) = Kz(t), & t \geq 0, \\ z(0) = z_0 \end{cases}$$

on a Hilbert space Z with $L : D(L) \subset Z \to Z$ and $K, G : D(L) \subset Z \to \mathbb{C}^m$. As is shown in Lemma 2.9 below, under suitable assumptions the operators A, B, C and D of the regular tuples exist and can be expressed in terms of L, G, and K. The benefit of using the framework of abstract boundary control systems is that (2.11) has a form which closely resembles both the wave part (2.7) and the Coleman–Gurtin part (2.8) with suitable choices of a differential operator $L : D(L) \subset Z \to Z$ and boundary trace operators $G, K : D(L) \subset Z \to \mathbb{C}$. We call (2.11) a *boundary control system* if the operator L, G and K form a *boundary node* defined as below.

Definition 2.6. The triple (G, L, K) in (2.11) is said to be an *(internally well-posed) boundary node* on the Hilbert spaces $(\mathbb{C}^m, Z, \mathbb{C}^m)$ (or sometimes, for short, on Z) if the linear operators $L : D(L) \subset Z \to Z$ and $G, K : D(L) \subset Z \to \mathbb{C}^m$ have the following properties:

  (a) The restriction $L|_{\mathrm{Ker}(G)} : \mathrm{Ker}(G) \subset Z \to Z$ generates a C_0-semigroup on Z;
  (b) $G, K \in \mathcal{L}(D(L), \mathbb{C}^m)$;
  (c) $\mathrm{Ran}(G) = \mathbb{C}^m$.

The boundary node is *impedance passive* if

$$\mathrm{Re}\langle Lz, z \rangle_Z \leq \mathrm{Re}\langle Gz, Kz \rangle_{\mathbb{C}^m}, \qquad z \in D(L).$$

The *transfer function* $P : \mathbb{C}_+ \to \mathbb{C}^{m \times m}$ of an impedance passive boundary node of the form (2.11) is defined so that, for $\lambda \in \mathbb{C}_+$ and $U \in \mathbb{C}^m$,

$$P(\lambda)U = Kz,$$

where $z \in D(L)$ satisfies $(\lambda - L)z = 0$ and $Gz = U$.

Remark 2.7. Conditions (a) and (b) in Definition 2.6 imply that $\mathrm{Ker}(G)$ is a complete finite-codimensional subspace of $D(L)$ (equipped with the graph norm of L). This in particular implies that $D(L)$ is a Hilbert space or, equivalently, that L is a closed operator. Moreover, $D(L)$ is densely and continuously embedded in Z.

Remark 2.8. In defining the transfer function of a boundary node, we do not distinguish between P in Definition 2.6 and its analytic extensions to domains containing $\mathbb{C}_+$. The existence and uniqueness of the solution $z \in D(L)$ of the 'abstract boundary value problem' $(\lambda - L)z = 0$ and $Gz = U$ for any $U \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}_+$ follow from [7, Thm. 2.9].

The next lemma collects results from [22, 28, 29, 30] to show how an impedance passive boundary node (G, L, K) on a Hilbert space Z gives rise to a regular tuple (A, B, C, D) on the same space.

Lemma 2.9. *Let (G, L, K) be an impedance passive boundary node on the Hilbert spaces $(\mathbb{C}^m, Z, \mathbb{C}^m)$. Assume that the transfer function P of the boundary node satisfies*

$$\sup_{s \in \mathbb{R}} \|P(\sigma + is)\| < \infty$$

for some $\sigma \geq 0$ and that $P(\lambda)$ converges to a limit as $\lambda \to \infty$ through the positive reals. Then there exists an impedance passive regular tuple (A, B, C, D) on Z such that

$$A = L|_{\mathrm{Ker}(G)},$$
$$Lz = A_{-1}z + BGz,$$
$$Kz = C_{\Lambda}z + DGz,$$
$$D = \lim_{\substack{\lambda \to \infty \\ \lambda > 0}} P(\lambda) \in \mathbb{C}^{m \times m}$$

for $z \in D(L)$. Furthermore, $Z^B = D(L)$, $\mathrm{Ran}(B) \cap Z = \{0\}$ and P coincides with the transfer function of the regular tuple (A, B, C, D) on $\rho(A) \cap \mathbb{C}_+$.

Proof. By [22, Thm. 2.3 and Prop. 2.5], the boundary node (G, L, K) defines a 'system node' S_{node} in the sense of [22, Def. 2.1] or [28, Def. 2.1]. By definition, the system node S_{node} is a linear block operator

$$S_{\mathrm{node}} = \begin{pmatrix} A\&B \\ C\&D \end{pmatrix} : D(S_{\mathrm{node}}) \subset Z \times \mathbb{C}^m \to Z \times \mathbb{C}^m$$

with components $C\&D : D(S_{\mathrm{node}}) \subset Z \times \mathbb{C}^m \to \mathbb{C}^m$ and

$$A\&B : D(S_{\mathrm{node}}) \subset Z \times \mathbb{C}^m \to Z, \qquad A\&B \begin{pmatrix} z \\ U \end{pmatrix} = A_{-1}z + BU, \qquad \begin{pmatrix} z \\ U \end{pmatrix} \in D(S_{\mathrm{node}}),$$

where $A : D(A) \subset Z \to Z$ is the generator of a C_0-semigroup on Z and $B \in \mathcal{L}(\mathbb{C}^m, Z_{-1})$. The result [22, Thm. 2.3(ii)] in particular shows that $A = L|_{\mathrm{Ker}(G)}$ and that the 'control operator' $B \in \mathcal{L}(\mathbb{C}^m, Z_{-1})$ of the system node S_{node} satisfies $Lz = A_{-1}z + BGz$ for $z \in D(L)$. Moreover, by [22, Thm. 2.3(v)] we have $Z \cap \mathrm{Ran}(B) = \{0\}$ and, letting $\lambda_0 \in \rho(A) \cap \mathbb{C}_+$,

$$D(L) = D(A) + \mathrm{Ran}(R(\lambda_0, A_{-1})B) = Z^B,$$

while by [22, Thm. 2.3(iv)] the 'combined observation and feedthrough operator' $C\&D$ of S_{node} is given by

$$C\&D \begin{pmatrix} z \\ U \end{pmatrix} = Kz$$

for all $z \in D(L)$ satisfying $Gz = U$. This further implies that the 'observation operator' $C \in \mathcal{L}(D(A), \mathbb{C}^m)$ of S_{node} satisfies $Cz = Kz$ for all $z \in D(A) = \mathrm{Ker}(G)$. Moreover, the transfer function P_{node} of the system node [28, Def. 2.1] then has the form

$$(2.12) \qquad P_{\mathrm{node}}(\lambda)U = C\&D \begin{pmatrix} R(\lambda, A_{-1})BU \\ U \end{pmatrix} = KR(\lambda, A_{-1})BU$$

for all $U \in \mathbb{C}^m$ and $\lambda \in \rho(A) \cap \mathbb{C}_+$. But if we write $z = R(\lambda, A_{-1})BU \in Z^B = D(L)$ then [22, Thm. 2.3(v)] implies that $Gz = GR(\lambda, A_{-1})BU = U$, and thus

$$(\lambda - L)z = (\lambda - A_{-1})z - BGz = (\lambda - A_{-1})R(\lambda, A_{-1})BU - BU = 0.$$

This shows that in fact $P_{\mathrm{node}}(\lambda)U = Kz = P(\lambda)U$ for all $U \in \mathbb{C}^m$ and $\lambda \in \rho(A) \cap \mathbb{C}_+$. By [28, Thm. 4.2] the system node S_{node} is impedance passive if (and only if)

$$\mathrm{Re}\langle A_{-1}z + BU, z\rangle_Z \leq \mathrm{Re}\left\langle C\&D \begin{pmatrix} z \\ U \end{pmatrix}, U \right\rangle_{\mathbb{C}^m}$$

for all $z \in D(L)$ and $U \in \mathbb{C}^m$ satisfying $Gz = U$. This property holds since for any $z \in D(L)$ and $U \in \mathbb{C}^m$ such that $Gz = U$ we have $A_{-1}z + BU = A_{-1}z + BGz = Lz$ and $C\&D(z, U)^T = Kz$, and thus

$$(2.13) \quad \mathrm{Re}\langle A_{-1}z + BU, z\rangle_Z = \mathrm{Re}\langle Lz, z\rangle_Z \leq \mathrm{Re}\langle Kz, Gz\rangle_{\mathbb{C}^m} = \mathrm{Re}\left\langle C\&D\begin{pmatrix} z \\ U \end{pmatrix}, U\right\rangle_{\mathbb{C}^m}$$

by impedance passivity of the boundary node. Furthermore, our assumption that P (and thus also P_{node}) is uniformly bounded on a vertical line in $\overline{\mathbb{C}_+}$ together with [28, Thm. 5.1] shows that S_{node} is well-posed in the sense of [28, Def. 2.1] (or [29, Def. 4.4]). In particular, the operators $B \in \mathcal{L}(\mathbb{C}^m, Z_{-1})$ and $C \in \mathcal{L}(D(A), \mathbb{C}^m)$ are admissible with respect to the semigroup generated by A [29, Prop. 4.9].

Our assumption that $P(\lambda)$ converges to a well-defined limit as $\lambda \to \infty$ with $\lambda > 0$ together with [29, Thm. 5.6] (or [30, Thm. 5.8]) implies that the system node S_{node} is 'regular' in the sense of [29, Def. 5.2]. If we define $D = \lim_{\lambda \to \infty, \lambda > 0} P(\lambda)$, then [29, Thm. 5.5] shows that $\mathrm{Ran}(R(\lambda, A_{-1})B) \subset D(C_\Lambda)$ and the transfer function P has the form

$$P(\lambda) = C_\Lambda R(\lambda, A_{-1})B + D, \qquad \lambda \in \mathbb{C}_+.$$

Thus (A, B, C, D) is regular in the sense of Definition 2.5. Finally, let $z \in D(L) = Z^B$ be arbitrary. Then there exist $z_0 \in D(A) = \mathrm{Ker}(G)$, $\lambda_0 \in \rho(A) \cap \mathbb{C}_+$ and $U \in \mathbb{C}^m$ such that $z = z_0 + R(\lambda_0, A_{-1})BU$. By [22, Thm. 2.3(v)] we have $GR(\lambda_0, A_{-1})B = I$, and hence $Gz = U$. Now a direct computation using $Cz_0 = Kz_0$ and (2.12) shows that

$$\begin{aligned} Kz &= Kz_0 + KR(\lambda_0, A_{-1})BU = Cz_0 + P(\lambda_0)U \\ &= C_\Lambda z_0 + C_\Lambda R(\lambda_0, A_{-1})BU + DU \\ &= C_\Lambda z + DGz; \end{aligned}$$

see also [30, Rem. 4.11]. Since the same computation also shows that

$$C\&D\begin{pmatrix} z \\ U \end{pmatrix} = Kz = C_\Lambda z + DU$$

for $z \in D(L)$ and $U \in \mathbb{C}^m$ satisfying $Gz = U$, the estimate in (2.13) implies that the regular tuple (A, B, C, D) is impedance passive. $\square$

Remark 2.10. In the study of our wave-heat system we shall require only the case $m = 1$ of the general framework set out above. However, as already mentioned in the Introduction, the same framework with $m > 1$ can be used in an analogous way to analyse more complicated coupled systems, such as for instance the wave-heat-wave system.

