IOS superposition theorem for infinite-dimensional systems

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Journal paper:
Bachmann, Dashkovskiy, Mironchenko. Characterization of input-to-output stability for infinite-dimensional systems, IEEE TAC, 2026.

www.mironchenko.com

Control system

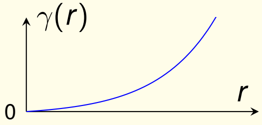
Definition (Control system)

$\Sigma := (X, \mathcal{U}, \phi, Y, h)$ consisting of:

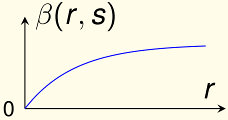
1. State space X
2. Input value space U
3. Input space $\mathcal{U}$ (normed linear subspace of $\{u \,|\, u : \mathbb{R}_+ \to U\}$),
4. Flow map $\phi \colon \mathbb{R}_+ \times X \times \mathcal{U} \to X$, $(x, u) \mapsto \phi(\cdot, x, u)$.
5. Output-value space $(Y, \|\cdot\|_Y)$.
6. Output map $h \colon X \to Y$.

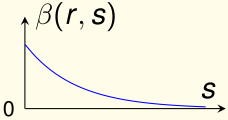
- We assume:
  Σ satisfies the identity, causality, and cocycle properties.
- $y(t, x, u) := h(\phi(t, x, u))$, $(x, u) \in X \times \mathcal{U}$.

Comparison functions



$$\mathcal{K}_\infty = \{\gamma \in C(\mathbb{R}_+, \mathbb{R}_+) \colon \gamma \text{ is strictly increasing, unbounded, } \gamma(0) = 0\}.$$





$$\begin{aligned}\mathcal{KL} := \{\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \colon\ & \forall s \geq 0 \colon \beta(\cdot, s) \in \mathcal{K}_\infty \text{ and} \\ & \forall r > 0 \colon \beta(r, \cdot) \text{ strictly decreasing to zero}\}.\end{aligned}$$

Input-to-output stability

Definition (Sontag, 1989; Jiang, Teel, Praly, 1994)

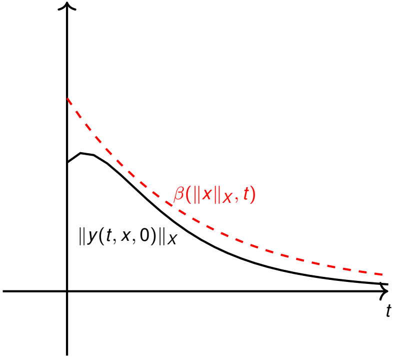
$$\textbf{IOS} \quad :\Leftrightarrow \quad \|y(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

increasing in $\|x\|_X$

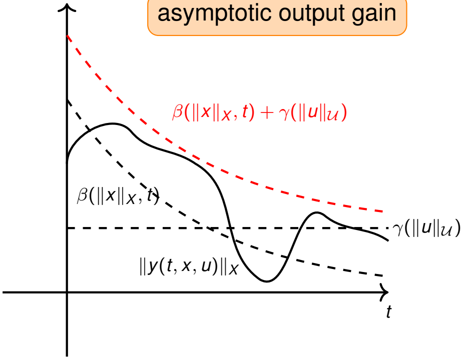
decreasing to 0 in t

$\gamma(0) = 0$, increasing

asymptotic output gain



(a) $u \equiv 0$



(b) $u \not\equiv 0$

Input-to-output stability

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increasing in $\|x\|_X$

decreasing to 0 in t

$\gamma(0) = 0$, increasing

asymptotic output gain

Possible choices of y

- $y =$ state $\quad \Rightarrow \quad$ IOS = input-to-state stability (ISS)
- $y =$ selected coordinates of the state $\quad \Rightarrow \quad$ IOS = partial (robust) stability
- $y =$ tracking error
- $y =$ observation error

IOS of ODEs

ISS and IOS characterizations for ODEs

- Sontag, Wang. *On characterizations of the input-to-state stability property*. SCL, 1995.
- Sontag, Wang. *New characterizations of input-to-state stability*. IEEE TAC, 1996.
- Ingalls, Sontag, Wang. *Generalizations of asymptotic gain characterizations of ISS to input-to-output stability*. ACC, 2001.

