Data-driven reduced-order PDE control using DMDc

Recent Trends in Control and Estimation of Distributed Parameter Systems

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with acknowledgment to: Paulo Henrique Biazetto

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Thoughts on data in control systems

Increasing role of data-centric methods in science / engineering / industry due to

- methodological advances in statistics, optimization, and machine learning (ML);
- unprecedented availability of brute force: deluge of data and computational power;
- ... and frenzy surrounding big data and ML.

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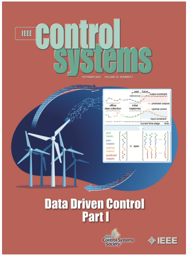
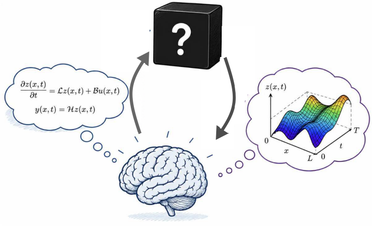


Scientific landscape

long and rich history (auto-tuning, system identification, adaptive control, RL, ...) and vast and fragmented research landscape



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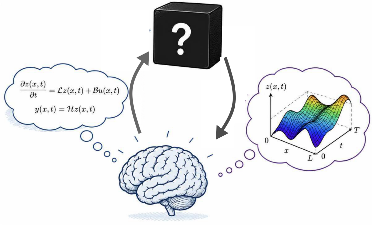

Direct data-driven control
Data are used directly to design the control system.

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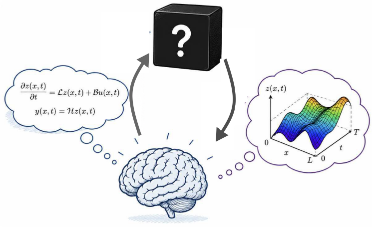


Direct data-driven control

Data are used directly to design the control system.

Indirect data-driven control

Data are first used to identify a dynamical model of the system.

Today's menu

1. $\{$*Linear PDE control*$\} \cap \{$*data-driven modeling*$\}$: DMDc for PDEs

2. From boundary-controlled PDEs to finite-dimensional spectral data

3. Reduced-order identification: DMDc + observer-based control

4. Towards nonlinear PDEs: Koopman/EDMD extensions

Foundational papers

P. J. Schmid and J. Sesterhenn, “Dynamic mode decomposition of numerical and experimental data,” presented at the *61st Annual Meeting of the APS Division of Fluid Dynamics*, San Antonio, Texas, November 25, 2008.

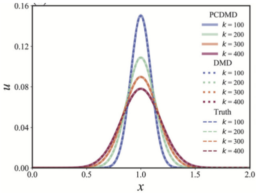
C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson, “Spectral analysis of nonlinear flows,” *Journal of Fluid Mechanics*, vol. 641, pp. 115–127, 2009.

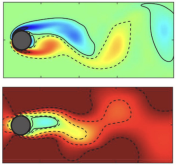
P. J. Schmid, “Dynamic mode decomposition of numerical and experimental data,” *Journal of Fluid Mechanics*, vol. 656, pp. 5–28, 2010.

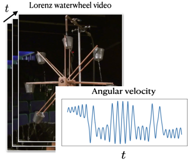
J. H. Tu, C. W. Rowley, D. M. Luchtenburg, S. L. Brunton, and J. N. Kutz, “On dynamic mode decomposition: Theory and applications,” *Journal of Computational Dynamics*, vol. 1, no. 2, pp. 391–421, 2014.

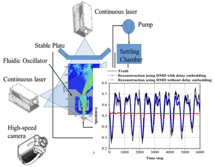
J. L. Proctor, S. L. Brunton, and J. N. Kutz, “Dynamic mode decomposition with control,” *SIAM Journal on Applied Dynamical Systems*, vol. 15, no. 1, pp. 142–161, 2016.