2.2. **The wave-part.** We now show that the wave part (2.7) can be written in the form (2.11) for some operators G_1, L_1 and K_1 defining a boundary node, and that this representation also defines a regular tuple (A_1, B_1, C_1, D_1) via Lemma 2.9. Boundary control systems and regular tuples associated with one-dimensional and multidimensional wave equations are rather well understood; see for instance [22, Sec. 5], [29, Ex. 5.8] as well as [19, 21, 34]. To prove this property for the wave part, we begin by identifying the

operators L_1, G_1, and K_1 of the boundary node (G_1, L_1, K_1). We can write (2.7) as a first order system

$$\begin{cases} \begin{pmatrix} u_t(x,t) \\ v_t(x,t) \end{pmatrix} = \begin{pmatrix} v(x,t) \\ u_{xx}(x,t) \end{pmatrix}, & x \in (-1,0), \ t > 0, \\ v(0,t) = U_1(t), \quad Y_1(t) = u_x(0,t), \quad u(-1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (-1,0). \end{cases}$$

If, for $t \geq 0$, we consider $z_1(t) = (u(\cdot, t), v(\cdot, t))^T$ to be the state of an abstract differential equation of the form (2.11) on the Hilbert space $Z_1 = H_l^1(-1,0) \times L^2(-1,0)$, then natural choices for the operators G_1, L_1 and K_1 of the boundary node (G_1, L_1, K_1) are

$$L_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}, \qquad D(L_1) = (H^2(-1,0) \cap H_l^1(-1,0)) \times H_l^1(-1,0),$$

$$G_1 \begin{pmatrix} u \\ v \end{pmatrix} = v(0) \qquad \text{and} \qquad K_1 \begin{pmatrix} u \\ v \end{pmatrix} = u'(0)$$

for all $(u,v)^T \in D(L_1)$. In particular, the boundary condition at $x = -1$ is part of the definition of $D(L_1)$, and the condition at $x = 0$ is determined by G_1.

Proposition 2.11. *The tuple (G_1, L_1, K_1) is an impedance passive boundary node on $(\mathbb{C}, Z_1, \mathbb{C})$ and defines an impedance passive regular tuple (A_1, B_1, C_1, D_1). In particular, $D_1 = 1 \in \mathbb{C}$ and the operator*

$$A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}, \quad D(A_1) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in H^2(-1,0) \times H_l^1(-1,0) \ \middle|\ u(-1) = v(0) = 0 \right\}$$

is skew-adjoint with compact resolvent. The spectrum of A_1 consists of simple eigenvalues, namely $\sigma(A_1) = \{ ik\pi \mid k \in \mathbb{Z} \}$. Writing $\{ \psi_k \mid k \in \mathbb{Z} \}$ for the corresponding set of orthonormal eigenvectors, the operator C_1 satisfies $|C_1 \psi_k| = 1$ for all $k \in \mathbb{Z}$.

Proof. It is easy to show that the restriction $A_1 = L_1|_{\mathrm{Ker}(G_1)} : D(A_1) \subset Z_1 \to Z_1$ with the above domain is skew-adjoint and has compact resolvent. In particular, A_1 generates a unitary group on Z_1. It is also straightforward to show that $G_1, K_1 \in \mathcal{L}(D(L_1), \mathbb{C})$, and certainly $\mathrm{Ran}(G_1) = \mathbb{C}$. Thus (G_1, L_1, K_1) is a boundary node on $(\mathbb{C}, Z_1, \mathbb{C})$ in the sense of Definition 2.6. If $z = (u,v)^T \in D(L_1)$ then using $v(-1) = 0$ we readily see, using integration by parts, that

$$\mathrm{Re}\langle L_1 z, z \rangle_{Z_1} = \mathrm{Re}\langle v(0), u'(0) \rangle_{\mathbb{C}} = \mathrm{Re}\langle G_1 z, K_1 z \rangle_{\mathbb{C}}.$$

Thus (G_1, L_1, K_1) is impedance passive.

To show that the wave part also defines a regular tuple, we compute the transfer function P_1 of the boundary node (G_1, L_1, K_1). By definition, if $\lambda \in \mathbb{C}_+$ then $P_1(\lambda) = K_1 z$, where $z = (u,v)^T \in D(L_1)$ is such that

$$\begin{cases} (\lambda - L_1)z = 0 \\ \qquad G_1 z = 1 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \lambda u(x) = v(x), & x \in (-1,0), \\ \lambda v(x) = u''(x), & x \in (-1,0), \\ u(-1) = 0, \quad v(0) = 1. \end{cases}$$

We have

$$u(x) = \frac{\sinh(\lambda(x+1))}{\lambda \sinh(\lambda)},$$

and hence $P_1(\lambda) = u'(0) = \coth(\lambda)$. Since $\sup_{\tau \in \mathbb{R}} |\coth(1 + i\tau)| < \infty$ and $P_1(\lambda) \to 1$ as $\lambda \to \infty$ with $\lambda > 0$, Lemma 2.9 shows that the wave part defines an impedance passive regular tuple (A_1, B_1, C_1, D_1) on Z_1 and that $D_1 = \lim_{\lambda \to \infty} P_1(\lambda) = 1$.

The eigenvalues of A_1 are $ik\pi$ for $k \in \mathbb{Z}$, and the corresponding orthonormal eigenvectors are given by

$$\psi_0 = \begin{pmatrix} x+1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi_k = \frac{1}{k\pi} \begin{pmatrix} \sin(k\pi(x+1)) \\ ik\pi \sin(k\pi(x+1)) \end{pmatrix} \qquad \text{for } k \neq 0.$$

Since $\{\, \psi_k \mid k \in \mathbb{Z} \,\} \subset D(A_1) = \mathrm{Ker}(G_1)$, we have $C_1\psi_k = K_1\psi_k = (-1)^k$ for all $k \in \mathbb{Z}$. $\quad\square$

2.3. Coleman–Gurtin part.

As our next step we show that the Coleman–Gurtin part, too, defines an impedance passive regular tuple. Based on the structure (2.8) we may consider $z_2(t) = (w(\cdot, t), \eta^t(\cdot, \cdot))^T$ for $t \geq 0$ to be the state of the boundary node on the Hilbert space $Z_2 = L^2(0,1) \times \mathcal{M}$, and we may choose the operators $L_2 : D(L_2) \subset Z_2 \to Z_2$ and $G_2, L_2 : D(L_2) \subset Z_2 \to \mathbb{C}$ as

$$L_2\begin{pmatrix} w \\ \eta \end{pmatrix} = \begin{pmatrix} \phi'' \\ T\eta + w \end{pmatrix}, \qquad D(L_2) = \left\{ \begin{pmatrix} w \\ \eta \end{pmatrix} \in H_r^1(0,1) \times D(T) \,\middle|\, \phi \in H^2(0,1) \right\},$$

$$G_2\begin{pmatrix} w \\ \eta \end{pmatrix} = -\phi'(0) \qquad \text{and} \qquad K_2\begin{pmatrix} w \\ \eta \end{pmatrix} = w(0)$$

for all $(w, \eta)^T \in D(L_2)$.

Proposition 2.12. *The tuple (G_2, L_2, K_2) is an impedance passive boundary node on $(\mathbb{C}, Z_2, \mathbb{C})$ and defines an impedance passive regular tuple (A_2, B_2, C_2, D_2). In particular, $D_2 = 0$ and the transfer function P_2 of the regular tuple is given by*

$$P_2(\lambda) = \frac{\tanh\sqrt{\lambda/\ell(\lambda)}}{\sqrt{\lambda\ell(\lambda)}}, \qquad \lambda \in \mathbb{C}_+,$$

where $\ell : \overline{\mathbb{C}_+} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\ell(\lambda) = 1 + \frac{1}{\lambda} \int_0^\infty \mu(s)(1 - e^{-\lambda s})\, ds.$$

The proof of Proposition 2.12 requires the following lemma.

Lemma 2.13. *The operator $A_2 = L_2|_{\mathrm{Ker}(G_2)}$ satisfies $\mathrm{Ran}(I - A_2) = Z_2$.*

Proof. We begin by showing that, for every $\hat{\eta} \in \mathcal{M}$, the function $\hat{\xi}$ defined by

$$\hat{\xi}(x, s) = \int_0^s e^{-(s-\sigma)}\hat{\eta}(x, \sigma)\, d\sigma, \qquad x \in (0,1), \ s > 0,$$

belongs to $\mathcal{M}$ and satisfies the estimate $\|\hat{\xi}\|_{\mathcal{M}} \leq \|\hat{\eta}\|_{\mathcal{M}}$. To this end, we introduce the auxiliary function K defined by

$$K(s) = \int_0^s e^{-(s-\sigma)}\sqrt{\mu(\sigma)}\|\hat{\eta}(\sigma)\|_{H_r^1(0,1)}\, d\sigma, \qquad s > 0.$$

By Young's inequality for convolutions we have $\|K\|_{L^2(0,\infty)} \leq \|\hat{\eta}\|_{\mathcal{M}}$. Hence, using the monotonicity of μ, we obtain $\|\hat{\xi}\|_{\mathcal{M}} \leq \|K\|_{L^2(0,\infty)} \leq \|\hat{\eta}\|_{\mathcal{M}}$, as desired.

