Digital model predictive control for linear distributed parameter systems

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Introduction

- We consider digital model predictive control (MPC) for linear distributed parameter systems by utilizing Cayley-Tustin time discretization.
- Thanks to the Cayley transform, the discrete-time model predictive controller corresponds to a sampled version of continuous-time one.
- Hence, we get a digital controller for continuous-time systems by solving a discrete-time model predictive control problem.
- After introducing the Cayley transform and discrete-time MPC, we will see how the Cayley transform can be computed for boundary-controlled PDEs.
- A 1D heat equation with boundary control and measurement is considered as an illustrative example of Cayley transform and digital MPC.

Cayley-Tustin time discretization

Discrete-time MPC

Boundary control systems

Example: 1D heat equation

Conclusions

Tustin time discretization

- Consider a linear system in state-space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

- Given a time discretization parameter $h > 0$, Tustin discretization of the system is given by

$$\begin{aligned} \frac{x(jh) - x((j-i)h)}{h} &\approx A\frac{x(jh) + x((j-1)h)}{2} + Bu(jh), \\ y(jh) &\approx C\frac{x(jh) + x((j-1)h)}{2} + Du(jh). \end{aligned}$$

Cayley transform

- Rewriting the discretized system gives discrete-time dynamics

$$x(k) = A_\delta x(k-1) + B_\delta u(k),$$
$$y(k) = C_\delta x(k-1) + D_\delta u(k),$$

where $\delta = 2/h$, $u(k)/\sqrt{h}$ approximates $u(jh)$, and $y(k)/\sqrt{h}$ approximates $y(jh)$.

- The mapping from the continuous-time system to the discrete-time system is called the Cayley transform, which is given, for $\delta > 0$ sufficiently large, by

$$\begin{bmatrix} A_\delta & B_\delta \\ C_\delta & D_\delta \end{bmatrix} = \begin{bmatrix} -I + 2\delta(\delta - A)^{-1} & \sqrt{2\delta}(\delta - A)^{-1}B \\ \sqrt{2\delta}C(\delta - A)^{-1} & G(\delta) \end{bmatrix},$$

where $G(s) = C(s - A)^{-1}B + D$ is the transfer function of (A, B, C, D).

Some properties of Cayley transform

- The discrete- and continuous-time transfer functions are connected by

$$G_d(z) = zC_d(I - zA_d)^{-1}B_d + D_d = G\left(\frac{1-z}{1+z}\delta\right).$$

- It is a convergent time discretization scheme in the input/output sense [HM07].
- It preservers several system-theoretic properties between discrete- and continuous-time [CO97].
- The above includes optimizability and optimal cost. In particular, the discrete-time solution coincides with sampled continuous-time solution [OC04].

Infinite-horizon optimal control problem

- We consider quadratic optimal control problems of the form

$$
\begin{aligned}
\min_u \sum_{k=k_0}^{\infty} & y_k^T Q y_k + u_k^T R u_k + (u_k - u_{k-1})^T V (u_k - u_{k-1}), \\
\text{s.t. } x_k &= A_\delta x_{k-1} + B_\delta u_k, \\
y_k &= C_\delta x_{k-1} + D_\delta u_k, \\
u_{\min} &\leq u_k \leq u_{\max},
\end{aligned}
$$

  where $Q, R > 0, V \geq 0$ and $u_{\min} < 0 < u_{\max}$.
- Additional affine input and output constraints could be considered as well.

Truncation to finite horizon

- The infinite-horizon problem on $k \in [k_0, \infty)$ is truncated to a finite-horizon problem on $k \in [k_0, k_0 + N_h)$ for prediction horizon N_h.
- In order to do this, we assume $u_k = 0$ for $k \geq k_0 + N_h$, and introduce a terminal penalty $\langle x_{k+N_h-1}, \Pi x_{k+N_h-1} \rangle$.
- Π is the unique, self-adjoint solution of the discrete-time Lyapunov equation

$$A_\delta^* \Pi A_\delta - \Pi = -C_\delta^* Q C_\delta,$$

which is given by $\Pi x = \lim_{n\to\infty} \sum_{\ell=0}^{n} (A_\delta^*)^\ell C_\delta^* Q C_\delta A_\delta^\ell x$.

- Due to the Cayley transform, Π coincides with the solution of the continuous-time Lyapunov equation $A^*\Pi + \Pi A = -C^* Q C$ on $\mathcal{D}(A)$.

