Adaptive robust output regulation control design

Sepideh Afshar, Lassi Paunonen

***Abstract*— In this paper we consider controller design for robust output tracking and disturbance rejection for linear distributed parameter systems. In output regulation the frequencies of the reference and disturbance signals are typically assumed to be known in advance. In this paper we propose a new control design for robust output regulation for signals with unknown frequencies. Our controller is based on a time-dependent internal model where the frequencies are updated based on an adaptive estimator. We use the main results to design a controller for output tracking of an electromagnetic system which models magnetic drug delivery.**

I. Introduction

The problem of output regulation, defining a control input such that the output of the system converges to a reference signal, is encountered in many applications. This problem has been studied extensively for finite-dimensional control systems [1], [2], [3], as well as for linear distributed parameter systems (DPS) and controlled partial differential equations [4], [5]. In robust output regulation the convergence of the output to the reference $y_{ref}(t)$ is required happen even in the presence of small perturbations and uncertainties in the parameters of the system.

In output regulation, the reference and disturbance signals $y_{ref}(t)$ and $w_{dist}(t)$ are typically assumed be linear combinations of sinusoidal signals with known frequencies, and the knowledge of these frequencies is essential in the controller design. In particular, the *internal model principle* by Francis and Wonham [6] and Davison [7] states that in order to solve the robust output regulation problem a controller needs to include the (complex) frequencies $\{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$ of the signals $y_{ref}(t)$ and $w_{dist}(t)$ as eigenvalues with sufficiently high multiplicities. The internal model principle is also valid for linear distributed parameter systems [8], [9] and it has been used extensively in robust controller design for PDE systems [10], [11], [12], [13], [14].

In this paper we focus on a situation where the frequencies of $y_{ref}(t)$ and $w_{dist}(t)$ are instead *unknown*, and they need to be recovered based on measurements of the reference signal. For this control problem we propose a solution which is based on using an adaptive estimator to find convergent estimates $\{i\omega_k(t)\}_{k=1}^q$ for the unknown frequencies $\{i\omega_k\}_{k=1}^q$, and constructing a linear controller based on a time-dependent internal model which utilizes those estimates.

Gordon Center for Medical Imaging, Harvard Medical School, Massachusetts General Hospital, Boston, MA, USA (SA). Mathematics, Faculty of Information Technology and Communication Sciences, Tampere University, Tampere, Finland (LP). safshar1@mgh.harvard.edu, lassi.paunonen@tuni.fi

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In the final part of the paper we use our main results to design a controller for robust output tracking of an electromagnetic system which models magnetic drug delivery.

Output regulation for signals with unknown frequencies has been studied in several references for finite-dimensional systems, e.g., [3], [15] and for DPS in [16], [17], where the system is transformed into a canonical form for adaptive control design. Our approach does not use such a transformation and because of this we avoid posing some limiting structural assumptions on the control system.

Notation. If X and Y are Hilbert spaces, then the space of bounded linear operators $A : X \to Y$ is denoted by $\mathcal{L}(X,Y)$. The domain and kernel of an operator $A : D(A) \subset X \to Y$ are denoted by $D(A)$ and $\mathcal{N}(A)$, respectively. The resolvent operator of $A : D(A) \subset X \to X$ is defined as $R(\lambda, A) = (\lambda I - A)^{-1}$ for those $\lambda \in \mathbb{C}$ for which the inverse is bounded. By $L^p(0,\tau;X)$ and $L^\infty(0,\tau;X)$ we denote, respectively, the spaces of p-integrable and essentially bounded measurable functions $f : (0,\tau) \to X$.

II. Robust output regulation problem

In this paper we consider a linear DPS of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{II.1}$$

on a Hilbert space X, where $x(t) \in X$, $u(t) \in \mathbb{C}^p$, $y(t) \in \mathbb{C}^p$, and $w_{dis}(t) \in U_d$ are the system's state, input signal, output signal, and external disturbance, respectively. In particular, the number of outputs of the system is $p \in \mathbb{N}$. The operator $A : D(A) \subset X \to X$ is assumed to generate a strongly continuous semigroup on X, $B \in \mathcal{L}(\mathbb{C}^p, X)$, $B_d \in \mathcal{L}(U_d, X)$ for some Hilbert space U_d, $C \in \mathcal{L}(X, \mathbb{C}^p)$, and $D \in \mathbb{C}^{p\times p}$ are linear operators; B is the input operator and C is the output operator. Furthermore, define the state of an exosystem by $v(t) \in \mathbb{C}^q$; the reference signal $y_{ref}(t)$ and the disturbance signal $w_{dist}(t)$ are generated by the exosystem

$$\begin{aligned} \dot{v}(t) &= Sv(t), \qquad v(0) = v_0 \in \mathbb{C}^q \\ w_{dist}(t) &= Ev(t) \\ y_{ref}(t) &= -Fv(t). \end{aligned} \tag{II.2}$$

where $S = \operatorname{diag}(i\omega_1, \dots, i\omega_q)$ with unknown distinct eigenvalues $\{i\omega_k\}_{k=1}^q$, $E \in \mathcal{L}(\mathbb{C}^q, U_d)$, and $F \in \mathbb{C}^{p\times q}$.

We consider non-autonomous dynamic error feedback controllers of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1(t)z(t) + \mathcal{G}_2(t)(y(t) - y_{ref}(t)), \\ u(t) &= K(t)z(t) \end{aligned} \tag{II.3}$$

on a Hilbert space Z, with initial condition $z(0) = z_0 \in Z$.

