Model Predictive Control of Discrete-time Linear Port-Hamiltonian Boundary Control Systems

Recent Trends in Control and Estimation of Distributed Parameter Systems

ECC 2026 · Workshop · July 7–10, Reykjavík, Iceland

Outline

- ☑ Introduction & Motivations
- ☑ Discrete-time port-Hamiltonian BCSs
- ☑ Explicit solution of the discrete-time dynamics
- ☑ MPC design for discrete-time port-Hamiltonian BCSs
- ☑ Numerical example
- ☑ Conclusions & Future activities

Introduction & Motivations

- This talk deals with *discrete-time BCSs* in *port-Hamiltonian* form
  - We present a *parametrized* expression of the state evolution, used to….
  - Design a simple *model predictive control* (MPC) law, by exploiting the passivity of the discrete-time system
- This work shares several ideas with

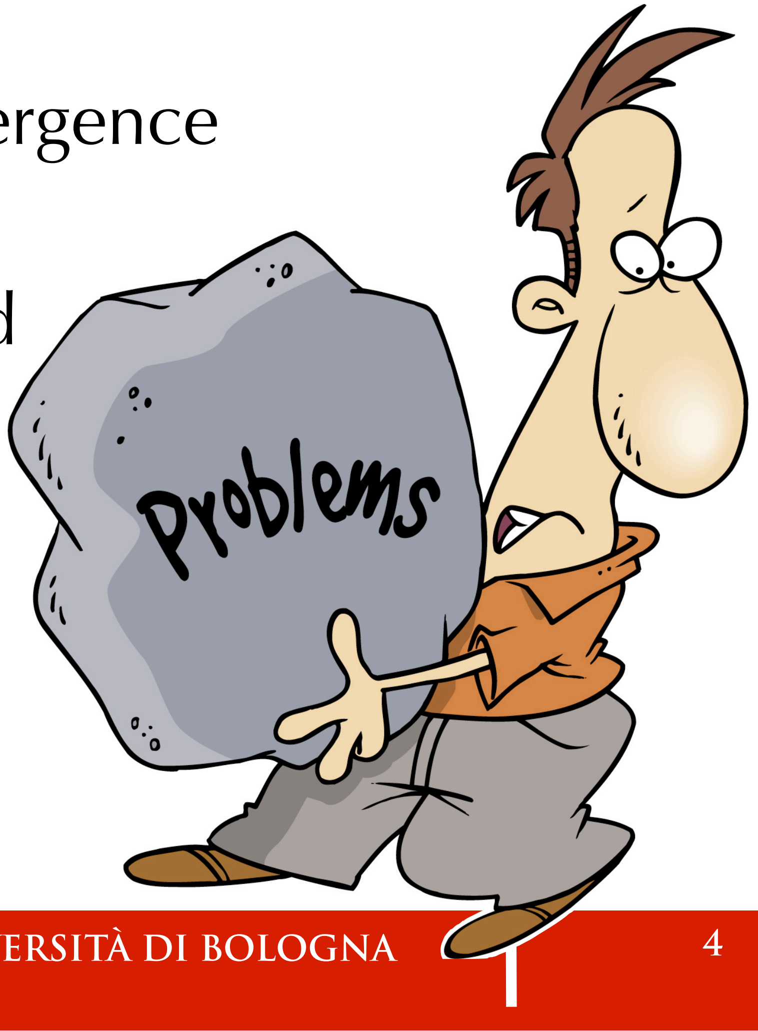
> S. Dubljevic and J.-P. Humaloja, *"Model predictive control for regular linear systems,"* Automatica, vol. 119, p. 109066, Sep. 2020

in particular the fact that the MPC design is based on a *discrete-time approximation* of the plant

  - In the linear case, the discrete-time approximations are *equivalent*
  - Here, the dynamics is described by a **BVP** to be solved at each iteration to have the state evolution that is a constraint in the optimization problem
  - We derive a *parametrization* of the trajectory and of the control input in terms of a finite number of parameters which become the *design variables*

Introduction & Motivations

- The parametrization is used for the *MPC design*
  - The approach relies on standard arguments and the optimization problem includes *input and output constraints*
  - To achieve asymptotic stability, a *terminal cost* and a *terminal constraint* are introduced
  - The passivity of the BCS, the *damping injection law* and the terminal constraint play a fundamental role in deriving conditions for constraint satisfaction
  - The *terminal cost* is necessary to have asymptotic convergence by relying on passivity arguments
  - The optimization problem is a quadratically constrained quadratic programming *(QCQP) problem*
  - **Feasibility is not guaranteed** when output constraints are present, and is assumed at the beginning of the transient