Formulation of quadratic minimization problem

- Denoting $U_k = (u_{k+j})_{j=0}^{N_h-1}$, we get a quadratic problem

$$\min_{U_k} U_k^T H U_k + 2U_k^T (P x_{k-1} - V_0 u_{k-1}),$$

$$\text{s.t.} \begin{bmatrix} I_{mN_h} \\ -I_{mN_h} \end{bmatrix} U_k \leq \begin{bmatrix} U_{\max} \\ -U_{\min} \end{bmatrix},$$

where $H = (h_{ij}) \in \mathbb{R}^{mN_h \times mN_h}$ satisfies $H > 0$ with

$$h_{ij} = \begin{cases} D_\delta^* Q D_\delta + B_\delta^* \Pi B_\delta + R + 2V, & i = j \\ D_\delta^* Q C_\delta A_\delta^{i-j-1} B_\delta + B_\delta^* \Pi A_\delta^{i-j} B_\delta - V, & i = j+1 \\ D_\delta^* Q C_\delta A_\delta^{i-j-1} B_\delta + B_\delta^* \Pi A_\delta^{i-j} B_\delta, & i > j+1 \\ h_{ji}^*, & i < j \end{cases} \quad \text{and } V_0 = \begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Formulation of quadratic minimization problem

- Moreover, P in the quadratic problem is given by

$$P = \begin{bmatrix} D_\delta^* Q C_\delta + B_\delta^* \Pi A_\delta \\ D_\delta^* Q C_\delta A_\delta + B_\delta^* \Pi A_\delta^2 \\ \vdots \\ D_\delta^* Q C_\delta A_\delta^{N_h-2} + B_\delta^* \Pi A_\delta^{N_h-1} \end{bmatrix}.$$

- Hence, in order to formulate the quadratic control problem, we need both $(A_\delta, B_\delta, C_\delta, D_\delta)$ and $(A_\delta^*, B_\delta^*, C_\delta^*, D_\delta^*)$, which also enable us to approximate Π.
- The terminal penalty Π can be omitted, if the optimal control problem has the turnpike property [TZ25].

Model predictive control algorithm

- At step k, the MPC algorithm solves U_k^* from

$$\min_{U_k} U_k^T H U_k + 2U_k^T(Px_{k-1} - V_0 u_{k-1}),$$

  and sets the first component u_k^* of U_k^* as input to the system.
- The state and output get updated according to the dynamics

$$\begin{aligned} x_k &= A_\delta x_{k-1} + B_\delta u_k^*, \\ y_k &= C_\delta x_{k-1} + D_\delta u_k^*, \end{aligned}$$

  and U_k^* is solved again for $k \to k+1$.
- Alternatively, $u_k^*/\sqrt{h}$ can be set as the continuous-time input for $t \in [kh, (k+1)h)$, which gives a digital model predictive controller.

On convergence and underlying assumptions

- If the system is (approximately) controllable and observable, the MPC closed-loop converges asymptotically to $u_k = y_k = 0$ as $k \to \infty$.
- Existence of Π requires that (A_δ, C_δ) is infinite-time admissible.
- If A_δ has finitely many unstable eigenvalues, one can impose a terminal constraint to force the unstable modes to zero by the end of the prediction horizon.
- For unstable systems, one can combine MPC with state feedback $u_k^f = K_\delta x_{k-1}$, and solve MPC for the prestabilized system $x_k = (A_\delta + B_\delta K_\delta)x_{k-1} + B_\delta u_k$, where the input constraint becomes $u_{\min} \leq K_\delta x_{k-1} + u_k \leq u_{\max}$.
- Alternatively, $u_k = K_\delta x_{k-1}$ can be used as terminal assumption for $k \geq k_0 + N_h$, in which case the terminal penalty Π is solved from

$$(A_\delta + B_\delta K_\delta)^* \Pi (A_\delta + B_\delta K_\delta) - \Pi = -(C_\delta + D_\delta K_\delta)^* Q (C_\delta + D_\delta K_\delta).$$

Boundary control systems

- A boundary control system is of the form

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), \\ \mathcal{B}x(t) &= u(t), \\ y(t) &= \mathcal{C}x(t). \end{aligned}$$

- The triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a boundary control system on (Hilbert spaces) (X, U, Y) if
  - $\mathcal{A}|_{\mathcal{N}(\mathcal{B})} =: A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X,
  - $\mathcal{B} \in \mathcal{L}(\mathcal{D}(\mathcal{A}), U)$ has a right inverse $B_R \in \mathcal{L}(U, X)$,
  - $\mathcal{C} \in \mathcal{L}(\mathcal{D}(\mathcal{A}), Y)$.
- We will denote by X_{-1} the completion of X with respect to the norm $\|(s - A)^{-1} \cdot \|_X$, for any $s \in \rho(A)$, and by A_{-1} the unique extension of A to X.