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ISS characterizations for infinite-dimensional systems

- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems*, TAC, 2018.

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Applications

- Non-coercive Lyapunov theory for nonlinear systems
- Small-gain theorems for nonlinear (in)finite networks with ∞-dim components
- Relations of ISS and energy-based stability notions
- Lyapunov-Krasovskii theory of time-delay systems

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Today: IOS of infinite-dimensional systems

Motivation: Uniform global asymptotic stability for ODEs

$$\dot{x} = f(x), \quad f \text{ is Lipschitz.} \qquad \text{(ODE)}$$

Definition (UGAS (= IOS with $y = \phi$ and no inputs))

(ODE) is uniformly globally asymptotically stable (UGAS) if $\exists \beta \in \mathcal{KL}$:

$$x \in \mathbb{R}^n, \quad t \geq 0 \quad \Rightarrow \quad |\phi(t,x)| \leq \beta(|x|, t).$$

Proposition (UGAS Superposition Theorem for ODEs)

(ODE) is UGAS $\quad \Leftrightarrow \quad$ the following properties hold:

- (ODE) is forward complete: all solutions exist on $\mathbb{R}_+$.
- (ODE) is locally stable: $\quad \exists \sigma \in \mathcal{K}_\infty$ and $r > 0$:

$$|x| \leq r, \quad t \geq 0 \quad \Rightarrow \quad |\phi(t,x)| \leq \sigma(|x|).$$

- (ODE) is globally attractive:

$$x \in \mathbb{R}^n \quad \Rightarrow \quad \lim_{t \to \infty} |\phi(t,x)| = 0.$$

Can this result be extended to ∞-dim systems?

$$X = l_2 := \Big\{ (x_i)_{i=0}^{\infty} \subset \mathbb{R} : \sum_{i=0}^{\infty} |x_i|^2 < \infty \Big\}.$$

Consider the following system on X, with $a > 0$ small enough.

$$\dot{x}_i = -\frac{1}{i+1} x_i, \qquad i = 0, 1, \ldots.$$

- All modes converge to 0.
- But the higher is the mode, the slower it converges
- There is no uniform decay rate for all initial conditions.

This system is stable and globally attractive, but not UGAS!

Conjecture: Uniform global asymptotic stability for ∞-dim systems

Definition (UGAS (= IOS with $y = \phi$ and no inputs))

Σ is uniformly globally asymptotically stable (UGAS) if $\exists \beta \in \mathcal{KL}$:

$$x \in X, \quad t \geq 0 \quad \Rightarrow \quad \|\phi(t,x)\|_X \leq \beta(\|x\|_X, t).$$

Conjecture (UGAS Superposition Theorem for ∞-dim systems)

Σ is UGAS $\quad \Leftrightarrow \quad$ the following properties hold:

- Σ is forward complete: all solutions exist on $\mathbb{R}_+$.
- Σ is locally stable: $\quad \exists \sigma \in \mathcal{K}_\infty$ and $r > 0$:

$$\|x\|_X \leq r, \quad t \geq 0 \quad \Rightarrow \quad \|\phi(t,x)\|_X \leq \sigma(\|x\|_X).$$

- Σ is uniformly globally attractive (UGATT):

$$R > 0 \quad \Rightarrow \quad \lim_{t \to \infty} \sup_{\|x\|_X \leq R} \|\phi(t,x)\|_X = 0.$$

Example (Bachmann, Dashkovskiy, M. [TAC, 2026]; M., Wirth [TAC, 2018])

I want to construct a system

$$\dot{x} = f(x) \qquad (\Sigma)$$

- f is Lipschitz continuous on bounded balls
- Σ is forward complete: All solutions exist on $\mathbb{R}_+$
- Σ is locally exponentially stable
- Σ is uniformly globally attractive: For any $\varepsilon > 0$, there is $T = T(\varepsilon) > 0$:

$$x \in X, \quad t \geq T \quad \Rightarrow \quad \|\phi(t,x)\|_X \leq \varepsilon.$$

- Σ does not have bounded finite-time reachability sets:

$\exists \tau > 0, C > 0$: $\phi\big([0,\tau] \times \{x \in X : \|x\|_X \leq C\}\big)$ is an unbounded set.