We assume the unbounded part of $\mathcal{G}_1(\cdot)$ does not depend on t, i.e., $\mathcal{G}_1(t) = \mathcal{G}_1^\infty + \Delta_\mathcal{G}(t)$ where $\mathcal{G}_1^\infty : D(\mathcal{G}_1^\infty) \subset Z \to Z$ generates a strongly continuous semigroup on Z and $\Delta_\mathcal{G}(\cdot) \in L^\infty(0,\infty;\mathcal{L}(Z))$. Moreover, we assume $\mathcal{G}_2(\cdot) \in L^\infty(0,\infty;\mathcal{L}(\mathbb{C}^p,Z))$ and $K(\cdot) \in L^\infty(0,\infty;\mathcal{L}(Z,\mathbb{C}^p))$.

Define $x_e = (x,z)^T \in X_e := X \times Z$. The closed-loop system of the plant and the controller takes the form

$$\begin{aligned} x_e(t) &= A_e(t)x_e(t) + B_e(t)v(t) \\ e(t) &= C_e(t)x_e(t) + D_e v(t) \end{aligned} \tag{II.4}$$

where $e(t) = y(t) - y_{ref}(t)$, $D_e = F$,

$$A_e(t) = \begin{bmatrix} A & BK(t) \\ \mathcal{G}_2(t)C & \mathcal{G}_1(t) + \mathcal{G}_2(t)DK(t) \end{bmatrix}$$
$$B_e(t) = \begin{bmatrix} E \\ \mathcal{G}_2(t)F \end{bmatrix}, \; C_e(t) = \begin{bmatrix} C, DK(t) \end{bmatrix}.$$

Our assumptions on the controller imply that $A_e(t) = A_e^\infty + \Delta_e(t)$ where $A_e^\infty : D(A_e^\infty) \subset X_e \to X_e$ generates a strongly continuous semigroup $T_e^\infty(t)$ on X_e and $\Delta_e(\cdot) \in L^\infty(0,\infty;\mathcal{L}(X_e))$. Under these assumptions the closed-loop has a well-defined mild solution $x_e(t)$ and error $e(t)$ defined by the *evolution family* $U_e(t,s)$ associated to the family $(A_e(t))_{t\geq 0}$ of operators [18, Ch. 5 & Rem. 5.3.2].

The robust output regulation problem is defined as follows.

The Robust Output Regulation Problem. *The dynamic error feedback controller* (II.3) *needs to be defined in such a way that the following are satisfied:*

(a) *The closed-loop system* (II.4) *is exponentially stable.*
(b) *For all initial states $v_0 \in \mathbb{C}^q$, $x_0 \in X$ and $z_0 \in Z$ the regulation error satisfies*

$$\|y(t) - y_{ref}(t)\| \to 0. \tag{II.5}$$

(c) *If (A,B,B_d,C,D,E,F) are perturbed to $(\tilde{A},\tilde{B},\tilde{B}_d,\tilde{C},\tilde{D},\tilde{E},\tilde{F})$ in such a way that the perturbed closed-loop system remains stable, then for all initial states $v_0 \in \mathbb{C}^q$, $x_0 \in X$ and $z_0 \in Z$ the regulation error converges to zero.*

In part (a) the exponential stability of the closed-loop system (II.4) is required in the sense that there exists $M_e, \alpha > 0$ such that $\|U_e(t,s)\| \leq M_e e^{-\alpha(t-s)}$ for all $t \geq s$. Similarly in part (c) the perturbations $(\tilde{A},\tilde{B},\tilde{B}_d,\tilde{C},\tilde{D},\tilde{E},\tilde{F})$ are required to preserve this stability of the non-autonomous closed-loop system. Because of this, the class of tolerated perturbations also depends on the constructed controller.

Throughout this paper, we consider controllers whose parameters converge to constant operators as $t \to \infty$ in the sense that the following property is satisfied. This especially covers the situation where the internal model is constructed using frequency estimates that converge asymptotically to the true frequencies. In Section IV we will present a controller for which this property follows immediately from the convergence of the frequency estimation algorithm in Section III-B.

Property II.1. For some $\mathcal{G}_2^\infty \in \mathcal{L}(\mathbb{C}^p, Z)$ and $K^\infty \in \mathcal{L}(Z,\mathbb{C}^p)$ we have $\|\mathcal{G}_2(t) - \mathcal{G}_2^\infty\|_{\mathcal{L}(\mathbb{C}^p,Z)} \to 0$, $\|K(t) - K^\infty\|_{\mathcal{L}(Z,\mathbb{C}^p)} \to 0$, and $\|\Delta_\mathcal{G}(t)\|_{\mathcal{L}(Z)} \to 0$ as $t \to \infty$. ▲

Property II.1 also implies that $\Delta_e(t) \to 0$, $B_e(t) - B_e^\infty \to 0$, $C_e(t) - C_e^\infty \to 0$, and $D_e(t) - D_e^\infty \to 0$ as $t \to \infty$ where $(A_e^\infty, B_e^\infty, C_e^\infty, D_e^\infty)$ is the closed-loop system of the form (II.4) with an autonomous controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$.