Next, for an arbitrarily given $\hat{z} = (\hat{w}, \hat{\eta})^T \in Z_2$, we consider the equation $(I - A_2)z = \hat{z}$ in the unknown $z = (w, \eta)^T \in D(A_2)$. Componentwise, we get the system

(2.14)
$$\begin{cases} w(x) - \phi''(x) = \hat{w}(x), & x \in (0,1), \\ \eta(x,s) - T\eta(x,s) - w(x) = \hat{\eta}(x,s), & x \in (0,1), \ s > 0. \end{cases}$$

Integrating the second identity and using $\eta(x,0) = 0$, we find

(2.15)
$$\eta(x,s) = (1 - e^{-s})w(x) + \hat{\xi}(x,s), \qquad x \in (0,1), \ s > 0.$$

From the definition of ϕ we infer that $w(x) = \phi(x)/\ell(1) - \hat{\varrho}(x)$, where

$$\hat{\varrho}(x) = \frac{1}{\ell(1)} \int_0^\infty \mu(s)\hat{\xi}(x,s)\,ds, \qquad x \in (0,1).$$

Substituting into the first equation in (2.14), we arrive at

$$\frac{\phi(x)}{\ell(1)} - \phi''(x) = \hat{w}(x) + \hat{\varrho}(x), \qquad x \in (0,1).$$

The general solution of this equation above with the boundary condition $\phi(1) = 0$ (coming from the fact that $w(1) = \eta(1,s) = 0$) can be written as

(2.16)
$$\phi(x) = b \sinh\left(\frac{1-x}{\sqrt{\ell(1)}}\right) - \Phi(x), \qquad x \in (0,1),$$

where $b \in \mathbb{C}$ and

$$\Phi(x) = \sqrt{\ell(1)} \int_x^1 \sinh\left(\frac{r-x}{\sqrt{\ell(1)}}\right)(\hat{w}(r) + \hat{\varrho}(r))\,dr, \qquad x \in (0,1).$$

Accordingly, we have

(2.17)
$$w(x) = \frac{b}{\ell(1)} \sinh\left(\frac{1-x}{\sqrt{\ell(1)}}\right) - \frac{\Phi(x)}{\ell(1)} - \hat{\varrho}(x), \qquad x \in (0,1).$$

We now claim that $\phi \in H^2(0,1)$ and $w \in H_r^1(0,1)$. By (2.16) and (2.17), the claim follows provided that $\hat{\varrho} \in H_r^1(0,1)$. But the latter is true, since

$$\|\hat{\varrho}\|_{H_r^1(0,1)} \leq \frac{\sqrt{\kappa}}{\ell(1)} \|\hat{\xi}\|_{\mathcal{M}} \leq \frac{\sqrt{\kappa}}{\ell(1)} \|\hat{\eta}\|_{\mathcal{M}}$$

by our earlier estimate, where $\kappa = \int_0^\infty \mu(s)\,ds$ denotes the total mass of μ. Next we show that the function η given by (2.15) belongs to $\mathcal{M}$. Since we already know that $\hat{\xi} \in \mathcal{M}$, we only need to prove that the map $s \mapsto (1 - e^{-s})w$ lies in $\mathcal{M}$. But this follows from the estimate

$$\int_0^\infty \mu(s)|1 - e^{-s}|^2 \|w\|_{H_r^1(0,1)}^2 \, ds \leq \kappa \|w\|_{H_r^1(0,1)}^2.$$

Since $\eta, \hat{\eta}, w \in \mathcal{M}$, we also have $\eta_s = \hat{\eta} + w - \eta \in \mathcal{M}$. Finally, by monotonicity of μ,

$$\|\eta(s)\|_{H_r^1(0,1)} \leq |1 - e^{-s}| \|w\|_{H_r^1(0,1)} + \frac{e^s}{\mu(s)} \left(\int_0^s \mu(\sigma)\,d\sigma\right)^{\frac{1}{2}} \|\hat{\eta}\|_{\mathcal{M}} \to 0$$

as $s \to 0$, and we have thus proved that $\eta \in D(T)$.

It remains only to show that the constant b in (2.16) may be chosen in such a way that $\phi'(0) = 0$, but a straightforward calculation yields

$$b = \left[\frac{1}{\sqrt{\ell(1)}} \cosh\left(\frac{1}{\sqrt{\ell(1)}}\right) \right]^{-1} \int_0^1 \cosh\left(\frac{r}{\sqrt{\ell(1)}}\right) \left(\hat{w}(r) + \hat{\varrho}(r)\right) \mathrm{d}r,$$

and this completes the proof.   □

Proof of Proposition 2.12. We begin by showing that (G_2, L_2, K_2) is an impedance passive boundary node in the sense of Definition 2.6. We note first that $\mathrm{Ran}(G_2) = \mathbb{C}$. If $z = (w, \eta)^T \in D(L_2)$, then using $w(1) = 0$ and (2.1) we readily get

$$\mathrm{Re}\langle L_2 z, z \rangle_{Z_2} = \mathrm{Re}\langle w(0), -\phi'(0) \rangle_{\mathbb{C}} - \|w\|^2_{H^1_r(0,1)} + \mathrm{Re}\langle T\eta, \eta \rangle_{\mathcal{M}}$$
$$\leq \mathrm{Re}\langle w(0), -\phi'(0) \rangle_{\mathbb{C}} = \mathrm{Re}\langle G_2 z, K_2 z \rangle_{\mathbb{C}}.$$

This estimate already shows that (G_2, L_2, K_2) is impedance passive if it is a boundary node in the sense of Definition 2.6. Moreover, the same estimate shows that $\mathrm{Re}\langle L_2 z, z \rangle \leq 0$ for $z \in \mathrm{Ker}(G_2)$, and thus the restriction $A_2 = L_2|_{\mathrm{Ker}(G_2)}$ is dissipative. By Lemma 2.13 we also have $\mathrm{Ran}(I - A_2) = Z_2$, and therefore A_2 generates a contraction semigroup on Z_2 by the Lumer–Phillips theorem. In order to prove that (G_2, L_2, K_2) is an impedance passive boundary node it remains to verify that $G_2, K_2 \in \mathcal{L}(D(L_2), \mathbb{C})$. Recall that the norm on $D(L_2)$ is taken to be the graph norm of L_2, that is to say

$$\|z\|^2_{D(L_2)} = \|L_2 z\|^2_{Z_2} + \|z\|^2_{Z_2} = \|\phi''\|^2_{L^2(0,1)} + \|T\eta + w\|^2_{\mathcal{M}} + \|w\|^2_{L^2(0,1)} + \|\eta\|^2_{\mathcal{M}},$$

for $z = (w, \eta)^T \in D(L_2) \subset Z_2$. Note first that

$$|G_2 z| = |\phi'(0)| \lesssim \|\phi'\|_{H^1(0,1)} \lesssim \|\phi'\|_{L^2(0,1)} + \|z\|_{D(L_2)}.$$

By interpolation and an application of Young's inequality we have

$$\|\phi'\|_{L^2(0,1)} \lesssim \|\phi\|_{L^2(0,1)} + \|z\|_{D(L_2)},$$

and hence $|G_2 z| \lesssim \|\phi\|_{L^2(0,1)} + \|z\|_{D(L_2)}$. In order to show that $G_2 \in \mathcal{L}(D(L_2), \mathbb{C})$, it remains to control the term $\|\phi\|_{L^2(0,1)}$. To this end we observe that, by definition of ϕ,

$$\|\phi\|_{L^2(0,1)} \lesssim \|z\|_{D(L_2)} + \int_0^\infty \mu(s)\|\eta(s)\|_{H^1_r(0,1)} \, \mathrm{d}s \lesssim \|z\|_{D(L_2)} + \|\eta\|_{\mathcal{M}} \lesssim \|z\|_{D(L_2)},$$

and hence boundedness of G_2 follows. In order to show that $K_2 \in \mathcal{L}(D(L_2), \mathbb{C})$ we first note that, since $w(1) = 0$, we have $|K_2 z| = |w(0)| \lesssim \|w\|_{H^1_r(0,1)}$. Next, the definition of ϕ implies

$$\|w\|_{H^1_r(0,1)} \lesssim \|\phi'\|_{L^2(0,1)} + \|\eta\|_{\mathcal{M}} \lesssim \|\phi'\|_{L^2(0,1)} + \|z\|_{D(L_2)}.$$

The term $\|\phi'\|_{L^2(0,1)}$ can be estimated as before, and the boundedness of K_2 follows. Thus (G_2, L_2, K_2) is a boundary node on $(\mathbb{C}, Z_2, \mathbb{C})$ in the sense of Definition 2.6.

The transfer function P_2 of the boundary node is defined, for $\lambda \in \mathbb{C}_+$, by $P_2(\lambda) = K_2 z$, where $z = (w, \eta)^T \in D(L_2)$ solves the problem $(\lambda - L_2)z = 0$ and $G_2 z = 1$. Arguing as in

the proof of Lemma 2.13, the first component of z can be written as

$$w(x) = \frac{\sinh\left(\sqrt{\lambda/\ell(\lambda)}(1-x)\right)}{\sqrt{\lambda\ell(\lambda)}\cosh\sqrt{\lambda/\ell(\lambda)}}, \qquad x \in (0,1),$$

and hence

$$P_2(\lambda) = w(0) = \frac{\tanh\sqrt{\lambda/\ell(\lambda)}}{\sqrt{\lambda\ell(\lambda)}}, \qquad \lambda \in \mathbb{C}_+.$$

We now show that $P_2(\lambda) \to 0$ uniformly in $\operatorname{Im}\lambda$ as $\operatorname{Re}\lambda \to \infty$. This implies in particular that $P_2(\lambda) \to 0$ as $\lambda \to \infty$ through the reals and that there exists $\sigma > 0$ such that $\sup_{s\in\mathbb{R}}|P_2(\sigma + is)| < \infty$. Thus by Lemma 2.9 the Coleman–Gurtin part defines an impedance passive regular tuple (A_2, B_2, C_2, D_2) with $D_2 = \lim_{\lambda\to\infty} P_2(\lambda) = 0$. In order to prove the required uniform decay estimate, observe first that $\ell(\lambda) = 1 + O(|\lambda|^{-1})$ and hence $\lambda\ell(\lambda) = \lambda + O(1)$ and $\lambda/\ell(\lambda) = \lambda + O(1)$ as $|\lambda| \to \infty$ in the right half-plane. In particular, $|\lambda\ell(\lambda)| \geq \frac{1}{2}\operatorname{Re}\lambda$ for $\operatorname{Re}\lambda$ sufficiently large. Moreover,

$$\operatorname{Re}\sqrt{\lambda/\ell(\lambda)} = \operatorname{Re}\sqrt{\lambda} + O(1)$$

as $|\lambda| \to \infty$ with $\lambda \in \mathbb{C}_+$, and for $\lambda \in \mathbb{C}_+$ we have $\operatorname{Re}\sqrt{\lambda} \geq \sqrt{\operatorname{Re}\lambda}$. It follows that $\operatorname{Re}\sqrt{\lambda/\ell(\lambda)} \geq \frac{1}{2}\sqrt{\operatorname{Re}\lambda}$ for $\operatorname{Re}\lambda$ sufficiently large. Thus

$$|P_2(\lambda)| \leq \frac{1}{|\sqrt{\lambda\ell(\lambda)}|}\left|1 - \frac{2}{e^{2\sqrt{\lambda/\ell(\lambda)}} + 1}\right| \leq \sqrt{\frac{2}{\operatorname{Re}\lambda}}\left(1 + \frac{2}{e^{\sqrt{\operatorname{Re}\lambda}} - 1}\right)$$

when $\operatorname{Re}\lambda$ is sufficiently large, and the claim follows. $\qquad\square$

2.4. **Proof of Theorem 2.2.** Theorem 2.2 is an immediate corollary of the following more detailed proposition.