UGATT but not BRS

Example (Bachmann, Dashkovskiy, M. [TAC, 2026]; M., Wirth [TAC, 2018])

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X has to be infinite-dimensional! And f has to be nonlinear!

For nonlinear ∞-dim systems, FC does not imply BRS

$$X = l_2 := \Big\{(x_i)_{i=0}^{\infty} \subset \mathbb{R} : \sum_{i=0}^{\infty} |x_i|^2 < \infty\Big\}.$$

Consider the following system on X, with $a > 0$ small enough.

$$\begin{aligned} \dot{x}_0 &= -x_0 - a x_0^3, \\ \dot{x}_i &= -x_i + x_i|x_i|(x_0 - 1) - \frac{1}{i^2}x_i^3, \qquad i \in \mathbb{N}. \end{aligned}$$

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Literature for those who like counterexamples

- Bachmann, Dashkovskiy, M. *Characterization of input-to-output stability for infinite-dimensional systems*, IEEE TAC, 2026.
- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems*, IEEE TAC, 2018.
- Mancilla-Aguilar, Haimovich. *Forward completeness does not imply bounded reachability sets and global asymptotic stability is not necessarily uniform for time-delay systems*. Automatica, 2024.
- Chaillet, Wirth, M., Brivadis. *For time-invariant delay systems, global asymptotic stability does not imply uniform global attractivity.* L-CSS, 2024.

Theorem: Uniform global asymptotic stability for ∞-dim systems

UGAS Superposition Theorem for ∞-dim systems I

Σ is UGAS $\quad\Leftrightarrow\quad$ the following properties hold:

- Σ has bounded reachability sets: all solutions exist on $\mathbb{R}_+$, and

$$\tau \geq 0, \quad R > 0 \quad \Rightarrow \quad \sup_{\|x\|_X < R,\ t < \tau} \|\phi(t,x)\|_X < \infty.$$

- Σ is locally stable: $\exists \sigma \in \mathcal{K}_\infty$ and $r > 0$:

$$\|x\|_X \leq r, \quad t \geq 0 \quad \Rightarrow \quad \|\phi(t,x)\|_X \leq \sigma(\|x\|_X).$$

- Σ is uniformly globally attractive (UGATT):

$$R > 0 \quad \Rightarrow \quad \lim_{t\to\infty} \sup_{\|x\|_X \leq R} \|\phi(t,x)\|_X = 0.$$

Theorem: Uniform global asymptotic stability for ∞-dim systems

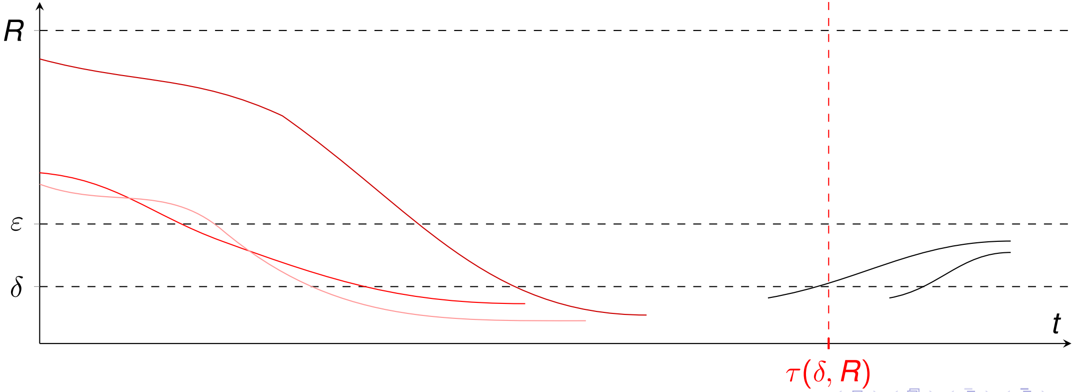
Σ is locally stable: $\exists \sigma \in \mathcal{K}_\infty$ and $r > 0$:

$$\|x\|_X \le r, \quad t \ge 0 \quad \Rightarrow \quad \|\phi(t,x)\|_X \le \sigma(\|x\|_X).$$