III. Main results

We denote by $\sigma^0 = (i\omega_k)_{k=1}^q \in \mathbb{C}^q$ the true frequencies of $y_{ref}(t)$ and $w_{dist}(t)$. Our main aim is to study internal model based controllers where the correct frequencies σ_0 are replaced by on-line estimates $\{i\omega_k(t)\}_{k=1}^q$ of σ^0. A suitable adaptive estimator for the frequencies is presented in Section III-B. To justify the validity our general approach, we will first show in Section III-A that if the estimates $\{i\omega_k(t)\}_{k=1}^q$ converge to the correct frequencies $\{i\omega_k\}_{k=1}^q$ as $t \to \infty$ and if the controller stabilizes the non-autonomous closed-loop system (II.4), then the controller solves the robust output regulation problem.

More generally, it may be the case that the on-line estimates $\{i\omega_k(t)\}_{k=1}^q$ do not converge exactly to σ^0, but only their approximate values. In the situation where $\omega_k(t) \to \omega_k^\infty$ for all k as $t \to \infty$, our results show that if the limits $\sigma^\infty := (i\omega_k^\infty)_{k=1}^q \in \mathbb{C}^q$ are sufficiently close to the true frequencies $\sigma^0 = (i\omega_k)_{k=1}^q$, then the tracking error $e(t) = y(t) - y_{ref}(t)$ will become small as $t \to \infty$.

In both cases (when either $\sigma^\infty = \sigma^0$ or $\|\sigma^\infty - \sigma^0\|_{\mathbb{C}^q}$ is small) we assume that the asymptotic limit $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ of the controller has *an internal model* of the limit frequencies $\sigma^\infty = (i\omega_k^\infty)_{k=1}^q \in \mathbb{C}^q$ in the following sense. Here p is the number of outputs of the plant.

Definition III.1 ([8, Def. 6.1])**.** The autonomous controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ is said to *have an internal model* of constant frequencies $\sigma^\infty = (i\omega_k^\infty)_{k=1}^q \in \mathbb{C}^q$ if $\dim \mathcal{N}(i\omega_k^\infty - \mathcal{G}_1^\infty) \geq p$ for all $k \in \{1,\ldots,q\}$.

Our results are not restricted to controllers with time-dependent frequencies in the internal model, but also other parameters of the controller are allowed to vary with time. This is also typically necessary for achieving closed-loop stability. Theorem III.2 also shows that in order to achieve exponential closed-loop stability of the non-autonomous system, it is sufficient that the asymptotic limit $(A_e^\infty, B_e^\infty, C_e^\infty, D_e^\infty)$ of the closed-loop system is exponentially stable as an autonomous system. This is a consequence of Property II.1, and it can be utilized in the controller design.

A. Output Regulation for Converging Frequencies

Theorem III.2. *Assume the controller has Property II.1. Furthermore, assume $(\mathcal{G}_1(t), \mathcal{G}_2(t), K(t))$ are such that the following are satisfied.*

- *The semigroup $T_e^\infty(t)$ generated by A_e^∞ is exponentially stable.*
- *The controller $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ has an internal model of the limit frequencies $\sigma^\infty = (i\omega_k^\infty)_{k=1}^q$ in the sense of Definition III.1.*

Then the controller solves the robust output tracking problem in the sense that for any $\delta > 0$ there exists $\gamma > 0$ such that

$$\text{if} \quad \|\sigma^\infty - \sigma^0\| \le \gamma, \qquad \text{then}$$
$$\limsup_{t\to\infty} \|y(t) - y_{ref}(t)\| \le \delta \|v_0\|$$

for all initial states $x_0 \in X$, $z_0 \in Z$, and $v_0 \in \mathbb{C}^q$. Moreover, if $\sigma^\infty = \sigma^0$, then $\|y(t) - y_{ref}(t)\| \to 0$ as $t \to \infty$ for all initial states $x_0 \in X$, $z_0 \in Z$, and $v_0 \in \mathbb{C}^q$.

The proof is based on the following two lemmas.

Lemma III.3. *Denote $A_e(t) = A_e^\infty + \Delta_e(t)$ and assume the controller has Property II.1. The evolution family $U_e(t,s)$ is exponentially stable if and only if the semigroup generated by A_e^∞ is exponentially stable.*

Proof. For any $\varepsilon > 0$ we can choose $t_1 \ge 0$ such that $\|\Delta(t)\| \le \varepsilon$ for $t \ge t_1$. Thus if A_e^∞ generates an exponentially stable semigroup the result [19, Thm. 4.2] implies that the evolution family $U_e(t,s)$, $t \ge s \ge t_1$ associated to the family $(A_e(t))_{t \ge t_1}$ is exponentially stable, and therefore the same clearly holds for the evolution family $U_e(t,s)$. Writing $A_e^\infty = A_e(t) - \Delta_e(t)$ we can similarly use [19, Thm. 4.2] to deduce that the exponential stability of $U_e(t,s)$ implies the exponential stability of the semigroup generated by A_e^∞. □

The output maps of the time-dependent closed-loop system $(A_e(t), B_e(t), C_e(t), D_e(t))$ and the autonomous system $(A_e^\infty, B_e^\infty, C_e^\infty, D_e^\infty)$ are denoted, respectively, by

$$(\mathbb{F}_s v)(t) = C_e(t) \int_s^t U_e(t,r) B_e(r) v(r) dr + D_e(t) v(t)$$
$$(\mathbb{F}_s^\infty v)(t) = C_e^\infty \int_s^t T_e^\infty(t-r) B_e^\infty v(r) dr + D_e^\infty v(t)$$

for $v \in L^1_{\text{loc}}(0,\infty;\mathbb{C}^q)$, where $T_e^\infty(t)$ is the semigroup generated by A_e^∞.