Proposition 2.14. *The operator $\mathbb{A}$ has the form*

$$(2.18) \qquad \mathbb{A} = \begin{pmatrix} A_{1,-1} & B_1 C_{2\Lambda} \\ -B_2 C_{1\Lambda} & A_{2,-1} - B_2 D_1 C_{2\Lambda} \end{pmatrix},$$

$$(2.19) \qquad D(\mathbb{A}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z_1^{B_1} \times Z_2^{B_2} \,\middle|\, \begin{array}{l} A_{1,-1}z_1 + B_1 C_{2\Lambda}z_2 \in Z_1, \\ A_{2,-1}z_2 - B_2(C_{1\Lambda}z_1 + D_1 C_{2\Lambda}z_2) \in Z_2 \end{array} \right\},$$

where (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are the impedance passive regular tuples associated to the wave part and the Coleman–Gurtin part, respectively. The operator $\mathbb{A}$ generates a contraction semigroup on the space $\mathcal{H}$.

Proof. By definition, we have $\mathcal{H} = Z_1 \times Z_2$. Let (G_1, L_1, K_1) and (G_2, L_2, K_2) be the boundary nodes associated to the wave part and the Coleman–Gurtin part, respectively, as defined in Propositions 2.11 and 2.12. If we write $z_1 = (u, v)^T$ and $z_2 = (w, \eta)^T$, the operator $\mathbb{A}$ and its domain may be written as

$$\mathbb{A}\begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad D(\mathbb{A}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in D(L_1) \times D(L_2) \,\middle|\, \begin{array}{l} G_1 z_1 = K_2 z_2, \\ G_2 z_2 = -K_1 z_1 \end{array} \right\}.$$

Since $D_2 = 0$, Lemma 2.9 implies that $L_1 = A_{1,-1} + B_1 G_1$, $K_1 = C_{1\Lambda} + D_1 G_1$ on $D(L_1)$, and $L_2 = A_{2,-1} + B_2 G_2$, $K_2 = C_{2\Lambda}$ on $D(L_2)$. It follows that, for $(z_1, z_2)^T \in D(\mathbb{A})$, $G_1 z_1 = K_2 z_2 = C_{2\Lambda} z_2$ and

$$G_2 z_2 = -K_1 z_1 = -C_{1\Lambda} z_1 - D_1 G_1 z_1 = -C_{1\Lambda} z_1 - D_1 C_{2\Lambda} z_2,$$

and hence

$$L_1 z_1 = A_{1,-1} z_1 + B_1 G_1 z_1 = A_{1,-1} z_1 + B_1 C_{2\Lambda} z_2,$$
$$L_2 z_2 = A_{2,-1} z_2 + B_2 G_2 z_2 = (A_{2,-1} - B_2 D_1 C_{2\Lambda}) z_2 - B_2 C_{1\Lambda} z_1.$$

These formulas show that the operator $\mathbb{A}$ has the desired form (2.18) on $D(\mathbb{A})$ and that the inclusion "$\subset$" holds in (2.19). It therefore remains to show that the inclusion "$\supset$" holds in (2.19). To this end, assume that $(z_1, z_2)^T \in Z_1^{B_1} \times Z_2^{B_2}$ is such that $A_{1,-1} z_1 + B_1 C_{2\Lambda} z_2 \in Z_1$ and $A_{2,-1} z_2 - B_2(C_{1\Lambda} z_1 + D_1 C_{2\Lambda} z_2) \in Z_2$. Then $(z_1, z_2)^T \in D(L_1) \times D(L_2)$ by Lemma 2.9, so it suffices to show that $G_1 z_1 = K_2 z_2$ and $G_2 z_2 = -K_1 z_1$. Lemma 2.9 also implies that $Z_k \cap \mathrm{Ran}(B_k) = \{0\}$ for $k = 1, 2$. We have

$$Z_1 \ni A_{1,-1} z_1 + B_1 C_{2\Lambda} z_2 = L_1 z_1 - B_1 G_1 z_1 + B_1 C_{2\Lambda} z_2 = L_1 z_1 + B_1(-G_1 z_1 + K_2 z_2),$$

and since $L_1 z_1 \in Z_1$ we see that $G_1 z_1 = K_2 z_2$. Since $G_1 z_1 = K_2 z_2 = C_{2\Lambda} z_2$ and $C_{1\Lambda} z_1 + D_1 C_{2\Lambda} z_2 = K_1 z_1$ we find similarly that

$$Z_2 \ni A_{2,-1} z_2 - B_2(C_{1\Lambda} z_1 + D_1 C_{2\Lambda} z_2) = L_2 z_2 - B_2(G_2 z_2 + K_1 z_1),$$

which implies that $G_2 z_2 = -K_1 z_1$, as required.

Finally, since $\mathbb{A}$ has the form in Proposition 2.14 where (A_k, B_k, C_k, D_k) for $k = 1, 2$ are impedance passive regular tuples with $D_1 = 1 \geq 0$ and $D_2 = 0$, the operator $\mathbb{A}$ generates a contraction semigroup by [24, Lem. 4.2]. $\qquad\square$

3. Resolvent Estimates

We now study the behaviour of the resolvent operator $R(is, \mathbb{A})$ as $s \to \pm\infty$. In Section 3.1, we establish an asymptotic upper bound on $\|R(is, \mathbb{A})\|$, and then in Section 3.2 we shall show this upper bound to be optimal.

3.1. **Upper bound.** Our main result here is the following.

Theorem 3.1. *Assume that* (1.5) *holds. Then the operator $\mathbb{A}$ satisfies $\sigma(\mathbb{A}) \subset \mathbb{C}_-$ and*

$$\|R(is, \mathbb{A})\| = O(|s|^{1/2}), \qquad s \to \pm\infty.$$

The proof of Theorem 3.1 is based on the following abstract result from [24], which we state in the special case where A_1 has compact resolvent and the eigenvalues of A_1 have a uniform gap (but are not necessarily simple).

Theorem 3.2 ([24, Thm. 3.7])**.** *Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be impedance passive regular tuples on Z_1 and Z_2, respectively, with $\mathbb{C}^{m \times m} \ni D_1 \geq 0$ and $D_2 = 0 \in \mathbb{C}^{m \times m}$. Assume that A_1 is skew-adjoint with compact resolvent and spectrum $\sigma(A_1) = \{is_k \mid k \in \mathbb{Z}\}$, that the eigenvalues of A_1 satisfy $\inf_{k \neq l} |s_k - s_l| > 0$, and that the semigroup generated by A_2 is exponentially stable. In addition, assume that there exists a constant*

$\gamma_0 > 0$ *such that* $\|C_1 z\|_{\mathbb{C}^m} \geq \gamma_0 \|z\|_{Z_1}$ *for all* $z \in \mathrm{Ker}(is_k - A_1)$ *and* $k \in \mathbb{N}$, *and that there exists a non-increasing function* $\nu : \mathbb{R}_+ \to (0, 1]$ *such that*

$$\mathrm{Re}\langle P_2(is)U, U \rangle_{\mathbb{C}^m} \geq \nu(|s|)\|U\|_{\mathbb{C}^m}^2, \qquad U \in \mathbb{C}^m, \ s \in \mathbb{R}.$$

Then the block operator $\mathbb{A}$ *defined by*

$$\mathbb{A} = \begin{pmatrix} A_{1,-1} & B_1 C_{2\Lambda} \\ -B_2 C_{1\Lambda} & A_{2,-1} - B_2 D_1 C_{2\Lambda} \end{pmatrix},$$

$$D(\mathbb{A}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z_1^{B_1} \times Z_2^{B_2} \left| \begin{array}{l} A_{1,-1}z_1 + B_1 C_{2\Lambda}z_2 \in Z_1, \\ A_{2,-1}z_2 - B_2(C_{1\Lambda}z_1 + D_1 C_{2\Lambda}z_2) \in Z_2 \end{array} \right. \right\},$$

satisfies $i\mathbb{R} \subset \rho(\mathbb{A})$ *and*

$$\|R(is, \mathbb{A})\| = O(\nu(|s|)^{-1}), \qquad s \to \pm\infty.$$

Proof. The theorem was proved in more general form in [24, Thm. 3.7]. The regular tuples (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) correspond to (A_c, B_c, C_c, D_c) and (A, B, C, D) in [24], respectively. Similarly, the transfer functions P_1 and P_2 correspond to the transfer functions G and P in [24]. The current statement follows from [24, Thm. 3.7] if we let $\Omega_\varepsilon = \mathbb{R}$, in which case the required condition (2) of [24, Thm. 3.5] is trivially satisfied. The non-increasing function ν corresponds to the function η in [24, Thm. 3.7]. Moreover, as explained in [24, Rem. 3.8], the assumption that the eigenvalues of A_1 satisfy the uniform gap condition $\inf_{k \neq l} |s_k - s_l| > 0$ implies that it is possible to choose δ and γ in [24, Thm. 3.7] to be constant functions. With these choices [24, Thm. 3.7] implies that $i\mathbb{R} \subset \rho(\mathbb{A})$ and there exists a constant $M > 0$ such that $\|R(is, \mathbb{A})\| \leq M/\nu(|s|)$ for all $s \in \mathbb{R}$. $\qquad \square$

We begin by showing that the semigroup generated by the operator $A_2 = L_2|_{\mathrm{Ker}(G_2)} : D(A_2) \subset Z_2 \to Z_2$ introduced in Section 2.3 is exponentially stable whenever condition (1.5) is satisfied.

Lemma 3.3. *If condition* (1.5) *holds, the contraction semigroup generated by* A_2 *is exponentially stable.*

Proof. Recall that A_2 is the infinitesimal generator of a contraction semigroup on $Z_2 = L^2(0,1) \times \mathcal{M}$, as was shown in the proof of Proposition 2.12. We prove that $i\mathbb{R} \subset \rho(A_2)$ and $\sup_{r \in \mathbb{R}} \|R(ir, A_2)\| < \infty$. The claim then follows from the Gearhart-Prüss theorem [16, Thm. V.1.11]. To this end we begin by introducing the space $\mathcal{N} = L_g^2(0, \infty; H_r^1(0,1))$ of $H_r^1(0,1)$-valued functions on $(0, \infty)$ which are square-integrable with respect to the measure $g(s)\mathrm{d}s$, endowed with the inner product

$$\langle \eta, \xi \rangle_{\mathcal{N}} = \int_0^\infty g(s)\langle \eta(s), \xi(s) \rangle_{H_r^1(0,1)} \, \mathrm{d}s.$$

By [17, Rem. 2.3], condition (1.5) is equivalent to the estimate $g(s) \leq \Theta\mu(s)$ for some $\Theta > 0$ and all $s > 0$. It follows that

(3.1) $$\|\eta\|_{\mathcal{N}} \leq \sqrt{\Theta}\|\eta\|_{\mathcal{M}}, \qquad \eta \in \mathcal{M},$$

and hence $\mathcal{M} \subset \mathcal{N}$ with continuous inclusion. Next, given $\hat{z} = (\hat{w}, \hat{\eta})^T \in Z_2$ and $r \in \mathbb{R}$, we consider the resolvent equation $(ir - A_2)z = \hat{z}$ in the unknown $z = (w, \eta)^T \in D(A_2)$.