Definition (Weak UGATT)

Σ is weakly uniformly globally attractive (weakly UGATT): $\quad \forall \delta > 0,\, R > 0 \;\exists\; \tau = \tau(\delta, R)$

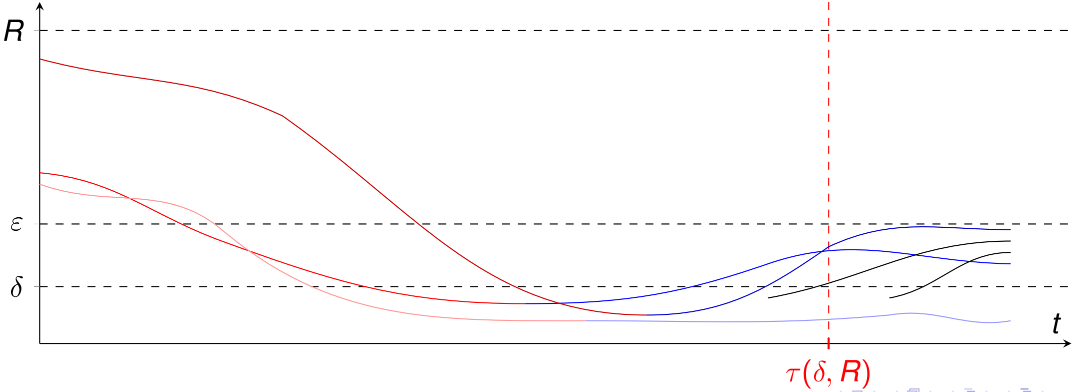
$$\|x\| \le R \quad \Rightarrow \quad \exists t \le \tau(\delta, R) : \quad \|\phi(t,x)\|_X < \delta.$$



Definition (Weak UGATT)

Σ is **weakly uniformly globally attractive (weakly UGATT)**: $\forall \delta > 0,\ R > 0\ \exists\ \tau = \tau(\delta, R)$

$$\|x\| \leq R \quad \Rightarrow \quad \exists t \leq \tau(\delta, R): \quad \|\phi(t,x)\|_X < \delta.$$



Theorem: Uniform global asymptotic stability for ∞-dim systems

UGAS Superposition Theorem for ∞-dim systems

$$\text{UGAS} \quad \Leftrightarrow \quad \text{BRS} \quad \wedge \quad \text{LS} \quad \wedge \quad \text{weak UGATT}$$

Theorem: Uniform global asymptotic stability for ∞-dim systems

UGAS Superposition Theorem for ∞-dim systems

$$\text{UGAS} \quad \Leftrightarrow \quad \text{BRS} \quad \wedge \quad \text{LS} \quad \wedge \quad \text{weak UGATT}$$

Proved in (as a corollary of the main result which is the criteria for ISS)

- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems*, TAC, 2018.

How to check these conditions?

- Lyapunov criteria for BRS
  - Bachmann, M. *Lyapunov characterization of boundedness of reachability sets for infinite-dimensional systems.* Submitted to IEEE TAC, 2026.
- Lyapunov criteria for LS: very classical
- Lyapunov criteria for weak UGATT
  - M., Wirth. *Non-coercive Lyapunov functions for infinite-dimensional systems*, JDE, 2019.
  - Jacob, M., Partington, Wirth. *Noncoercive Lyapunov functions for input-to-state stability of infinite-dimensional systems.*, SICON, 2020.