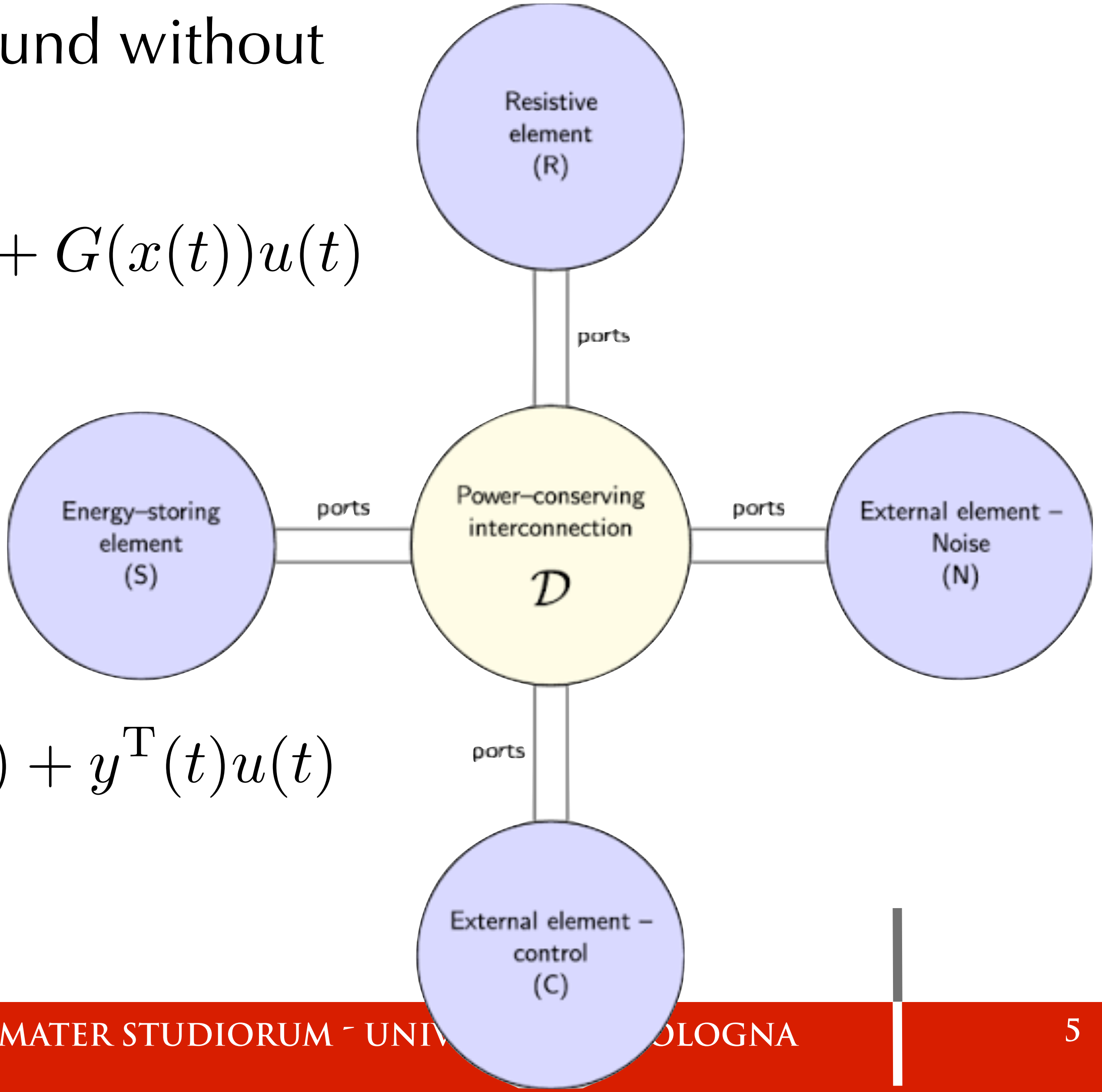


☑ What is a *port-Hamiltonian system?*

- It is modeling framework for complex physical systems that "tracks" *their energy*
- It breaks systems *into parts* that store energy (like springs or capacitors), parts that lose energy (like friction or resistors), and interconnections that move energy around without losing it.

$$\begin{cases} \dot{x}(t) = \big[J(x(t)) - R(x(t))\big]\nabla H(x(t)) + G(x(t))u(t) \\ y(t) = G^{\mathrm{T}}(x(t))\nabla H(x(t)) \\ x(0) = x_0 \end{cases}$$

$$\dot{H}(x(t)) = -\nabla^{\mathrm{T}} H(x(t))R(x(t))\nabla H(x(t)) + y^{\mathrm{T}}(t)u(t)$$



- This work is that it is heavily based on a *discrete-time* approximation of a *linear, port-Hamiltonian BCS*
  - To present the framework, we start from the *finite-dimensional* case, so we provide the discrete-time formulation of the port-Hamiltonian system

$$\begin{cases} \dot{x}(t) = F(x(t))\nabla H(x(t)) + G(x(t))u(t) \\ y(t) = G^{\mathrm{T}}(x(t))\nabla H(x(t)) \\ x(0) = x_0 \end{cases}$$

$F(x) + F^{\mathrm{T}}(x) \leq 0$

  - Along system trajectories we have that

$$\dot{H}(x(t)) = \nabla^{\mathrm{T}} H(x(t))F(x(t))\nabla H(x(t)) + y^{\mathrm{T}}(t)u(t)$$

so if H is lower-bounded, we have *passivity*

  - To get the discrete-time dynamics, the time-derivative is approximated by the *finite difference* as

$$\dot{x}(t_k) \simeq \frac{1}{\tau}(x_{k+1} - x_k)$$

$x_k = x(t_k)$

$t_k = k\tau$ with $\tau > 0$ and $k \in \mathbb{N}$

...but what about the gradient operator?