Taking the inner product of this equation with z in Z_2 and using $w(1) = \phi'(0) = 0$ and (2.1), we get

$$\|w\|^2_{H^1_r(0,1)} - \frac{1}{2}\int_0^\infty \mu'(s)\|\eta(s)\|^2_{H^1_r(0,1)}\,\mathrm{d}s = \mathrm{Re}\langle(ir - A_2)z, z\rangle_{Z_2} = \mathrm{Re}\langle\hat{z}, z\rangle_{Z_2}.$$

Since $\mu'(s) \leq 0$ for almost all $s > 0$, we have

$$(3.2) \qquad\qquad \|w\|^2_{L^2(0,1)} \leq \|w\|^2_{H^1_r(0,1)} \leq \|z\|_{Z_2}\|\hat{z}\|_{Z_2}.$$

The resolvent equation may be rewritten in component form as

$$\begin{cases} irw(x) - \phi''(x) = \hat{w}(x), & x \in (0,1), \\ ir\eta(x,s) - T\eta(x,s) - w(x) = \hat{\eta}(x,s), & x \in (0,1), \ s > 0. \end{cases}$$

Recalling that $\mathcal{M} \subset \mathcal{N}$, we may take the inner product in $\mathcal{N}$ of the second equation above with η. Taking the real part of the resulting expression, we obtain

$$(3.3) \qquad\qquad -\mathrm{Re}\langle T\eta, \eta\rangle_{\mathcal{N}} = \mathrm{Re}\langle w, \eta\rangle_{\mathcal{N}} + \mathrm{Re}\langle\hat{\eta}, \eta\rangle_{\mathcal{N}}.$$

Integrating by parts with respect to s and employing a limiting argument (cf. (2.1)) yields

$$-\mathrm{Re}\langle T\eta, \eta\rangle_{\mathcal{N}} = -\frac{1}{2}\int_0^\infty g'(s)\|\eta(s)\|^2_{H^1_r(0,1)}\,\mathrm{d}s = \frac{1}{2}\|\eta\|^2_{\mathcal{M}}.$$

Hence, (3.1), (3.2) and (3.3) imply that

$$\frac{1}{2}\|\eta\|^2_{\mathcal{M}} \leq \sqrt{\Theta}\|w\|_{H^1_r(0,1)}\|\eta\|_{\mathcal{M}} + \Theta\|\hat{\eta}\|_{\mathcal{M}}\|\eta\|_{\mathcal{M}} \leq \frac{1}{4}\|\eta\|^2_{\mathcal{M}} + 2\Theta\|z\|_{Z_2}\|\hat{z}\|_{Z_2},$$

and combining this with (3.2) we readily arrive at

$$(3.4) \qquad\qquad \|z\|_{Z_2} \leq (1 + 8\Theta)\|\hat{z}\|_{Z_2}.$$

The desired result now follows at once. Indeed, since A_2 the generator of a contraction semigroup on Z_2, we have $\mathbb{C}_+ \subset \rho(A_2)$. Hence $\sigma(A_2) \cap i\mathbb{R}$ is contained in the topological boundary of $\sigma(A_2)$, and thus in the approximate point spectrum of A_2. However, (3.4) shows that no purely imaginary number can be an approximate eigenvalue of A_2, since otherwise there would exist a sequence of unit vectors $z_n \in D(A_2)$ with $(ir - A_2)z_n \to 0$ in Z_2 as $n \to \infty$, which contradicts (3.4). It follows that $i\mathbb{R} \subset \rho(A_2)$, and now (3.4) yields the bound $\sup_{r\in\mathbb{R}}\|R(ir, A_2)\| \leq 1 + 8\Theta$. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By Propositions 2.11 and 2.12, (A_k, B_k, C_k, D_k) for $k = 1, 2$ are impedance passive regular tuples with $D_1 = 1 \in \mathbb{C}$ and $D_2 = 0 \in \mathbb{C}$. Moreover, by Proposition 2.11 the operator A_1 is skew-adjoint with compact resolvent and spectrum $\sigma(A_1) = \{ik\pi \mid k \in \mathbb{Z}\}$ consisting of simple eigenvalues. Furthermore, $|C_1\psi| = \|\psi\|$ for all $\psi \in \mathrm{Ker}(ik\pi - A_1)$ and $k \in \mathbb{Z}$. By Lemma 3.3 the semigroup generated by A_2 is exponentially stable. Due to the structure of $\mathbb{A}$ described in Theorem 2.2 we may derive the desired resolvent estimate using Theorem 3.2 provided we can find a non-increasing

function $\nu : \mathbb{R}_+ \to (0,1]$ such that $\operatorname{Re} P_2(is) \geq \nu(|s|)$ for all $s \in \mathbb{R}$. We shall show that there exists a constant $c_0 \in (0,1]$ such that

$$(3.5) \qquad \operatorname{Re} P_2(is) \geq \frac{c_0}{1 + |s|^{1/2}}, \qquad s \in \mathbb{R}.$$

We begin by observing that by exponential stability of the semigroup generated by A_2 the transfer function P_2 of the Coleman–Gurtin part (A_2, B_2, C_2, D_2) extends analytically across the imaginary axis and, in particular, satisfies

$$P_2(is) = \frac{\tanh \sqrt{is/\ell(is)}}{\sqrt{is\ell(is)}}, \qquad s \neq 0,$$

where we recall that, for $s \neq 0$,

$$(3.6) \qquad \ell(is) = 1 + \frac{1}{is} \int_0^\infty \mu(r)(1 - e^{-isr}) \, \mathrm{d}r.$$

Integration by parts yields

$$(3.7) \qquad \ell(is) = 1 + \int_0^\infty g(r)e^{-isr} \, \mathrm{d}r.$$

This expression shows in particular that we may indeed define $\ell(is)$ and hence $P_2(is)$ in a natural way also for $s = 0$, by setting $\ell(0) = 2$ and $P_2(0) = 1/2$. In particular, both $s \mapsto \ell(is)$ and $s \mapsto P_2(is)$ are continuous on $\mathbb{R}$. We now prove that $\operatorname{Re} \ell(is) \geq 1$ for all $s \in \mathbb{R}$. Note first that $\operatorname{Re} \ell(is) = \operatorname{Re} \ell(-is)$ for all $s \in \mathbb{R}$ and, as has just been noted, that $\ell(0) = 2$. For $s > 0$, we see from (3.6) that

$$\operatorname{Re} \ell(is) = 1 + \frac{1}{s} \int_0^\infty \mu(r) \sin(rs) \, \mathrm{d}r = 1 + \frac{1}{s} \sum_{n=0}^\infty \int_{2n\pi/s}^{2(n+1)\pi/s} \mu(r) \sin(rs) \, \mathrm{d}r$$

$$= 1 + \frac{1}{s} \sum_{n=0}^\infty \int_0^{\pi/s} \left(\mu\left(\frac{2n\pi}{s} + r\right) - \mu\left(\frac{(2n+1)\pi}{s} + r\right) \right) \sin(rs) \, \mathrm{d}r.$$

By monotonicity of μ and non-negativity of $\sin(t)$ for $0 \leq t \leq \pi$ all of the integrands are non-negative, and hence $\operatorname{Re} \ell(is) \geq 1$ for all $s \in \mathbb{R}$. Next we prove the asymptotic estimate

$$(3.8) \qquad P_2(is) = \frac{1 \mp i}{\sqrt{2}|s|^{1/2}} + O(|s|^{-3/2}), \qquad s \to \pm\infty.$$

Note first that, by integrability of μ, $\ell(is) = 1 + O(|s|^{-1})$ and hence also $\ell(is)^{-1} = 1 + O(|s|^{-1})$ as $|s| \to \infty$. Thus

$$(3.9) \qquad \sqrt{is/\ell(is)} = \sqrt{is(1 + O(|s|^{-1}))} = |s|^{1/2}\frac{1 \pm i}{\sqrt{2}}(1 + O(|s|^{-1})), \qquad s \to \pm\infty,$$

and similarly

$$(3.10) \qquad \frac{1}{\sqrt{is\ell(is)}} = \left(|s|^{1/2}\frac{1 \pm i}{\sqrt{2}}(1 + O(|s|^{-1})) \right)^{-1} = \frac{1 \mp i}{\sqrt{2}|s|^{1/2}}(1 + O(|s|^{-1}))$$

as $s \to \pm\infty$. The estimate (3.9) yields

$$\tanh \sqrt{is/\ell(is)} = 1 - \frac{2}{e^{2\sqrt{is/\ell(is)}} + 1} = 1 + O\left(e^{-\sqrt{2}|s|^{1/2}}\right), \qquad |s| \to \infty,$$

and combining this with (3.10) we quickly obtain (3.8). It follows that (3.5) holds for some $c_0 \in (0,1]$ and for $|s|$ sufficiently large. Hence by continuity of the map $s \mapsto \operatorname{Re} P_2(is)$ on $\mathbb{R}$ it suffices, in order to prove (3.5), to show that $\operatorname{Re} P_2(is) > 0$ for all $s \in \mathbb{R}$. First, from (3.7) we see that

$$\operatorname{Im} \ell(is) = -\int_0^\infty g(r) \sin(rs) \, \mathrm{d}r, \qquad s \in \mathbb{R},$$

which implies, in particular, that

$$|\operatorname{Im} \ell(is)| < \int_0^\infty g(r) \, \mathrm{d}r = 1, \qquad s \in \mathbb{R}.$$

Let us denote by Σ the sector $\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi/4\}$. Since $\operatorname{Re} \ell(is) \geq 1$ and $|\operatorname{Im} \ell(s)| < 1$ we see that $\ell(is) \in \Sigma$ for all $s \in \mathbb{R}$, and because Σ is invariant under the inversion $z \mapsto z^{-1}$ we also have $\ell(is)^{-1} \in \Sigma$ for all $s \in \mathbb{R}$. Fix $s > 0$ and let $\theta = \arg \ell(is)$. Here and in what follows we take arg to be the principal value of the argument, so that $|\arg \lambda| \leq \pi$ for all $\lambda \in \mathbb{C}$. Then $\arg \sqrt{is/\ell(is)} = \frac{\pi}{4} - \frac{\theta}{2}$ and $\arg \sqrt{is\ell(is)} = \frac{\pi}{4} + \frac{\theta}{2}$. Let $a, b > 0$ be such that $\sqrt{is/\ell(is)} = a + ib$. Then

$$\tanh \sqrt{is/\ell(is)} = \frac{\sinh(2a) + i\sin(2b)}{\cosh(2a) + \cos(2b)}.$$

Using that $|\sin(2b)| < 2b$ and $\sinh(2a) > 2a$ together with monotonicity of the arctangent, we find that

$$\left| \arg \tanh \sqrt{is/\ell(is)} \right| = \left| \tan^{-1}\left(\frac{\sin(2b)}{\sinh(2a)} \right) \right| < \tan^{-1}\left(\frac{b}{a} \right) = \arg \sqrt{is/\ell(is)} = \frac{\pi}{4} - \frac{\theta}{2}.$$

Since $\arg P_2(is) = \arg \tanh \sqrt{is/\ell(is)} - \arg \sqrt{is\ell(is)}$, we obtain $-\pi/2 < \arg P_2(is) < \pi/4$. In particular, $\operatorname{Re} P_2(is) > 0$. An analogous argument applies when $s < 0$, and thus there exists $c_0 \in (0,1]$ such that (3.5) holds. Hence if we let $\nu(r) = c_0/(1 + \sqrt{r})$ for $r \geq 0$, then (3.5) yields $\operatorname{Re} P_2(is) \geq \nu(|s|)$ for all $s \in \mathbb{R}$. It follows from Theorem 3.2 that $\sigma(\mathbb{A}) \subset \mathbb{C}_-$ and $\|R(is, \mathbb{A})\| = O(|s|^{1/2})$ as $s \to \pm\infty$, as required. $\qquad\square$

3.2. **Optimality.** The following result shows that the resolvent estimate in Theorem 3.1 is optimal.