Lemma III.4. *Assume Property II.1 holds. Denote $A_e(t) = A_e^\infty + \Delta_e(t)$ and define $\mathbb{F}_s$ and $\mathbb{F}_s^\infty$ as above. Then*

$$\|(\mathbb{F}_s v)(t) - (\mathbb{F}_s^\infty v)(t)\| \to 0,$$

as $t \to \infty$ for any $s \ge 0$ and any continuous and uniformly bounded $v \in \mathrm{BUC}([s,\infty),\mathbb{C}^q)$.

Proof. Let $s \ge 0$ and $v \in \mathrm{BUC}([s,\infty),\mathbb{C}^q)$. The formulas of $\mathbb{F}_s$ and $\mathbb{F}_s^\infty$ imply

$$\begin{aligned}
&\|(\mathbb{F}_s v)(t) - (\mathbb{F}_s^\infty v)(t)\| \\
&\le \|C_e(t) - C_e^\infty\| \|\int_s^t U_e(t,r) B_e(r) v(r) dr\| \\
&+ \|C_e^\infty\| \|\int_s^t [U_e(t,r) - T_e^\infty(t-r)] B_e(r) v(r) dr\| \\
&+ \|C_e^\infty\| \|\int_s^t T_e^\infty(t-r)(B_e(r) - B_e^\infty) v(r) dr\|.
\end{aligned} \tag{III.1}$$

The first and last terms on the right-hand side of (III.1) converge by assumption and boundedness of $U_e(t,r)$, $B_e(r)$, C_e^∞, and $T_e^\infty(t-r)$; thus, it is sufficient to consider the second term. The evolution family $U_e(t,s)$ and the semigroup $T_e^\infty(t)$ are related by the "variation of parameters formula"

$$\begin{aligned}
U_e(t,r)x_e &= T_e^\infty(t-r)x_e \\
&+ \int_r^t U_e(t,\tau)\Delta_e(\tau) T_e^\infty(\tau - r) x_e d\tau,
\end{aligned} \tag{III.2}$$

for all $x_e \in X_e$. Denote $f(t) = B_e(t)v(t)$ and $g(\tau) = \int_s^\tau T_e^\infty(\tau - r) f(r) dr$ for brevity. Then (III.2) implies

$$\int_s^t [U_e(t,r) - T_e^\infty(t-r)] f(r) dr = \int_s^t U_e(t,\tau)\Delta_e(\tau) g(\tau) d\tau.$$

Stability of $T_e^\infty(t)$, $\Delta_e(t) \to 0$, and $f(\cdot) \in L^\infty(0,\infty;X_e)$ first imply that $\Delta_e(\tau)g(\tau) \to 0$ as $\tau \to \infty$. The exponential stability of $U_e(t,s)$ then finally implies that the right-hand side of the above identity converges to zero as $t \to \infty$. □

Proof of Theorem III.2. We present the proof for the nominal parameters (A, B, B_d, C, D, E, F). The proof is completely analogous for the perturbed parameters $(\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F})$, because Lemma III.3 implies that the exponential stability of the perturbed non-autonomous system implies that the limit operator $\tilde{A}_e^\infty$ generates an exponentially stable semigroup.

The state of the exosystem is of the form $v(t) = e^{St}v_0 = (e^{i\omega_k t} v_{0k})_{k=1}^q = \sum_{k=1}^q e^{i\omega_k t} v_{0k} \mathbf{e}_k$, where $\mathbf{e}_k \in \mathbb{C}^q$ denotes the kth Euclidean basis vector. Our aim is to show that

$$\left\| e(t) - \sum_{k=1}^q e^{i\omega_k t} v_{0k} P_e^\infty(i\omega_k)\mathbf{e}_k \right\| \to 0, \tag{III.3}$$

as $t \to \infty$ where $P_e^\infty(\lambda) = C_e^\infty R(\lambda, A_e^\infty) B_e^\infty + D_e^\infty$. If we can show this, the claim of the theorem follows from the fact that since $(\mathcal{G}_1^\infty, \mathcal{G}_2^\infty, K^\infty)$ contains an internal model of the frequencies σ^∞, we must have $P_e^\infty(i\omega_k^\infty)\mathbf{e}_k = 0$ for all $k \in \{1,\ldots,q\}$. But since $P_e^\infty(\cdot)$ is continuous on $i\mathbb{R}$, we have that the norms $\|P_e^\infty(i\omega_k)\mathbf{e}_k\|$ are small for every $k \in \{1,\ldots,q\}$ provided that $\|\sigma^0 - \sigma^\infty\| \le \gamma$ with $\gamma > 0$ sufficiently small. More precisely, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \|e(t)\| \le \sum_{k=1}^q \|v_{0k} P_e^\infty(i\omega_k)\mathbf{e}_k\| \\
&\le \left(\sum_{k=1}^q \|v_{0k}\|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^q \|P_e^\infty(i\omega_k)\mathbf{e}_k\|^2\right)^{\frac{1}{2}} \le \delta \|v_0\|,
\end{aligned}$$

where we have chosen $\delta^2 = \sum_{k=1}^q \|P_e^\infty(i\omega_k)\mathbf{e}_k\|^2$ and $\delta \to 0$ as $\gamma \to 0$. In particular, if $\sigma^\infty = \sigma^0$ we clearly have $\delta = 0$ since in this case $P_e^\infty(i\omega_k)\mathbf{e}_k = 0$ for all $k \in \{1,\ldots,q\}$.