Introduction & Motivations

Definition. Let $\Phi : \mathbb{R}^p \to \mathbb{R}$ be a continuously differentiable function. A *discrete gradient* $\bar{\nabla}\Phi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ is a continuous map such that, for all $\varphi, \varphi_+ \in \mathbb{R}^p$, we have

$$(\varphi_+ - \varphi)^{\mathrm{T}} \bar{\nabla}\Phi(\varphi, \varphi_+) = \Phi(\varphi_+) - \Phi(\varphi),$$

$$\lim_{\varphi_+ \to \varphi} \bar{\nabla}\Phi(\varphi, \varphi_+) = \nabla\Phi(\varphi)$$

- Examples are the *mean value*

$$\bar{\nabla}\Phi(\varphi, \varphi_+) = \int_0^1 \nabla\Phi\big((1-\sigma)\varphi + \sigma\varphi_+\big)\, \mathrm{d}\sigma$$

or the Gonzales discrete gradients

- Note that

$$\Phi(\varphi) = \frac{1}{2}\varphi^{\mathrm{T}} Q \varphi \qquad \bar{\nabla}\Phi(\varphi, \varphi_+) = \frac{1}{2} Q\,(\varphi + \varphi_+)$$

Introduction & Motivations

- The *discrete-time formulation* of the port-Hamiltonian dynamics is

$$\begin{cases} x_{k+1} = x_k + \tau \bar{F}(\hat{x}_k)\bar{\nabla} H(\hat{x}_k) + \tau \bar{G}(\hat{x}_k) u_k \\ y_k = \bar{G}^{\mathrm{T}}(\hat{x}_k)\bar{\nabla} H(\hat{x}_k) \end{cases}$$

$$\hat{x}_k = \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} \in \mathbb{R}^{2n}$$

where $\bar{F}$ and $\bar{G}$ are the approximations of F and G, i.e.:

$$\bar{F}(x, x_+) + \bar{F}^{\mathrm{T}}(x, x_+) \leq 0 \qquad \bar{G}(x, x) = G(x)$$

$$\bar{F}(x, x) = F(x)$$

- With simple calculations, we get that

$$H(x_{k+1}) - H(x_k) = \tau \bar{\nabla}^{\mathrm{T}} H(\hat{x}_k)\bar{F}(\hat{x}_k)\bar{\nabla} H(\hat{x}_k) + \tau y_k^{\mathrm{T}} u_k$$

- In the *linear case*, it is easy to see that

$$\begin{cases} x_{k+1} = x_k + \frac{\tau}{2} F Q (x_k + x_{k+1}) + \tau G u_k \\ y_k = \frac{1}{2} G^{\mathrm{T}} (x_k + x_{k+1}) \end{cases}$$

In the *linear case*, this discretization is equivalent to the *Cayley transform*, i.e. the following approximation has been made

$$\frac{1}{2}\left[\dot{x}(t_{k+1}) + \dot{x}(t_k)\right] \simeq \frac{1}{\tau}(x_{k+1} - x_k)$$

The drawback of this discrete-time system is that the dynamics is in *implicit form.* This is the price to pay to have "for free" the *energy-balance relation*

Discrete-time port-Hamiltonian BCSs

- We refer to the class of *linear port-Hamiltonian BCSs* in the form

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}x(t,z)\big) + P_0 \mathcal{L}x(t,z)$$

$P_1 = P_1^{\mathrm{T}}$

$P_0 + P_0^{\mathrm{T}} \leq 0$

where $z \in [a,b]$, $x \in L^2([a,b],\mathbb{R}^n)$,

$$H(x(t)) = \int_a^b \underbrace{\tfrac{1}{2}x^{\mathrm{T}}(t,z)\mathcal{L}x(t,z)}_{=h(x(t,z))}\,\mathrm{d}z$$

and $e(t,z) = \mathscr{L}x(t,z) \equiv \nabla h(x(t,z))$ denotes the co-energy variable

- The *boundary variables* are defined as

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R}\begin{pmatrix} e(b) \\ e(a) \end{pmatrix}$$

$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$

and the *input* and *output* as

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \in \mathbb{R}^n$$

$\tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$
$W\Sigma W^{\mathrm{T}} = 0$

$W\Sigma\tilde{W}^{\mathrm{T}} = I$

Discrete-time port-Hamiltonian BCSs

- Along system's trajectories we have that

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = \int_a^b e^{\mathrm{T}}(t,z)\frac{\partial x}{\partial t}(t,z)\,\mathrm{d}z$$

$$\leq 0 \qquad = \int_a^b e^{\mathrm{T}}(t,z)P_0 e(t,z)\,\mathrm{d}z + e_\partial^{\mathrm{T}}(t)f_\partial(t)$$

$$\leq y^{\mathrm{T}}(t)u(t)$$

  - The BCS is *well-posed* in the sense of the semigroup theory

- The goal is to get a *discrete-time formulation* of the BCS that preserves
  - The *dependance on z* of the energy variable x, and…
  - The *passivity* property

- We start by approximating the *time derivative* as

$$\frac{\partial x}{\partial t}(t_k,z) \simeq \tfrac{1}{\tau}\left[x_{k+1}(z) - x_k(z)\right]$$

and the *co-energy variable* with the discrete gradient of $h(x)$

$$\mathcal{L}x(t_k,z) \simeq \bar{\nabla}h(x_k,x_{k+1}) = \tfrac{1}{2}\mathcal{L}(x_k + x_{k+1}) = e_k$$