Theorem 3.4. *Suppose that $\sigma(\mathbb{A}) \subset \mathbb{C}_-$. Then*

$$\limsup_{s \to \infty} s^{-1/2} \|R(is, \mathbb{A})\| > 0.$$

Proof. For $n \geq 1$ let $\hat{z}_n = (\hat{u}_n, \hat{v}_n, 0, 0)^T \in \mathcal{H}$, where $\hat{u}_n(x) = \sin(2\pi nx)/2\pi n$ and $\hat{v}_n(x) = \cos(2\pi nx)$. In particular, $\|\hat{z}_n\|_{\mathcal{H}} = 1$ for all $n \geq 1$. Since $\sigma(\mathbb{A}) \subset \mathbb{C}_-$ by assumption, the

equation $(2\pi ni - \mathbb{A})z_n = \hat{z}_n$ has a unique solution $z_n = (u_n, v_n, w_n, \eta_n)^T \in D(\mathbb{A})$ for each $n \geq 1$. The components satisfy the system

(3.11)
$$\begin{cases} 2\pi ni\, u_n(x) - v_n(x) = \hat{u}_n(x), & x \in (-1, 0), \\ 2\pi ni\, v_n(x) - u_n''(x) = \hat{v}_n(x), & x \in (-1, 0), \\ 2\pi ni\, w_n(x) - \phi_n''(x) = 0, & x \in (0, 1), \\ 2\pi ni\, \eta_n(x, s) - T\eta_n(x, s) - w_n(x) = 0, & x \in (0, 1),\ s > 0, \end{cases}$$

where, as before,

$$\phi_n(x) = w_n(x) + \int_0^\infty \mu(s)\eta_n(x, s)\, \mathrm{d}s, \qquad x \in (0, 1).$$

Let us introduce the auxiliary functions $U_n^\pm = \frac{1}{2}(v_n \pm u_n')$ on $(-1, 0)$. Then, using the first two equations in (3.11), it is readily seen that

$$(U_n^+)'(x) = 2\pi ni\, U_n^+(x) - \cos(2\pi nx), \qquad (U_n^-)'(x) = -2\pi ni\, U_n^-(x)$$

for all $x \in (-1, 0)$, and solving these subject to $U_n^+(-1) + U_n^-(-1) = v_n(-1) = 0$ yields

$$U_n^+(x) = \mathrm{e}^{2\pi ni(x+1)}U_n^+(-1) - \int_{-1}^x \mathrm{e}^{2\pi ni(x-\tau)}\cos(2\pi n\tau)\, \mathrm{d}\tau,$$

$$U_n^-(x) = -\mathrm{e}^{-2\pi ni(x+1)}U_n^+(-1)$$

for all $x \in (-1, 0)$. Since $v_n = U_n^+ + U_n^-$ and $u_n' = U_n^+ - U_n^-$, it follows that

(3.12) $$v_n(x) = \left(\mathrm{e}^{2\pi ni(x+1)} - \mathrm{e}^{-2\pi ni(x+1)}\right)U_n^+(-1) - \int_{-1}^x \mathrm{e}^{2\pi ni(x-\tau)}\cos(2\pi n\tau)\, \mathrm{d}\tau,$$

(3.13) $$u_n'(x) = \left(\mathrm{e}^{2\pi ni(x+1)} + \mathrm{e}^{-2\pi ni(x+1)}\right)U_n^+(-1) - \int_{-1}^x \mathrm{e}^{2\pi ni(x-\tau)}\cos(2\pi n\tau)\, \mathrm{d}\tau$$

for all $x \in (-1, 0)$. In particular, (3.12) yields

(3.14)
$$\begin{aligned} \|v_n\|_{L^2(-1,0)}^2 &\geq \frac{1}{2}\int_{-1}^0 \left|\left(\mathrm{e}^{2\pi ni(x+1)} - \mathrm{e}^{-2\pi ni(x+1)}\right)U_n^+(-1)\right|^2 \mathrm{d}x \\ &\quad - \int_{-1}^0 \left|\int_{-1}^x \mathrm{e}^{2\pi ni(x-\tau)}\cos(2\pi n\tau)\, \mathrm{d}\tau\right|^2 \mathrm{d}x \geq |U_n^+(-1)|^2 - \frac{1}{3}. \end{aligned}$$

It moreover follows from (3.12) and (3.13) that $v_n(0) = -1/2$ and $u_n'(0) = 2U_n^+(-1) - 1/2$. Integrating the fourth equation in (3.11) and using the fact that $\eta_n(x, 0) = 0$ yields

$$\eta_n(x, s) = \frac{1 - \mathrm{e}^{-2\pi nis}}{2\pi ni}w_n(x), \qquad x \in (0, 1),\ s > 0.$$

Hence $\phi_n = \alpha_n w_n$, where

(3.15) $$\alpha_n = 1 + \frac{1}{2\pi ni}\left(\kappa - \int_0^\infty \mu(s)\mathrm{e}^{-2\pi nis}\, \mathrm{d}s\right)$$

with $\kappa = \int_0^\infty \mu(s)\, \mathrm{d}s$, and in particular $\phi_n'(0) = \alpha_n w_n'(0)$. Note also that $\alpha_n \to 1$ as $n \to \infty$, so by considering only sufficiently large values of $n \geq 1$ we may assume that

$\alpha_n \neq 0$. Now using (3.15) in the third equation in (3.11) we find that

$$w_n'(x) = \frac{2\pi n i}{\alpha_n} \int_0^x w_n(\tau)\, \mathrm{d}\tau + w_n'(0), \qquad x \in (0,1).$$

Let us set $\sigma_n = \sqrt{2\pi n i/\alpha_n}$ and

$$W_n^{\pm}(x) = \frac{1}{2}\left(w_n(x) \mp \sigma_n \int_0^x w_n(\tau)\, \mathrm{d}\tau\right), \qquad x \in (0,1).$$

Then $(W_n^{\pm})' = \frac{1}{2}w_n'(0) \mp \sigma_n W_n^{\pm}$, and solving these differential equations subject to $W_n^+(1) + W_n^-(1) = w_n(1) = 0$ yields

$$W_n^+(x) = \mathrm{e}^{-\sigma_n(x-1)}W_n^+(1) + \frac{1 - \mathrm{e}^{-\sigma_n(x-1)}}{2\sigma_n}w_n'(0),$$

$$W_n^-(x) = -\mathrm{e}^{\sigma_n(x-1)}W_n^+(1) - \frac{1 - \mathrm{e}^{\sigma_n(x-1)}}{2\sigma_n}w_n'(0)$$

for all $x \in (0,1)$. Since $W_n^+(0) = W_n^-(0)$, it follows that

$$W_n^+(1) = \frac{\mathrm{e}^{\sigma_n} + \mathrm{e}^{-\sigma_n} - 2}{2(\mathrm{e}^{\sigma_n} + \mathrm{e}^{-\sigma_n})\sigma_n}w_n'(0).$$

Thus, using the relation $w_n(0) = W_n^+(0) + W_n^-(0)$ we see, after a few elementary manipulations, that

$$(3.16) \qquad\qquad w_n(0) = -\frac{\tanh \sigma_n}{\sigma_n}\, w_n'(0).$$

Observe in particular that $\tanh \sigma_n \to 1$ as $n \to \infty$. Combining the coupling conditions $v_n(0) = w_n(0)$, $u_n'(0) = \phi_n'(0)$ with the identities $v_n(0) = -1/2$, $u_n'(0) = 2U_n^+(-1) - 1/2$ obtained above and using the fact that $\phi_n'(0) = \alpha_n w_n'(0)$, it follows from (3.16) that

$$\begin{cases} 2w_n'(0)\tanh \sigma_n = \sigma_n, \\ \qquad 2w_n'(0)\alpha_n = 4U_n^+(-1) - 1. \end{cases}$$

Now the definition of σ_n implies that

$$U_n^+(-1) = \frac{1}{4} + \frac{\alpha_n \sigma_n}{4\tanh \sigma_n} \sim \frac{\sigma_n}{4} \sim \frac{\sqrt{2\pi n i}}{4}$$

as $n \to \infty$, and hence, using (3.14),

$$\|z_n\|_{\mathcal{H}}^2 \geq \|v_n\|_{L^2(-1,0)}^2 \geq |U_n^+(-1)|^2 - \frac{1}{3} \geq \frac{\pi n}{16}$$

for all sufficiently large $n \geq 1$. The result now follows from the fact that $\|\hat{z}_n\|_{\mathcal{H}} = 1$ for all $n \geq 1$. $\qquad\square$

Remark 3.5. An alternative approach to proving optimality of the resolvent bound in Theorem 3.1 is to give a precise description of the part of the spectrum of $\mathbb{A}$ lying in a neighbourhood of the imaginary axis and then to bound the resolvent norm from below by means of the elementary estimate $\|R(\lambda, \mathbb{A})\| \geq \mathrm{dist}(\lambda, \sigma(\mathbb{A}))^{-1}$ for $\lambda \in \rho(\mathbb{A})$. Our approach is shorter and more direct. The required description of the spectrum of $\mathbb{A}$ may nevertheless be found in the appendix.