To prove (III.3), note that the exponential stability of the semigroup $T_e^\infty(t)$ generated by A_e^∞ implies the well-known property that for every $k \in \{1,\ldots,q\}$,

$$\begin{aligned}
&\lim_{t\to\infty} e^{-i\omega_k t} (\mathbb{F}_0^\infty e^{i\omega_k \cdot} \mathbf{e}_k)(t) \\
&= \lim_{t\to\infty} e^{-i\omega_k t} C_e^\infty \int_0^t T_e^\infty(t-s) B_e^\infty e^{i\omega_k s} \mathbf{e}_k ds + D_e^\infty \mathbf{e}_k \\
&= \lim_{t\to\infty} C_e^\infty \int_0^t e^{-i\omega_k (t-s)} T_e^\infty(t-s) B_e^\infty \mathbf{e}_k ds + D_e^\infty \mathbf{e}_k \\
&= P_e^\infty(i\omega_k)\mathbf{e}_k
\end{aligned}$$

(since $R(\lambda, A_e^\infty)$ is the Laplace transform of $T_e^\infty(t)$), and thus

$$\left\| (\mathbb{F}_0^\infty v)(t) - \sum_{k=1}^{q} e^{i\omega_k t} P_e^\infty(i\omega_k) v_{0k} \right\| \to 0,$$

as $t \to \infty$. Since $v(\cdot)$ is continuous and uniformly bounded, Lemma III.4 implies $\|(\mathbb{F}_0 v)(t) - (\mathbb{F}_0^\infty v)(t)\| \to 0$ as $t \to \infty$, and since

$$e(t) = C_e(t) U_e(t,0) x_{e0} + (\mathbb{F}_0 v)(t)$$

where $U_e(t,s)$ is exponentially stable, we have that (III.3) holds. □

B. Adaptive exosystem identification

Estimation of the exosystem's frequencies is an important part of our controller design. In this section we propose an adaptive approach to estimate $\{i\omega_k\}_{k=1}^q$ as new information on the reference signals is measured. The proposed observer is different from the proposed techniques in literature including [20], [21] in the sense that it is designed for output dimension $p \geq 1$. It also involves the derivatives of the output signal to improve the settling time and stability properties of the observer.

Define $\eta = [\eta_1, \eta_2, \cdots, \eta_p]^T$. A new system, which includes a p-copy of exosystem, are defined as

$$\dot{\eta} = \begin{bmatrix} S & 0 & \cdots & 0 \\ 0 & S & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S \end{bmatrix} \eta. \tag{III.4}$$

In Section III.B we assume $S \in \mathbb{C}^{2q+1}$ is diagonal with simple eigenvalues such that 0 *is an eigenvalue of S, and* $-i\omega_k$ *is an eigenvalue of S whenever* $i\omega_k$ *is an eigenvalue.* With suitable choice of the initial condition $\eta(0)$, the component reference signals $y_{ref,k}(t)$ have the forms

$$y_{ref,k} = F_k \eta_k \tag{III.5}$$

where $F_k : \mathbb{R}^{2q+1} \to \mathbb{R}$ is a linear operator for $k = 1, \cdots, p$ with nonzero components associated with the state $\eta_k(t)$ and zero components elsewhere. It can be shown that (III.4) and (III.5) can be transformed into

$$\dot{\bar{\eta}} = \begin{bmatrix} \bar{S} & 0 & \cdots & 0 \\ 0 & \bar{S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{S} \end{bmatrix} \bar{\eta} \tag{III.6}$$

where

$$\bar{S} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & a_1 & 0 & \cdots & a_q & 0 \end{bmatrix} \tag{III.7}$$

where $a_1 \neq 0$. Moreover, in this representation we have

$$y_{ref,k} = \bar{\eta}_{k1}, \quad \text{for } k = 1, \cdots, p,$$

where we have denoted $\bar{\eta} = [\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_p]^T$ and $\bar{\eta}_k = [\bar{\eta}_{k1}, \bar{\eta}_{k2}, \cdots, \bar{\eta}_{k(2q+1)}]^T$.

In (III.7), the variables a_k for $k = 1, \cdots, q$ are functions of the frequencies ω_k which are unknown. Let the unknown variables a_k be estimated by $\hat{a}_k$ and define $\hat{a} = [\hat{a}_1, \cdots, \hat{a}_q]^T$. Furthermore, define the variables

$$\theta_k = \bar{\eta}_{k(2q+1)} + b_0 \bar{\eta}_{k1} + b_1 \bar{\eta}_{k2} + \cdots + b_{2q-1} \bar{\eta}_{k2q} \tag{III.8}$$

for $k = 1, \cdots, p$ where the parameters b_i are defined such that the companion matrix associated to the polynomial

$$p_0(\lambda) = \lambda^{2q} + b_{2q-1}\lambda^{2q-1} + \cdots + b_0$$

is Hurwitz. Denote the observer state by $\hat{\eta} = [\hat{\eta}_1, \cdots, \hat{\eta}_p]^T$ with $\hat{\eta}_k = [\hat{\eta}_{k1}, \cdots, \hat{\eta}_{k(2q)}, \hat{\theta}_k]^T$; the observer is defined as

$$\begin{aligned} \dot{\hat{\eta}} &= \begin{bmatrix} \hat{S} & 0 & \cdots & 0 \\ 0 & \hat{S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{S} \end{bmatrix} \hat{\eta} + \begin{bmatrix} B_1(\theta_1 - \hat{\theta}_1) \\ B_2(\theta_2 - \hat{\theta}_2) \\ \vdots \\ B_p(\theta_p - \hat{\theta}_p) \end{bmatrix} \\ \dot{\hat{a}} &= h(y_{ref,1}, \cdots, y_{ref,p}, \hat{\eta}) \end{aligned} \tag{III.9}$$