Discrete-time port-Hamiltonian BCSs

- Putting everything together, we get the *discrete-time* system

$$\frac{1}{\tau}\left[x_{k+1}(z) - x_k(z)\right] = P_1 \frac{\mathrm{d}}{\mathrm{d}z} e_k(z) + P_0 e_k(z)$$

- The boundary *input* and *outputs* are defined as follows

$$\begin{pmatrix} u_k \\ 0 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix}$$

$$\begin{pmatrix} y_k \\ \bar{y}_k \end{pmatrix} = \begin{pmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{pmatrix} \begin{pmatrix} f_{\partial k} \\ e_{\partial k} \end{pmatrix}$$

$u_k, y_k \in \mathbb{R}^m$
$\bar{y}_k \in \mathbb{R}^{n-m}$

- The definition of u_k assures the *well-posedness* iff at each k for a given x_k there exists only one x_{k+1} that satisfies the previous ODE
  - The following *"boundary conditions"* are imposed on x_{k+1}:

$$W R \begin{pmatrix} \mathcal{L} x_{k+1}(\ell) \\ \mathcal{L} x_{k+1}(0) \end{pmatrix} = 2 \begin{pmatrix} u_k \\ 0 \end{pmatrix} - W R \begin{pmatrix} \mathcal{L} x_k(\ell) \\ \mathcal{L} x_k(0) \end{pmatrix}$$

  - It has been shown that the resulting *boundary-value problem* has a *unique solution* for every x_k and u_k

Discrete-time port-Hamiltonian BCSs

Proposition. The static feedback action

$$u_k = -K y_k, \quad K = K^{\mathrm{T}} > 0$$ *damping injection*

asymptotically stabilizes the discrete-time dynamics if K is such that

$$y_k^{\mathrm{T}} u_k \leq -\kappa_\ell \left\| e_k(\ell) \right\|^2 \qquad \text{or} \qquad y_k^{\mathrm{T}} u_k \leq -\kappa_0 \left\| e_k(0) \right\|^2$$

for some $\kappa_0, \kappa_\ell > 0$ and all $k \in \mathbb{N}$

Proof. We really on the *energy-multiplier method* and on the Lyapunov function $V_\varepsilon(x) = H(x) + \varepsilon W(x)$, where

$$W(x) = \frac{1}{2} \int_0^\ell \mu(z) x^{\mathrm{T}}(z) P_1^{-1} x(z) \, \mathrm{d}z$$

Since for some $\varepsilon_0 > \varepsilon$ we have that $\varepsilon_0 \, | \, W(x) \, | \leq H(x)$, we get that

$$\left(1 - \varepsilon_0^{-1} \varepsilon\right) H(x) \leq V_\varepsilon(x) \leq \left(1 + \varepsilon_0^{-1} \varepsilon\right) H(x)$$

and also that

$$\begin{aligned} V_\varepsilon(x_{k+1}) - V_\varepsilon(x_k) \leq & - \tau \kappa_K \left\| y_k \right\|^2 - \tau \varepsilon \bar{\kappa}_\mu \left\| \bar{y}_k \right\|^2 \\ & - \tau \varepsilon \kappa_\mu \left\| e_k \right\|^2 \end{aligned}$$

with $\kappa_K, \kappa_\mu, \bar{\kappa}_\mu > 0$

Explicit solution of the discrete-time dynamics

- The damping injection law does not allow to explicitly *handle constraints* on the input and on the output
  - This motivates the design of *MPC laws*
  - Note that the *passivity* and the *stability results* are independent from τ, while the accuracy increases when $\tau > 0$

- The *receding-horizon control law* that regulates the state x_k of the BCS equipped with the input u_k by iteratively solving an *optimization problem* over an *finite horizon* $N \in \mathbb{N}$
  - To simplify its solution, we *"get rid"* of the state x_k, and express the evolution thanks to a certain number of parameters
  - Note that the dynamics can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}e_k}{\mathrm{d}z}(z) &= P_1^{-1}\left(\tfrac{2}{\tau}\mathcal{L}^{-1} - P_0\right)e_k(z) - \tfrac{2}{\tau}P_1^{-1}x_k(z)\\ &= \Pi e_k(z) + \Phi x_k(z)\end{aligned}$$

Explicit solution of the discrete-time dynamics

$$\frac{de_k}{dz} = \Pi e_k(z) + \Phi x_k(z)$$

- The idea is to *iteratively solve* the previous ODE with a free "initial condition" which corresponds to $e_k(0)$
  - Such a solution is employed to *compute* x_{k+1} which appears as a "forcing action" in the same ODE, but at step $k+1$