Definition (Output-uniform local stability (OULS))

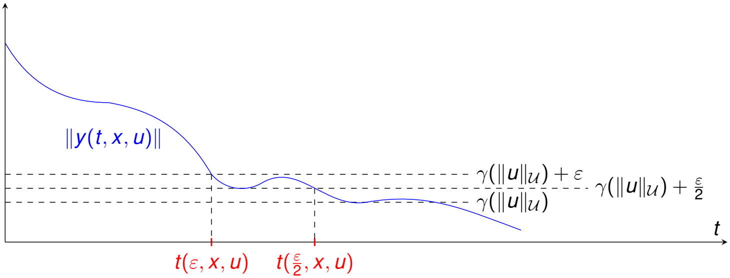
Σ is OULS $\quad := \quad \exists \sigma, \gamma \in \mathcal{K}_\infty$, $r > 0$:

$$\|x\|_X \leq r, \quad \|u\|_{\mathcal{U}} \leq r, \quad t \geq 0 \quad \Rightarrow \quad \|y(t,x,u)\|_Y \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}})$$

Definition (Output-uniform asymptotic gain property (OUAG))

$\exists\gamma \in \mathcal{K}_\infty$: $\forall \varepsilon, r, s > 0$: $\exists \tau = \tau(\varepsilon, r, s) \in \mathbb{R}_+$:

$$\|x\|_X \leq r,\ \|u\|_{\mathcal{U}} \leq s,\ t \geq \tau \quad \Rightarrow \quad \|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$



IOS superposition theorem I

Theorem (IOS superposition theorem; Bachmann, Dashkovskiy, M., IEEE TAC, 2026)

Let $\Sigma := (X, \mathcal{U}, \phi, Y, h)$*. Then,*

$$\textit{IOS} \quad \Leftrightarrow \quad y(\cdot) \textit{ is a bounded map } \wedge \textit{ OULS } \wedge \textit{ OUAG.}$$

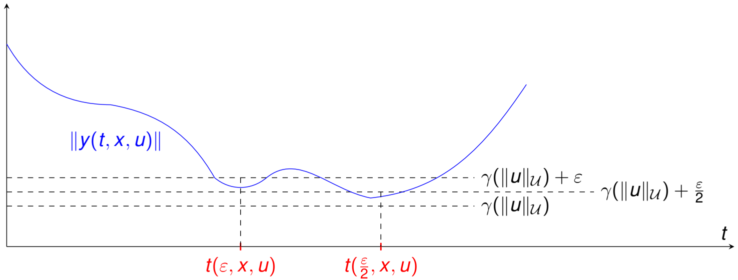
The condition that $y(\cdot)$ is a bounded map, cannot be dropped.

Uniform weak attractivity property

Definition (Output-uniform limit property (OULIM))

$\exists \gamma \in \mathcal{K}_\infty$: $\forall \varepsilon, r, s > 0$: $\exists \tau = \tau(\varepsilon, r, s) \in \mathbb{R}_+$:

$$\|x\|_X \leq r,\ \|u\|_{\mathcal{U}} \leq s \quad \Rightarrow \quad \exists t \leq \tau : \|y(t,x,u)\|_Y \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$



IOS superposition conjecture

We would like to prove:

$$\text{IOS} \quad \Leftrightarrow \quad y(\cdot) \text{ is a bounded map } \wedge \text{ OULS } \wedge \text{ OULIM}.$$

IOS superposition conjecture

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This indeed works, if $y = \phi$:

Corollary (Superposition theorem for systems with full-state output)

Let $\Sigma := (X, \mathcal{U}, \phi, Y, h)$ have full-state output ($y = \phi$). Then:

$$\text{ISS} \quad \Leftrightarrow \quad \text{BRS} \wedge \text{ULS} \wedge \text{ULIM.}$$

IOS superposition conjecture

We would like to prove:

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Let $\Sigma := (X, \mathcal{U}, \phi, Y, h)$ have full-state output ($y = \phi$). Then:

$$\mathit{ISS} \quad \Leftrightarrow \quad \mathit{BRS} \wedge \mathit{ULS} \wedge \mathit{ULIM}.$$

$y \neq \phi \quad \Rightarrow \quad$ this criterion is not true.