where $B_k = [0, 0, \cdots, k_0]^T$, $h(\cdot) : \mathbb{R}^{p+p(2q+1)} \to \mathbb{R}^q$ is a vector-valued function constructing the update rule and

$$\hat{S} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -b_0 & -b_1 & \cdots & -b_{2q-1} & 1 \\ \bar{b}_0 & \bar{b}_1 & \cdots & \bar{b}_{2q-1} & \bar{b}_{2q} \end{bmatrix}.$$

where $\bar{b}_0 = -b_0 b_{2q-1}$,$\bar{b}_{2i-1} = \hat{a}_{2i-1} + b_{2i-2} - b_{2q-1} b_{2i-1}$ for for $i = 1, \cdots, q$, $\bar{b}_{2i} = b_{2i-1} - b_{2q-1} b_{2i}$ for $i = 1, \cdots, q-1$, and $\bar{b}_{2q} = b_{2q-1}$. Furthermore, $k_0 > 0$ is a filtering gain. The following theorem introduces an update rule for identifying these variables which can be used to define the frequencies in time. Note that by definition of θ_k, the time derivatives of the reference signal are required to calculate the error $\theta_k - \hat{\theta}_k$. These derivatives can be estimated via a high gain observer introduced in [22].

Theorem III.5. *Let the observer dynamics for the unknown frequencies be defined by* (III.9) *and let* $\gamma > 0$*. Assume further that the measurement vector* $\theta(t) = (\theta_1(t), \cdots, \theta_p(t))$ *satisfies the "persistent excitation" criterion. For the choice of* $h(\cdot) : \mathbb{R}^{p+p(2q+1)} \to \mathbb{R}^q$

$$h = \frac{-1}{\gamma} \sum_{k=1}^{p} (\theta_k - \hat{\theta}_k) \begin{bmatrix} \hat{\eta}_{k2} \\ \hat{\eta}_{k4} \\ \vdots \\ \hat{\eta}_{k(2q)} \end{bmatrix}, \tag{III.10}$$

there exists a parameter vector $\bar{a} = [\bar{a}_1, \cdots, \bar{a}_q]^T$ *for which the system* (III.6) *has a solution* $\bar{\eta}$*, such that* $[\hat{\eta}^T, \hat{a}^T]^T \to [\bar{\eta}^T, \bar{a}^T]^T$*, and thus* $\omega_k(t) \to \omega_k$ *for* $k = 1 \cdots q$*, asymptotically and locally exponentially.*

Proof. Let P_0 be the solution to $P_0 A_0 + A_0 P_0 = -I$ where A_0 is the companion matrix corresponding to the polynomial

$p_0(\cdot)$. Also define $e_k = [\eta_{k1} - \hat{\eta}_{k1}, \cdots, \eta_{k(2q)} - \hat{\eta}_{k(2q)}]^T$. Define a continuous Lyapunov function as

$$V = \sum_{k=1}^{p}(V_k + \frac{1}{2}(\theta_k - \hat{\theta}_k)^2) + \frac{1}{\gamma}e_a^T e_a \tag{III.11}$$

where $e_a = a - \hat{a}$, and $V_k = e_k^T P_0 e_k$. Differentiating both sides of (III.11) with respect to time and employing (III.10) lead to

$$\dot{V} = -\frac{1}{2}e_k^T e_k - \bar{\gamma}\sum_{k=1}^{p}(\theta_k - \hat{\theta}_k)^2 \tag{III.12}$$

where $\bar{\gamma}$ is a positive real number. The asymptotic convergence follows from La Salle theorem and the local exponential convergence follows from the same argument as in [22, Section VI.B]. □

IV. Simulation results

In this section, we consider the output regulation design for a simplified magnetic system. Here, an observer dynamics is embedded in the control dynamics. Define K_{21}, L such that $A + BK_{21}$ and $A + LC$ are exponentially stable. Furthermore, set $G_1(t) = \text{diag}(\hat{S}(t), \cdots, \hat{S}(t))$ as defined in (III.9) and $G_2(t)$ to be full rank, and let $P(t)$ solve the Sylvester differential equation

$$\dot{P}(t) + G_1(t)P(t) = (A + BK_{21})G_1(t) + G_2(t)C.$$

Finally, define $B_1(t) = P(t)B$; choose $K_1(t)$ such that $G_1(t) + B_1(t)K_1(t)$ is exponentially stable. Define $K_2(t) = K_{21} + K_1(t)P(t)$. The controller parameters are chosen $(\mathcal{G}_1(t), \mathcal{G}_2(t), K(t))$ as

$$\begin{aligned} \mathcal{G}_1(t) &= \begin{bmatrix} G_1(t) & 0 \\ BK_1(t) & A + BK_2(t) + LC \end{bmatrix} \\ \mathcal{G}_2(t) &= \begin{bmatrix} 0 \\ -G_2(t) \\ L \end{bmatrix}, \quad K(t) = [K_1(t), K_2(t)]. \end{aligned} \tag{IV.1}$$

It can be shown that under certain natural assumptions the closed loop system obtained using the introduced control parameters is exponentially stable and the controller has Property II.1.