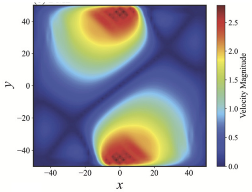
Works very well accross case studies

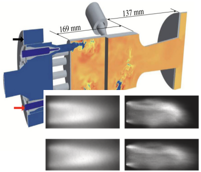


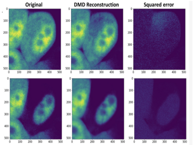


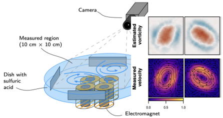












... especially when first-principles models are hard or impossible to get

Problem Formulation

We consider linear boundary control systems of the form

$$\begin{aligned} \dot{z}(t) &= \mathfrak{A}z(t), \qquad z(0) = z_0, \\ \mathfrak{B}z(t) &= u(t), \end{aligned}$$

where $\mathfrak{A} \in \mathscr{L}(\mathscr{Z}, \mathscr{H})$, and $\mathfrak{B} \in \mathscr{L}(\mathscr{Z}, \mathscr{U})$.

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Assumption

The operators $\mathfrak{A}$ and $\mathfrak{B}$ satisfy

1. $\mathfrak{B} \in \mathscr{L}(\mathscr{Z}, \mathscr{U})$ *is surjective, and its kernel is dense in* $\mathscr{H}$.
2. *There exists* $\mu \in \mathbb{R}$ *such that* $\ker(\mu I - \mathfrak{A}) \cap \ker \mathfrak{B} = \{0\}$, *and the operator* $\mu I - \mathfrak{A} : \mathscr{Z} \to \mathscr{H}$ *is surjective.*
3. *For every initial condition* $z_0 \in \mathscr{Z}$ *satisfying the compatibility condition* $\mathfrak{B}z_0 = 0$, *the homogeneous boundary-value problem admits a unique solution* $z \in C^1([0, T]; \mathscr{H}) \cap C([0, T]; \mathscr{Z})$, *which depends continuously on* z_0.

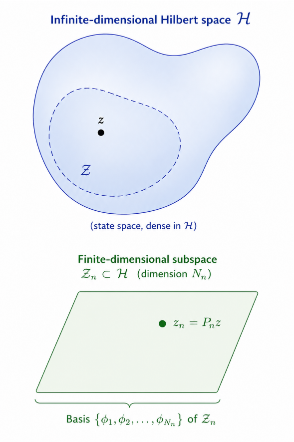
Finite-dimensional approximation

Choose a finite-dimensional subspace

$$\mathscr{Z}_n = \mathrm{span}\{\text{basis functions}\},\ N_n = \dim(\mathscr{Z}_n)$$

and project the PDE state onto it:

$$z_n(t) = P_n z(t).$$



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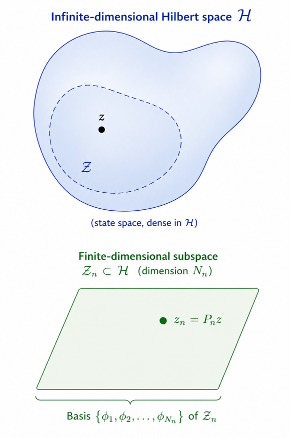
$$\dot{z}_n(t) = A_n z_n(t) + B_n u(t), \qquad z_n(0) = P_n z_0.$$



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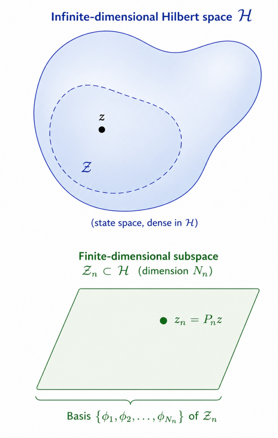
As the approximation order increases,

$$\sup_{t\in[0,T]} \|z_n(t) - z(t)\|_{\mathscr{H}} \to 0.$$



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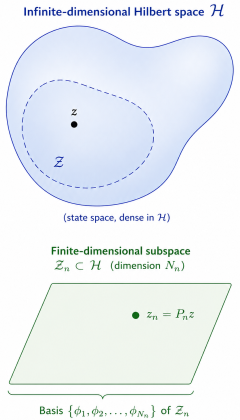


Goal

Use data from this finite-dimensional system to identify a DMDc model.