- **Proposition.** Given $x_0 \in X$ and for all $k = 0, \ldots, N$, there exists a unique set of *constants* $\bar{e}_k \in \mathbb{R}^n$ such that for every u_k

$$e_k(z) = \mathrm{e}^{\Pi z}\left[\bar{e}_k + \sum_{i=0}^{k-1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\Gamma_j(z)\bar{e}_{k-i-1} + \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}X_j(z)\right]$$

$$x_{k+1} = 2\mathscr{L}^{-1}e_k - x_k$$

If we let

$$\gamma(z) = 2\mathrm{e}^{-\Pi z}\Phi\mathcal{L}^{-1}\mathrm{e}^{\Pi z}$$

then we have that…

Explicit solution of the discrete-time dynamics

- … for any $z \in [0, \ell]$:

$$X_0(z) = \int_0^z \mathrm{e}^{-\Pi\sigma}\Phi x_0(\sigma)\,\mathrm{d}\sigma$$

$$\Gamma_0(z) = 2\int_0^z \mathrm{e}^{-\Pi\sigma}\Phi\mathcal{L}^{-1}\mathrm{e}^{\Pi\sigma}\,\mathrm{d}\sigma = \int_0^z \gamma(\sigma)\,\mathrm{d}\sigma$$

These quantities can be computed in advance…

and

$$X_i(z) = \int_0^z \gamma(\sigma)X_{i-1}(\sigma)\,\mathrm{d}\sigma \qquad \Gamma_i(z) = \int_0^z \gamma(\sigma)\Gamma_{i-1}(\sigma)\,\mathrm{d}\sigma$$

- The result is proved by induction
  - For example, for $k = 0$, we have

$$e_0(z) = \mathrm{e}^{\Pi z}\left[\bar{e}_0 + X_0(z)\right]$$

where $\bar{e}_0$ is the *unique solution* for given x_0 and u_0 of

$$\begin{pmatrix} u_0 \\ 0 \end{pmatrix} = WR\begin{pmatrix} \mathrm{e}^{\Pi\ell} \\ I \end{pmatrix}\bar{e}_0 + WR\begin{pmatrix} \mathrm{e}^{\Pi\ell} \\ 0 \end{pmatrix}X_0(\ell)$$

The "next" state can be computed since $x_{k+1} = 2\mathcal{L}^{-1}e_k - x_k$

MPC design

- We drive the state x_k to the origin via a *receding-horizon* control law
  - At each sampling instant t, we determine a sequence of future controls of length N and only the *first one* is applied; at $t+1$, the sequence is computed again based on a *"new"* measure
- The control input is determined by solving an *optimal control problem* at time $t \in \mathbb{N}$, with initial state $x_0 = x_t$
  - Such a problem is formulated as

$$\min J(\hat{x}_t, \hat{u}_t) + J'(x_{N+1})$$

$\hat{x}_t = (x_0, \ldots, x_N)$
$\hat{u}_t = (u_0, \ldots, u_N)$

where

$$J(\hat{x}_t, \hat{u}_t) = \sum_{k=0}^{N} \left[\|x_k\|_Q^2 + \|u_k\|_R^2 \right] \quad + \quad J'(x_{N+1}) = \kappa_\Omega V_\varepsilon(x_{N+1})$$

$\kappa_\Omega > 0$

subject to the system dynamics, $x_0 = x_t$ with $t \in \mathbb{N}$, and

$$u_k \in U, \text{ with } k = 0, \ldots, N$$

$$y_k \in Y \text{ and } \bar{y}_k \in \bar{Y}, \text{ with } k = 0, \ldots, N$$

$$x_{N+1} \in X_\Omega$$

MPC design

$\hat{e}_t = (\bar{e}_0, \ldots, \bar{e}_N)$

- The constraints associated to the *BCS dynamics* can be written as

$$e_k(z) = \Upsilon_k(z)\hat{e}_t + \Xi_k(z;x_0) \qquad + \qquad x_{k+1}(z) = \Upsilon'_k(z)\hat{e}_t + \Xi'_k(z;x_0)$$

and this implies that

$$\begin{pmatrix} u_k \\ 0 \end{pmatrix} = WR\begin{pmatrix} \Upsilon_k(\ell) \\ \Upsilon_k(0) \end{pmatrix}\hat{e}_t + WR\begin{pmatrix} \Xi_k(\ell;x_0) \\ \Xi_k(0;x_0) \end{pmatrix}$$

$$\begin{pmatrix} y_k \\ \bar{y}_k \end{pmatrix} = \tilde{W}R\begin{pmatrix} \Upsilon_k(\ell) \\ \Upsilon_k(0) \end{pmatrix}\hat{e}_t + \tilde{W}R\begin{pmatrix} \Xi_k(\ell;x_0) \\ \Xi_k(0;x_0) \end{pmatrix}$$

- We map the optimization problem to a *QCQP,* with design variable $\hat{e}_t$
  - The *terminal constraint* is based on an L^2-norm and it introduces a quadratic constraint on $\hat{e}_t$; for some $\delta' > \delta > 0$, we let

$$X_\Omega = \{x \in X \mid H(x) \leq \delta\} \qquad + \qquad X'_\Omega = \{x \in X \mid H(x) \leq \delta'\}$$

  - The condition $x_{N+1} \in X_\Omega$ means that, at the end of the prediction horizon, the *stored energy* is "small"
  - Finally, also the *terminal cost* is quadratic in x_{N+1} or in $\hat{e}_t$