The magnetic drug delivery system considered in this section control the distribution of magnetic nanoparticles in a fluidic environment; the area of interest is located inside a electromagnetic actuator. The electromagnetic structure is composed of four gradient electromagnets and two Helmholtz coils. For details on the actuator configuration, please refer to [23], [24]. The current of the electromagnets are denoted by $I_1(t)$ and $I_3(t)$ in $x-$direction as well as $I_3(t)$ and $I_4(t)$ in $y-$direction; furthermore, the current of the uniform coils are denoted by $I_5(t)$ in $x-$direction and $I_6(t)$ in $y-$direction. The particle distribution dynamics can be represented by

$$\dot{c} = -\nabla.(-D\nabla + \kappa c\nabla(H^T H)) \tag{IV.2}$$

where D is the diffusion coefficient, κ is a coefficient defined by the magnetic properties of the nanoparticles and their size, and H is the magnetization vector and a linear function of the current vector. The boundary conditions are set to be Dirichlet with zero concentration values at boundaries.

Since H is a linear function of the current vector, the second term on the right hand side (RHS) of (IV.2) is quadratic function of currents and can be written as $H^T H = QI$ where Q is vector function of spatial variables and

$$\begin{aligned} I = [&I_1 I_1, I_1 I_2, \cdots, I_1 I_6, \\ &I_2 I_2, \cdots, I_2 I_6, \cdots, I_6 I_6]^T. \end{aligned}$$

In this paper, only four components of the vector I which play more dominant role in magnetic actuation compared to other components are considered in control design; based the information provided in [25], these components are found to be $I_5 I_1$, $I_5 I_3$, $I_6 I_2$, and $I_6 I_4$. In addition, the second term in RHS of (IV.2) is linearized around the initial condition which is a constant.

Equation (IV.2) is solved over a 2D working space of size 2cm×2cm. The specification of the electromagnetic system can be found in [23]. The diffusion coefficient is set as $D = 1 \times 10^{-9}$ m^2/s. The particle size is 500 nm in radius. The concentration is normalized such that the initial condition is $c(0) = 1$. The observer and controller initial conditions are zero. The equations are scaled in time by dividing the time variable by $t_0 = 5 \times 10^5$. The equations are approximated with a finite-dimensional ones using the Finite Element Method with square elements and piecewise linear basis functions. The domain of interest is divided into 15×15 elements. The output operator is defined as

$$y(t) = \begin{bmatrix} \int_{x=-L_0}^{L_0} xc(x,y)dxdy \\ \int_{y=-L_0}^{L_0} yc(x,y)dxdy \\ \int_{x=-L_0}^{L_0} \Pi(x)c(x,y)dxdy \\ \int_{x=-L_0}^{L_0} \Pi(-x)c(x,y)dxdy \end{bmatrix}$$

where $\Pi(\cdot)$ is a Heaviside function. In the new time scale $\bar{t} = t/t_0$, the reference signals are defined as

$$\begin{aligned} y_{ref}(\bar{t}) &= [y_{r1}(\bar{t}), y_{r2}(\bar{t}), y_{r3}(\bar{t}), y_{r4}(\bar{t})]^T \\ &= \begin{bmatrix} .005\sin(20\bar{t}) + .005\sin(60\bar{t}) \\ .005\cos(20\bar{t}) + .005\cos(60\bar{t}) \\ 7.1 \times 10^{-4}(1 + \text{sign}(y_{r1}(\bar{t})))/4 \\ 7.1 \times 10^{-4} - y_{r3}(\bar{t}) \end{bmatrix}. \end{aligned} \tag{IV.3}$$

The controller introduced in (IV.1) is used to force the system (IV.2) follow the reference signals. The number of unknown frequencies is two. The observer (III.9) is solved to update the unknown frequencies simultaneously with (IV.1). The gains K_{21}, $K_1(t)$, and L are defined using MATLAB "lqr" function. For K_{21} and L, all the weight matrices are set to identity. In order to find $K_1(t)$, the LQR problem is solved to stabilize $G_1(t)+5I_{20\times 20}$ with a state weight matrix $Q_0 = 100I_{20\times 20}$ and a control weight matrix $R_0 = I_{4\times 4}$.

The simulation results for frequencies and the output regulation error as functions of time are shown in Figures 1 and 2. It is evident from these figures that the output regulation errors go to small values exponentially. The nonzero

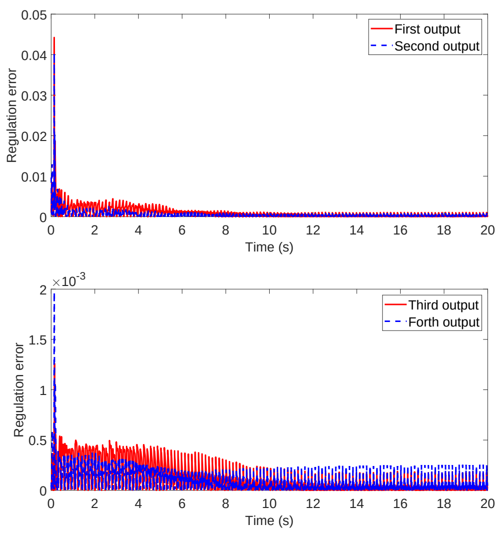

Fig. 1. The components of the regulation error $y(t) - y_{ref}(t)$ as functions of time. The errors converge to small values at an exponential rate.