4. Energy Decay

In this last main section we convert the resolvent estimate obtained in Theorem 3.1 into a decay rate for the semigroup $(S(t))_{t\geq0}$ generated by $\mathbb{A}$. In particular, we shall show that $(S(t))_{t\geq0}$ is semi-uniformly polynomially stable. The key to this is the following well-known theoretical result due to Borichev and Tomilov [6, Thm. 2.4].

Theorem 4.1. *Let A be the generator of a bounded C_0-semigroup $(T(t))_{t\geq0}$ on a Hilbert space Z, and suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$. For each fixed $\alpha > 0$ the following statements are equivalent:*

> (i) $\|R(is, A)\| = O(|s|^\alpha)$ *as* $|s| \to \infty$;
> (ii) $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$ *as* $t \to \infty$;
> (iii) $\|T(t)z\|_Z = o(t^{-1/\alpha})$ *as* $t \to \infty$ *for every* $z \in D(A)$.

A C_0-semigroup satisfying the equivalent conditions of Theorem 4.1 is said to be *polynomially stable (with parameter α.)* By [4, Prop. 1.3] the implication (ii) $\implies$ (i) holds much more generally and even for C_0-semigroups on Banach spaces, whereas passing from (i) to (ii) in general requires a logarithmic correction factor in the Banach space setting, as is shown in [4, Thm. 1.5] and [6, Thm. 4.1]. For C_0-semigroups on Hilbert spaces the implication (i) $\implies$ (ii) has recently been extended beyond the case of polynomial resolvent growth in [26, Thm. 3.2].

From now on we consider the C_0-semigroup $(S(t))_{t\geq0}$ generated by the operator $\mathbb{A}$ associated with system (1.1). Since the orbits of the semigroup $(S(t))_{t\geq0}$ with initial values $z_0 \in D(\mathbb{A})$ correspond to classical solutions of the abstract Cauchy problem (2.5), we may interpret parts (ii) and (iii) of Theorem 4.1 as statements about (uniform) rates of energy decay of classical solutions to our problem (1.1).

Theorem 4.2. *Assume that (1.5) holds. Then the semigroup $(S(t))_{t\geq0}$ generated by the operator $\mathbb{A}$ is polynomially stable with parameter $1/2$. In particular, for any vector $z_0 = (u, v, w, \eta)^T \in D(\mathbb{A})$, the associated classical solution of (2.5) satisfies $\|S(t)z_0\|_{\mathcal{H}} = o(t^{-2})$ as $t \to \infty$.*

Proof. This follows immediately from Theorems 3.1 and 4.1. $\qquad\square$

Our next result shows that optimality of the resolvent bound in Theorem 3.1, as established in Theorem 3.4, implies optimality of the decay rate in Theorem 4.2.

Proposition 4.3. *Let $(S(t))_{t\geq0}$ be the C_0-semigroup generated by $\mathbb{A}$. Given any function $r : \mathbb{R}_+ \to (0,\infty)$ such that $r(t) = o(t^{-2})$ as $t \to \infty$, there exists a vector $z_0 = (u, v, w, \eta)^T \in D(\mathbb{A})$ such that*

$$(4.1) \qquad \limsup_{t\to\infty} \frac{\|S(t)z_0\|_{\mathcal{H}}}{r(t)} = \infty.$$

In other words, for any such function r there exist initial data giving rise to a classical solution of (2.5) whose energy decays strictly more slowly than $r(t)$ as $t \to \infty$.

Proof. Replacing $r(t)$ by $\sup_{s\geq t} r(s)$ for $t \geq 0$ if necessary, we may assume that r is non-increasing. Suppose, for the sake of a contradiction, that (4.1) is false for all $z_0 \in D(\mathbb{A})$. Since $1 \in \rho(\mathbb{A})$ by contractivity of the semigroup $(S(t))_{t\geq0}$ and since $(I - \mathbb{A})^{-1}$

maps $\mathcal{H}$ onto $D(\mathbb{A})$, we then have $\sup_{t\geq 0} r(t)^{-1}\|S(t)(I-\mathbb{A})^{-1}z_0\|_{\mathcal{H}} < \infty$ for all $z_0 \in \mathcal{H}$. Thus $\sup_{t\geq 0} r(t)^{-1}\|S(t)(I-\mathbb{A})^{-1}\| < \infty$ by the uniform boundedness principle, and we may let $C = \sup_{t\geq 0} r(t)^{-1}\|S(t)(I-\mathbb{A})^{-1}\|$, a positive real number. Note in particular that $\|S(t)(I-\mathbb{A})^{-1}\| \to 0$ as $t \to \infty$, and hence $\sigma(\mathbb{A}) \subset \mathbb{C}_-$ by [4, Prop. 1.3]. On the other hand, it follows straightforwardly from Theorem 3.4 and [10, Prop. 5.4] that $\limsup_{t\to\infty} t^2\|S(t)(I-\mathbb{A})^{-1}\| > 0$, so we may find a sequence $(t_n)_{n\geq 1}$ of positive real numbers such that $t_n \to \infty$ as $n \to \infty$ and a constant $c > 0$ such that $t_n^2\|S(t_n)(I-\mathbb{A})^{-1}\| \geq c$ for all $n \geq 1$. Thus $Cr(t_n) \geq \|S(t_n)(I-\mathbb{A})^{-1}\| \geq ct_n^{-2}$ for all $n \geq 1$, which contradicts the assumption that $r(t) = o(t^{-2})$ as $t \to \infty$. $\square$

APPENDIX: THE SPECTRUM OF $\mathbb{A}$

In this appendix we describe, using similar techniques as in [14], the spectrum of $\mathbb{A}$ near the imaginary axis. We shall assume throughout that (1.5) holds. For $\delta > 0$ as in (1.5), we introduce the vertical strip $\Pi_\delta = \{\lambda \in \mathbb{C} : -\frac{\delta}{2} < \operatorname{Re}\lambda \leq 0\}$, and we denote by Z_ℓ the zero set of the map $\ell : \Pi_\delta \setminus \{0\} \to \mathbb{C}$, noting that, by (1.5),

$$\ell(\lambda) = 1 + \frac{1}{\lambda}\int_0^\infty \mu(s)(1 - e^{-\lambda s})\,\mathrm{d}s$$

is indeed well-defined for every $\lambda \in \Pi_\delta \setminus \{0\}$. We also consider the set

$$\Sigma = \left\{ \lambda \in \Pi_\delta \setminus (Z_\ell \cup \{0\}) \,\Big|\, \sqrt{\ell(\lambda)\lambda}\sinh\lambda\cosh\sqrt{\tfrac{\lambda}{\ell(\lambda)}} + \cosh\lambda\sinh\sqrt{\tfrac{\lambda}{\ell(\lambda)}} = 0 \right\}.$$

Theorem A.1. *The spectrum $\sigma(\mathbb{A})$ of the operator $\mathbb{A}$ satisfies $\sigma(\mathbb{A}) \cap \Pi_\delta = \Sigma \cup Z_\ell$.*

In the proof of this theorem we shall make use of the following technical lemma whose proof is similar to the argument in the first part of the proof of Lemma 2.13 and consequently omitted.

Lemma A.2. *For any $\hat{\eta} \in \mathcal{M}$ and $\lambda \in \Pi_\delta$, the function $\xi_{\hat{\eta},\lambda}$ defined by*

$$\hat{\xi}_{\hat{\eta},\lambda}(x,s) = \int_0^s e^{-\lambda(s-\sigma)}\hat{\eta}(x,\sigma)\,\mathrm{d}\sigma, \qquad x \in (0,1),\ s > 0,$$

belongs to $\mathcal{M}$, and $\|\hat{\xi}_{\hat{\eta},\lambda}\|_{\mathcal{M}} \leq \frac{2\sqrt{C}}{\delta + 2\operatorname{Re}\lambda}\|\hat{\eta}\|_{\mathcal{M}}$.

Proof of Theorem A.1. We divide the proof into three steps.

Step 1. We first show that $0 \notin \sigma(\mathbb{A})$, which is to say that for every $\hat{z} = (\hat{u},\hat{v},\hat{w},\hat{\eta})^T \in \mathcal{H}$ the equation $\mathbb{A}z = \hat{z}$ has a unique solution $z = (u,v,w,\eta)^T \in D(\mathbb{A})$. Componentwise, we obtain

$$\begin{cases} v(x) = \hat{u}(x), & x \in (-1,0), \\ u''(x) = \hat{v}(x), & x \in (-1,0), \\ \phi''(x) = \hat{w}(x), & x \in (0,1), \\ T\eta(x,s) + w(x) = \hat{\eta}(x,s), & x \in (0,1),\ s > 0. \end{cases}$$

Integrating the last equation and using $\eta(x,0) = 0$, we obtain

(A.1) $$\eta(x,s) = sw(x) - \hat{\xi}_{\hat{\eta},0}(x,s), \qquad x \in (0,1),\ s > 0.$$

Solving for u and ϕ by using the conditions $u(-1) = 0$ and $\phi(1) = 0$ we find that

$$u(x) = a(x+1) + x\int_{-1}^{x} \hat{v}(r)\,\mathrm{d}r - \int_{-1}^{x} r\hat{v}(r)\,\mathrm{d}r, \qquad x \in (-1,0),$$

$$\phi(x) = b(1-x) - x\int_{x}^{1} \hat{w}(r)\,\mathrm{d}r + \int_{x}^{1} r\hat{w}(r)\,\mathrm{d}r, \qquad x \in (0,1),$$

for some $a, b \in \mathbb{C}$. Note in particular that $u \in H^2(-1,0)$ and $\phi \in H^2(0,1)$. Next, recalling the definition of ϕ, we find

$$w(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^{\infty} \mu(s)\hat{\xi}_{\hat{\eta},0}(x,s)\,\mathrm{d}s, \qquad x \in (0,1).$$

From Lemma A.2 we obtain

$$\left\|\int_0^{\infty} \mu(s)\hat{\xi}_{\hat{\eta},0}(x,s)\,\mathrm{d}s\right\|_{H_r^1(0,1)} \leq \sqrt{\kappa}\|\hat{\xi}_{\hat{\eta},0}\|_{\mathcal{M}} \leq 2\delta^{-1}\sqrt{\kappa C}\|\hat{\eta}\|_{\mathcal{M}},$$

where $\kappa = \int_0^{\infty} \mu(s)\,\mathrm{d}s$. In particular, we have $w \in H_r^1(0,1)$. Since the map $s \mapsto s^2\mu(s)$ is an element of $L^1(0,\infty)$, it follows from Lemma A.2 that the function η defined in (A.1) belongs to $\mathcal{M}$. Note also that $\eta_s = w - \hat{\eta} \in \mathcal{M}$. Finally, by monotonicity of μ, we have

$$\|\eta(s)\|_{H_r^1(0,1)} \leq s\|w\|_{H_r^1(0,1)} + \frac{1}{\mu(s)}\left(\int_0^{s} \mu(\sigma)\,\mathrm{d}\sigma\right)^{1/2}\|\hat{\eta}\|_{\mathcal{M}} \to 0, \qquad s \to 0,$$

which implies that $\eta \in D(T)$. It remains only to show that the constants $a, b \in \mathbb{C}$ may be chosen in such a way that the coupling conditions $v(0) = w(0)$ and $u'(0) = \phi'(0)$ are satisfied. Straightforward computations show that these conditions are equivalent to

$$b = 2\hat{u}(0) - \int_0^{\infty} \mu(s)\hat{\xi}_{\hat{\eta},0}(0,s)\,\mathrm{d}s - \int_0^{1} r\hat{w}(r)\,\mathrm{d}r, \qquad a = -b - \int_{-1}^{0} \hat{v}(r)\,\mathrm{d}r - \int_0^{1} \hat{w}(r)\,\mathrm{d}r.$$

Since all of the integrals are finite, we may indeed find suitable constants $a, b \in \mathbb{C}$.