Proposition. The initial optimization problem with the *input and output constraints* reformulated as

$$u_m \le u_k \le u_M$$

$$y_m \le y_k \le y_M$$

$$\bar{y}_m \le \bar{y}_k \le \bar{y}_M$$

is a QCQP problem in the design variable $\bar{e}_t$

QCQP = Quadratically Constrained Quadratic Programming

- The *computations* related to the parametrization are *performed in advance,* so the complexity of the MPC scheme is rather low
  - This is immediate for the functions Υ_k and Υ'_k

$e_k(z) = \Upsilon_k(z)\hat{e}_t + \Xi_k(z; x_0)$

- If x_0 is represented by a *sequence of samples,* also the operators Ξ_k and Ξ'_k admit a matrix representation that can be computed off-line

The *stability proof* requires to define the set

$$X_Q = \left\{ x \in X \mid \|x\|_Q^2 \le \delta_Q \right\}$$

$\delta_Q > 0$

$$X_\Omega \subset X_Q \subset X'_\Omega$$

Proposition. Let us assume that there exists $\delta_U, \delta_Y, \delta_{\bar{Y}} > 0$ such that

$$\left\{u \in \mathbb{R}^m \mid \|u\|^2 \leq \delta_U\right\} \subseteq U$$

$$\left\{y \in \mathbb{R}^m \mid \|y\|^2 \leq \delta_Y\right\} \subseteq Y$$

$$\left\{\bar{y} \in \mathbb{R}^{n-m} \mid \|\bar{y}\|^2 \leq \delta_{\bar{Y}}\right\} \subseteq \bar{Y}$$

If the prediction interval N is sufficiently long so that the *optimization problem is feasible,* the receding horizon control law renders the BCS asymptotically stable in the nominal case if there exists a *stabilizing damping injection gain K* and

$$\kappa_\Omega \geq (\tau \kappa_K)^{-1} \lambda_{\max}(KRK)$$

$$\delta' \leq \frac{\tau}{1+\varepsilon_0^{-1}\varepsilon} \min\left(\kappa_K \lambda_{\max}^2(K)\delta_U, \kappa_K \delta_Y, \varepsilon \bar{\kappa}_\mu \delta_{\bar{Y}}\right)$$

☑ **Proof.** (Almost) By construction, if $x_t \in X \backslash X_Q$, the state reaches X_Q in finite time. Thus, we have to study what happens when $x_t \in X_Q$

- In this case, we employ the *sub-optimal* control sequence
$$\hat{u}_t^{\text{sub}} = (u_t^{\text{sub}}, \ldots, u_{t+N}^{\text{sub}})$$
obtained thanks to the *damping injection* law, an so $x_{t+N+1} \in X_\Omega$ if N is sufficiently large
- The only thing that must be checked is the *input and output constraints are satisfied:* this follows from the passivity of the BCS and from the fact that for the damping injection law we have that
$$V_\varepsilon(x_{k+1}) - V_\varepsilon(x_k) \leq -\tau\kappa_K \left\|y_k\right\|^2 - \tau\varepsilon\bar{\kappa}_\mu \left\|\bar{y}_k\right\|^2 - \tau\varepsilon\kappa_\mu \left\|e_k\right\|^2$$
and also
$$\left(1 - \varepsilon_0^{-1}\varepsilon\right) H(x) \leq V_\varepsilon(x) \leq \left(1 + \varepsilon_0^{-1}\varepsilon\right) H(x)$$
- The damping injection control law provides a sub-optimal input that is only *instrumental* for the proof, and the gain K is related to the design parameter δ' associated with the *terminal constraint*

Two limitations!!

- The lack of a well-defined *lower bound on the prediction horizon N*
  - The first issue is due to the fact that, unlike the continuous-time case, no *exponential stability results* are available for the discrete-time BCS
  - In practice, N may be selected using the *continuous-time result,* or tuned according to the *propagation speed* of the underlying hyperbolic PDE so to have enough time to see the effect of the control input and of the current state on the output

- No *feasibility characterization* for the optimization problem
  - *Input constraints* can be handled by choosing a sufficiently long prediction horizon, whereas *output constraints* depend on the initial condition
  - When the system is far from equilibrium, i.e. the stored energy is large, the simultaneous presence of input and output constraints may render the optimization problem infeasible, especially during the *initial transient*

Numerical example

☑ The MPC strategy damps the longitudinal vibrations in a beam

- The state variable is $x = (q, p)$, being q and p the beam deformation and linear momentum density, and

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix} + \mathcal{L} = \begin{pmatrix} ES_b & 0 \\ 0 & \frac{1}{\rho S_b} \end{pmatrix}$$

- The *co-energy variables* are the velocity of the cross section $v = \dfrac{p}{\rho S_b}$ and the axial stress $\sigma = ES_b q$
- *Boundary actuation* and *sensing* are defined as

$$\begin{aligned} u_k &= \sigma_k(\ell) = \tfrac{1}{2} ES_b\big[q_{k+1}(\ell) + q_k(\ell)\big] \\ y_k &= v_k(\ell) = \tfrac{1}{2\rho S_b}\big[p_{k+1}(\ell) + p_k(\ell)\big] \\ 0 &= v_k(0) = \tfrac{1}{2\rho S_b}\big[p_{k+1}(0) + p_k(0)\big] \\ -\bar{y}_k &= \sigma_k(0) = \tfrac{1}{2} ES_b\big[q_{k+1}(0) + q_k(0)\big] \end{aligned}$$