Local stability for systems with inputs

Definition (Output-uniform global stability (OUGS))

$$\Sigma \text{ is OUGS} \quad := \quad \exists \sigma, \gamma \in \mathcal{K}_\infty :$$

$$x \in X, \quad u \in \mathcal{U}, \quad t \geq 0 \quad \Rightarrow \quad \|y(t,x,u)\|_Y \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}})$$

Definition (Output Lagrange stability (OL))

$$\Sigma \text{ is OL} \quad := \quad \exists \sigma, \gamma \in \mathcal{K}_\infty :$$

$$x \in X, \quad u \in \mathcal{U}, \quad t \geq 0 \quad \Rightarrow \quad \|y(t,x,u)\|_Y \leq \sigma(\|y(0,x,u)\|_Y) + \gamma(\|u\|_{\mathcal{U}})$$

$$y = \phi \quad \Rightarrow \quad \text{OUGS} = \text{OL}.$$

$y \neq \phi \quad \Rightarrow \quad$ OUGS $\neq$ OL, a major difficulty in analysis of IOS in compare to ISS.

IOS superposition theorem II

Theorem (IOS superposition theorem; Bachmann, Dashkovskiy, M., IEEE TAC, 2026)

Let $\Sigma := (X, \mathcal{U}, \phi, Y, h)$, *and let*

- *h be* $\mathcal{K}$*-bounded:* $\exists \sigma_1, \gamma_1 \in \mathcal{K}_\infty$:

$$x \in X \quad \Rightarrow \quad \|h(x)\|_Y \leq \sigma_1(\|x\|_X).$$

Then,

$$\text{IOS} \wedge \text{OL} \quad \Leftrightarrow \quad \text{OULIM} \wedge \text{OL}.$$

IOS superposition theorem II

Theorem (IOS superposition theorem; Bachmann, Dashkovskiy, M., IEEE TAC, 2026)

Let $\Sigma := (X, \mathcal{U}, \phi, Y, h)$*, and let*

- *h be* $\mathcal{K}$*-bounded:* $\exists \sigma_1, \gamma_1 \in \mathcal{K}_\infty$:

$$x \in X \quad \Rightarrow \quad \|h(x)\|_Y \leq \sigma_1(\|x\|_X).$$

Then,

$$IOS \wedge OL \quad \Leftrightarrow \quad OULIM \wedge OL.$$

Potential applications

- IOS Lyapunov-Krasovskii theory for delay systems
- Non-coercive Lyapunov theory
- IOS small-gain theory
- Relations of IOS and energy-based stability notions

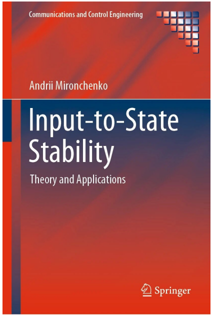
Conclusions

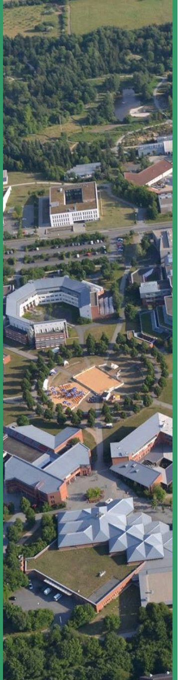
Main results

- IOS $\Leftrightarrow$ OUAG $\wedge$ OULS $\wedge$ $y(\cdot)$ is a bounded map.
- IOS $\wedge$ OL $\Leftrightarrow$ OULIM $\wedge$ OL $\wedge$ h is K-bounded.
- Boundedness of reachability sets is important!

More theory, examples & applications:

- Bachmann, Dashkovskiy, M. *Characterization of input-to-output stability for infinite-dimensional systems*, IEEE TAC, 2026.
- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems*, IEEE TAC, 2018.
- Bachmann, M. *Lyapunov characterization of boundedness of reachability sets for infinite-dimensional systems.* Submitted to IEEE TAC, 2026.
- Bachmann, Dashkovskiy, M. *An IOS small-gain theorem for networks of distributed parameter systems with outputs*. Submitted to CDC 2026.





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Lucas Brivadis *(L2S, CNRS)*
Emilia Fridman *(U Tel Aviv)*
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Iasson Karafyllis *(NTU Athens)*
Rami Katz *(U Tel Aviv)*
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