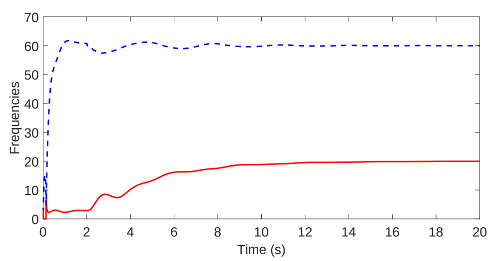

Fig. 2. The unknown reference frequencies estimated by the observer (III.9).

errors are due to numerical errors in the computations. Furthermore, Figure 2 shows that the estimated references converge reasonably fast to true values.

V. Conclusion and Future Works

In this paper we proposed a robust output regulation approach for DPSs with unknown exosystems. The controller design consists of an observer to update the parameters of the exosystem and an internal model based robust output regulator. It was shown that the for converging unknown parameters, the output of the controlled system converges to the reference signal $y_{ref}(t)$. A robust controller satisfying the conditions of Theorem III.2 was designed for an electromagnetic system. The simulation results showed a fast convergence of output regulation errors to small values as well as the convergence of the estimated frequencies. This confirms the performance of the proposed observer. Extending the controller design procedure for more general systems including control affine or semi-linear systems is an important topic for future research.

References

[1] E. Davison, "The output control of linear time-invariant multivariable systems with unmeasurable arbitrary disturbances," *IEEE Transactions on Automatic Control*, vol. 17, no. 5, pp. 621–630, 1972.

[2] J. Huang, *Nonlinear Output Regulation, Theory and Applications*. SIAM, Philadelphia, 2004.

[3] A. Serrani, A. Isidori, and L. Marconi, "Semiglobal nonlinear output regulation with adaptive internal model," *IEEE Trans. Automat. Control*, vol. 46, no. 8, pp. 1178–1194, 2001.

[4] J. M. Schumacher, "Finite-dimensional regulators for a class of infinite-dimensional systems," *Systems & Control Letters*, vol. 3, no. 1, pp. 7–12, 1983.

[5] C. I. Byrnes, I. G. Laukó, D. S. Gilliam, and V. I. Shubov, "Output regulation for linear distributed parameter systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 12, pp. 2236–2252, 2000.

[6] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *Applied mathematics and optimization*, vol. 2, no. 2, pp. 170–194, 1975.

[7] E. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE transactions on Automatic Control*, vol. 21, no. 1, pp. 25–34, 1976.

[8] L. Paunonen and S. Pohjolainen, "Internal model theory for distributed parameter systems," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4753–4775, 2010.

[9] L. Paunonen and S. Pohjolainen, "The internal model principle for systems with unbounded control and observation," *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3967–4000, 2014.

[10] T. Hämäläinen and S. Pohjolainen, "A finite-dimensional robust controller for systems in the CD-algebra," *IEEE Transactions on Automatic control*, vol. 45, no. 3, pp. 421–431, 2000.

[11] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica*, vol. 39, no. 9, pp. 1555–1569, 2003.

[12] E. Immonen, "On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control," *SIAM Journal on Control and Optimization*, vol. 45, no. 6, pp. 2065–2093, 2007.

[13] T. Hämäläinen and S. Pohjolainen, "Robust regulation of distributed parameter systems with infinite-dimensional exosystems," *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 4846–4873, 2010.

[14] L. Paunonen, "Controller design for robust output regulation of regular linear systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 2974–2986, 2016.

[15] R. Marino and P. Tomei, "Output regulation for linear systems via adaptive internal model," *IEEE Transactions on Automatic Control*, vol. 48, pp. 2199–2202, Dec 2003.

[16] X. Wang, H. Ji, and C. Wang, "Output regulation for a class of infinite dimensional systems with an unknown exosystem," *Asian Journal of Control*, vol. 16, no. 5, pp. 1548–1552, 2014.

[17] X. Wang, H. Ji, and J. Sheng, "Output regulation problem for a class of SISO infinite dimensional systems via a finite dimensional dynamic control," *J. Syst. Sci. Complex.*, vol. 27, no. 6, pp. 1172–1191, 2014.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.

[19] S. Clark, Y. Latushkin, S. Montgomery-Smith, and T. Randolph, "Stability radius and internal versus external stability in Banach spaces: an evolution semigroup approach," *SIAM J. Control Optim.*, vol. 38, no. 6, pp. 1757–1793, 2000.

[20] G. Obregon-Pulido, B. Castillo-Toledo, and A. Loukianov, "A globally convergent estimator for n-frequencies," *IEEE Transactions on Automatic Control*, vol. 47, no. 5, pp. 857–863, 2002.

[21] R. Marino and P. Tomei, "Global estimation of n unknown frequencies," *IEEE Transactions on Automatic Control*, vol. 47, pp. 1324–1328, Aug 2002.

[22] A. Serrani, A. Isidori, and L. Marconi, "Semi-global nonlinear output regulation with adaptive internal model," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1178–1194, 2001.

[23] S. Afshar, M. B. Khamesee, and A. Khajepour, "Optimal configuration for electromagnets and coils in magnetic actuators," *IEEE Transactions on Magnetics*, vol. 49, no. 4, pp. 1372–1381, 2013.

[24] S. Afshar, M. B. Khamesee, and A. Khajepour, "MIMO regulation control design for magnetic steering of a ferromagnetic particle inside a fluidic environment," *International Journal of Control*, vol. 88, no. 10, pp. 1942–1962, 2015.

[25] S. Afshar, "Modeling and control of a magnetic drug delivery system," Master's thesis, University of Waterloo, 2012.