$\ell = 1$
$D = 0$
$\mathscr{L} = I$

- The sampling period is $\tau = 0.05$ and we have chosen $K = 1$
- For simplicity, no constraints are imposed on $\bar{y}_k$

Numerical example

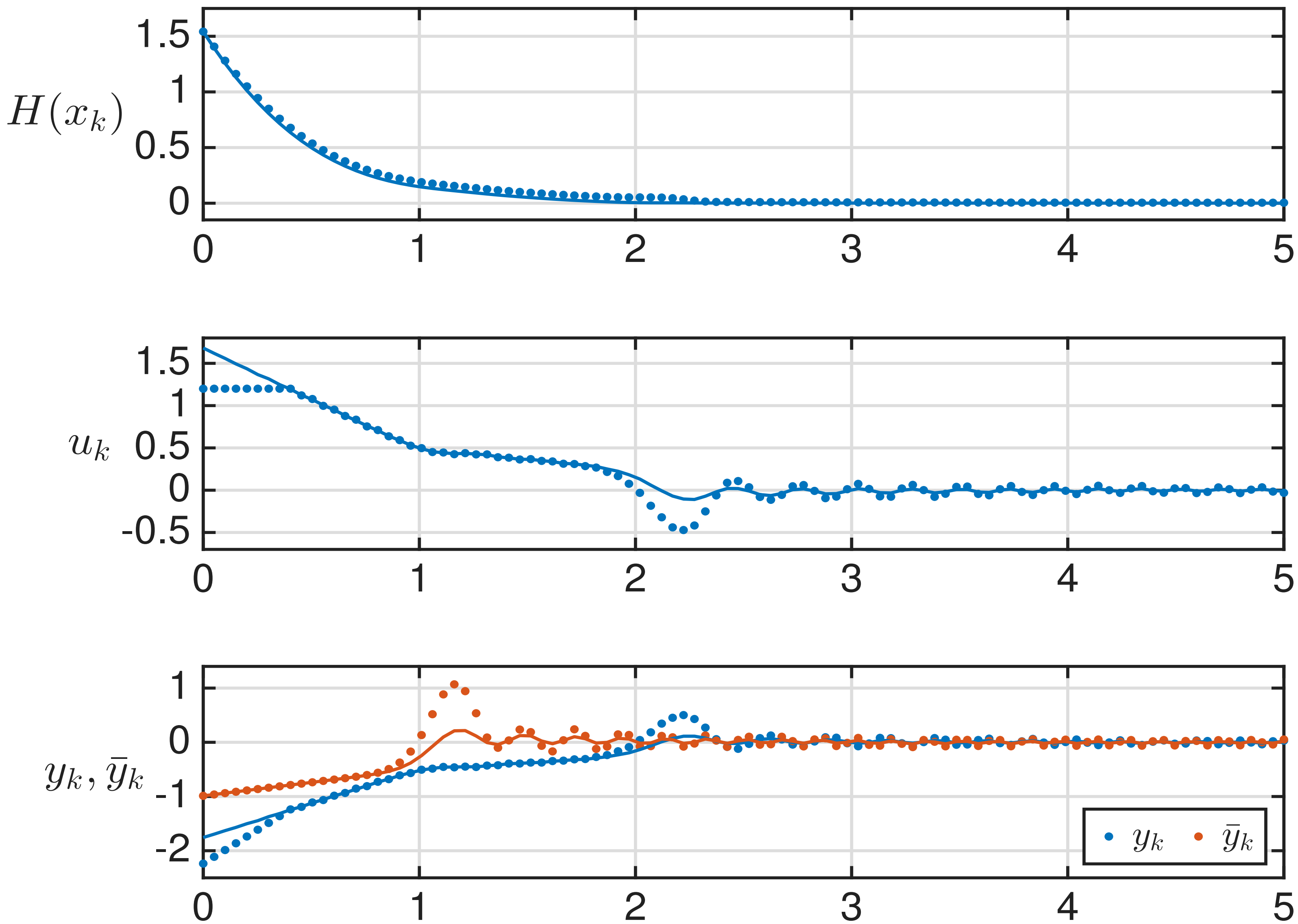
- Some information about the main **parameters**
  - Based on empirical considerations, the *prediction horizon* is $N = 30$, thus it is possible to have a prediction horizon of 1.5s
  - The weighting matrices are $Q = 10I$ and $R = 1$
  - The terminal cost $\kappa_\Omega V_\varepsilon(x)$ is with $\mu(z) = \bar{\mu}_0 + \bar{\mu}_1 z$, but since there is no constraint on $\bar{y}_k$, we can set $\bar{\mu}_0 = 0$
  - It is easy to see that $|W(x)| \leq \bar{\mu}_1 H(x)$, so a simple choice is to have $\bar{\mu}_1 = 1$ and $\varepsilon_0 = 0.05$
  - Finally, we can set $\kappa_\Omega = 50$, while the last parameters δ and δ' depend on the *input and output constraints*

$$V_\varepsilon(x) = H(x) + \varepsilon W(x)$$

$$W(x) = \frac{1}{2}\int_0^\ell \mu(z) x^{\mathrm{T}}(z) P_1^{-1} x(z)\,\mathrm{d}z$$