Boundary control systems

- There exists an operator $B \in \mathcal{L}(U, X_{-1})$ such that a boundary control system can be written equivalently as

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t),$$
$$y(t) = \mathcal{C}x(t).$$

- In fact, B is connected to $\mathcal{B}$ through $\mathcal{B}(s - A_{-1})^{-1}B = I$ for every $s \in \rho(A)$, so that $B_R = (s - A_{-1})^{-1}B$ is then a right inverse of $\mathcal{B}$.
- We will look next into computing the Cayley transform for boundary control systems, which does not explicitly require the operator B.

Cayley transform for boundary control systems

- In order to compute $A_\delta = -I + (\delta - A)^{-1}$, taking the Laplace transform of $\dot{x}(t) = Ax(t)$ gives a boundary value problem $sx(s) - x(0) = Ax(s)$.
- Fixing $s = \delta \in \rho(A)$, the solution to the boundary value problem is $x(\delta) = (\delta - A)^{-1}x(0)$, where $x \in \mathcal{D}(A)$ gives the boundary conditions.
- Once we have computed $(\delta - A)^{-1}$, $C_\delta = \sqrt{2\delta}\mathcal{C}(\delta - A)^{-1}$ is obtained by simply applying the boundary trace $\mathcal{C}$ to $(\delta - A)^{-1}$ and multiplying by $\sqrt{2\delta}$.
- This process would be the same for any PDE, whereas in the computation of B_δ and D_δ we get to utilize the boundary control system structure.

Cayley transform for boundary control systems

- In order to compute $B_\delta = \sqrt{2\delta}(\delta - A_{-1})^{-1}B$, we have by [TW09, Rem. 10.1.5] that $f = (\delta - A_{-1})^{-1}Bv$ is the unique solution to the boundary value problem

$$\delta f = \mathcal{A} f, \qquad \mathcal{B} f = v.$$

- Hence, $B_\delta v = \sqrt{2\delta} f$.
- In order to compute $D_\delta = G(\delta)$, we have by [CM03, Thm 2.9] that the transfer function of a boundary control system is $G(s) = \mathcal{C}(s - A_{-1})^{-1}B$.
- Since we solved $f = (\delta - A_{-1})^{-1}Bv$, we have $D_\delta v = G(\delta)v = \mathcal{C} f$.

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Example: 1D heat equation

Conclusions

Example: 1D heat equation

- Consider a 1D diffusion-reaction PDE on $\zeta \in [0,1]$ as

$$\begin{aligned} \dot{x}(t,\zeta) &= x''(t,\zeta) + rx(t,\zeta), \\ x'(t,0) &= 0, \\ x(t,1) &= u(t), \\ y(t) &= x(t,0). \end{aligned}$$

- This can be written as a boundary control system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ on $(L^2(0,1;\mathbb{R}), \mathbb{R}, \mathbb{R})$ by defining $\mathcal{A}x = x'' + rx$ with domain $\mathcal{D}(\mathcal{A}) = \{x \in H^2(0,1;\mathbb{R}) : x'(0) = 0\}$, $\mathcal{B}x = x(1)$, and $\mathcal{C}x = x(0)$.
- We assume that $r < (\pi/2)^2$ so that $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ generates a stable C_0-semigroup.

Cayley transform for heat equation

- Let us compute the resolvent by solving $sx(s,\zeta) - x(0,\zeta) = Ax(s,\zeta)$, i.e.,

$$x''(s,\zeta) = (s-r)x(s,\zeta) - x(0,\zeta).$$

- Rewriting the above for $X = (x, x')^T$, we get the first-order ODE

$$X'(s,\zeta) = \begin{bmatrix} 0 & 1 \\ s-r & 0 \end{bmatrix} X(s,\zeta) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(0,\zeta) =: \bar{A}X(s,\zeta) - \bar{B}(0,\zeta)$$