Step 2. We prove that $Z_\ell \subset \sigma(\mathbb{A})$ by showing that $\lambda - \mathbb{A}$ is not onto for $\lambda \in Z_\ell$. Pick any $\hat{w} \in L^2(0,1) \setminus H^1(0,1)$ and set $\hat{z} = (0,0,\hat{w},0)^T \in \mathcal{H}$. If $\lambda - \mathbb{A}$ were onto, then there would exist $z = (u,v,w,\eta)^T \in D(\mathbb{A})$ such that $\lambda z - \mathbb{A}z = \hat{z}$. In component form, the problem becomes

$$\begin{cases} \lambda u(x) = v(x), & x \in (-1,0), \\ \lambda v(x) = u''(x), & x \in (-1,0), \\ \lambda w(x) - \phi''(x) = \hat{w}(x), & x \in (0,1), \\ \lambda\eta(x,s) - T\eta(x,s) = w(x), & x \in (0,1),\ s > 0. \end{cases}$$

Integrating the last equation with $\eta(x,0) = 0$ we get $\eta(x,s) = \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda s})w(x)$ for $x \in (0,1)$ and $s > 0$. Since $\ell(\lambda) = 0$ a short calculation yields $\phi = 0$, and now the third equation implies that $w = \lambda^{-1}\hat{w} \notin H_r^1(0,1)$. This is the desired contradiction.

Step 3. Let $\lambda \in \Pi_\delta \setminus (Z_\ell \cup \{0\})$ be arbitrary. In the light of Steps 1 and 2, the result will be proved once we have shown that $\lambda \in \Sigma \iff \lambda \in \sigma(\mathbb{A})$. To this end, let us fix

an arbitrary $\hat{z} = (\hat{u}, \hat{v}, \hat{w}, \hat{\eta})^T \in \mathcal{H}$. Our goal is to show that the equation $\lambda z - \mathbb{A}z = \hat{z}$ admits a unique solution $z = (u, v, w, \eta)^T \in D(\mathbb{A})$ if and only if

$$(A.2) \qquad \sqrt{\ell(\lambda)\lambda} \sinh \lambda \cosh \sqrt{\tfrac{\lambda}{\ell(\lambda)}} + \cosh \lambda \sinh \sqrt{\tfrac{\lambda}{\ell(\lambda)}} \neq 0.$$

In component form, our problem becomes

$$\begin{cases} \lambda u(x) - v(x) = \hat{u}(x), & x \in (-1, 0), \\ \lambda v(x) - u''(x) = \hat{v}(x), & x \in (-1, 0), \\ \lambda w(x) - \phi''(x) = \hat{w}(x), & x \in (0, 1), \\ \lambda \eta(x, s) - T\eta(x, s) - w(x) = \hat{\eta}(x, s), & x \in (0, 1), \ s > 0. \end{cases}$$

Integrating the last equation and using $\eta(x, 0) = 0$ we find

$$(A.3) \qquad \eta(x, s) = \frac{1 - e^{-\lambda s}}{\lambda} w(x) + \hat{\xi}_{\hat{\eta}, \lambda}(x, s), \qquad x \in (0, 1), \ s > 0.$$

Recalling the definition of ϕ, we have

$$(A.4) \qquad w(x) = \frac{\phi(x)}{\ell(\lambda)} - \frac{1}{\ell(\lambda)} \int_0^\infty \mu(s) \hat{\xi}_{\hat{\eta}, \lambda}(x, s) \, ds, \qquad x \in (0, 1).$$

Using the boundary condition $u(-1) = 0$ we obtain

$$u(x) = a(\lambda) \sinh(\lambda(x + 1)) - U(\lambda, x), \qquad x \in (-1, 0),$$

where $a(\lambda) \in \mathbb{C}$ and

$$U(\lambda, x) = \frac{1}{\lambda} \int_{-1}^x \sinh(\lambda(x - r))(\hat{v}(r) + \lambda \hat{u}(r)) \, dr, \qquad x \in (-1, 0).$$

Once u has been found, v is determined by the first equation of our system. It is straightforward to check that $u \in H^2(-1, 0)$ and $v \in H_l^1(-1, 0)$. Let us introduce the auxiliary function

$$\hat{\varrho}_{\hat{\eta}, \lambda}(x) = \frac{\lambda}{\ell(\lambda)} \int_0^\infty \mu(s) \hat{\xi}_{\hat{\eta}, \lambda}(x, s) \, ds, \qquad x \in (0, 1).$$

The general solution for ϕ subject to the boundary condition $\phi(1) = 0$ may be written as

$$(A.5) \qquad \phi(x) = -b(\lambda) \sinh \left(\sqrt{\tfrac{\lambda}{\ell(\lambda)}} (1 - x) \right) - \Phi(\lambda, x), \qquad x \in (0, 1),$$

where $b(\lambda) \in \mathbb{C}$ and

$$\Phi(\lambda, x) = \sqrt{\frac{\ell(\lambda)}{\lambda}} \int_x^1 \sinh \left(\sqrt{\tfrac{\lambda}{\ell(\lambda)}} (r - x) \right) (\hat{w}(r) + \hat{\varrho}_{\hat{\eta}, \lambda}(r)) \, dr, \qquad x \in (0, 1).$$

By (A.4), we also have

$$(A.6) \qquad w(x) = -\frac{b(\lambda)}{\ell(\lambda)} \sinh \left(\sqrt{\tfrac{\lambda}{\ell(\lambda)}} (1 - x) \right) - \frac{\Phi(\lambda, x)}{\ell(\lambda)} - \frac{\hat{\varrho}_{\hat{\eta}, \lambda}(x)}{\lambda}, \qquad x \in (0, 1).$$

Once w has been found, η is determined by (A.3). We now show that $\phi \in H^2(0, 1)$, $w \in H_r^1(0, 1)$ and $\eta \in D(T)$. In fact, it follows from (A.5) and (A.6) that $\phi \in H^2(0, 1)$

and $w \in H_r^1(0,1)$ provided that $\hat{\varrho}_{\hat{\eta},\lambda} \in H_r^1(0,1)$. The latter follows from Lemma A.2, which also yield the bounds

$$\left\| \frac{\lambda}{\ell(\lambda)} \int_0^\infty \mu(s)\hat{\xi}_{\hat{\eta},\lambda}(x,s)\,\mathrm{d}s \right\|_{H_r^1(0,1)} \leq \left| \frac{\lambda}{\ell(\lambda)} \right| \sqrt{\kappa} \|\hat{\xi}_{\hat{\eta},\lambda}\|_{\mathcal{M}} \leq \left| \frac{\lambda}{\ell(\lambda)} \right| \frac{2\sqrt{\kappa C}}{\delta + 2\,\mathrm{Re}\,\lambda} \|\hat{\eta}\|_{\mathcal{M}}.$$

In order to prove that $\eta \in D(T)$, we first show that $\eta \in \mathcal{M}$. Since $\hat{\xi}_{\hat{\eta},\lambda} \in \mathcal{M}$ by Lemma A.2, we only need to show that the map $s \mapsto (1 - \mathrm{e}^{-\lambda s})w$ lies in $\mathcal{M}$. To this end, note that

$$\int_0^\infty \mu(s)|1 - \mathrm{e}^{-\lambda s}|^2 \|w\|_{H_r^1(0,1)}^2 \,\mathrm{d}s \leq 2\kappa\|w\|_{H_r^1(0,1)}^2 + \|w\|_{H_r^1(0,1)}^2 \int_0^\infty \mu(s)\mathrm{e}^{-2(\mathrm{Re}\,\lambda)s}\,\mathrm{d}s.$$

Since $-2\,\mathrm{Re}\,\lambda < \delta$, (1.5) implies that $\int_0^\infty \mu(s)\mathrm{e}^{-2(\mathrm{Re}\,\lambda)s}\,\mathrm{d}s < \infty$, and hence $\eta \in \mathcal{M}$. Thus $\eta_s = \hat{\eta} + w - \lambda\eta \in \mathcal{M}$ as well. Finally, by monotonicity of μ, we have

$$\|\eta(s)\|_{H_r^1(0,1)} \leq \frac{|1 - \mathrm{e}^{-\lambda s}|}{|\lambda|}\|w\|_{H_r^1(0,1)} + \frac{\mathrm{e}^{-(\mathrm{Re}\,\lambda)s}}{\mu(s)}\left(\int_0^s \mu(\sigma)\,\mathrm{d}\sigma\right)^{1/2}\|\hat{\eta}\|_{\mathcal{M}} \to 0$$

as $s \to 0$. Thus $\eta \in D(T)$. It remains only to show that the coefficients $a(\lambda), b(\lambda) \in \mathbb{C}$ may be chosen in such a way that the coupling conditions $v(0) = w(0)$ and $u'(0) = \phi'(0)$ are satisfied. It is straightforward to show that these conditions are equivalent to the matrix equation

$$\begin{pmatrix} \ell(\lambda)\lambda\sinh\lambda & \sinh\sqrt{\frac{\lambda}{\ell(\lambda)}} \\ \ell(\lambda)\lambda\cosh\lambda & -\sqrt{\ell(\lambda)\lambda}\cosh\sqrt{\frac{\lambda}{\ell(\lambda)}} \end{pmatrix} \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} = \begin{pmatrix} \hat{f}(\lambda) \\ \hat{g}(\lambda) \end{pmatrix},$$

where

$$\hat{f}(\lambda) = \ell(\lambda)\left(\hat{u}(0) + \lambda U(\lambda, 0) - \frac{\hat{\varrho}_{\hat{\eta},\lambda}(0)}{\lambda}\right) - \Phi(\lambda, 0), \qquad \hat{g}(\lambda) = \ell(\lambda)(U_x(\lambda, 0) - \Phi_x(\lambda, 0)).$$

Hence we may uniquely determine $a(\lambda), b(\lambda)$ if and only if the determinant of the matrix appearing on the left-hand side is non-zero, which in turn is equivalent to (A.2). $\qquad\square$

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