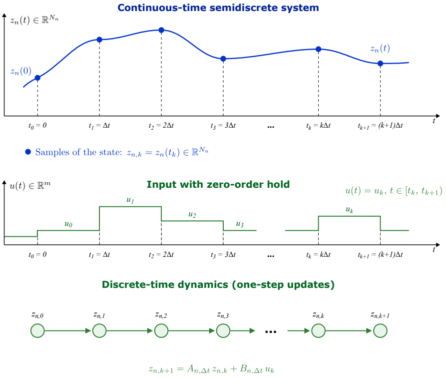
Sampled semidiscrete dynamics

Start from the finite-dimensional model:

$$\dot{z}_n(t) = A_n z_n(t) + B_n u(t).$$

Sample the system at $t_k = k\Delta t$ and assume a zero-order hold:

$$u(t) = u_k, \qquad t \in [t_k, t_{k+1}).$$



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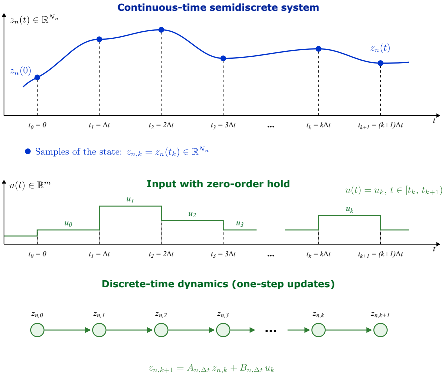


The exact one-step dynamics are

$$z_{n,k+1} = A_{n,\Delta t} z_{n,k} + B_{n,\Delta t} u_k.$$

This is the discrete-time model identified by DMDc.

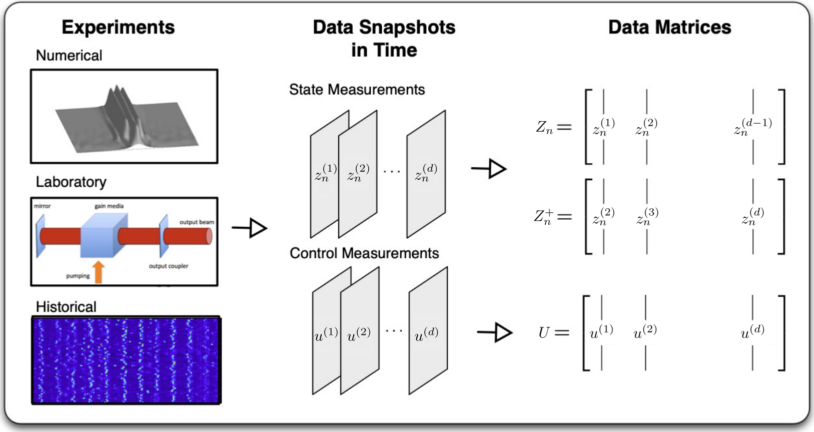
DMDc Identification

Suppose that d state–input–successor triplets

$$\left\{\left(z_n^{(j)}, u^{(j)}, z_n^{(j)+}\right)\right\}_{j=1}^{d}$$

are available, where

$$z_n^{(j)+} = F_n^{\Delta t}\left(z_n^{(j)}, u^{(j)}\right).$$



DMDc identification

Given the snapshot matrices, DMDc identifies the discrete-time model

$$z_{n,k+1} \approx A_{n,\Delta t} z_{n,k} + B_{n,\Delta t} u_k.$$

The matrices are obtained from

$$\min_{A,B} \left\| Z_n^+ - A Z_n - B U \right\|_F^2 .$$

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The resulting predictor is

$$\widehat{z}_{n,k+1} = \widehat{A}_{n,\Delta t} \widehat{z}_{n,k} + \widehat{B}_{n,\Delta t} u_k.$$

Reduced-order DMDc

Consider the singular value decomposition

$$Z_n = \Phi_n \Sigma_n V_n^\top.$$

Let $\Phi_{n,r} = \Phi_n(:, 1 : r) \in \mathbb{R}^{N_n \times r}$, $\Phi_{n,r}^\top \Phi_{n,r} = I_r$, where $r \ll N_n$.
The reduced coordinates are defined by