- The solution to the first-order ODE is

$$X(s,\zeta) = e^{\bar{A}\zeta}X(s,0) - \int_0^\zeta e^{\bar{A}(\zeta-\eta)}\bar{B}x(0,\eta)d\eta.$$

- Using the boundary condition $x'(0) = 0$ of $\mathcal{D}(A)$ and

$$e^{\bar{A}\zeta} = \begin{bmatrix} \cosh(\sqrt{s-r}) & \sinh(\sqrt{s-r})/\sqrt{s-r} \\ \sqrt{s-r}\sinh(\sqrt{s-r}) & \cosh(\sqrt{s-r}) \end{bmatrix},$$

we can solve

$$x(s,\zeta) = \cosh(\sqrt{s-r}\zeta)x(s,0) - \int_0^\zeta \frac{\sinh(\sqrt{s-r}(\zeta-\eta))}{\sqrt{s-r}} x(0,\eta)d\eta.$$

- Using the second boundary condition $x(1) = 0$ of $\mathcal{D}(A)$, we can solve

$$x(s,0) = \frac{1}{\cosh(\sqrt{s-r})} \int_0^1 \frac{\sinh(\sqrt{s-r}(1-\eta))}{\sqrt{s-r}} x(0,\eta)d\eta.$$

Cayley transform for heat equation: A_δ, C_δ

- Thus, inserting $s = \delta$ we have $x(\delta) = (\delta - A)^{-1}x(0)$ as

$$x(\delta,\zeta) = \frac{\cosh(\sqrt{\delta - r}\zeta)}{\cosh(\sqrt{\delta - r})}\int_0^1 \frac{\sinh(\sqrt{\delta - r}(1-\eta))}{\sqrt{\delta - r}}x(0,\eta)d\eta$$

$$-\int_0^\zeta \frac{\sinh(\sqrt{\delta - r}(\zeta-\eta))}{\sqrt{\delta - r}}x(0,\eta)d\eta.$$

- For C_δ, we apply $\sqrt{2\delta}\mathcal{C}x = \sqrt{2\delta}x(0)$ to $x(\delta,\cdot)$, which gives

$$C_\delta x(0,\cdot) = \sqrt{2\delta}\frac{\cosh(\sqrt{\delta - r}\zeta)}{\cosh(\sqrt{\delta - r})}\int_0^1 \frac{\sinh(\sqrt{\delta - r}(1-\eta))}{\sqrt{\delta - r}}x(0,\eta)d\eta.$$

Cayley transform for heat equation: B_δ, D_δ

- We now solve $f = (\delta - A_{-1})^{-1} B v$ from $\delta f = \mathcal{A} f, \mathcal{B} f = v$, that is,

$$\delta f = f'' + r f, \qquad f'(0) = 0, \qquad f(1) = v.$$

- The solution is given by (`dsolve` in MATLAB)

$$f = \frac{\cosh(\sqrt{\delta - r}\zeta)}{\cosh(\sqrt{\delta - r})} v.$$

- Now, $B_\delta v = \sqrt{2\delta} f$ and $D_\delta v = \mathcal{C} f$. Setting $v = 1$, we get

$$B_\delta = \sqrt{2\delta}\frac{\cosh(\sqrt{\delta - r}\zeta)}{\cosh(\sqrt{\delta - r})}, \qquad D_\delta = \frac{1}{\cosh(\sqrt{\delta - r})}.$$

The terminal penalty Π

- We already saw the expression $\Pi x = \lim_{n\to\infty} \sum_{\ell=0}^{n} (A_\delta^*)^\ell C_\delta^* Q C_\delta A_\delta^\ell x$, so we could approximate Π by truncating the series expression to some finite $n = N$.
- Alternatively, we van utilize the Riesz spectral property of A and solve Π from the continuous-time equation $A^*\Pi + \Pi A = -C^*QC$ on $\mathcal{D}(A)$, where $C = \mathcal{C}|_{\mathcal{D}(A)}$.
- Applying the continuous-time equation to an eigenfunction ϕ_k of A, and using $A\phi_k = \lambda_k \phi_k$ for the corresponding eigenvalue λ_k, we get

$$\Pi\phi_k = (-\lambda_k - A^*)^{-1} C^* Q C \phi = (C(-\bar{\lambda}_k - A)^{-1})^* Q C \phi_k,$$

$$\Rightarrow \quad \Pi x = \sum_k \langle x, \tilde{\phi} \rangle (C(-\bar{\lambda}_k - A)^{-1})^* Q C \phi_k.$$

- This is a general formula for Π for any Riesz spectral A.