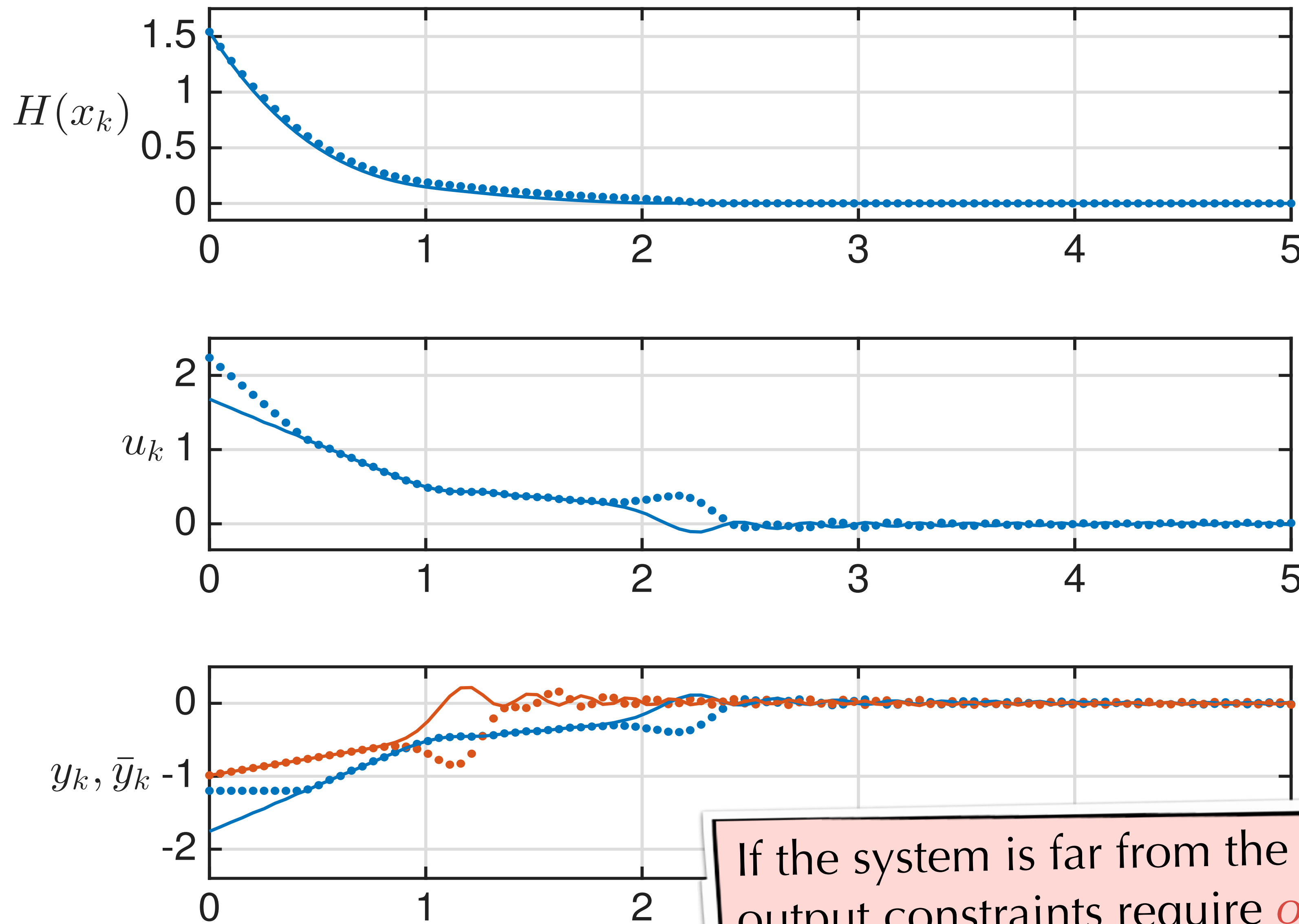
Numerical example

- Transient response with *only input constraints* and $u_M = -u_m = 1.2$
  - An admissible choice for δ' is 0.05, while we let $\delta = 0.045$
  - We have verified that values of N larger than $25 \div 30$ do not give a substantial improvement



Numerical example

- Transient response with only output constraints and $y_M = -y_m = 1.2$
  - The previous choice for the parameters is still valid
  - If we would have imposed an *input constraint* e.g. with $u_M = 2.0$, the optimization problem would **not** have been feasible



If the system is far from the equilibrium, input and output constraints require *opposite strategies* to be met. Thus, it is preferable to rely on *soft constraints*

Conclusions & Future activities

- In this talk, a discrete-time approximation of a linear port-Hamiltonian BCS is the reference model within an *MPC scheme*
  - The control law is obtained by solving an optimization problem where the plant dynamics are imposed as constraints
  - To simplify the solution, a *parametrization* of the state trajectory is derived
  - The controller is able to *handle constraints* on the control input, as well as on the collocated and non-collocated outputs

- Future research will focus on
  - The introduction of *soft constraints* to improve feasibility properties
  - On the extension to more challenging scenarios, such as output tracking or *nonlinear BCSs*
  - On the *stable coupling* between the discrete-time MPC controller and the continuous-time plant

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Alessandro Macchelli

Department of Electrical, Electronic, and Information Engineering (DEI)

alessandro.macchelli@unibo.it

http://www.unibo.it