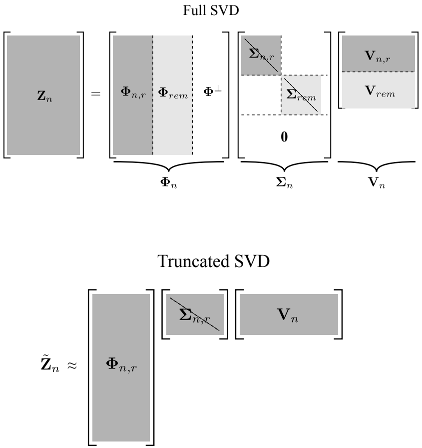
$$q_r = \Phi_{n,r}^\top z_n,$$

and the semidiscrete state is approximated by

$$z_n \approx \Phi_{n,r} q_r.$$

The projected snapshot matrices are

$$Q_n := \Phi_{n,r}^\top Z_n,$$
$$Q_n^+ := \Phi_{n,r}^\top Z_n^+.$$



Reduced-order DMDc

Reduced-order DMDc identifies the discrete-time matrices

$$\widehat{A}_{n,r,\Delta t} \in \mathbb{R}^{r\times r}, \qquad \widehat{B}_{n,r,\Delta t} \in \mathbb{R}^{r\times m},$$

by solving

$$\min_{A_r, B_r} \left\| Q_n^+ - A_r Q_n - B_r U \right\|_F^2 .$$

The corresponding minimum-norm solution is

$$\begin{bmatrix} \widehat{A}_{n,r,\Delta t} & \widehat{B}_{n,r,\Delta t} \end{bmatrix} = Q_n^+ \begin{bmatrix} Q_n \\ U \end{bmatrix}^\dagger .$$

Identification and projection errors

Theorem

Let $z(t)$ be the solution of the boundary control system, $z_n(t)$ the solution of its semidiscrete approximation, and $\widehat{q}_{n,r}(t)$ the solution of the identified continuous-time reduced-order model.

Assume that the columns of $\Phi_{n,r}$ are orthonormal with respect to the $\mathscr{H}$-inner product, and define

$$q_{n,r}(t) = \Phi_{n,r}^* z_n(t).$$

Then

$$\|z(t) - \Phi_{n,r}\widehat{q}_{n,r}(t)\|_{\mathscr{H}} \leq \underbrace{\|z(t) - z_n(t)\|_{\mathscr{H}}}_{\text{spatial discretization error}} + \underbrace{\|z_n^{\perp}(t)\|_{\mathscr{H}}}_{\text{projection / reduction error}} + \underbrace{\|q_{n,r}(t) - \widehat{q}_{n,r}(t)\|}_{\text{identification / prediction error}} .$$

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Importance for control

Accurate identification of the reduced model is not sufficient by itself. Closed-loop guarantees must also account for the spatial discretization and the dynamics neglected by the reduced basis.

Application: reaction–diffusion equation

PDE formulation
Consider the following PDE

$$\begin{aligned} z_t(t,x) &= z_{xx}(t,x) + \lambda z(t,x), \\ z_x(t,0) &= 0, \\ z(t,1) &= u(t), \\ z(0,x) &= z_0(x). \end{aligned}$$

where

$$x \in (0,1), \qquad t \geq 0$$

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\end{aligned}
$$

where

$$x \in (0,1), \qquad t \geq 0$$

Abstract form
Let

$$
\begin{aligned}
\mathscr{H} &= L^2(0,1),\\
\mathscr{Z} &= \{z \in H^2(0,1) : z_x(0) = 0\}.
\end{aligned}
$$

Define

$$\mathfrak{A}z = z_{xx} + \lambda z, \qquad \mathfrak{B}z = z(1)$$

Then, the PDE system can be rewriten as

$$\dot{z}(t) = \mathfrak{A}z(t), \qquad \mathfrak{B}z(t) = u(t).$$

Spectral approximation

Spectral representation

Consider the eigenfunctions ϕ_k:

$$\phi_k(x) = \sqrt{2}\cos(\nu_k x), \quad \nu_k = \left(k + \frac{1}{2}\right)\pi.$$

The first $n+1$ modes are

$$z_n(t,x) = \sum_{k=0}^{n} z_k(t)\phi_k(x),$$

where

$$z_k(t) = \langle z(t,\cdot), \phi_k \rangle_{L^2} \,.$$

The modal components can be collected in

$$Z_n(t) = \begin{bmatrix} z_0(t) & \cdots & z_n(t) \end{bmatrix}^\top.$$

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Modal dynamics

Projecting the PDE onto each eigenfunction gives

$$\dot{z}_k(t) = \left(\lambda - \nu_k^2\right) z_k(t) + \beta_k u(t),$$

Therefore,

$$\dot{Z}_n(t) = A_n Z_n(t) + B_n u(t),$$

with

$$A_n = \operatorname{diag}\left(\lambda - \nu_0^2, \ldots, \lambda - \nu_n^2\right),$$
$$B_n = \begin{bmatrix} \beta_0 & \cdots & \beta_n \end{bmatrix}^\top.$$