Cayley transform for Riesz spectral systems

- For Riesz spectral systems, the Cayley transform has spectral representation

$$A_\delta x = \sum_k \frac{\delta + \lambda_k}{\delta - \lambda_k} \langle x, \tilde{\phi} \rangle \phi_k, \qquad B_\delta = \sqrt{2\delta} \sum_k \frac{B^* \tilde{\phi}_k}{\delta - \bar{\lambda}_k} \phi_k,$$

$$C_\delta x = \sqrt{2\delta} \sum_k \frac{\mathcal{C}\phi_k}{\delta - \lambda_k} \langle x, \tilde{\phi} \rangle, \qquad D_\delta = \sum_k \frac{\mathcal{C}\phi_k B_k^* \tilde{\phi}_k}{\delta - \bar{\lambda}_k}.$$

- For boundary control systems, B^* can be computed from

$$\langle \mathcal{A} x_1, x_2 \rangle_X = \langle x_1, A^* x_2 \rangle_X + \langle \mathcal{B} x_1, B^* x_2 \rangle_U, \quad x_1 \in \mathcal{D}(\mathcal{A}), x_2 \in \mathcal{D}(A^*),$$

without knowing B explicitly.

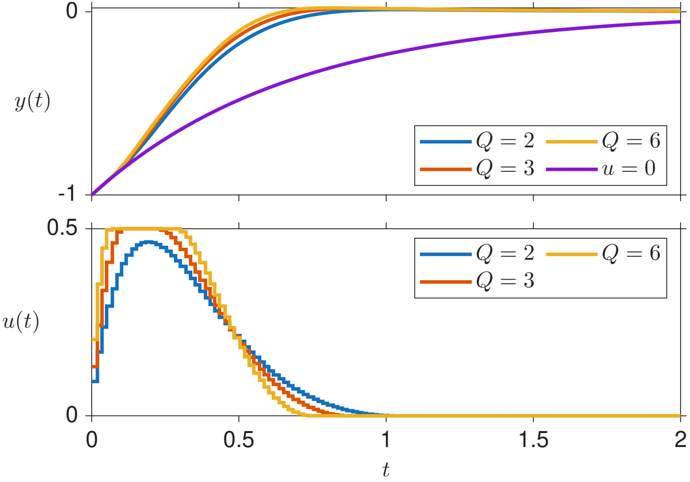
Simulation example

- Let us (approximately) solve optimal controls for the heat equation from

$$\min_{u} \sum_{k=0}^{\infty} y_k^T Q y_k + u_k^T R u_k + (u_k - u_{k-1})^T V (u_k - u_{k-1}),$$

  with input constraints $0 \leq u_k \leq 0.5$ and weights $R = 0, V = 50$, and $Q = 2, 3, 6$.
- We use MPC with horizon $H_h = 10$, and set $u_{-1} = 0$.
- The sampling time is taken as $h = 1/60$, and at every sampling instant $t = kh$, we solve u_k^* from the MPC problem, and set $u(t) = u_k^*/\sqrt{h}$ for $t \in [kh, (k+1)h)$.
- The initial condition for the heat equation is taken as $x = -\cos\left(\frac{\pi}{2}\zeta\right)$.

Simulation results



Remarks on simulation results

- The digital MPC improves the transient response of the stable heat equation.
- The output gets steered to zero (slightly) faster by increasing the weight Q.
- Alternatively, the output performance could be prioritized by keeping Q fixed and decreasing the penalty V on the change of input.
- Note that we can take $R = 0$ when $V > 0$, which may be a better input penalty in practice (e.g., considering potential actuator limitations).
- Input constraints can be easily incorporated into the MPC design.
- Other (e.g., output) constraints can be incorporated as well, but these may lead to feasibility issues when solving the quadratic minimization problem for the controls.

Concluding remarks

- We looked into digital MPC for linear systems by utilizing the Cayley transform.
- The disrete-time MPC problem can be solved efficiently by existing algorithms.
- For simple PDEs, the Cayley transform can be computed in closed form.
- For more general PDEs, numerical methods should be employed to approximately compute the Cayley transform (and particularly the resolvent operator).
- In the example of the 1D heat equation, the digital MPC improved transient response while satisfying input constraints.
- Other types of actuator limitations can be also taken into account in MPC, e.g. avoiding sudden changes in the input by penalizing the change of input.

Thank you!

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