Simulation setup for reduced-order DMDc

PDE parameter:

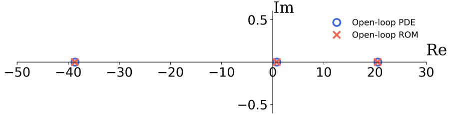
$$\lambda = 23.$$

Reference model for data generation:

$$N + 1 = 101 \quad \text{spectral modes.}$$

Training data:

200 simulations with different initial conditions and control inputs.
The control input was varied in amplitude.



DMDc identification

The snapshot data were used to construct the reduced basis and identify the matrices

$$\widehat{A}_{n,r,\Delta t}, \qquad \widehat{B}_{n,r,\Delta t}.$$

Reduced-order DMDc output feedback

The controller is implemented through a reduced-order observer:

$$\dot{q}_r(t) = \widehat{A}_r q_r(t) + \widehat{B}_r u(t) + L_r \left(y(t) - \widehat{C}_r q_r(t) \right).$$

The boundary input is computed from the estimated reduced state:

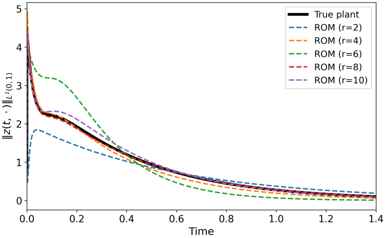
$$u(t) = K_r \widehat{q}_r(t).$$

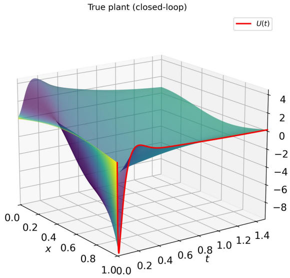
Data-driven design

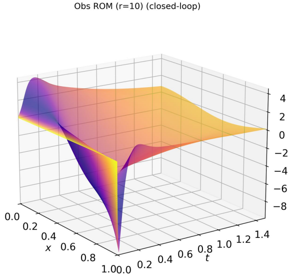
The gains K_r and L_r are designed only from the identified reduced-order DMDc model:

$$(\widehat{A}_r, \widehat{B}_r, \widehat{C}_r).$$

Simulation results







Application: Stabilization of Thermoacoustic Instabilities in a Rijke Tube

The system is written as

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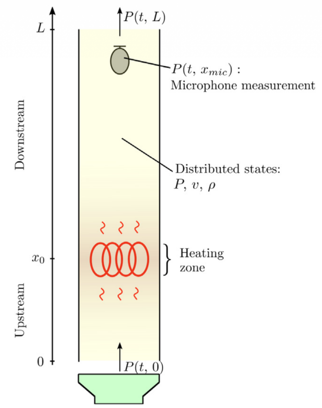

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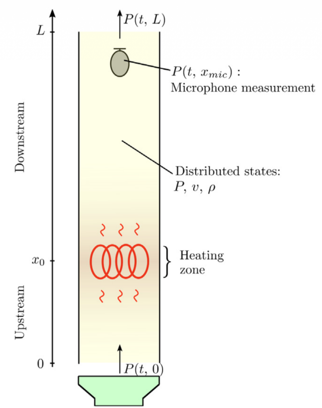
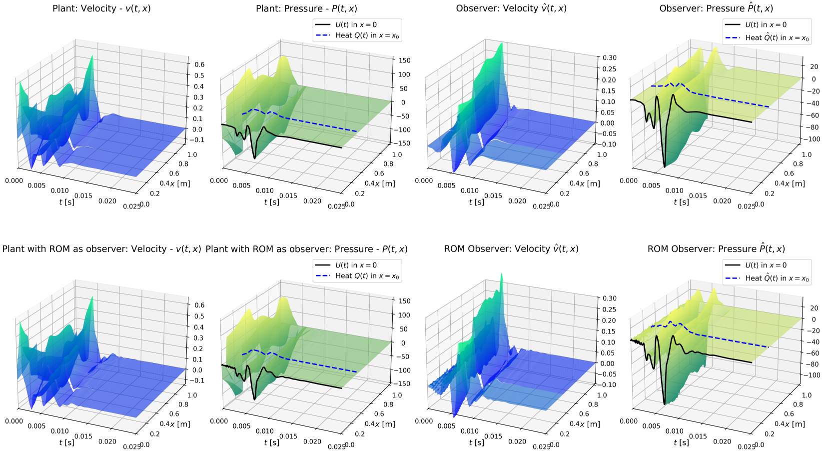
The observer-based backstepping controller design is computationally too complex for real-time implementation.



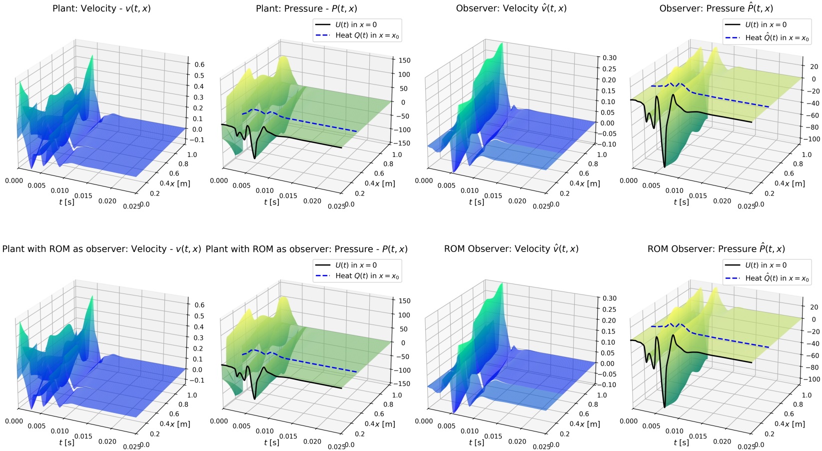
Simulation results

PDE with 601 states, ROM with N = 40 spectral modes.



Simulation results

PDE with 601 states, ROM with N = 140 spectral modes.



Modes	Speedup	$\|\mathbf{e_p}\|_\infty$	$\|\mathbf{e_p}(\mathbf{T})\|$	$\|\mathbf{e_o}\|_\infty$	$\|\mathbf{e_o}(\mathbf{T})\|$
40	28.91×	8.354426	0.049980	13.482175	0.058172
140	5.01×	0.519396	0.001949	0.856135	0.001856
240	1.60×	0.516804	0.000156	0.516795	0.000156

Towards Nonlinear PDEs

For nonlinear PDEs:

$$\dot{z}(t) = \mathscr{A} z(t) + \mathscr{N}(z(t)) + \mathscr{B} u(t),$$
$$y(t) = \mathscr{C} z(t).$$

New difficulties appear:

- modes are coupled by nonlinear terms;
- low-order dynamics may depend on the unresolved tail;
- DMDc is no longer sufficient;

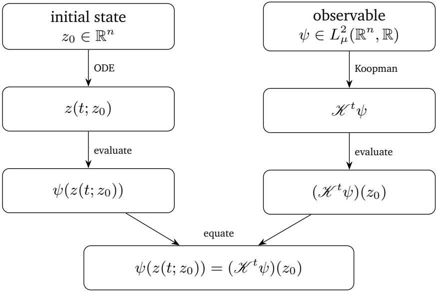
From DMDc to Koopman/EDMDc

Introduce observables:

$$\Psi(z_n) = \begin{bmatrix}\psi_1(z_n) & \cdots & \psi_N(z_n)\end{bmatrix}^\top .$$

Koopman theory allows to trade nonlinearity in exchange for infinite dimensionality based on the identity

$$(\mathscr{K}^t\psi)(z_n) = \psi(z_n(t; z_{n,0}))$$



Then identify a lifted linear model:

$$\Psi(Z_{k+1}) \approx A_\Psi \Psi(Z_k) + B_\Psi u_k.$$

Conclusions

Summary

- DMDc provides a systematic way to identify reduced-order models for control of distributed parameter systems.
- The reduced model enables the design of low-dimensional observer-based output-feedback controllers.
- Projection, model-reduction, and identification errors must be explicitly considered in the closed-loop analysis.

Future work

- **Input–output identification:** evelop methods that do not require full-state measurements.
- **Stability certificates:** derive less conservative conditions directly in terms of the data and identification residuals.
- **Data requirements:** characterize excitation, sample complexity, and robustness to noise.